Advanced Digital Signal Processing

高等數位訊號處理

授課者： 丁 建 均

Office：明達館723室， TEL： 33669652

E-mail: jjding@ntu.edu.tw

課程網頁：https://djj.ee.ntu.edu.tw/ADSP.htm

歡迎大家來修課，也歡迎有問題時隨時聯絡！

上課方式

(1) 錄影 ，影片將藉由 NTU Cool 下載　https://cool.ntu.edu.tw

(2) 現場 (週三下午 15:30~18:20，明達館205室)

1530 ~ 1620
1630 ~ 1720
1730 ~ 1820

作業和報告繳交方式：

用 NTU Cool 來繳交作業與報告的電子檔　https://cool.ntu.edu.tw

注意，Tutorial 一定要交Word 或 Latex 原始碼
Wiki 要寄編輯條目的連結給老師

上課時間：15 週

3n 週 出作業
3n+2 週交作業

2/19,

3/5 ~~2/26~~,

3/12 ~~3/5~~, 出 HW1

3/19 ~~3/12~~,

3/26 ~~3/19~~, 交 HW1

4/2 ~~3/26~~, 出 HW2

4/9 ~~4/2~~, ← 代課

4/16 ~~4/9~~, 交 HW2

~~代課~~

4/23 ~~4/16~~, 出 HW3

4/30 ~~4/23~~,

5/7 ~~4/30~~, 交 HW3

5/14 ~~5/7~~, 出 HW4

5/17 ~~5/14~~,

5/21, 交 HW4,

5/28, 出 HW5

6/4 之前, Oral Presentation

6/11, 交 HW5 及 term paper

原則上: $3n$-1 週出作業, $3n$+1 週繳交

- **評分方式：**

Basic: 15 scores

原則上每位同學都可以拿到 11 分以上，
另外，會有額外問答題，每位同學四次，每答對一次加0.8分
若有聽過期末口頭報告並參與票選，再加0.8分

Homework: 60 scores (5 times, 每 3 週一次)

請自己寫，和同學內容極高度相同 ，將扣 70% 的分數
就算寫錯但好好寫也會給 40~95% 的分數，
遲交分數打 8 折，不交不給分。 不知道如何寫，可用 E-mail 和我聯絡，或於上課時發問
禁止 Ctrl-C Ctrl-V 的情形。

Term paper 25 scores

Term paper 25 scores

方式有五種

(1) 書面報告

10頁以上(不含封面)，中英文皆可，11或12的字體，題目可選擇和信號處理(包括信號、通訊、影像、音訊、生醫訊號、經濟信號處理等等) 有關的任何一個主題。

包括 abstract, conclusion, 及 references，並且要分 sections。 儘量工整

鼓勵多做實驗及模擬，

有做模擬的同學請將程式附上來，會有額外加分。

嚴禁 Ctrl-C Ctrl-V 的情形，否則扣 70% 的分數

pw: gice 1997

(2) Tutorial (對既有領域做淺顯易懂的整理)

限二十個名額，和書面報告格式相同，但頁數限制為18頁以上(若為加強前人的 tutorial，則頁數為 (2/3)N + 13 以上，N 為前人 tutorial 之頁數)，題目由老師指定，以清楚且有系統的介紹一個主題的基本概念和應用為要求，為上課內容的進一步探討和補充，交 Word 檔。

選擇這個項目的同學，學期成績加 4分

or Latex 原始碼

(3) 口頭報告 full

限十個名額，每個人 15~40分鐘，題目可選擇和課程有關的任何一個主題。口頭報告的同學請在6月4日以前將影片錄好，並且把影片(或連結)寄給老師。有意願的同學，請儘早告知，以先登記的同學為優先。

選擇這個項目的同學，學期成績加 2分

口頭報告時，希望同學們至少能參與線上觀看，並將做為~~第五次作業的其中一題~~。

(4) 編輯 Wikipedia

中文或英文網頁皆可，至少 2 個條目，但不可同一個條目翻成中文和英文。總計 80行以上。限和課程相關者，自由發揮，越有條理、有系統的越好

選擇編輯 Wikipedia 的同學，請於 5月29日前，向我登記並告知我要編緝的條目(2 個以上)，若有和其他同學選擇相同條目的情形，則較晚向我登記的同學將更換要編緝的條目

書面報告和編輯 Wikipedia，期限是 6月11日

以上(1), (2), (3), (4) 不管選哪個題目，若有做實驗模擬，請附上程式碼，會有額外的加分 (鼓勵不強制)

Tutorial 可供選擇的題目(可以略做修改)

✓(1) Gini Index and Renyi Entropy

(2) Green Learning

✓(3) Learning-Based Signal Denoising

(4) Non-uniform Signal Interpolation

✓(5) SNAKE Algorithm for Image Segmentation

(6) Noise Estimation in Image Processing

(7) Signal Processing Techniques in Metaverse

(8) Autoregressive Integrated Moving Average (ARIMA)

✓(9) Anomaly Detection

✓(10) Mamba Model for Speech Analysis

✓(11) Mamba Model for Image Processing

(12) Inference Engine

Tutorial 可供選擇的題目(可以略做修改)

✓(13) 3D Point Cloud

✓(14) Large Language Model (LLM)

✓(15) Graph Cut for Image Segmentation

✓(16) Brain-to-Speech Technology

✓(17) Semantic Communication

✓(18) Knowledge Distillation

✓(19) Data Clustering

✓(20) Multi-modality Models

Matlab Program

Download: 請洽台大各系所

參考書目

洪維恩，Matlab 程式設計，旗標，台北市，2013 . (合適的入門書)

張智星，Matlab 程式設計入門篇，第四版，碁峰，2016.

預計看書學習所花時間： 3~5 天

Python Program

Download: https://www.python.org/

參考書目

葉難， Python程式設計入門，博碩，2015

黃健庭， Python程式設計：從入門到進階應用 ，全華，2020

The Python Tutorial   https://docs.python.org/3/tutorial/index.html

研究所和大學以前追求知識的方法有什麼不同？

研究所：觀念的學習，思考，創新

大學：記憶，熟練度，解題速度
以前

Question:

Fourier transform: $X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi f t}dt$
(FT)

Why should we use the Fourier transform?

advantages of the FT
(1) spectrum analysis
(2) convolution $\xrightarrow{FT}$ multiplication $Y(f)=X(f)H(f)$

$y(t)=x(t)*h(t)$
$=\int x(t-\tau)h(\tau)d\tau$

Any linear time-invariant (LTI) system can be expressed by convolution.

Is the Fourier transform the best choice in any condition?

$x(t)\,e^{-j2\pi ft} = x(t)\left(\cos(2\pi ft)-j\sin(2\pi ft)\right)$

1 complex mul. = 4 real muls.

$(a+ib)(c+id) = ac-bd+i(ad+bc)$

not rational number mul.
complex multi.

I. Introduction

Outline

- Introduction (1.3W)
- 1. Filter Design
  - FIR (2W)
  - IIR, Others (1.7W)
- 2. Homomorphic Signal Processing (1W)
- 3. Applications
  - Acoustics (1W)
  - Compression, Image (2 W)
- 4. Fast Algorithms
  - Basic (1W)
  - FFT and Convolution (2W)
  - Others (0.5W)
- 5. Orthogonal Transforms
  - Walsh Transform (0.8 W)
  - Number Theory (1 W)
  - OFDM, CDMA (0.7 W)

目標：

(1) 對 Digital Signal Processing 作更有系統且深入的了解

(2) 探討 Digital Signal Processing 的應用

(3) 學習 Digital Signal Processing 幾個重要子領域的基礎知識

Part 1: Filter

IIR: infinite impulse response h無限長

FIR: finite impulse response h有限長

- Filter 的分類

x : input

h : impulse response

y : output

$$y[n] = x[n] * h[n] = \sum_{\tau} x[n-\tau]\, h[\tau]$$

- filter
  - digital
    - IIR (考慮aliasing)
      - bilinear transform
      - impulse invariant
      - step invariant
    - FIR
      - MSE (mean square error)
      - minimax
      - frequency sampling
  - analog (技術較早開發)

$$\text{length}(y) = \text{length}(x) + \text{length}(h) - 1$$

IIR filter 的優點：(1) easy to design

(2) (sometimes) easy to implement (等比級數) (see page 16)

