XIV. Orthogonal Transform and Multiplexing

◉ 14-A Orthogonal and Dual Orthogonal

Any $M \times N$ discrete linear transform can be expressed as the matrix form:

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[M-1] \end{bmatrix}}_{\mathbf{Y}\,=} = \underbrace{\begin{bmatrix} \phi_0^*[0] & \phi_0^*[1] & \phi_0^*[2] & \cdots & \phi_0^*[N-1] \\ \phi_1^*[0] & \phi_1^*[1] & \phi_1^*[2] & \cdots & \phi_1^*[N-1] \\ \phi_2^*[0] & \phi_2^*[1] & \phi_2^*[2] & \cdots & \phi_2^*[N-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{M-1}^*[0] & \phi_{M-1}^*[1] & \phi_{M-1}^*[2] & \cdots & \phi_{M-1}^*[N-1] \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\mathbf{X}}$$

$$y[m] = \langle x[n], \phi_m[n] \rangle = \sum_{n=0}^{N-1} x[n]\phi_m^*[n]$$

inner product

Orthogonal: $\langle \phi_k[n], \phi_h[n] \rangle = \sum_{n=0}^{N-1} \phi_k[n] \phi_h^*[n] = 0$ **when** $\boldsymbol{k \neq h}$

orthogonal transforms 的例子：

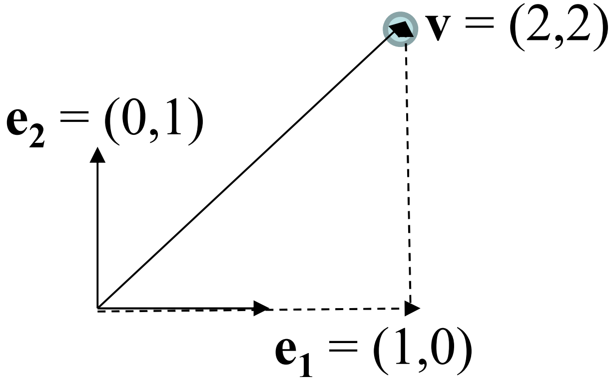
- discrete Fourier transform
- discrete cosine, sine, Hartley transforms
- Walsh Transform, Haar Transform
- discrete Legendre transform
- discrete orthogonal polynomial transforms

  Hahn, Meixner, Krawtchouk, Charlier

為什麼在信號處理上，我們經常用 orthogonal transform?

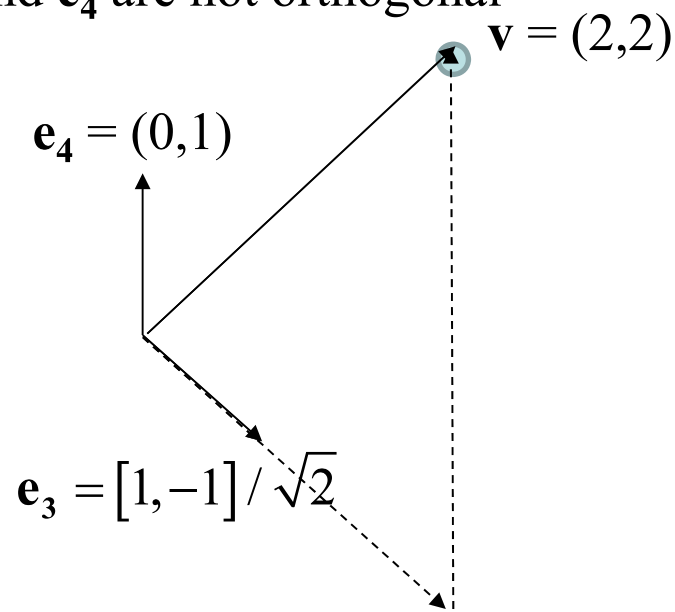
Orthogonal transform 最大的好處何在？

$\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal



$$\mathbf{v} = 2\mathbf{e}_1 + 2\mathbf{e}_2$$

$\mathbf{e}_3$ and $\mathbf{e}_4$ are not orthogonal



$$\mathbf{v} = 2\sqrt{2}\mathbf{e}_3 + 4\mathbf{e}_4$$

- If partial terms are used for reconstruction

for orthogonal case,

perfect reconstruction: $x[n] = \sum_{m=0}^{N-1} C_m^{-1} y[m]\phi_m[n]$

partial reconstruction: $x_K[n] = \sum_{m=0}^{K-1} C_m^{-1} y[m]\phi_m[n] \qquad K < N$

