

2. Digital Filter Design (A)

任何可以用來去除 noise 作用的 operation，皆被稱為 filter

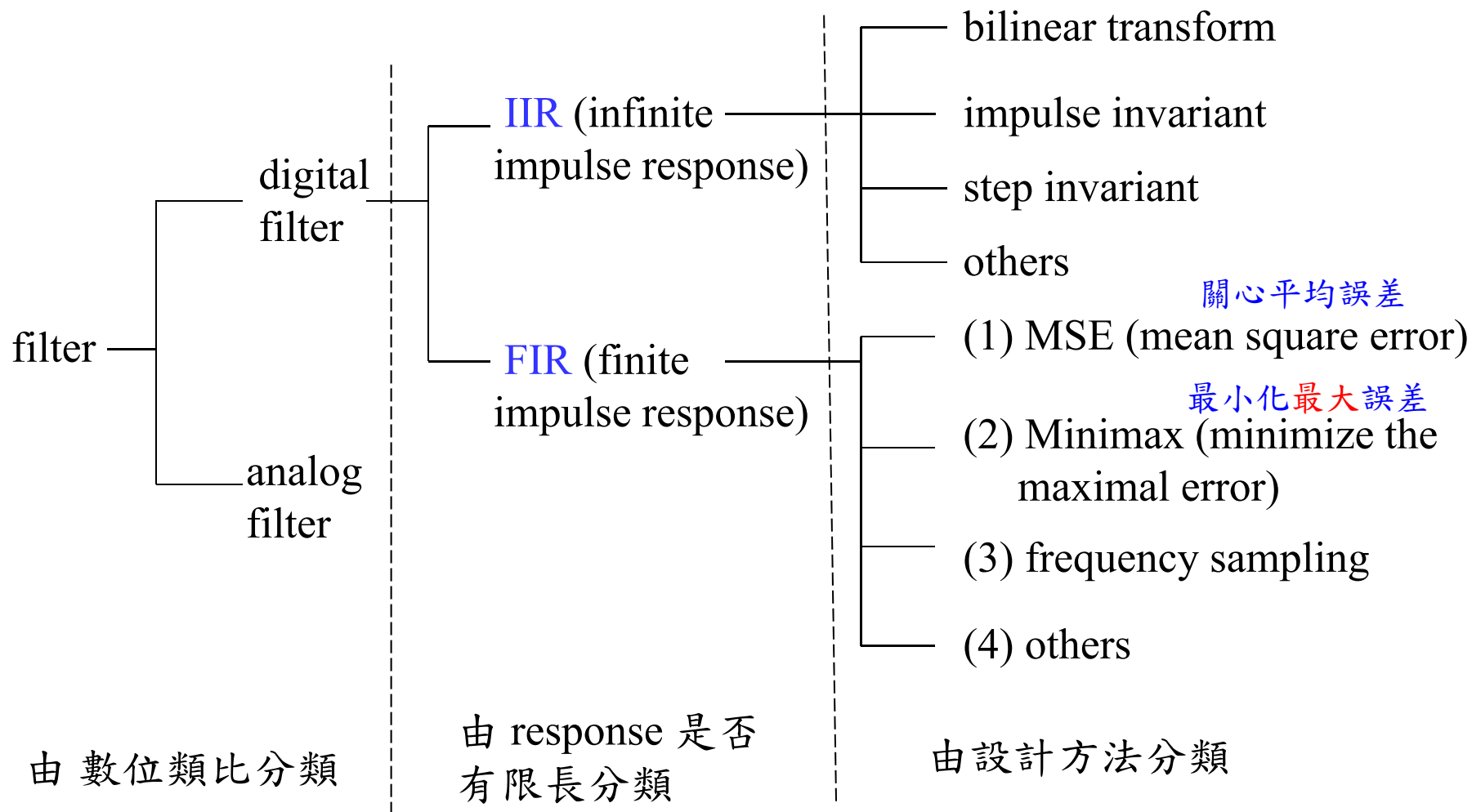
甚至有部分的 operation，雖然主要功用不是用來去除 noise，但是可以用 $\text{FT} + \text{multiplication} + \text{IFT}$ 來表示，也被稱作是 filter

$\frac{\text{FT} + \text{multiplication} + \text{IFT}}{\parallel}$
 convolution, LTI system

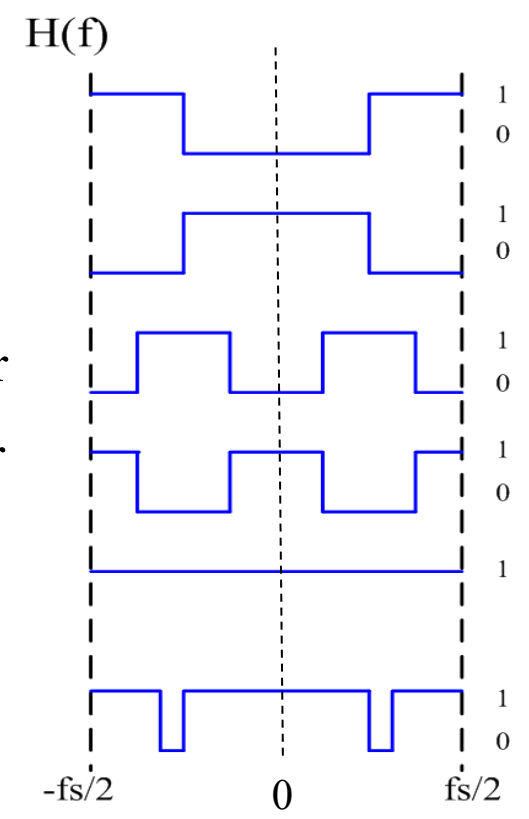
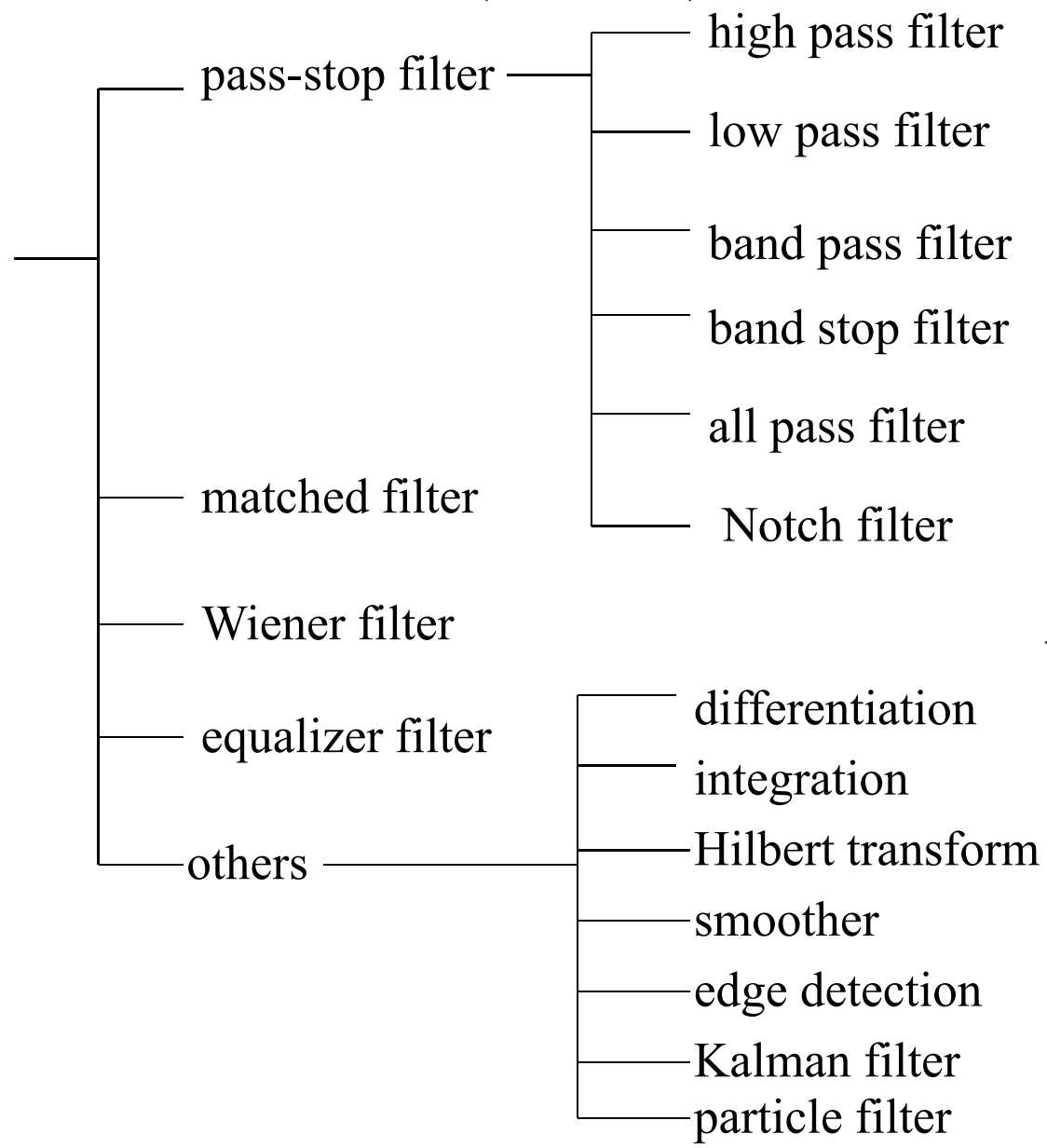
Reference