缺點：(1) usually hard to implement

(2) output has infinite length

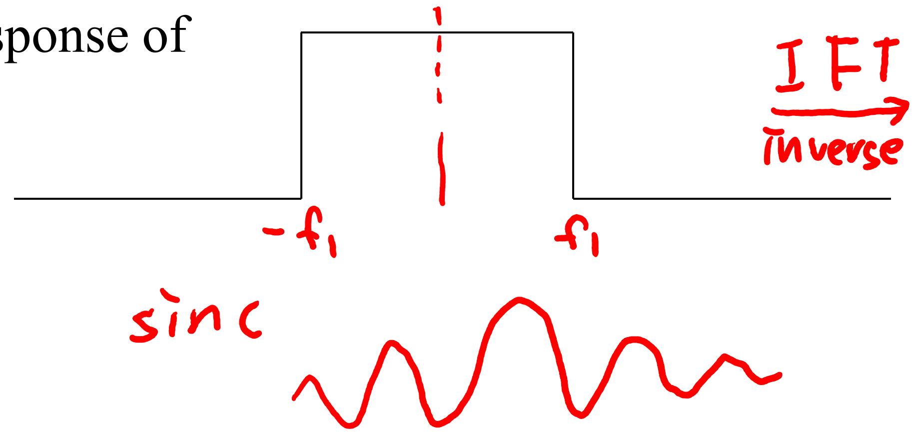
(3) may not be stable

FIR filter 的優點：(1) usually easier to implement

(2) output has finite length

(3) stable

缺點：An FIR filter is impossible to have the ideal frequency response of



$$\xrightarrow[\text{inverse FT}]{\text{IFT}} 2f_1 \,\mathrm{sinc}(2f_1 t) = \frac{2f_1 \sin(2\pi f_1 t)}{2\pi f_1 t}$$

An IIR Filter May Not be Hard to Implement

IIR

Ex : $h[n] = (0.9)^n$, for $n \geqq 0$, $h[n] = 0$, otherwise

$$y[n] = x[n] * h[n]$$

Z transform (page 29)

$$Y(z) = X(z)H(z)$$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} 0.9^n z^{-n}$$

$$= \frac{1}{1-0.9z^{-1}}$$

$$Y(z) = \frac{X(z)}{1-0.9z^{-1}}$$

$$Y(z) - 0.9z^{-1}Y(z) = X(z)$$

inverse z $\downarrow$ transform

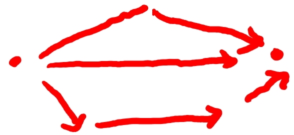
$$y[n] - 0.9\,y[n-1] = x[n]$$

$$y[n] = 0.9\,y[n-1] + x[n]$$

Part 2: Homomorphic Signal Processing

- 概念：把 convolution 變成 addition

$$y[n] = x[n] * h[n]$$
$$\downarrow FT$$
$$Y(f) = X(f)H(f)$$
$$\downarrow \log$$
$$\log(Y(f)) = \log(X(f)) + \log(H(f))$$



cepstrum

Part 3: Applications of DSP

filter design, data compression (image, video, text), acoustics (speech, music), image analysis (structural similarity, sharpness), 3D accelerometer

- **Part 4: Fast Algorithms**
- Basic Implementation Techniques

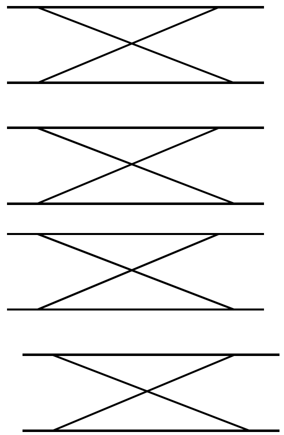
Example: one complex number multiplication

= ? Real number multiplication.

Trade-off: “Multiplication” takes longer than “addition”

- FFT and Convolution

Due to the Cooley-Tukey algorithm (butterflies),
the complexity of the FFT is:



The complexity of the convolution is: 3個 DFTs, $O(N \log_2 N)$

• Part 5: Orthogonal Transforms

DFT 的兩個主要用途:

Question: DFT 的缺點是什麼？ $DFT\left(x[n]\right)=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi mn}{N}}$

正負變化

- Walsh Transform
  (CDMA)

  Example:
  8-point Walsh transform

$$
\begin{array}{cc}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}
&
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}
\end{array}
$$

- Number Theoretic Transform

- Orthogonal Frequency-Division Multiplexing (OFDM)

- Code Division Multiple Access (CDMA)

Review 1: Four Types of the Fourier Transform

- **四種 Fourier transforms 的比較**

	time domain	frequency domain
(1) Fourier transform	continuous, aperiodic	continuous, aperiodic
(2) Fourier series	continuous, periodic (or continuous, only the value in a finite duration is known)	discrete, aperiodic
(3) discrete-time Fourier transform (DTFT)	discrete, aperiodic	continuous, periodic
(4) discrete Fourier transform (DFT)	discrete, periodic (or discrete, only the value in a finite duration is known)	discrete, periodic

$e^{j2\pi ft}$

$e^{-j2\pi ft}$

(1) Fourier Transform

$$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt \quad , \qquad x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$$

Alternative definitions $\omega=2\pi f$

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt \quad , \qquad x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

(2) Fourier series (suitable for period function)

$$X[m]=\int_{0}^{T}x(t)e^{-j\frac{2\pi m}{T}t}dt \qquad x(t)=T^{-1}\sum_{m=-\infty}^{\infty}X[m]e^{j\frac{2\pi m}{T}t}$$

T: 週期 $x(t)=x(t+T)$

Possible frequencies are to satisfy:

$$e^{j2\pi ft}=e^{j2\pi f(t+T)}=e^{j2\pi ft}e^{j2\pi fT}$$

頻率和 m 之間的關係：$f=\dfrac{m}{T}$

$\dfrac{1}{T}$ 整數倍

if $e^{j2\pi fT}=1$

$2\pi fT=2\pi m$

$f=\dfrac{m}{T}$

(1) $X(f) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi f t} dt$

(3) Discrete-time Fourier transform (DSP 常用) (DTFT)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n\Delta_t} \quad , \quad x[n] = \Delta_t \int_0^{1/\Delta_t} X(f) e^{j2\pi f n\Delta_t} df$$

$1/\Delta_t \rightarrow f_s$

$x[n] = x_a(n\Delta_t)$ $\underline{t = n\Delta_t}$

Δ_t : sampling interval

$$X(f+f_s) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi n\Delta_t(f+f_s)} = \sum_n x[n] e^{-j2\pi n\Delta_t f} e^{-j2\pi n \cancel{\Delta_t f_s}} = X(f)$$

Note: $X(f) = X(f+f_s)$

where $f_s = 1/\Delta_t$ (sampling frequency)

It can be rewritten as:

$\Delta_t df = dF$

$$X(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi nF} \qquad x[n] = \int_0^1 X(F) e^{j2\pi nF} dF = \int_{-1/2}^{1/2} X(F) e^{j2\pi nF} dF$$

$X(F) = X(F+1)$

$$F = \frac{f}{f_s} = f\Delta_t$$

(normalized frequency, see page 26)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n\Delta_t} \qquad x[n] = \frac{\Delta_t}{2\pi} \int_0^{2\pi/\Delta_t} X(\omega) e^{j\omega n\Delta_t} d\omega$$

(1) $X(f) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j2\pi f t}\, dt$

(4) Discrete Fourier transform (DFT) (DSP 常用)

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi mn}{N}}, \qquad x[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m] e^{j\frac{2\pi mn}{N}}$$

$t = n\Delta t$

$f = m\Delta f = \frac{m}{N\Delta t}$

$e^{-j2\pi f t} = e^{-j2\pi n\Delta t \frac{m}{N\Delta t}} = e^{-j\frac{2\pi nm}{N}}$

$dt\, df = \Delta t\, \Delta f = \frac{1}{N}$

$X[m+N] = \sum_n x[n] e^{-j\frac{2\pi}{N}(m+N)n} = \sum_n x[n]\, e^{-j\frac{2\pi}{N}mn} \times e^{-j2\pi n} = X[m]$