reconstruction error of partial reconstruction

$$\left\| x[n] - x_K[n] \right\|^2 = \sum_{n=0}^{N-1} \left\| \sum_{m=K}^{N-1} C_m^{-1} y[m]\phi_m[n] \right\|^2$$

$$= \sum_{n=0}^{N-1} \sum_{m=K}^{N-1} C_m^{-1} y[m]\phi_m[n] \sum_{m_1=K}^{N-1} C_{m_1}^{-1} y^*[m_1]\phi_{m_1}^*[n]$$

$$= \sum_{m=K}^{N-1} \sum_{m_1=K}^{N-1} C_m^{-1} y[m] C_{m_1}^{-1} y^*[m_1] \sum_{n=0}^{N-1} \phi_m[n]\phi_{m_1}^*[n]$$

$$= \sum_{m=K}^{N-1} \sum_{m_1=K}^{N-1} C_m^{-1} y[m] C_{m_1}^{-1} y^*[m_1] C_m \delta[m - m_1] = \sum_{m=K}^{N-1} C_m^{-1} \left| y[m] \right|^2$$

由於 $C_m^{-1} \left| y[m] \right|^2$ 一定是正的，可以保證 K 越大, reconstruction error 越小

For non-orthogonal case,

perfect reconstruction: $x[n] = \sum_{m=0}^{N-1} B[n,m]\, y[m]$ $\qquad$ $\mathbf{B} = \mathbf{A}^{-1}$

partial reconstruction: $x_K[n] = \sum_{m=0}^{K-1} B[n,m]\, y[m]$ $\qquad$ $K < N$

reconstruction error of partial reconstruction

$$\left\| x[n] - x_K[n] \right\|^2 = \sum_{n=0}^{N-1} \left\| \sum_{m=K}^{N-1} B[n,m]\, y[m] \right\|^2$$

$$= \sum_{n=0}^{N-1} \sum_{m=K}^{N-1} B[n,m]\, y[m] \sum_{m_1=K}^{N-1} B^*[n,m_1]\, y^*[m_1]$$

$$= \sum_{m=K}^{N-1} \sum_{m_1=K}^{N-1} y[m]\, y^*[m_1] \sum_{n=0}^{N-1} B[n,m]\, B^*[n,m_1]$$

由於 $y[m]y^*[m_1]\sum_{n=0}^{N-1} B[n,m]\, B^*[n,m_1]$ 不一定是正的，

無法保證 K 越大, reconstruction error 越小

◎ 14-B Frequency and Time Division Multiplexing

傳統 Digital Modulation and Multiplexing：使用 Fourier transform

- **Frequency-Division Multiplexing (FDM)**

$$z(t)=\sum_{n=0}^{N-1} X_n \exp\left(j2\pi f_n t\right) \qquad X_n = 0 \text{ or } 1$$

X_n can also be set to be -1 or 1

When (1) $t \in [0, T]$ (2) $f_n = n/T$

$$z(t)=\sum_{n=0}^{N-1} X_n \exp\left(j\frac{2\pi n t}{T}\right)$$

it becomes the **orthogonal frequency-division multiplexing (OFDM) in the continuous case.**

Furthermore, if the time-axis is also sampled

$$t = mT/N, \qquad m = 0, 1, 2, \ldots.., N-1$$

$t \in [0,T]$
sampling for t-axis

$$z\left(m\frac{T}{N}\right) = \sum_{n=0}^{N-1} X_n \exp\left(j\frac{2\pi nm}{N}\right)$$

(OFDM in the discrete case)

then the OFDM is equivalent to the transform matrix of the inverse discrete Fourier transform (IDFT), which is one of the discrete orthogonal transform.