- [1] A. V. Oppenheim and R. W. Schaffer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.
- [2] D. G. Manolakis and V. K. Ingle, *Applied Digital Signal Processing*, Cambridge University Press, Cambridge, UK. 2011.
- [3] B. A. Shenoi, *Introduction to Digital Signal Processing and Filter Design*, Wiley-Interscience, N. J., 2006.
- [4] A. A. Khan, *Digital Signal Processing Fundamentals*, Da Vinci Engineering Press, Massachusetts, 2005.
- [5] S. Winder, *Analog and Digital Filter Design*, 2nd Ed., Amsterdam, 2002.

◎ 2-A Classification for Filters



Classification for filters (依型態分)



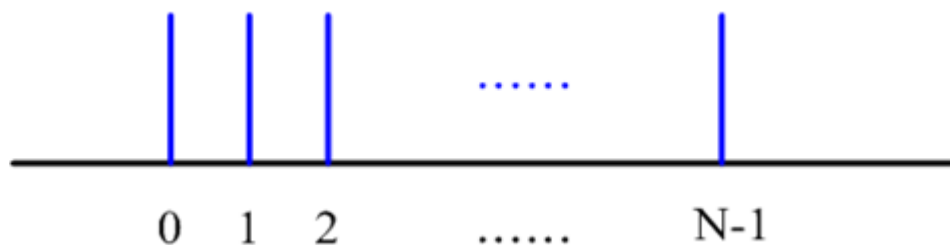
© 2-B FIR Filter Design

FIR filter: impulse response is nonzero at **finite number of points**

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N$$

($h[n]$ has N points, N is a finite number)

$h[n]$ is **causal**



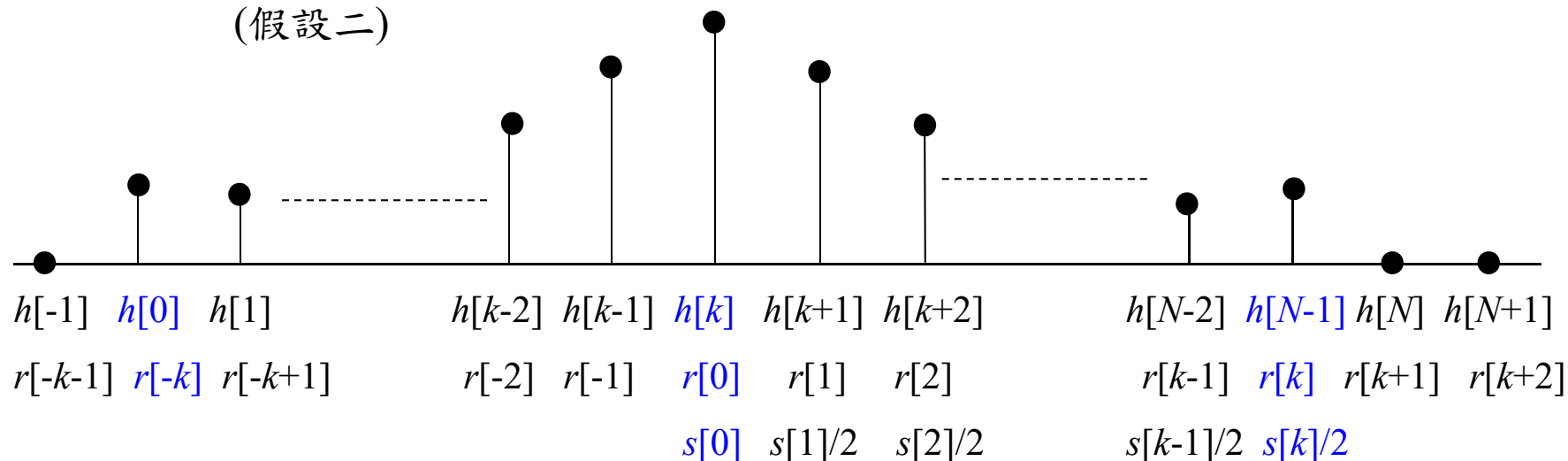
- FIR is more popular because its impulse response is finite.

(假設一)

Specially, when $h[n]$ is even symmetric $h[n] = h[N-1-n]$

and N is an odd number

(假設二)



(a) $r[n] = h[n + k]$, where $k = (N-1)/2$.

(b) $s[0] = r[0]$, $s[n] = 2r[n]$ for $0 < n \leq k$.

Impulse Response of the FIR Filter:

$$h[n] \quad (h[n] \neq 0 \text{ for } 0 \leq n \leq N-1)$$

$$r[n] = h[n + k], \quad k = (N-1)/2 \quad (r[n] \neq 0 \text{ for } -k \leq n \leq k, \text{ see page 41})$$

Suppose that the filter is **even**, $r[n] = r[-n]$.

$$\text{Set } s[0] = r[0], \quad s[n] = 2r[n] \text{ for } n \neq 0.$$

Then, the discrete-time Fourier transform of the filter is

$$H(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi F n} \quad (F = f \Delta_t \text{ is the **normalized frequency**)} \quad \text{See page 22}$$

$$H(F) = e^{-j2\pi F k} R(F) \quad R(F) = \sum_{n=-k}^k r[n] e^{-j2\pi F n}$$

$$= \sum_{n=-k}^{-1} r[n] e^{-j2\pi F n} + r[0] + \sum_{n=1}^k r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^k r[-n] e^{j2\pi F n} + r[0] + \sum_{n=1}^k r[n] e^{-j2\pi F n}$$

$$R(F) = \sum_{n=0}^k s[n] \cos(2\pi n F)$$

◎ 2-C Least MSE Form and Minimax Form FIR Filters

(1) least MSE (mean square error) form

(關心 平均 誤差)

$$\text{MSE} = f_s^{-1} \int_{-f_s/2}^{f_s/2} |H(f) - H_d(f)|^2 df, \quad f_s: \text{ sampling frequency}$$

$H(f)$: the spectrum of the filter we obtain

$H_d(f)$: the spectrum of the desired filter

(2) mini-max (minimize the maximal error) form

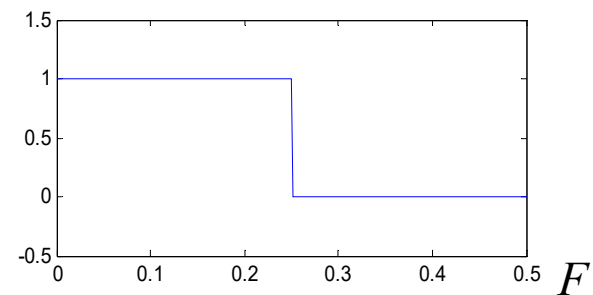
(關心 最大 誤差)

$$\text{maximal error: } \underset{f}{\text{Max}} |H(f) - H_d(f)|$$

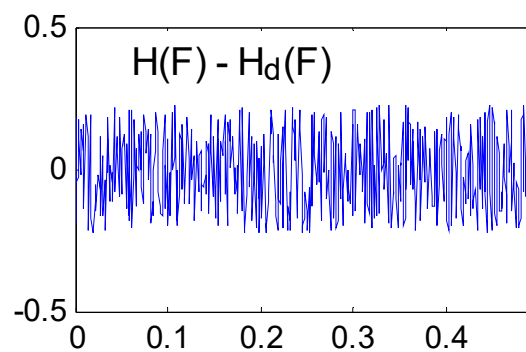
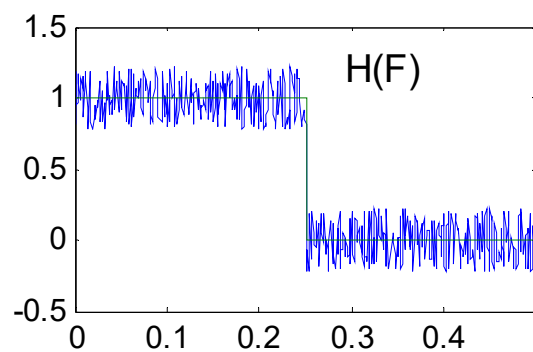
The transition band is always ignored