Note: $x[n] = x[n+N]$, $\qquad X[m] = X[m+N]$

If $x[n]$ was sampled from $x_c(t)$, $x[n] = x_c(n\Delta_t)$, then

$$x_c(n\Delta_t) = x_c(n\Delta_t + N\Delta_t)$$

頻率和 m 之間的關係：$f = \dfrac{m}{N\Delta_t} = \dfrac{m}{N} f_s$

i.e., $f = m\Delta_f \quad where \quad \Delta_f = \dfrac{1}{N\Delta_t}$

Review 2: Normalized Frequency

(1) Definition of **normalized frequency *F***:

$$F = \frac{f}{f_s} = f\,\Delta_t = \frac{\omega \Delta_t}{2\pi}$$

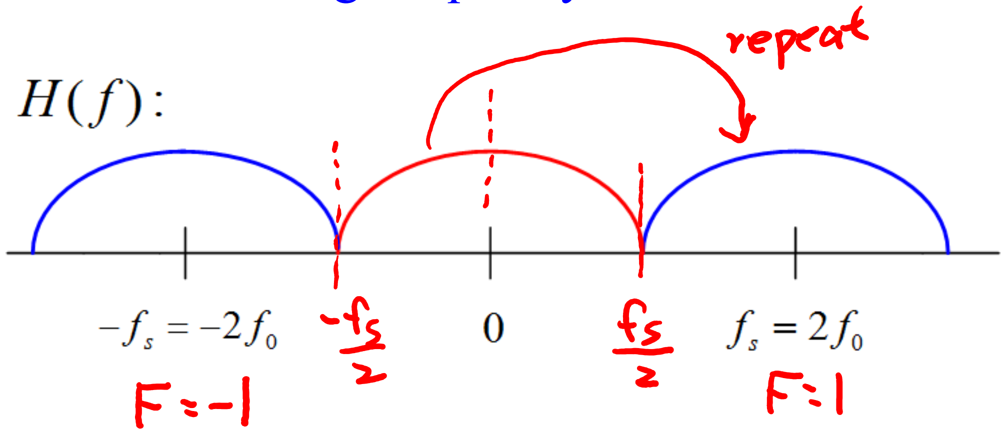
where $f_s = 1/\Delta_t$ (sampling frequency)

Δ_t : sampling interval

(2) folding frequency f_0

$$f_0 = \frac{f_s}{2}$$

若以 normalized frequency 來表示，
folding frequency = 1/2



$H(f) = H(f+f_s)$
$H(F) = H(F+1)$

For the discrete time Fourier transform

(1) $G(f) = G(f + f_s) \longrightarrow$ i.e., $G(F) = G(F + 1)$.

(2) If $g[n]$ is real $\longrightarrow$ $G(F) = G^*(-F)$ (* means conjugation)

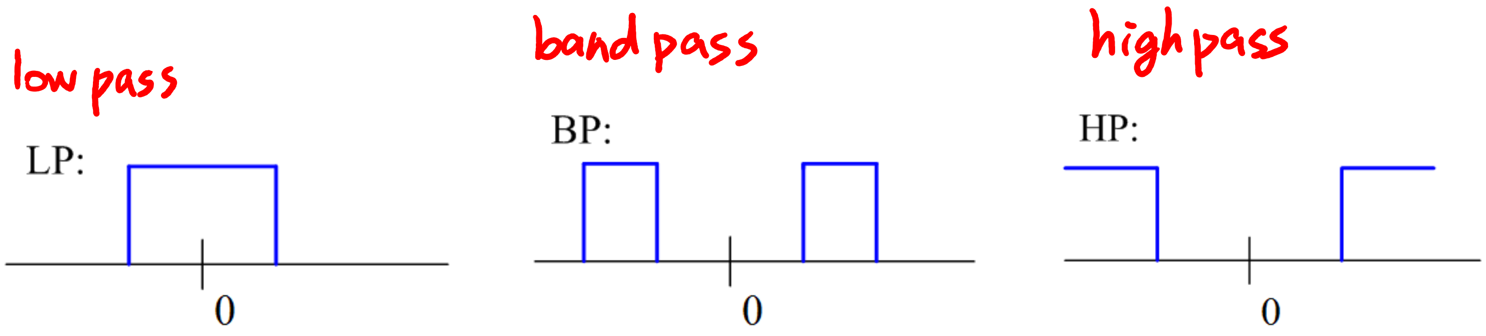
只需知道 $G(F)$ for $0 \le F \le ½$ (即 $0 < f < f_0$)

就可以知道全部的 $G(F)$

(3) If $g[n] = g[-n]$ (even) $\longrightarrow$ $G(F) = G(-F)$,

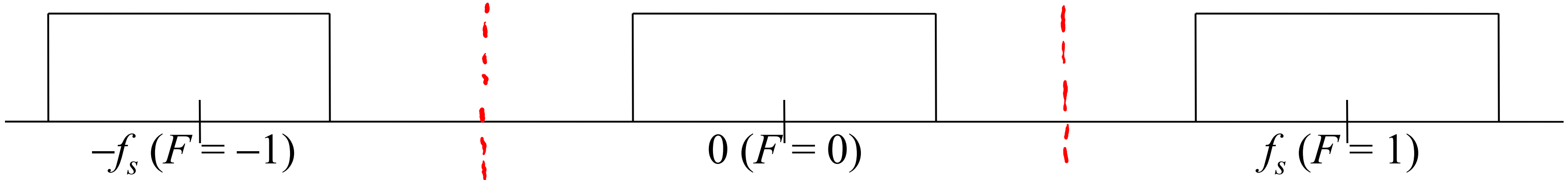
$g[n] = -g[-n]$ (odd) $\longrightarrow$ $G(F) = -G(-F)$

Analog filter: $H(f)$

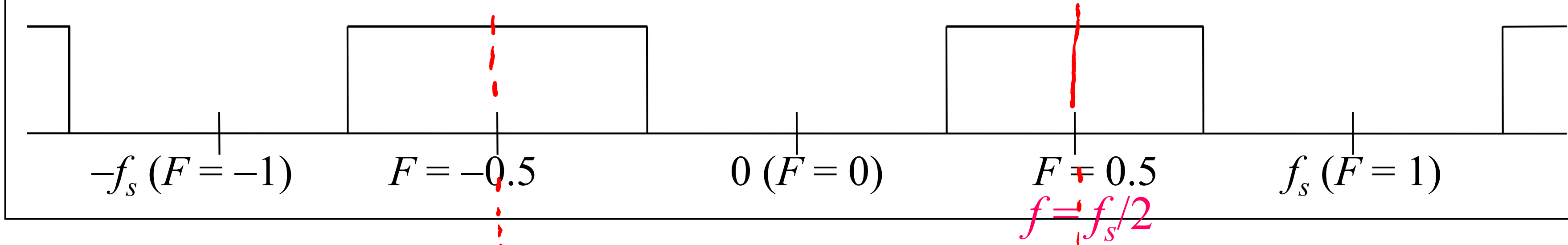


- Discrete time Fourier transform of the lowpass, highpass, and band pass filters

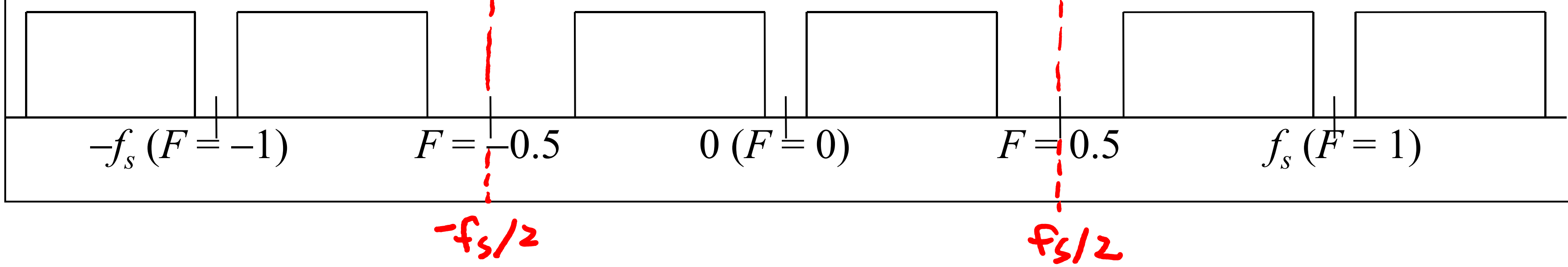
low pass filter（ **pass band 在 fs 的整數倍附近** ）



high pass filter



band pass filter



Review 3: Z Transform and Laplace Transform

- ***Z*-Transform**

suitable for discrete signals

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

Compared with the discrete time Fourier transform:

$$G(f) = \sum_{n=-\infty}^{\infty} g[n] e^{-j2\pi f n \Delta_t}$$

$z = e^{j2\pi f \Delta_t}$

$|z| = 1$

• Laplace Transform

suitable for continuous signals

One-sided form $$G(s)=\int_0^{\infty} g(t)e^{-st}dt$$

Two-sided form $$G(s)=\int_{-\infty}^{\infty} g(t)e^{-st}dt$$

Compared with the Fourier transform:

$$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$$

$s = j2\pi f$

pure imaginary

Review 4: IIR Filter Design

Two types of digital filter:

(1) IIR filter (infinite impulse response filter)

(2) FIR ~~filer~~ filter (finite impulse response filer)

There are 3 popular methods to design the IIR filter

- bilinear transform
- impulse invariant
- step invariant

Advantage:

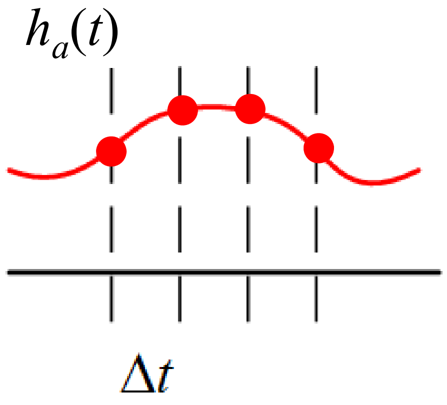
Disadvantage:

Method 1: Impulse Invariance

白話一點，就是直接做 sampling

analog filter $h_a(t)$ digital filter $h[n]$

$$\boxed{h[n] = h_a(n\Delta_t)}$$



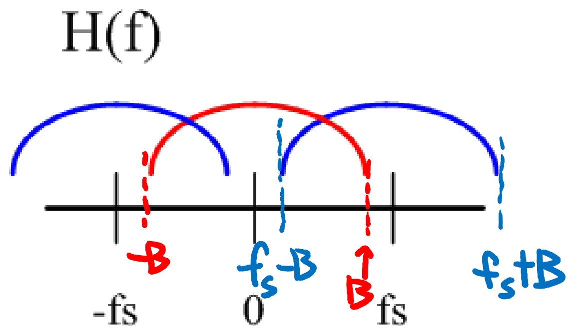
Advantage : Simple

Disadvantage : (1) infinite

(2) aliasing effect

$f_s - B < B$

$f_s < 2B$



$$f_s < 2B$$

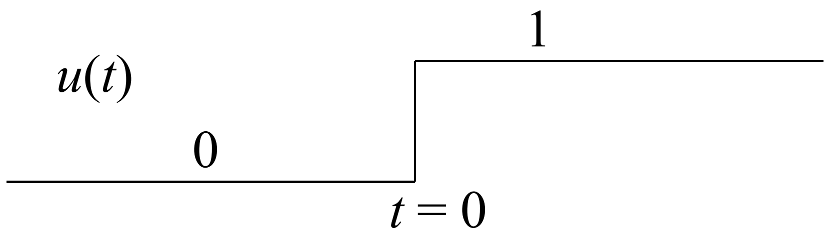
$$H(f) = f_s \sum_{n=-\infty}^{\infty} H_a(f + nf_s)$$

Method 2: Step Invariance

對 step function 的 response 作 sampling

analog filter $h_a(t)$ digital filter $h[n]$

step function (continuous form)



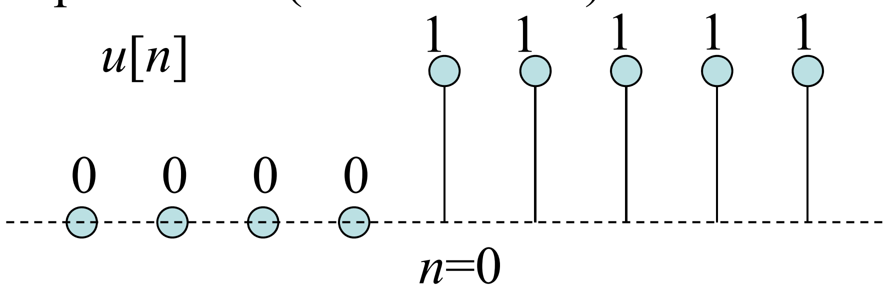
Laplace transform of $u(t)$:

$$\frac{1}{s}$$

Fourier transform of $u(t)$:

$$\frac{1}{j2\pi f}$$

step function (discrete form)



Z transform of $u[n]$:

$$\frac{1}{1-z^{-1}}$$

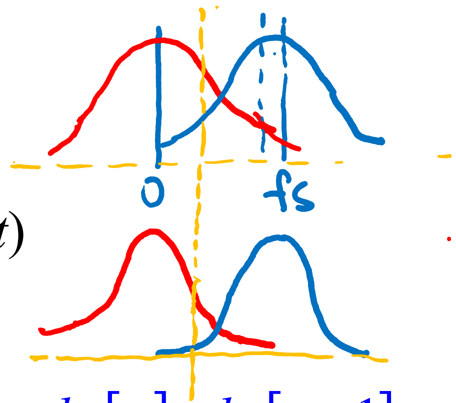
$u(t-\tau)=1$ if $t-\tau \geq 0$
$\tau \leq t$

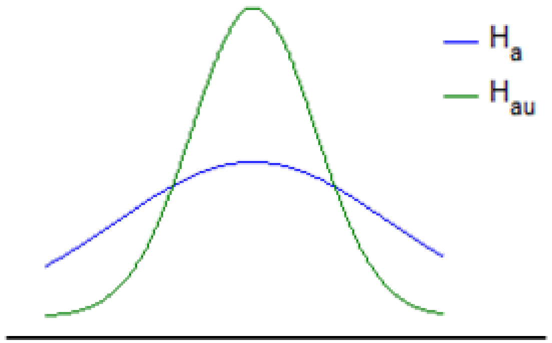
Step 1 Calculate the convolution of $h_a(t)$ and $u(t)$

$$h_{a,u}(t)=h_a(t)*u(t)=\int_{-\infty}^{\infty}h_a(\tau)u(t-\tau)d\tau=\int_{-\infty}^{t}h_a(\tau)d\tau$$

$$H_{a,u}(f)=\frac{H_a(f)}{j2\pi f}$$

(其實就是對 $h_a(t)$ 做積分)





Step 2 Perform sampling for $h_{a,u}(t)$

$$h_u[n]=h_{a,u}(n\Delta_t)$$

Step 3 Calculate $h[n]$ from $h[n]=h_u[n]-h_u[n-1]$

Note: Since $h_u[n]=h[n]*u[n]$

$$H_u(z)=\frac{1}{1-z^{-1}}H(z)$$

$$H(z)=\left(1-z^{-1}\right)H_u(z)$$

so $h[n]=h_u[n]-h_u[n-1]$

Advantage of the step invariance method:

＊主要 Advantage:

Disadvantage of the step invariance method:

較為間接，設計上稍微複雜

Method 3: Bilinear Transform

Suppose that we have known an analog filter $h_a(t)$ whose frequency response is $H_a(f)$.

To design the digital filter $h[n]$ with the frequency response $H(f)$,

$$\boxed{H\left(f_{new}\right) = H_a\left(f_{old}\right)}$$

$$f_{old} \in (-\infty, \infty)$$

$$f_{new} \in (-f_s/2,\ f_s/2)$$

$f_s = 1/\Delta_t$ (sampling frequency)

- The relation between f_{new} and f_{old} is determined by the mapping function

$$\boxed{s = c\frac{1-z^{-1}}{1+z^{-1}}}$$

s: index of the Laplace transform

z: index of the Z transform

c: some constant

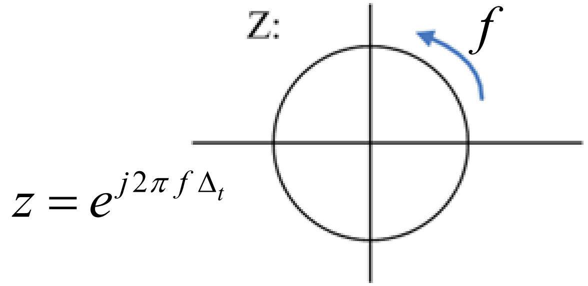
Step 1 Step 3

$$h_a(t) \xrightarrow{Laplace} H_a(s) \to H(z) = H_a\left(c\frac{1-z^{-1}}{1+z^{-1}}\right) \xrightarrow[Z-transform]{Inverse} h[n]$$

step 2

$s = c\dfrac{1-z^{-1}}{1+z^{-1}}$ $\quad s = j2\pi f_{old}$ $\quad z = e^{j2\pi f_{new}\Delta_t}$ 代入