Modulation: $$Y_m = z\left(m\frac{T}{N}\right) = \sum_{m=0}^{N-1} A[m,n]X_n$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & e^{j\frac{4\pi}{N}} & \cdots & e^{j\frac{2(N-1)\pi}{N}} \\ 1 & e^{j\frac{4\pi}{N}} & e^{j\frac{8\pi}{N}} & \cdots & e^{j\frac{4(N-1)\pi}{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\frac{2(N-1)\pi}{N}} & e^{j\frac{4(N-1)\pi}{N}} & \cdots & e^{j\frac{2(N-1)(N-1)\pi}{N}} \end{bmatrix}$$

Modulation: $$Y_m = \sum_{m=0}^{N-1} A[m,n] X_n$$

Demodulation: $$X_n = \frac{1}{N} \sum_{m=0}^{N-1} A^*[m,n] Y_m$$

Example: $N = 8$

$X_n = [1, 0, 1, 1, 0, 0, 1, 1]$ $\quad$ $(n = 0 \sim 7)$

- **Time-Division Multiplexing (TDM)**

$$z(0) = X_0, \quad z\left(\frac{T}{N}\right) = X_1, \quad z\left(2\frac{T}{N}\right) = X_2, \quad \cdots\cdots, \quad z\left((N-1)\frac{T}{N}\right) = X_{N-1}$$

$$y(m) = z\left(m\frac{T}{N}\right) = \sum_{m=0}^{N-1} A[m,n]X_n$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

(also a discrete orthogonal transform)

思考：

既然 time-division multiplexing 那麼簡單

那為什麼要使用 frequency-division multiplexing
和 orthogonal frequency-division multiplexing (OFDM)?

⦿ 14-C Code Division Multiple Access (CDMA)

除了 frequency-division multiplexing 和 time-division multiplexing，是否還有其他 multiplexing 的方式？

使用其他的 orthogonal transforms
即 code division multiple access (CDMA)

CDMA is an important topic in spread spectrum communication

參考資料

[1] M. A. Abu-Rgheff, *Introduction to CDMA Wireless Communications*, Academic, London, 2007

[2] 邱國書, 陳立民譯, "CDMA 展頻通訊原理", 五南, 台北, 2002.

CDMA 最常使用的 orthogonal transform 為 Walsh transform

$$\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\end{bmatrix}=\begin{bmatrix}1&1&1&1&1&1&1&1\\1&1&1&1&-1&-1&-1&-1\\1&1&-1&-1&-1&-1&1&1\\1&1&-1&-1&1&1&-1&-1\\1&-1&-1&1&1&-1&-1&1\\1&-1&-1&1&-1&1&1&-1\\1&-1&1&-1&-1&1&-1&1\\1&-1&1&-1&1&-1&1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\end{bmatrix}\begin{array}{l}\leftarrow \text{channel 1}\\\leftarrow \text{channel 2}\\\leftarrow \text{channel 3}\\\leftarrow \text{channel 4}\\\leftarrow \text{channel 5}\\\leftarrow \text{channel 6}\\\leftarrow \text{channel 7}\\\leftarrow \text{channel 8}\end{array}$$

channel 1

當有兩組人在同一個房間裡交談 (A 和B交談)，(C 和D交談) ，如何才能夠彼此不互相干擾？

(1) Different Time

(2) Different Tone

(3) Different Language

CDMA 分為：

(1) Orthogonal Type　　(2) Pseudorandom Sequence Type

Orthogonal Type 的例子：　兩組資料 [1, 0, 1]　　[1, 1, 0]

(1) 將 0 變為 −1　　　[1, −1, 1]　　[1, 1, −1]

(2) 1, −1, 1　modulated by [1, 1, 1, 1, 1, 1, 1, 1]　(channel 1)

$\rightarrow$ [1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1]

1, 1, −1　modulated by [1, 1, 1, 1, -1, -1, -1, -1] (channel 2)