Example:

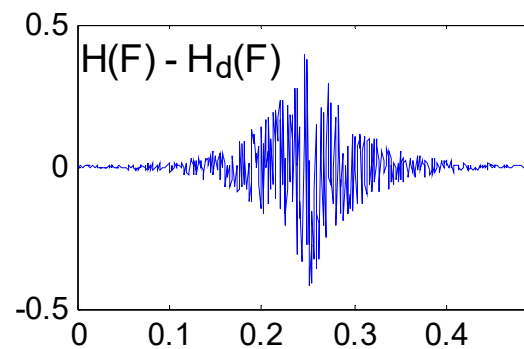
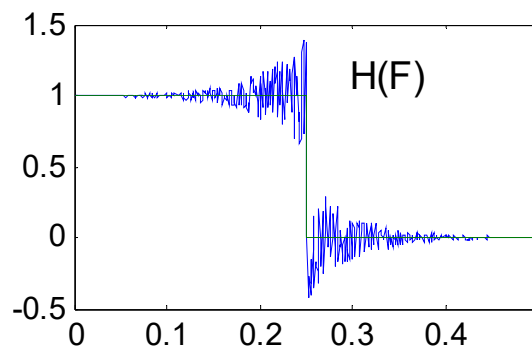
desired output $H_d(F)$



(A) larger MSE, but smaller maximal error



(B) smaller MSE, but larger maximal error



© 2-D Review: FIR Filter Design in the MSE Sense

$$R(F) = \sum_{n=0}^k s[n] \cos(2\pi n F)$$

$$MSE = f_s^{-1} \int_{-f_s/2}^{f_s/2} |R(f) - H_d(f)|^2 df = \int_{-1/2}^{1/2} |R(F) - H_d(F)|^2 dF \quad F = f / f_s$$

$$= \int_{-1/2}^{1/2} \left| \sum_{n=0}^k s[n] \cos(2\pi n F) - H_d(F) \right|^2 dF$$

$$= \int_{-1/2}^{1/2} \left(\sum_{n=0}^k s[n] \cos(2\pi n F) - H_d(F) \right) \left(\sum_{\tau=0}^k s[\tau] \cos(2\pi \tau F) - H_d(F) \right) dF$$

$$= \int_{-1/2}^{1/2} \sum_{n=0}^k s[n] \cos(2\pi n F) \sum_{\tau=0}^k s[\tau] \cos(2\pi \tau F) dF$$

$$- 2 \int_{-1/2}^{1/2} \sum_{n=0}^k s[n] \cos(2\pi n F) H_d(F) dF + \int_{-1/2}^{1/2} H_d^2(F) dF$$

$$\frac{\partial MSE}{\partial s[n]} = 2 \sum_{\tau=0}^k s[\tau] \int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF - 2 \int_{-1/2}^{1/2} H_d(F) \cos(2\pi n F) dF = 0$$

$$\frac{\partial MSE}{\partial s[n]} = 2 \sum_{\tau=0}^k s[\tau] \int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF - 2 \int_{-1/2}^{1/2} H_d(F) \cos(2\pi n F) dF = 0$$

From the facts that

$$\int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 0 \quad \text{when } n \neq \tau,$$

$$\int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 1/2 \quad \text{when } n = \tau, n \neq 0,$$

$$\int_{-1/2}^{1/2} \cos(2\pi n F) \cos(2\pi \tau F) dF = 1 \quad \text{when } n = \tau, n = 0.$$

Therefore,

$$\frac{\partial MSE}{\partial s[0]} = 2s[0] - 2 \int_{-1/2}^{1/2} H_d(F) dF = 0$$

$$\frac{\partial MSE}{\partial s[n]} = s[n] - 2 \int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F) dF = 0 \quad \text{for } n \neq 0.$$

Minimize MSE \rightarrow Make $\frac{\partial MSE}{\partial s[n]} = 0$ for all n 's

$$\therefore \boxed{s[0] = \int_{-1/2}^{1/2} H_d(F) dF}, \quad \boxed{s[n] = 2 \int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F) dF}.$$

Finally, set $h[k] = s[0]$,

$$h[k+n] = s[n]/2, \quad h[k-n] = s[n]/2 \quad \text{for } n = 1, 2, 3, \dots, k,$$

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N.$$

Then, $h[n]$ is the impulse response of the designed filter.

© 2-E FIR Filter Design in the Mini-Max Sense

It is also called “Remez-exchange algorithm”
or “Parks-McClellan algorithm”

References

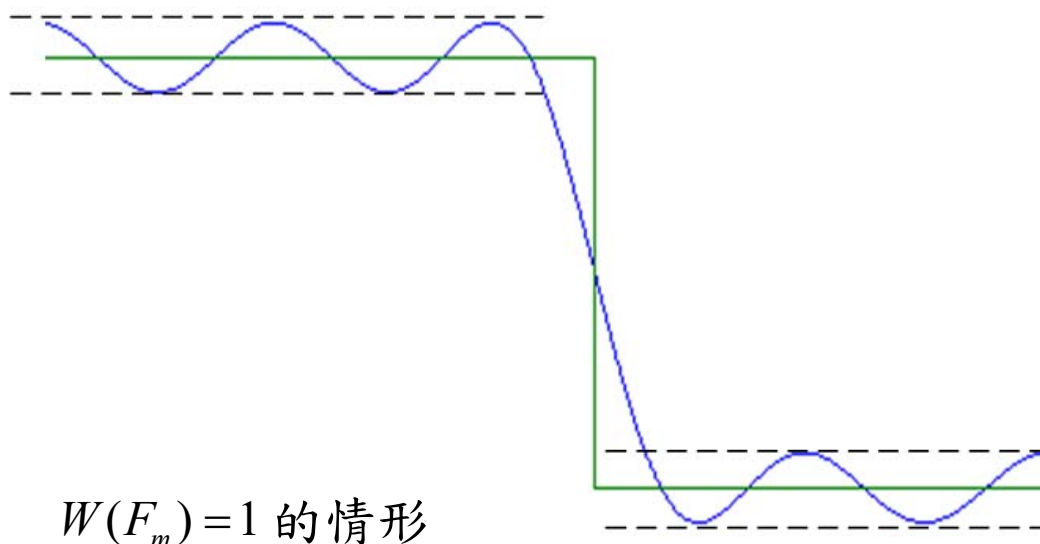
- [1] T. W. Parks and J. H. McClellan, “Chebychev approximation for nonrecursive digital filter with linear phase”, *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.
- [2] J. H. McClellan, T. W. Parks, and L. R. Rabiner “A computer program for designing optimum FIR linear phase digital filter”, *IEEE Trans. Audio-Electroacoustics*, vol. 21, no. 6, Dec. 1973.
- [3] F. Mintz and B. Liu, “Practical design rules for optimum FIR bandpass digital filter”, *IEEE Trans. ASSP*, vol. 27, no. 2, Apr. 1979.
- [4] E. Y. Remez, “General computational methods of Chebyshev approximation: The problems with linear real parameters,” AEC-TR-4491. ERDA Div. Phys. Res., 1962.