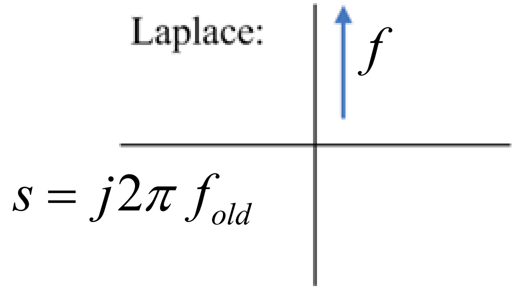
參考 page 29、page 30



$$j2\pi f_{old} = c\frac{1-e^{-j2\pi f_{new}\Delta_t}}{1+e^{-j2\pi f_{new}\Delta_t}} = c\frac{e^{j\pi f_{new}\Delta_t}-e^{-j\pi f_{new}\Delta_t}}{e^{j\pi f_{new}\Delta_t}+e^{-j\pi f_{new}\Delta_t}}$$

$$= c\frac{j\sin(\pi f_{new}\Delta_t)}{\cos(\pi f_{new}\Delta_t)}$$

$$2\pi f_{old} = c\tan(\pi f_{new}\Delta_t)$$



$$\boxed{f_{new} = \frac{1}{\pi\Delta_t}\operatorname{atan}\left(\frac{2\pi}{c}f_{old}\right) = \frac{f_s}{\pi}\operatorname{atan}\left(\frac{2\pi}{c}f_{old}\right)}$$

- Suppose that the Laplace transform of the analog filter $h_a(t)$ is $H_{a,L}(s)$

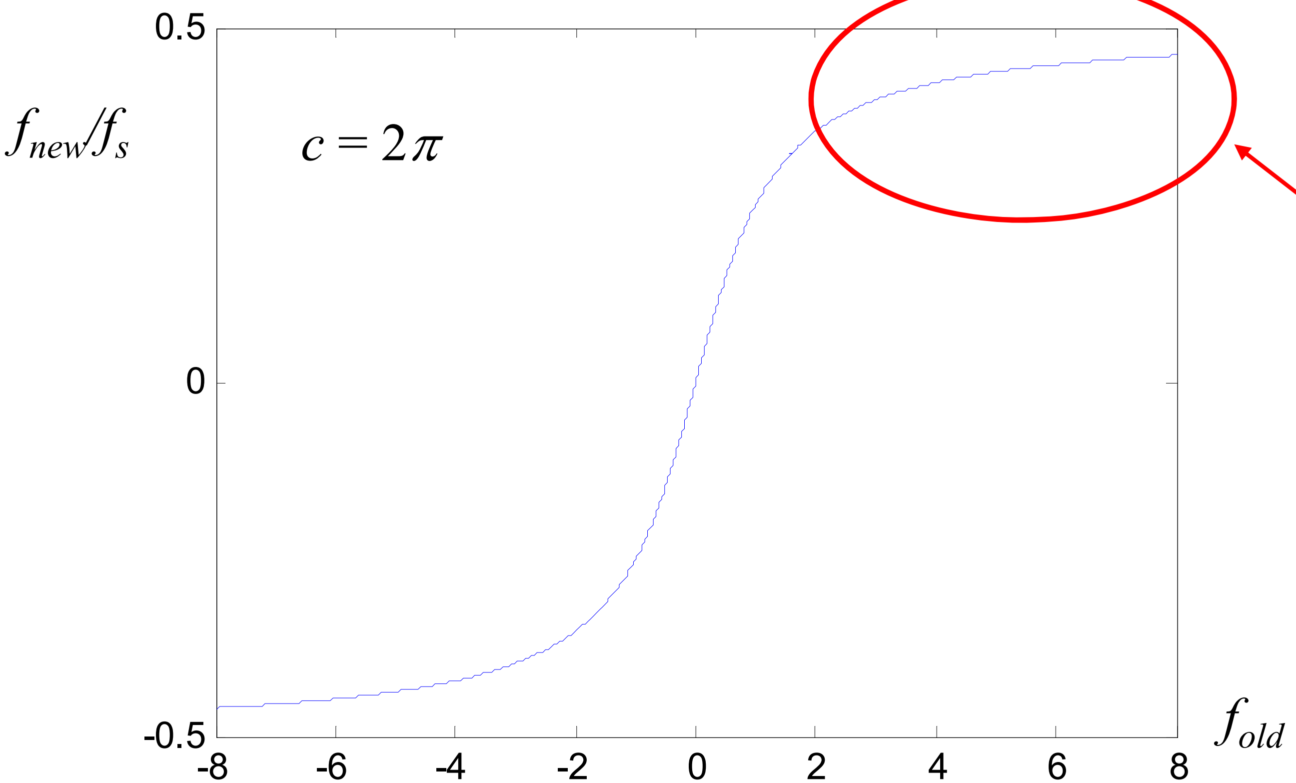
  The Z transform of the digital filter $h[n]$ is $H_z(z)$

$$\boxed{H_z(z) = H_{a,L}\left(c\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$f_{new} = \frac{f_s}{\pi}\,\mathrm{atan}\left(\frac{2\pi}{c} f_{old}\right)$$

$$\frac{f_s}{\pi}\left(-\frac{\pi}{2}\right) < f_{new} < \frac{f_s}{\pi}\left(\frac{\pi}{2}\right)$$
$$-\frac{f_s}{2} < f_{new} < \frac{f_s}{2}$$

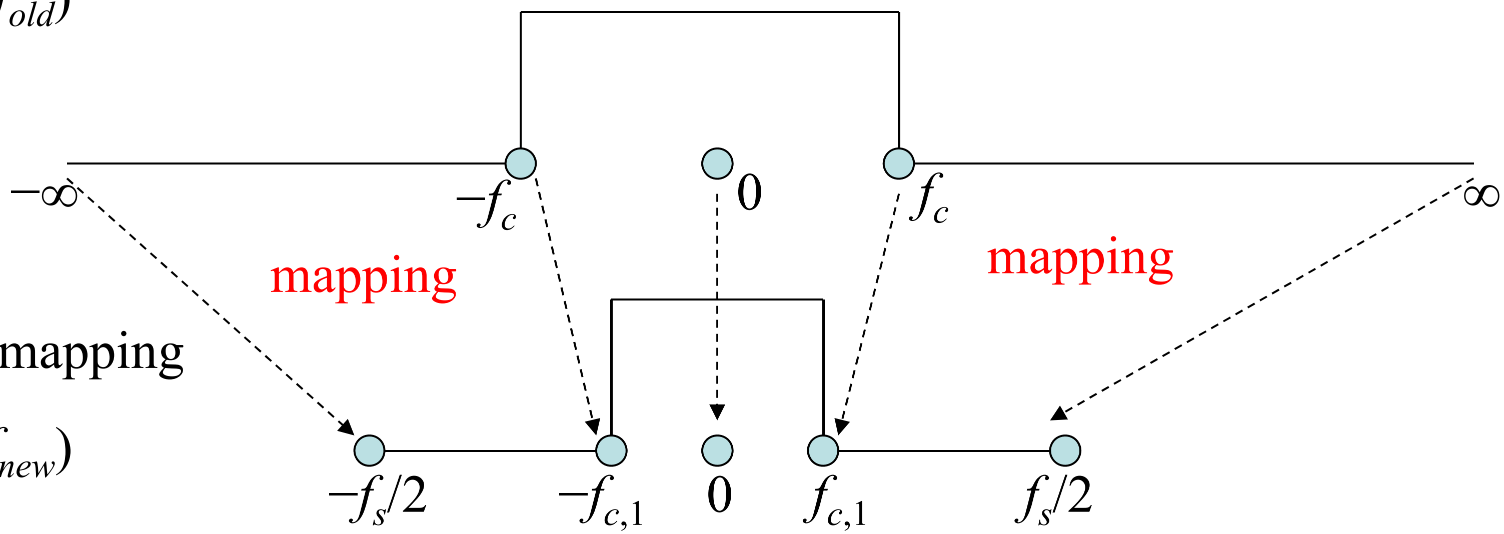
f_{old}	$-\infty$	0	∞	1
f_{new}	$-f_s/2$	0	$f_s/2$	$\frac{f_s}{\pi}\,\mathrm{atan}\left(\frac{2\pi}{c}\right)$



nonlinear distortion

analog filter

$H_a(f_{old})$



after mapping

$H(f_{new})$

$$f_{new} = \frac{f_s}{\pi} \operatorname{atan}\left(\frac{2\pi}{c} f_{old} \right)$$