$\rightarrow$ [1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1]

(3) 相合

[2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, -2, -2, 0, 0, 0, 0, 2, 2, 2, 2] demodulation

[2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, -2, -2, 0, 0, 0, 0, 2, 2, 2, 2]

[1, 1, 1, 1, 1, 1, 1, 1]   [1, 1, 1, 1, 1, 1, 1,1]

[1, 1, 1, 1, 1, 1, 1, 1]

內積 = 8

注意：

(1) 使用 N-point Walsh transform 時，總共可以有 N 個 channels

(2) 除了 Walsh transform 以外，其他的 orthogonal transform 也可以使用

(3) 使用 Walsh transform 的好處

- Orthogonal Transform 共通的問題:　需要同步 synchronization

$\mathbf{R_1} = [\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1]$

$\mathbf{R_2} = [\ 1,\ \ 1,\ \ 1,\ \ 1, -1, -1, -1, -1]$

$\mathbf{R_5} = [\ 1, -1, -1,\ \ 1,\ \ 1, -1, -1,\ \ 1]$

$\mathbf{R_8} = [\ 1, -1,\ \ 1, -1,\ \ 1, -1,\ \ 1, -1]$

但是某些 basis, 就算不同步也近似 orthogonal

$<\mathbf{R_1}[n], \mathbf{R_1}[n]> = 8$,　$<\mathbf{R_1}[n], \mathbf{R_k}[n]> = 0$ if $k \neq 1$

$<\mathbf{R_1}[n], \mathbf{R_k}[n-1]> = 2$ or 0　if $k \neq 1$.

這裡的shift為circular shift

CDMA 的優點：

(1) 運算量相對於 frequency division multiplexing 減少很多

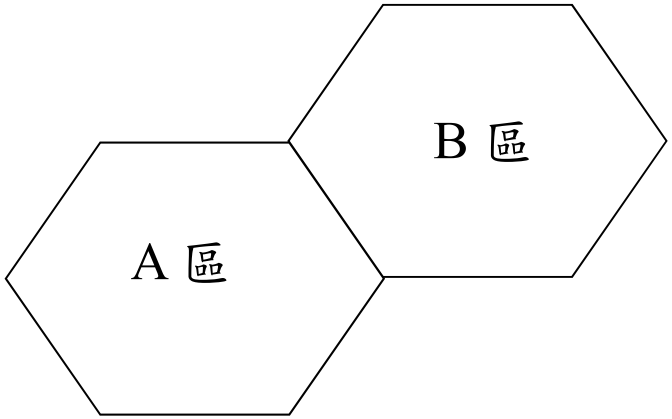
(2) 可以減少 noise 及 interference的影響

(3) 可以應用在保密和安全傳輸上

(4) 就算只接收部分的信號，也有可能把原來的信號 recover 回來

(5) 相鄰的區域的干擾問題可以減少

相鄰的區域，使用差距最大的「語言」，則干擾最少



假設 A 區使用的 orthogonal basis 為 $\phi_k[n]$, $k = 0, 1, 2, \ldots, N-1$

B 區使用的 orthogonal basis 為 $\mu_h[n]$, $h = 0, 1, 2, \ldots, N-1$

設法使 $\max\left(\left|\dfrac{\langle \phi_k[n], \mu_h[n] \rangle}{\langle \phi_k[n], \phi_h[n] \rangle}\right|\right)$ 為最小

$k = 0, 1, 2, \ldots, N-1$, $h = 0, 1, 2, \ldots, N-1$

附錄十七 常用的影像修飾方法

(1) Lightening and Darkening

Input $\longrightarrow$ YCbCr $\longrightarrow$ $Y_o = f(\mathrm{Y})$ $\longrightarrow$ Output