Suppose that:

- Two constraints
- ① Filter length = N , N is odd, $N = 2k+1$.
 - ② Frequency response of the **desired filter**: $H_d(F)$ is an **even function**
(F is the normalized frequency)
 - ③ The weighting function is $W(F)$

用 Mini-Max 方法所設計出的 filters，一定會滿足以下二個條件

- (1) 有 $k+2$ 個以上的 **extreme points** \rightarrow **Error 的 local maximal
local minimum**
- (2) 在 extreme points 上， $W(F_m) |R(F_m) - H_d(F_m)|$ 是定值



$W(F_m)=1$ 的情形

證明可參考

T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

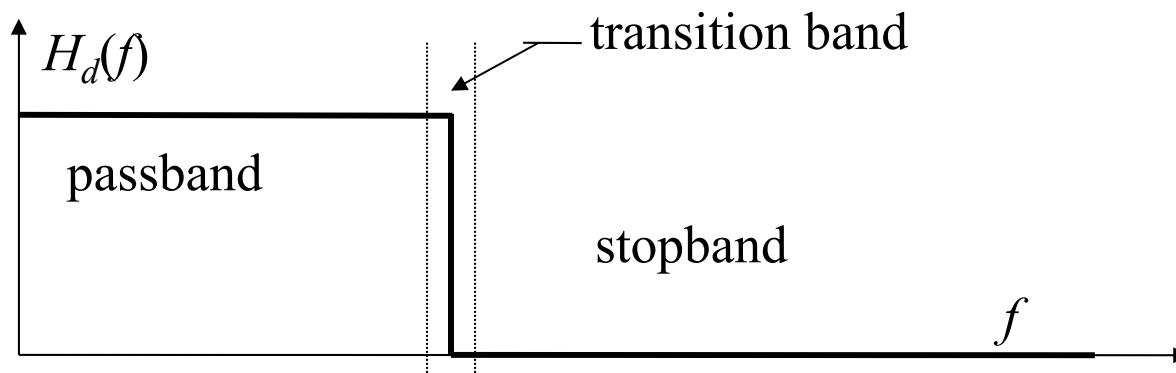
- Generalization for Mini-Max Sense by **weight function**

maximal error: $\text{Max}_f |H(f) - H_d(f)|$

weighted maximal error: $\text{Max}_f |W(f)[H(f) - H_d(f)]|$

where $W(f)$ is the **weight function**.

The weight function is designed according to which band is more important.



Q: How do we choose $W(f)$ When SNR \uparrow ?

Example: If we treat the passband the same important as the stopband.

$W(f) = 1$ in the passband, $W(f) = 1$ in the stopband

Q1: $W(f) = 1$ in the passband, $W(f) < 1$ in the stopband 代表什麼？

Q2: $W(f) < 1$ in the passband, $W(f) = 1$ in the stopband 代表什麼？

Q3: 如何用來壓縮特定區域 (如 transition band 附近) 的 error？

Q4: Weighting function 的概念可否用在 MSE sense ？

(Step 1): Choose **arbitrary** $k+2$ extreme frequencies in the range of $0 \leq F \leq 0.5$, (denoted by $F_0, F_1, F_2, \dots, F_{k+1}$)

Note: (1) Exclude the transition band.

(2) The extreme points cannot be all in the stop band.

Set E_1 (error) $\rightarrow \infty$

Extreme frequencies: the locations where the error is maximal.

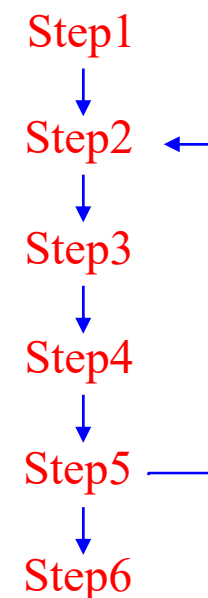
$$[R(F_0) - H_d(F_0)]W(F_0) = -e \quad [R(F_1) - H_d(F_1)]W(F_1) = e$$

$$[R(F_2) - H_d(F_2)]W(F_2) = -e \quad [R(F_3) - H_d(F_3)]W(F_3) = e$$

⋮
⋮

$$[R(F_{k+1}) - H_d(F_{k+1})]W(F_{k+1}) = (-1)^{k+2} e$$

(參考 page 52)



(Step 2): From page 42, $[R(F_m) - H_d(F_m)]W(F_m) = (-1)^{m+1}e$ (where $m = 0, 1, 2, \dots, k+1$) can be written as

$$\sum_{n=0}^k s[n] \cos(2\pi F_m n) + (-1)^m W^{-1}(F_m) e = H_d(F_m)$$

where $m = 0, 1, 2, \dots, k+1$.

Expressed by the matrix form:

$$\begin{bmatrix} 1 & \cos(2\pi F_0) & \cos(4\pi F_0) & \cdots & \cos(2\pi kF_0) & 1/W(F_0) \\ 1 & \cos(2\pi F_1) & \cos(4\pi F_1) & \cdots & \cos(2\pi kF_1) & -1/W(F_1) \\ 1 & \cos(2\pi F_2) & \cos(4\pi F_2) & \cdots & \cos(2\pi kF_2) & 1/W(F_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(2\pi F_k) & \cos(4\pi F_k) & \cdots & \cos(2\pi kF_k) & (-1)^k / W(F_k) \\ 1 & \cos(2\pi F_{k+1}) & \cos(4\pi F_{k+1}) & \cdots & \cos(2\pi kF_{k+1}) & (-1)^{k+1} / W(F_{k+1}) \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ \vdots \\ s[k] \\ e \end{bmatrix} = \begin{bmatrix} H_d[F_0] \\ H_d[F_1] \\ H_d[F_2] \\ \vdots \\ H_d[F_k] \\ H_d[F_{k+1}] \end{bmatrix}$$

Solve $s[0], s[1], s[2], \dots, s[k]$ from the above matrix

(performing the matrix inversion).

Square matrix

(Step 3): Compute $\text{err}(F)$ for $0 \leq F \leq 0.5$, exclude the transition band.

$$\text{err}(F) = [R(F) - H_d(F)]W(F) = \left\{ \sum_{n=0}^k s[n] \cos(2\pi n F) - H_d(F) \right\} W(F)$$

(Step 4): Find $k+2$ **local maximal (or minimal) points** of $\text{err}(F)$

local maximal point: if $q(\tau) > q(\tau + \Delta)$ and $q(\tau) > q(\tau - \Delta)$,

then τ is a local maximal of $q(x)$.

local minimal point: if $q(\tau) < q(\tau + \Delta)$ and $q(\tau) < q(\tau - \Delta)$,

then τ is a local minimal of $q(x)$.

Other rules: Page 58

Denote the local maximal (or minimal) points by $P_0, P_1, \dots, P_k, P_{k+1}$

These $k+2$ extreme points could include the boundary points of the transition band

(Step 5):

$$\text{Set } E_0 = \text{Max}(|\text{err}(F)|). \quad \begin{cases} E_0 : \text{現在的Max } |\text{err}(F)| \\ E_1 : \text{前一次iteration的Max } |\text{err}(F)| \end{cases}$$

(Case a) If $E_1 - E_0 > \Delta$, or $E_1 - E_0 < 0$ (or the first iteration) \rightarrow

set $F_n = P_n$ and $E_1 = E_0$, return to Step 2.

(Case b) If $0 \leq E_1 - E_0 \leq \Delta \rightarrow$ continue to Step 6.

(Step 6):

Set $h[k] = s[0]$,

$$h[k+n] = s[n]/2, \quad h[k-n] = s[n]/2 \quad \text{for } n = 1, 2, 3, \dots, k$$

(referred to page 41)

Then $h[n]$ is the impulse response of the designed filter.

◎ 2-F Mini-Max FIR Filter 設計時需注意的地方

(1) Extreme points 不要選在 transition band

Initial guess的extreme points只要注意別取在transition band裡，即能保證 converge，不同的guess會影響converge的速度但不影響結果

(2) E_1 (error of the previous iteration) $<$ E_0 (present error) 時，亦不為收斂

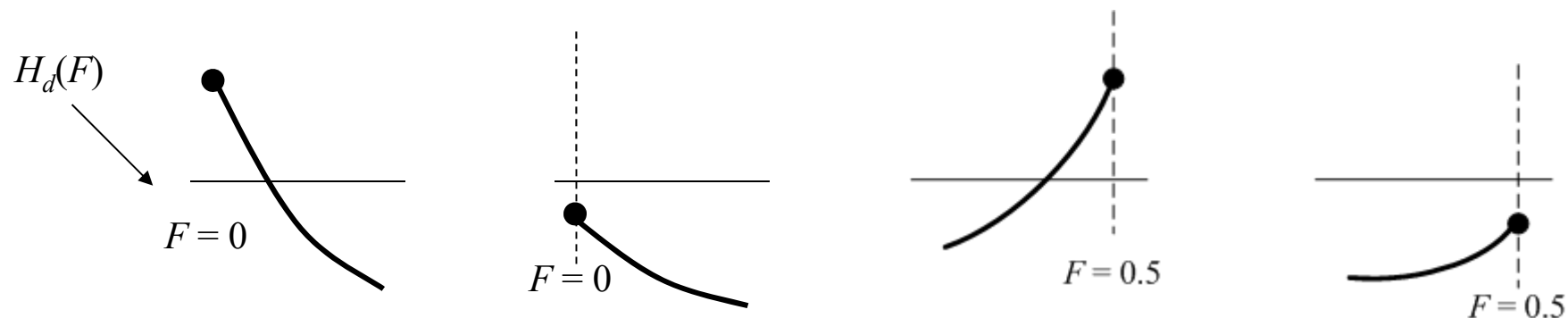
(3) 思考：page 53 中的 matrix operation，如何用一行的指令寫出？

(4) Extreme points 判斷的規則：

(a) The **local peaks** or **local dips** that are not at **boundaries** must be extreme points.

boundaries: $F = 0$, $F = 0.5$, 以及 transition band 的兩端

(b) For boundary points



Add a zero to the outside and conclude whether the point is a local maximum or a local minimum.