Advantage of the bilinear transform

$B = f_{c,1}$

$f_s > 2B = 2f_{c,1}$

Disadvantage of the bilinear transform

附錄一： 學習 DSP 知識把握的要點

(1) Concepts: 這個方法的核心概念、基本精神是什麼

(2) Comparison: 這方法和其他方法之間，有什麼相同的地方？
有什麼相異的地方

(3) Advantages: 這方法的優點是什麼

(3-1) Why? 造成這些優點的原因是什麼

(4) Applications: 這個方法要用來處理什麼問題，有什麼應用

(5) Disadvantages: 這方法的缺點是什麼

(5-1) Why? 造成這些缺點的原因是什麼

(6) Innovations: 這方法有什麼可以改進的地方
或是可以推廣到什麼地方

2. Digital Filter Design (A)

任何可以用來去除 noise 作用的 operation，皆被稱為 filter

甚至有部分的 operation，雖然主要功用不是用來去除 noise，但是可以用 FT + multiplication + IFT 來表示，也被稱作是 filter

||

convolution, LTI system

linear time invariant

Reference

[1] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

[2] D. G. Manolakis and V. K. Ingle, *Applied Digital Signal Processing*, Cambridge University Press, Cambridge, UK. 2011.

[3] B. A. Shenoi, *Introduction to Digital Signal Processing and Filter Design*, Wiley-Interscience, N. J., 2006.

[4] A. A. Khan, *Digital Signal Processing Fundamentals*, Da Vinci Engineering Press, Massachusetts, 2005.

[5] S. Winder, *Analog and Digital Filter Design*, 2nd Ed., Amsterdam, 2002.

◉ 2-A Classification for Filters

- filter
  - digital filter
    - IIR (infinite impulse response)
      - bilinear transform
      - impulse invariant
      - step invariant
      - others
    - FIR (finite impulse response)
      - 關心平均誤差
        (1) MSE (mean square error)
      - 最小化最大誤差
        (2) Minimax (minimize the maximal error)
      - (3) frequency sampling
      - (4) others
  - analog filter

由 數位類比分類

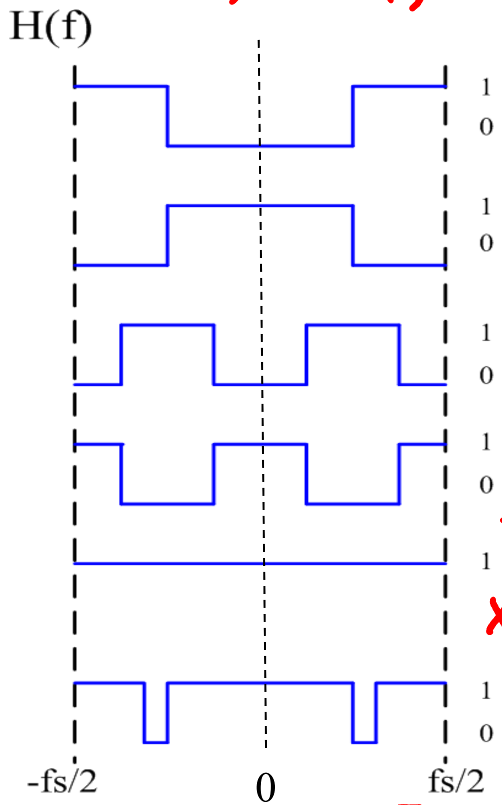
由 response 是否有限長分類

由設計方法分類

Classification for filters (依型態分)

- pass-stop filter
  - 1 0
  - high pass filter
  - low pass filter
  - band pass filter
  - band stop filter
  - all pass filter
  - Notch filter — = bandstop filter with very narrow stop band
- pattern recognition: matched filter
- noise removal: Wiener filter
- multi path: equalizer filter
- others
  - odd: differentiation — $H(f) = j2\pi f$
  - odd: integration — $H(f) = \frac{1}{j2\pi f}$
  - Hilbert transform — $H(f) =$ j, $-j$
  - smoother
  - edge detection
  - Kalman filter
  - particle filter
  - (Kalman filter, particle filter) prediction

$H(f) = H(-f)$



$F = -1/2$ $\qquad$ $F = 1/2$

$x[n] \to x[n-n_0]$

$X(F) \to e^{-j2\pi F n_0} X(F)$

$H(F) = e^{-j2\pi F n_0}$

$|H(F)| = 1$

⦿ 2-B FIR Filter Design

FIR filter: impulse response is nonzero at finite number of points

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N$$

($h[n]$ has N points, N is a finite number)

$h[n]$ is causal $h[n] = 0$ for $n < 0$

- FIR is more popular because its impulse response is finite.

If $h[n] = h[-n]$ then $H(f) = H(-f)$

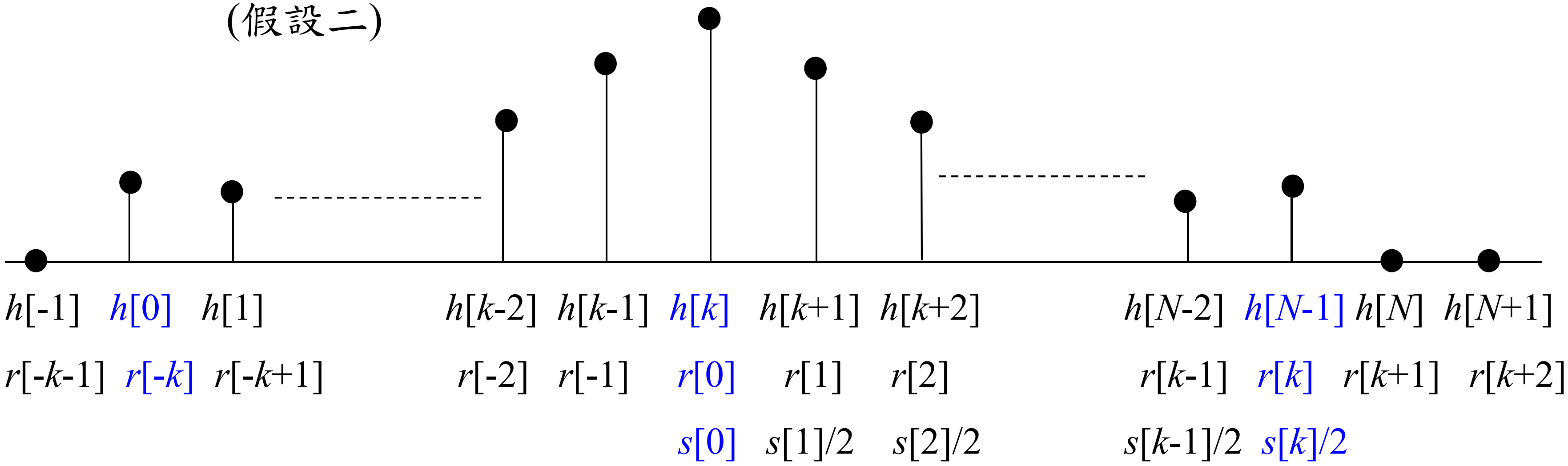
$H(f) = FT(h[n])$

(假設一)

Specially, when $h[n]$ is $\boxed{\text{even}}$ symmetric $h[n] = h[N-1-n]$

and $\boxed{N \text{ is an odd number}}$ $k = \frac{N-1}{2}$ is an integer

(假設二)



$r[n] = r[-n]$

(a) $r[n] = h[n + k]$, where $k = (N-1)/2$. $R(F) = e^{j2\pi kF} H(F)$

(b) $s[0] = r[0]$, $s[n] = 2r[n]$ for $0 < n \le k$.

Impulse Response of the FIR Filter:

$h[n]$ ($h[n] \neq 0$ for $0 \le n \le N-1$)

$r[n] = h[n+k]$, $k = (N-1)/2$ ($r[n] \neq 0$ for $-k \le n \le k$, see page 45)

Suppose that the filter is **even**, $r[n] = r[-n]$.

Set $s[0] = r[0]$, $s[n] = 2r[n]$ for $n \neq 0$.