RGB to YCbCr

Cb unchanged
Cr unchanged

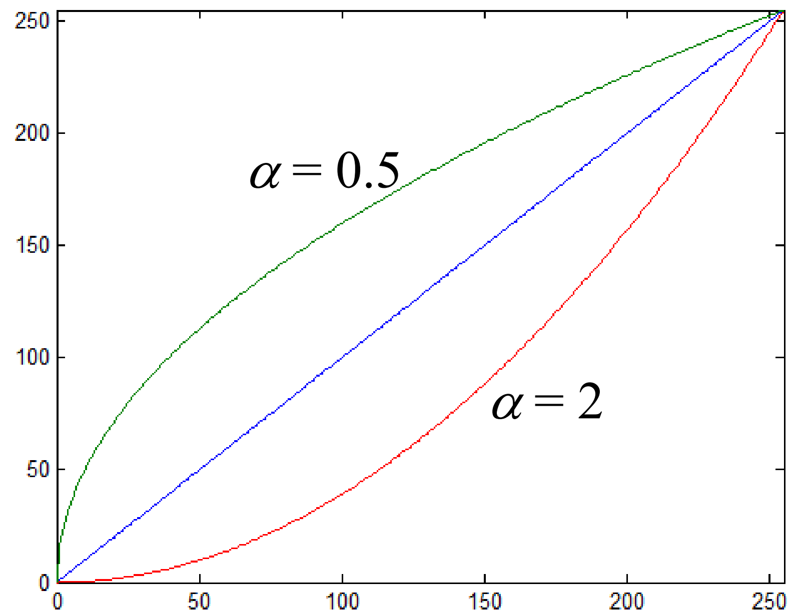
YCbCr to RGB

Example:

$$f(Y) = 255\left(\frac{Y}{255}\right)^{\alpha}$$

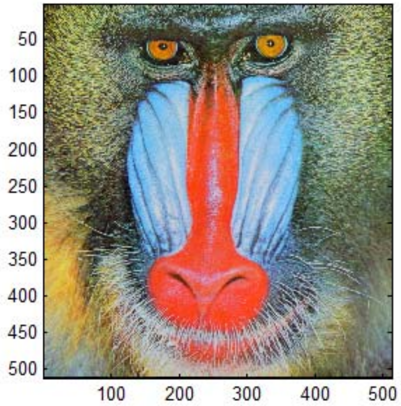
$\alpha < 1$: lightening

$\alpha > 1$: darening

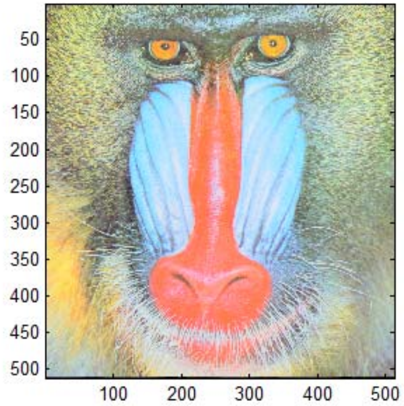


附錄十七　常用的影像修飾方法

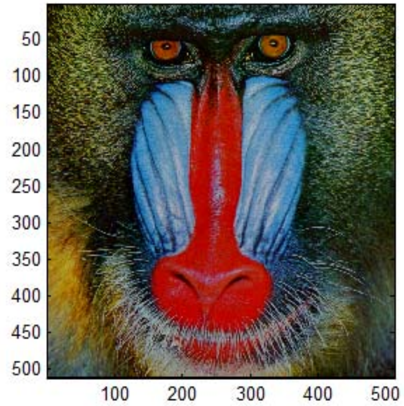
original image



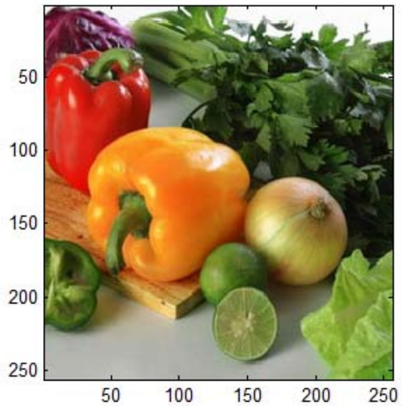
lighten



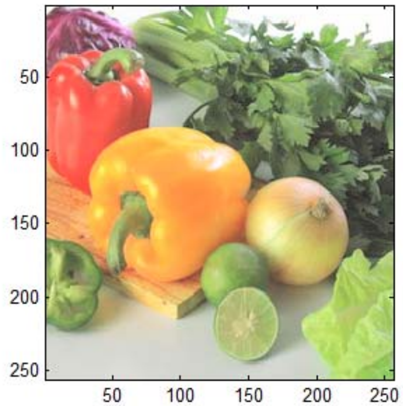
darken



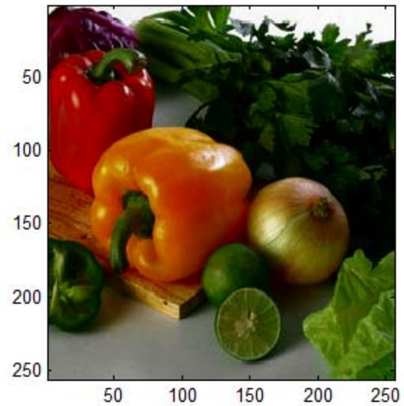
original image



lighten



darken



附錄十七　常用的影像修飾方法

(2) Morphology

(2-1) Erosion　　(去除區域外圍)

$$A[m,n] = A[m,n] \,\&\, A[m-1,n] \,\&\, A[m+1,n] \,\&\, A[m,n-1] \,\&\, A[m,n+1]$$

*	*	*	*	*	*	*
*	*	*	*	$\cancel{0}$	*	*