(5) 有時，會找到多於 $k+2$ 個 extreme points, 該如何選 $P_0, P_1, \dots, P_k, P_{k+1}$

(a) 優先選擇不在 boundaries 的 extreme points

(b) 其次選擇 boundary extreme points 當中 $|\text{err}(F)|$ 較大的，
直到湊足 $k+2$ 個 extreme points 為止

(c) 在 boundary extreme points 的 $|\text{err}(F)|$ 一樣大的情形，
優先選擇 transition bands 兩旁的點

© 2-G Examples for Mini-Max FIR Filter Design

- Example 1:** Design a 9-length highpass filter in the mini-max sense

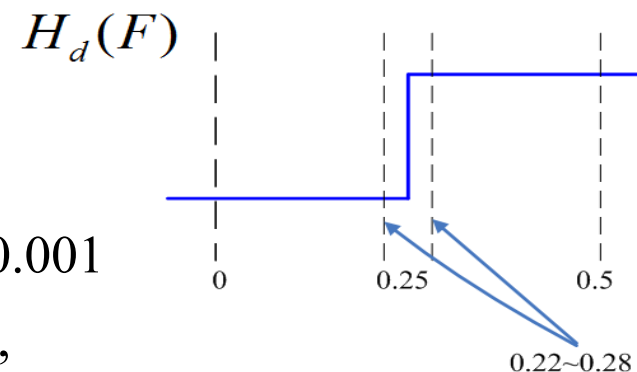
ideal filter: $H_d(F) = 0$ for $0 \leq F < 0.25$,

$H_d(F) = 1$ for $0.25 < F \leq 0.5$,

transition band: $0.22 < F < 0.28$ $\Delta = 0.001$

weighting function: $W(F) = 0.25$ for $0 \leq F \leq 0.22$,

$W(F) = 1$ for $0.28 \leq F \leq 0.5$,



(Step 1) Since $N = 9$, $k = (N-1)/2 = 4$, $k+2 = 6$,

→ **Choose 6 extreme frequencies**

(e.g., $F_0 = 0$, $F_1 = 0.1$, $F_2 = 0.2$, $F_3 = 0.3$, $F_4 = 0.4$, $F_5 = 0.5$)

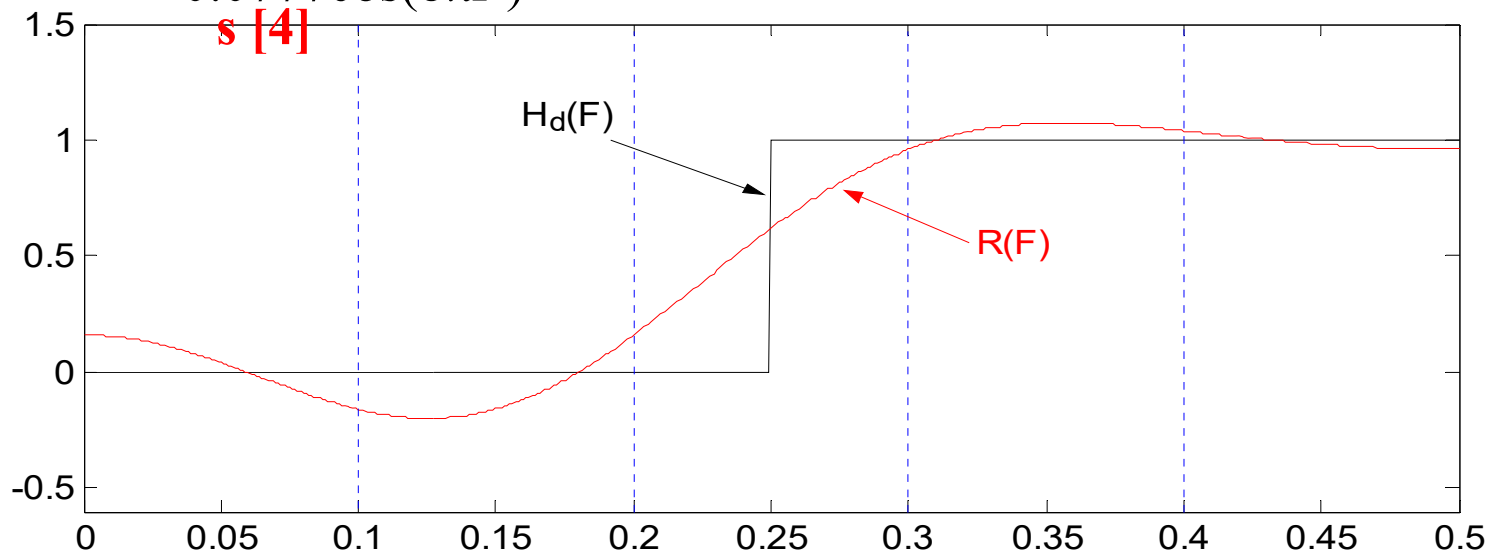
$$[R(F_n) - H_d(F_n)]W(F_n) = (-1)^{n+1}e, \quad n = 0, 1, 2, 3, 4, 5.$$

(Step 2)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

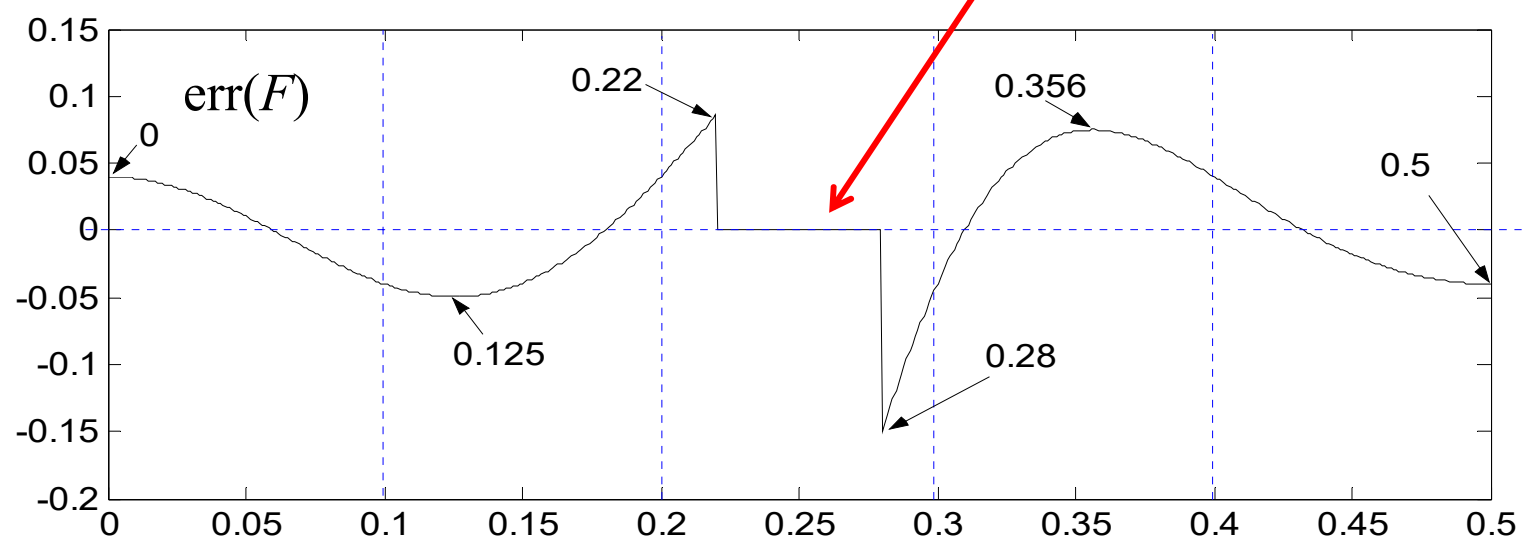
$$\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5120 \\ -0.6472 \\ -0.0297 \\ 0.2472 \\ 0.0777 \\ -0.040 \end{bmatrix}$$

$$R(F) = 0.5120 - 0.6472\cos(2\pi F) - 0.0297\cos(4\pi F) + 0.2472\cos(6\pi F) + 0.0777\cos(8\pi F)$$



(Step 3) $err(F) = [R(F) - H_d(F)]W(F)$

$W[F] = 0$ for $0.22 < F < 0.28$



(Step 4) Extreme points:

$$F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$$

(Step 5) $E_0 = \text{Max}[|\text{err}(F)|] = \underline{0.1501}$, return to Step 2.