Then, the discrete-time Fourier transform of the filter is

$$H(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi F n}$$

($F = f\Delta_t$ is the **normalized frequency**)

See page 26

$$R(F) = e^{j2\pi F k} H(F)$$

$$R(F) = \sum_{n=-k}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=-k}^{-1} r[n] e^{-j2\pi F n} + r[0] + \sum_{n=1}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^{k} r[-n] e^{j2\pi F n} + r[0] + \sum_{n=1}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^{k} r[n]\left(e^{j2\pi F n} + e^{-j2\pi F n}\right) + r[0]$$

$$\boxed{R(F) = \sum_{n=0}^{k} s[n] \cos(2\pi n F)}$$

$2r[n]\cos(2\pi n F)$

⦿ 2-C Least MSE Form and Minimax Form FIR Filters

(1) least MSE (mean square error) form

(關心 平均 誤差)

H(F) is replaced by R(F)

$$\text{MSE} = \int_{-1/2}^{1/2} \left| \cancel{H(F)} - H_d(F) \right|^2 dF$$

R(F)

L₂-norm problem

$H(F)$: the spectrum of the filter we obtain

$H_d(F)$: the spectrum of the desired filter

$F = f/f_s$: normalized frequency; f_s: sampling frequency

(2) mini-max (minimize the maximal error) form

(關心 最大 誤差)

L_∞-norm problem

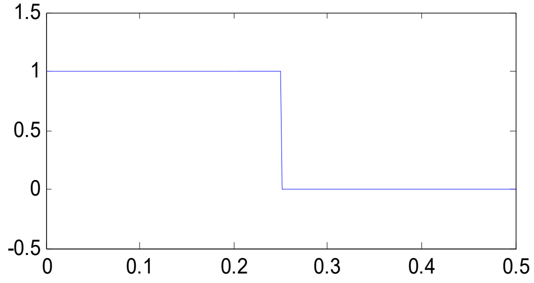
R(F)

maximal error: $\underset{F}{Max} \left| \cancel{H(F)} - H_d(F) \right|$

$-1/2 < F < 1/2$

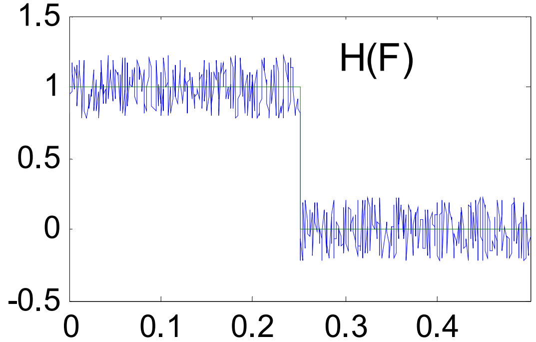
The transition band is always ignored

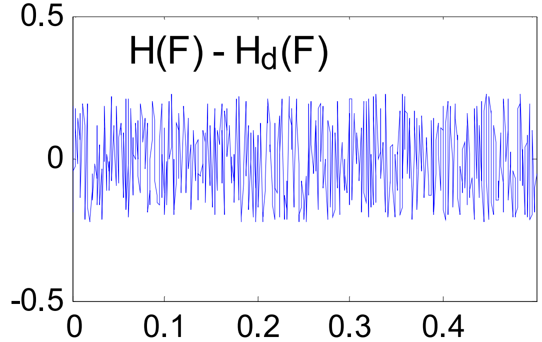
Example: desired output $H_d(F)$



F

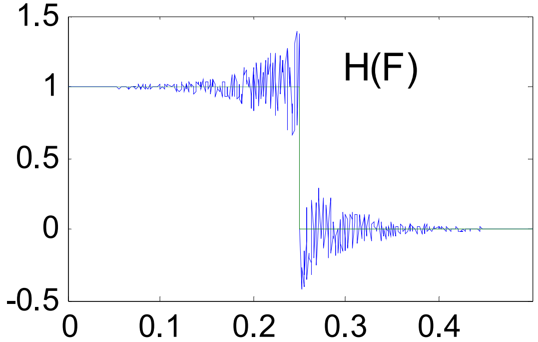
(A) larger MSE, but smaller maximal error

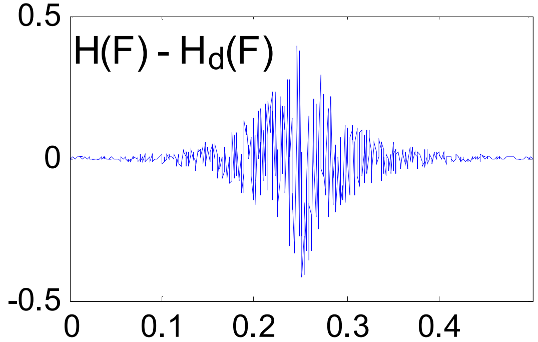




Max(|H(F)−Hd(F)|)
≅ 0.2

(B) smaller MSE, but larger maximal error





Max(|H(F)−Hd(F)|)
≅ 0.4

◉ 2-D Review: FIR Filter Design in the MSE Sense

$$R(F)=\sum_{n=0}^{k} s[n]\cos(2\pi nF)$$

$$MSE = f_s^{-1}\int_{-f_s/2}^{f_s/2}\left|R(f)-H_d(f)\right|^2 df = \int_{-1/2}^{1/2}\left|R(F)-H_d(F)\right|^2 dF$$

$F = f/f_s$

$$=\int_{-1/2}^{1/2}\left|\sum_{n=0}^{k} s[n]\cos(2\pi nF)-H_d(F)\right|^2 dF$$

$$\sum_{\tau=0}^{k} s[\tau]\int_{-\frac{1}{2}}^{\frac{1}{2}}\cos(2\pi nF)\cos(2\pi\tau F)\,dF - \int_{-1/2}^{1/2}\cos(2\pi nF)H_d(F)\,dF$$

$$=\int_{-1/2}^{1/2}\left(\sum_{v=0}^{k} s[v]\cos(2\pi vF)-H_d(F)\right)\left(\sum_{\tau=0}^{k} s[\tau]\cos(2\pi\tau F)-H_d(F)\right)dF$$

$v = n$

$\tau = n$

$$\sum_{v=0}^{k} s[v]\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi vF)\,dF - \int_{-\frac{1}{2}}^{\frac{1}{2}} H_d(F)\cos(2\pi nF)\,dF$$

$$\frac{\partial MSE}{\partial s[n]}=\int_{-1/2}^{1/2}\cos(2\pi nF)\left(\sum_{\tau=0}^{k} s[\tau]\cos(2\pi\tau F)-H_d(F)\right)dF$$

$$+\int_{-1/2}^{1/2}\left(\sum_{v=0}^{k} s[v]\cos(2\pi vF)-H_d(F)\right)\cos(2\pi nF)\,dF=0$$

$$\frac{\partial MSE}{\partial s[n]}=\cancel{2}\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF-\cancel{2}\int_{-1/2}^{1/2}H_d(F)\cos(2\pi nF)\,dF=0$$

$$\frac{\partial MSE}{\partial s[n]} = 2\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF - 2\int_{-1/2}^{1/2} H_d(F)\cos(2\pi nF)\,dF = 0$$

From the facts that orthogonal

$\cos a \cos b = \frac{1}{2}(\cos(a+b)+\cos(a-b))$

$$\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = 0 \qquad \text{when } n \neq \tau,$$

$$\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = 1/2 \qquad \text{when } n = \tau,\ n \neq 0,$$

$$\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = 1 \qquad \text{when } n = \tau,\ n = 0.$$

$$\int_{-\frac{1}{2}}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = \frac{1}{2}\int_{-1/2}^{1/2}\cos(2\pi(n+\tau)F)\,dF + \frac{1}{2}\int_{-1/2}^{1/2}\cos(2\pi(n-\tau)F)\,dF$$

Therefore,

$$= \frac{1}{4\pi(n+\tau)}\sin(2\pi(n+\tau)F)\Big|_{-1/2}^{1/2} + \frac{\sin(2\pi(n-\tau)F)}{4\pi(n-\tau)}\Big|_{-1/2}^{1/2}$$

$$= 0 \quad (\because \sin(\pi mF)=0 \text{ if } m \text{ is an integer}) \quad (n \neq \tau$$

$$\frac{\partial MSE}{\partial s[0]} = 2s[0] - 2\int_{-1/2}^{1/2} H_d(F)\,dF = 0$$

$$\frac{\partial MSE}{\partial s[n]} = s[n] - 2\int_{-1/2}^{1/2}\cos(2\pi nF) H_d(F)\,dF = 0 \qquad \text{for } n \neq 0.$$

If $n=\tau$, $\tau \neq 0$: $\int \cos(2\pi nF)\cos(2\pi\tau F)\,dF = 0 + \frac{1}{2}\int_{-1/2}^{1/2} 1\,dF = 1/2$

If $n=\tau=0$, $\int \cos(2\pi nF)\cos(2\pi\tau F)\,dF = 1/2 + 1/2 = 1$

Minimize MSE $\rightarrow$ Make $\dfrac{\partial MSE}{\partial s[n]} = 0$ for all n's

$$\therefore \quad \boxed{s[0] = \int_{-1/2}^{1/2} H_d(F)\,dF}\,, \qquad \boxed{s[n] = 2\int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F)\,dF}\,.$$

Finally, set $h[k] = \mathrm{s}[0]$,

$h[k+n] = \mathrm{s}[n]/2$, $h[k-n] = \mathrm{s}[n]/2$ for $n = 1, 2, 3, \ldots., k$,

$h[n] = 0$ for $n < 0$ and $n \geq N$.