*	*	$\cancel{0}$	$\cancel{0}$	0	$\cancel{0}$	*
*	$\cancel{0}$	0	0	0	$\cancel{0}$	*
*	$\cancel{0}$	0	0	0	$\cancel{0}$	*
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*	*	*	0	*	*	*
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Erosion for a Non-binary Image

$$A[m,n] = \min\{A[m,n],\, A[m-1,n],\, A[m+1,n],\, A[m,n-1],\, A[m,n+1]\}$$

附錄十七　常用的影像修飾方法

(2-2) Dilation　(擴大區域)

$$A[m,n] = A[m,n] \,||\, A[m-1,n] \,||\, A[m+1,n] \,||\, A[m,n-1] \,||\, A[m,n+1]$$

*	*	*	*	*	*	*	*
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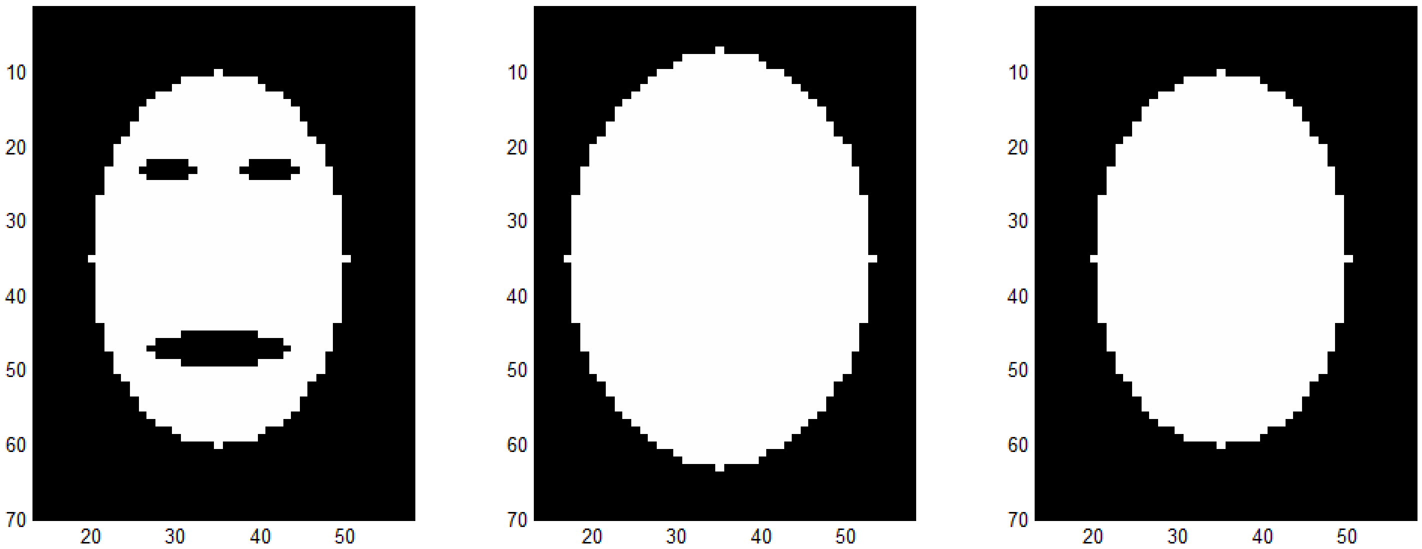
Dilation for a Non-binary Image