Second iteration

(Step 2) Using $F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$

$$\begin{aligned} \rightarrow s[0] &= 0.5018, s[1] = -0.6341, s[2] = -0.0194, s[3] = 0.3355, \\ s[4] &= 0.1385 \end{aligned}$$

(Step 3) $\text{err}(F) = [R(F) - H_d(F)]W(F)$,

(Step 4) extreme points : 0, 0.132, 0.22, 0.28, 0.336, 0.5

(Step 5) $E_0 = \text{Max}[|\text{err}(F)|] = \underline{0.0951}$, return to Step 2.

Third iteration

(Step 2), (Step 3), (Step 4), peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5) $E_0 = \underline{0.0821}$, return to Step 2.

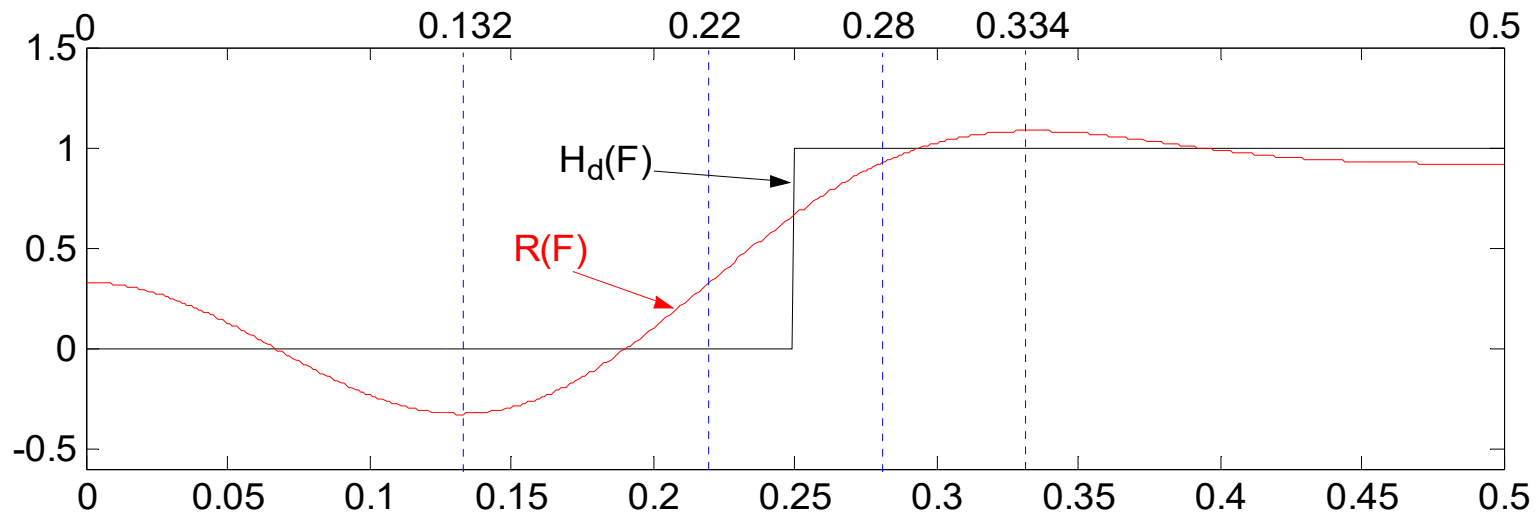
Fourth iteration

(Step 2), (Step 3) , (Step 4), peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5) $E_0 = \underline{0.0820}$, $E_1 - E_0 = 0.0001 \leq \Delta$, continues to Step 6.

(Step 6) From $s[0] = 0.4990$, $s[1] = -0.6267$, $s[2] = -0.0203$, $s[3] = 0.3316$,
 $s[4] = 0.1442$

$$\begin{aligned}
 h[4] &= s[0] = 0.4990, & h[3] &= h[5] = s[1]/2 = -0.3134, \\
 h[2] &= h[6] = s[2]/2 = -0.0101, & h[1] &= h[7] = s[3]/2 = 0.1658, \\
 h[0] &= h[8] = s[4]/2 = 0.0721.
 \end{aligned}$$



- **Example 2:** Design a 7-length digital filter in the mini-max sense

ideal filter: $H_d(F) = 1$ for $0 \leq F < 0.24$,

$H_d(F) = 0$ for $0.24 < F \leq 0.5$,

transition band: $0.21 < F < 0.27$

weighting function: $W(F) = 1$ for $0 \leq F \leq 0.21$,

$W(F) = 0.5$ for $0.27 \leq F \leq 0.5$,

(Step 1) Since $N = 7$, $k = (N-1)/2 = 3$, $k+2 = 5$,

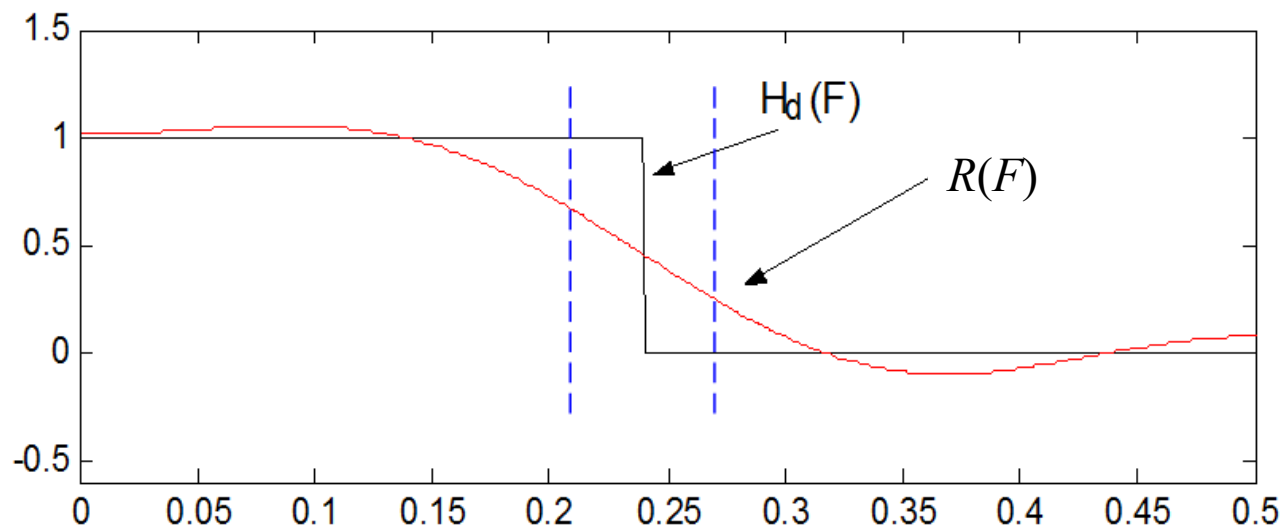
→ Choose 5 extreme frequencies

(e.g., $F_0 = 0.05$, $F_1 = 0.15$, $F_2 = 0.3$, $F_3 = 0.4$, $F_4 = 0.5$)

(Step 2)

$$\begin{bmatrix} 1 & 0.9511 & 0.8090 & 0.5878 & 1 \\ 1 & 0.5878 & -0.309 & -0.9511 & -1 \\ 1 & -0.309 & -0.809 & 0.809 & 2 \\ 1 & -0.809 & 0.309 & 0.309 & -2 \\ 1 & -1 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow s[0] = 0.4638, \quad s[1] = 0.6327, \quad s[2] = 0.0809, \quad s[3] = -0.1608, \quad e = -0.0364$$



After Step 2,

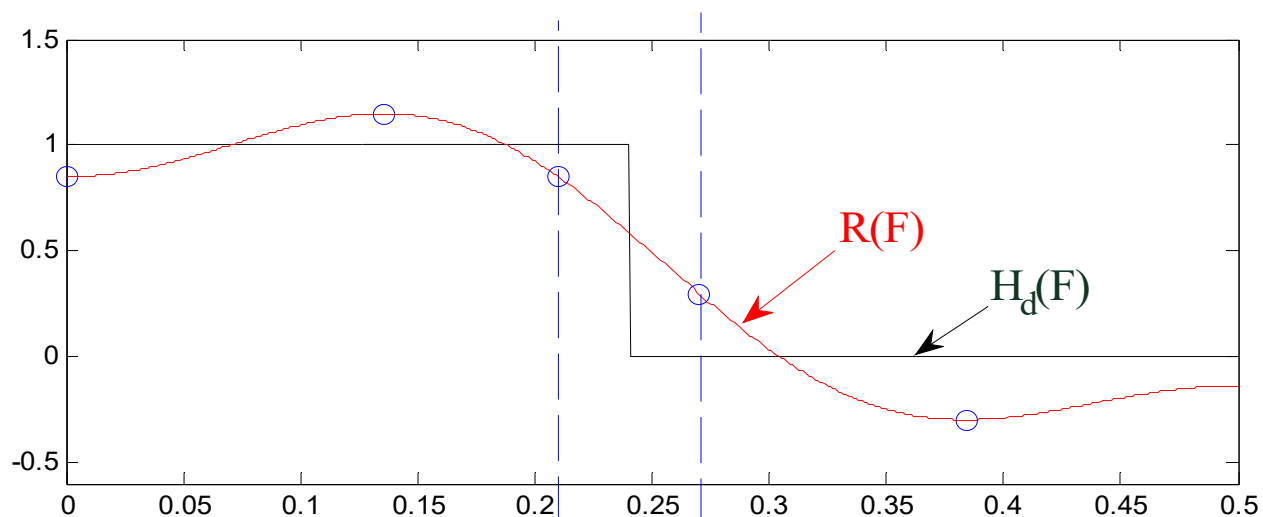
$$\text{(Step 3) } \text{err}(F) = [0.4638 + 0.6327 \cos(2\pi F) + 0.0809 \cos(4\pi F) - 0.1608 \cos(6\pi F) - H_d(F)]W(F)$$

(Step 4) extreme points: 0.089, 0.21, 0.27, 0.369, 0.5.

(Step 5) $E_0 = \text{Max}[|\text{err}(F)|] = 0.3396$, return to Step 2.

Iteration	1	2	3	4	5	6	7
Max[err(F)]	0.3396	0.2371	0.3090	0.1944	0.1523	0.1493	0.1493

After 7 times of iteration



$$s[0] = 0.4243, \quad s[1] = 0.7559, \quad s[2] = -0.0676, \quad s[3] = -0.2619, \quad e = 0.1493$$

(Step 6):

$$h[3] = 0.4243, \quad h[2] = h[4] = s[1]/2 = 0.3780,$$

$$h[1] = h[5] = s[2]/2 = -0.0338,$$

$$h[0] = h[6] = s[3]/2 = -0.1309, \quad h[n] = 0 \text{ for } n < 0 \text{ and } n > 6$$

附錄二：Spectrum Analysis for Sampled Signals

(學信號處理的人一定要會的基本常識)

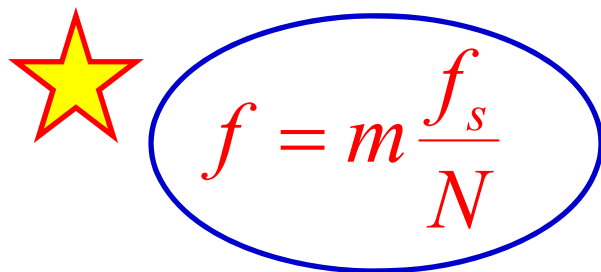
已知 $x[n]$ 是由一個 continuous signal $y(t)$ 取樣而得

$$x[n] = y(n\Delta_t)$$

$$\text{DFT: } X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N} \quad \text{FT: } Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} y(t) dt$$

Q: $x[n]$ 的 DFT 和 $y(t)$ 的 Fourier transform 之間有什麼關係?

Basic rule : 把間隔由 1 換成 f_s/N where $f_s = 1/\Delta_t$

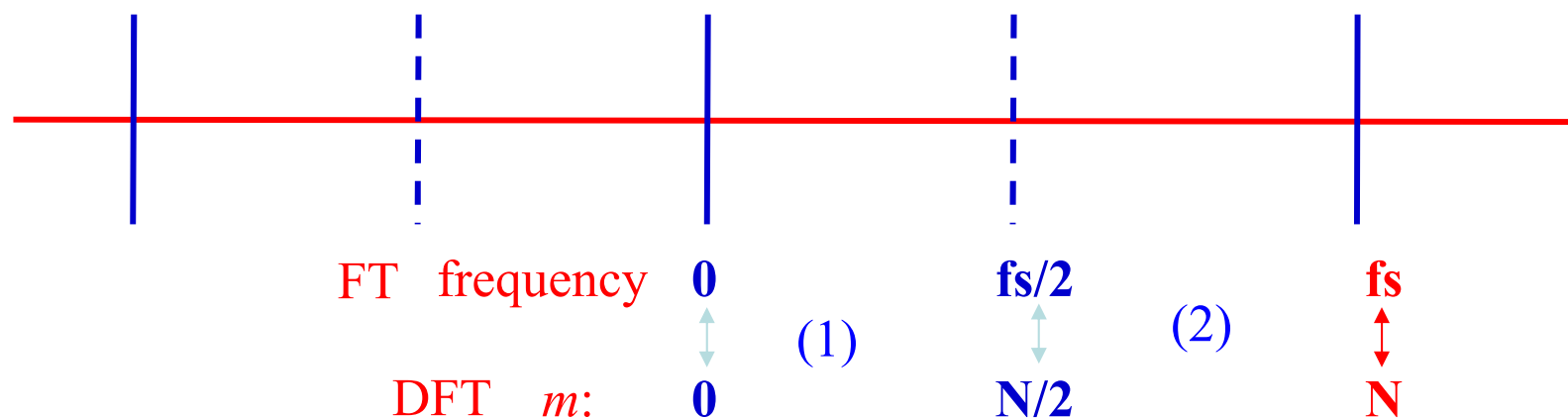


$$f = m \frac{f_s}{N}$$

(Very important)

$$(1) \quad X[m]\Delta_t \cong Y\left(m\frac{f_s}{N}\right) \quad f_s = 1/\Delta_t \quad \text{for } m \leq N/2$$

$$(2) \quad X[m]\Delta_t \cong Y\left((m-N)\frac{f_s}{N}\right) = Y\left(m\frac{f_s}{N} - f_s\right) \quad \text{for } m > N/2$$



If the sampling frequency is f_s , the FT output has the period of f_s

The DFT output has the period of N

Proof : $Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} y(t) dt$

用 $t = n\Delta_t$, $f = m\Delta_f$ 代入

$$Y(m\Delta_f) \cong \sum_n e^{-j2\pi m\Delta_f n\Delta_t} y(n\Delta_t) \Delta_t = \Delta_t \sum_n e^{-j2\pi m\Delta_f n\Delta_t} x[n]$$

$$\text{當 } \Delta_t \Delta_f = \frac{1}{N} \quad \text{i.e.,} \quad \Delta_f = \frac{1}{N\Delta_t} = \frac{f_s}{N}$$

$$\begin{aligned} Y\left(m \frac{f_s}{N}\right) &\cong \Delta_t \sum_n e^{-j2\pi \frac{mn}{N}} x[n] \\ &= \Delta_t DFT \{x[n]\} \end{aligned}$$