$$A[m,n] = Max\left\{A[m,n],\, A[m-1,n],\, A[m+1,n],\, A[m,n-1],\, A[m,n+1]\right\}$$

(2-3) Closing (Hole Filling)

closing = dilation k times + erosion k times

input　　　　dilation 3 times　　　　then erosion 3 times

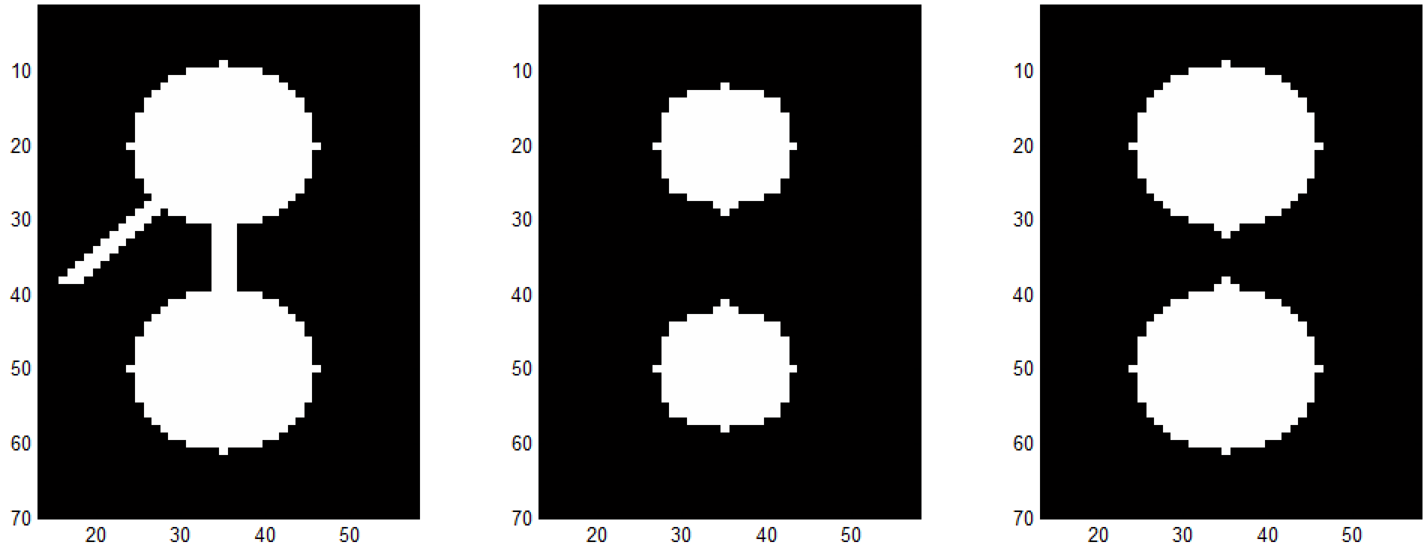


附錄十七　常用的影像修飾方法

(2-4) Opening

opening = erosion *k* times + dilation k times

input　　　erosion 3 times　　　then dilation 3 times



附錄十七　常用的影像修飾方法

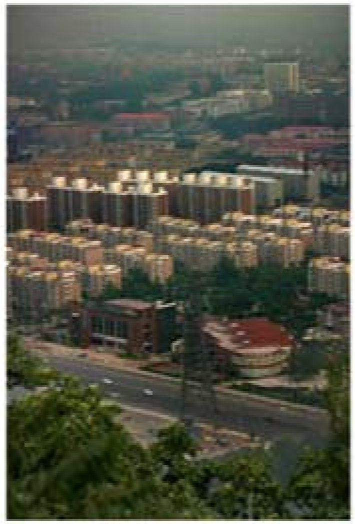
(3) Edge enhancement

$$input \;\; image + \alpha \left| edge \;\; detection \;\; output \right|$$

Original image

With edge enhancement

(4) Dehaze (除霧)



He, Kaiming, Jian Sun, and Xiaoou Tang. "Single image haze removal using dark channel prior." *IEEE Trans. Pattern Anal. Mach. Intell*., vol. 33, pp. 2341-2353 , 2011.

附錄十七　常用的影像修飾方法

Haze Model　$\mathbf{I}(\mathbf{x}) = \mathbf{J}(\mathbf{x})t(\mathbf{x}) + \mathbf{A}(1 - t(\mathbf{x}))$

$\mathbf{J}(\mathbf{x})$: scene, $\mathbf{I}(\mathbf{x})$: observed image

$\mathbf{t}(\mathbf{x})$: transmission, $\mathbf{A}$: intensity for the whole-haze case
$\mathbf{A}(1- \mathbf{t}(\mathbf{x}))$: airlight

定義 dark channel $\mathbf{J}^{dark}(\mathbf{x})$

$$J^{dark}(\mathbf{x}) = \min_{c \in \{r,g,b\}} \left(\min_{\mathbf{y} \in \Omega(\mathbf{x})} (J^c(\mathbf{y})) \right),$$

$\Omega(\mathbf{x})$: some patch (a small region)

Dark channel 為一個影像在一個小範圍區域當中，RGB 的最小值

一個正常影像的 dark channel 大多近於 0
一個受 haze 影響的影像，dark channel 常常不為 0

附錄十七　常用的影像修飾方法

Dehaze 的方法與流程

$$\mathbf{I}(\mathbf{x}) = \mathbf{J}(\mathbf{x})t(\mathbf{x}) + \mathbf{A}(1 - t(\mathbf{x}))$$

$$J^{dark}(\mathbf{x}) = \min_{c}(\min_{\mathbf{y}\in\Omega(\mathbf{x})}(J^c(\mathbf{y}))) = 0.$$

$$\min_{c}(\min_{\mathbf{y}\in\Omega(\mathbf{x})}(\frac{J^c(\mathbf{y})}{A^c})) = 0$$

$$\frac{\mathbf{I}(\mathbf{x})}{A^c} = \frac{\mathbf{J}(\mathbf{x})}{A^c}t(\mathbf{x}) + 1 - t(\mathbf{x})$$

$$\tilde{t}(\mathbf{x}) = 1 - \min_{c}(\min_{\mathbf{y}\in\Omega(\mathbf{x})}(\frac{I^c(\mathbf{y})}{A^c}))$$

find the transmission $t(\mathbf{x})$

A: the 95% largest intensity of $\mathbf{I}(\mathbf{x})$

$$\boxed{\mathbf{J}(\mathbf{x}) = \frac{\mathbf{I}(\mathbf{x})}{t(\mathbf{x})} + \mathbf{A}\left(1 - \frac{1}{t(\mathbf{x})}\right)}$$

recover the original image

He, Kaiming, Jian Sun, and Xiaoou Tang. "Single image haze removal using dark channel prior." *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 33, pp. 2341-2353 , 2011.

期末的勉勵

- 人生難免會有挫折，最重要的是，**我們面對挫折的態度是什麼**

- 長遠的願景要美麗，短期的目標要務實

祝各位同學暑假愉快！

各位同學在研究上或工作上，有任何和 digital signal processing 或 time frequency analysis 方面的問題，歡迎找我來一起討論。