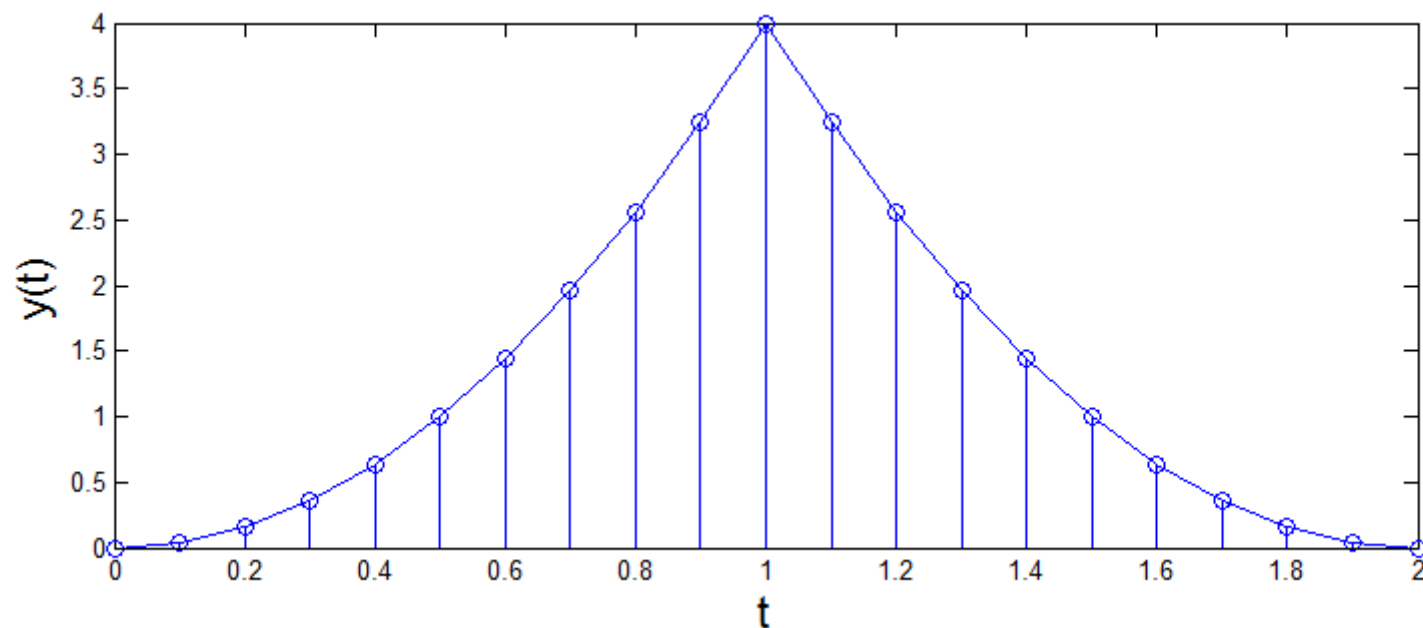
Example : 已知

$$y(t) = (2t)^2 \quad \text{for } 0 \leq t \leq 1 \quad y(t) = (4-2t)^2 \quad \text{for } 1 \leq t \leq 2$$

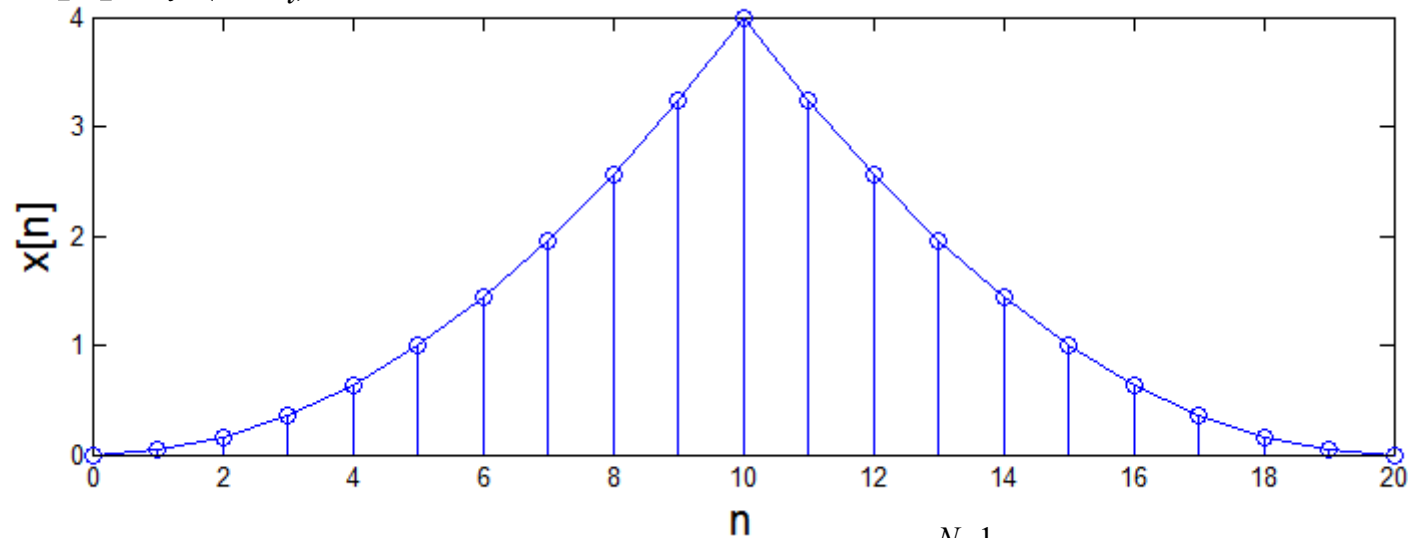
取樣間隔： $\Delta_t = 0.1$

$$x[n] = y(n \Delta_t) \text{ for } 0 \leq n \leq 20$$

如何用 DFT 來正確的畫出 $y(t)$ 的頻譜？



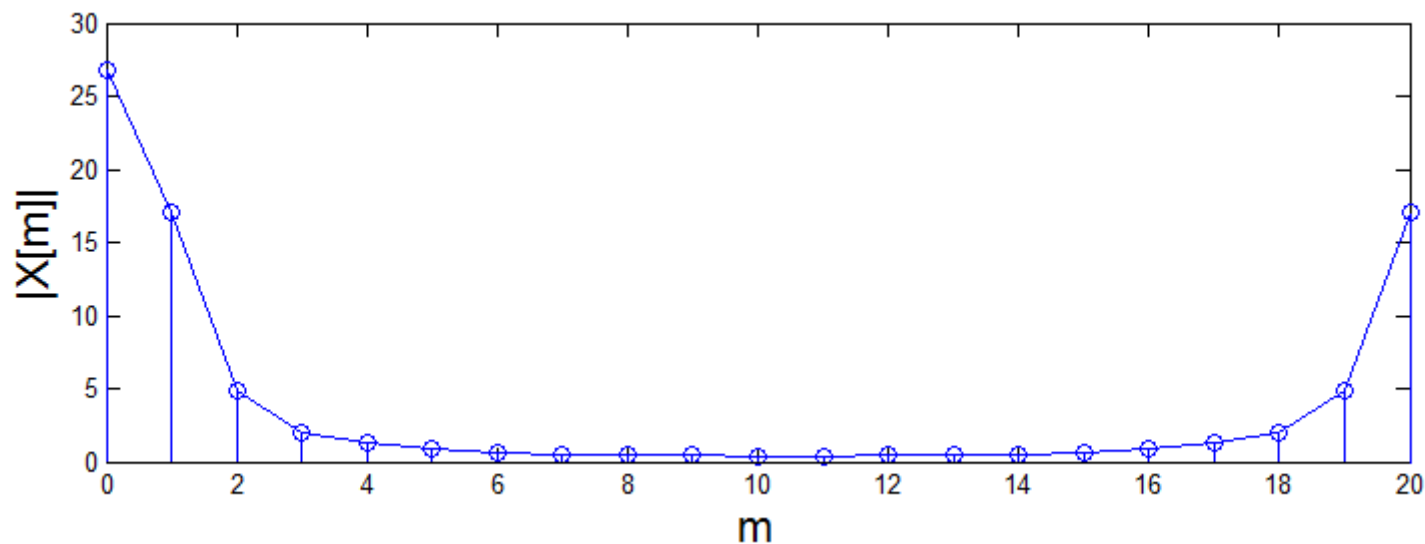
$$x[n] = y(n \Delta_t) \text{ for } 0 \leq n \leq 20$$



(Step 1) Perform the DFT for $x[n]$

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N}$$

$N = 21$



$$\text{(Step 2-1)} \quad Y\left(m \frac{f_s}{N}\right) \cong X[m] \Delta_t \quad \text{for } m \leq N/2$$

$$\text{(Step 2-2)} \quad Y\left((m-N) \frac{f_s}{N}\right) \cong X[m] \Delta_t \quad \text{for } m > N/2$$

In this example, $\frac{f_s}{N} = \frac{1}{N \Delta_t} = \frac{1}{21 \cdot 0.1} = 0.4762$