Then, $h[n]$ is the impulse response of the designed filter.

◉ 2-E FIR Filter Design in the Mini-Max Sense

It is also called "Remez-exchange algorithm"

or "Parks-McClellan algorithm"

References

[1] T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

[2] J. H. McClellan, T. W. Parks, and L. R. Rabiner "A computer program for designing optimum FIR linear phase digital filter", *IEEE Trans. Audio-Electroacoustics*, vol. 21, no. 6, Dec. 1973.

[3] F. Mintz and B. Liu, "Practical design rules for optimum FIR bandpass digital filter", *IEEE Trans. ASSP*, vol. 27, no. 2, Apr. 1979.

[4] E. Y. Remez, "General computational methods of Chebyshev approximation: The problems with linear real parameters," AEC-TR-4491. ERDA Div. Phys. Res., 1962.

Two constraints

Suppose that:

① Filter length = N, N is odd, $N = 2k+1$.

② Frequency response of the **desired filter**: $H_d(F)$ is an even function

(F is the normalized frequency)

③ The weighting function is $W(F)$

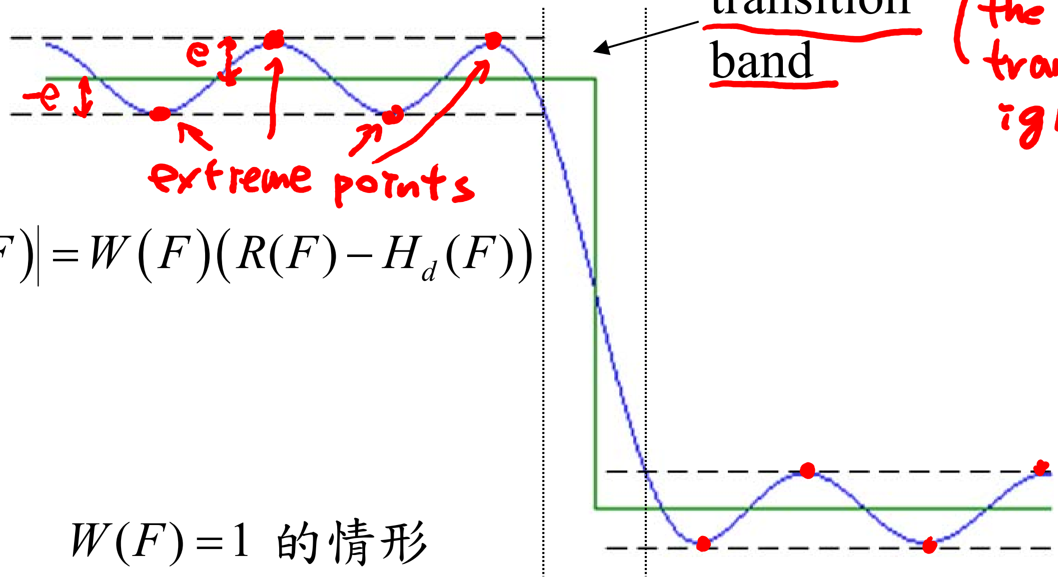
用Mini-Max方法所設計出的filters，一定會滿足以下二個條件

If $err(F) = W(F)\left(R(F) - H_d(F)\right)$

(1) $err(F)$有 $k+2$個以上的extreme points

Error 的 local maximal
local minimum

(2) 在extreme points上， $\left|err(F_m)\right| = \left|W(F_m)\left(R(F_m) - H_d(F_m)\right)\right|$是定值



(the error in the transition band is ignored)

$\left|err(F)\right| = W(F)\left(R(F) - H_d(F)\right)$

$W(F) = 1$ 的情形

證明可參考

T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

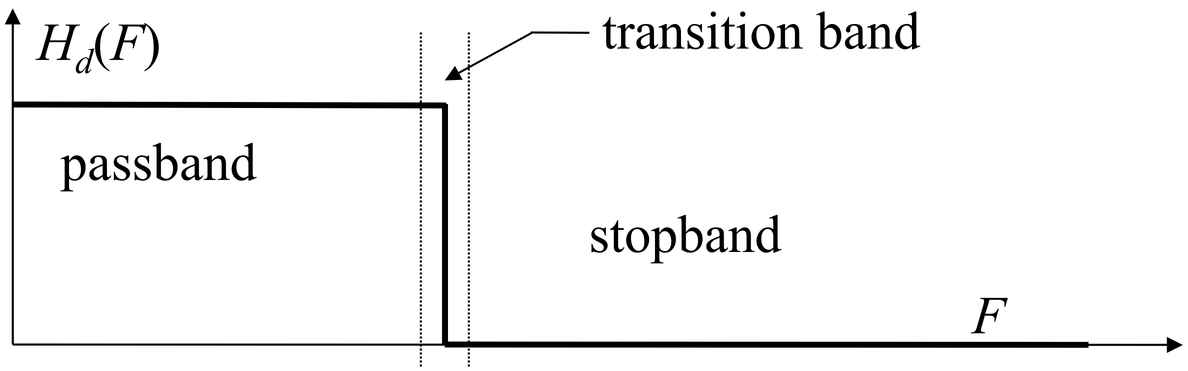
- Generalization for Mini-Max Sense by weight function

maximal error: $$\underset{F,\,F\notin \text{transition band}}{Max}\left|R(F)-H_d(F)\right|$$

weighted maximal error: $$\underset{F,\,F\notin \text{transition band}}{Max}\left|W(F)[R(F)-H_d(F)]\right|$$

where $W(F)$ is the weight function.

The weight function is designed according to which band is more important.



Q: How do we choose $W(F)$ when SNR $\uparrow$?

Example: If we treat the passband the same important as the stopband.
$W(F) = 1$ in the passband, $W(F) = 1$ in the stopband

Q1: $W(F) = 1$ in the passband, $W(F) < 1$ in the stopband 代表什麼？

Q2: $W(F) < 1$ in the passband, $W(F) = 1$ in the stopband 代表什麼？

Q3: 如何用來壓縮特定區域 (如 transition band 附近) 的 error？

Q4: Weighting function 的概念可否用在 MSE sense ？

$$
\begin{tikzcd}[row sep=small]
\text{guess } F_m & \text{Step1} \arrow[d] \\
s[n] & \text{Step2} \arrow[d] \\
err(F) & \text{Step3} \arrow[d] \\
\text{new } F_m & \text{Step4} \arrow[d] \\
E_1\text{-}E_0 & \text{Step5} \arrow[d, "0 \le E_1 - E_0 \le \Delta"] \arrow[uuu, bend right, swap] \\
h[n] & \text{Step6}
\end{tikzcd}
$$

◉ 2-F Mini-Max Design Process

(Step 1): Choose arbitrary k+2 extreme frequencies

(denoted by F_0, F_1, F_2, ….., F_{k+1})

Note: (1) $0 \le F \le 0.5$,

(2) Exclude the transition band.

(3) The extreme points cannot be all in the stop band.

Set E_1 (error) $\rightarrow \infty$

Extreme frequencies:

The locations where the error is maximal.

$$[R(F_0) - H_d(F_0)]W(F_0) = -e \qquad [R(F_1) - H_d(F_1)]W(F_1) = e$$

$$[R(F_2) - H_d(F_2)]W(F_2) = -e \qquad [R(F_3) - H_d(F_3)]W(F_3) = e$$

$$\vdots$$

$$[R(F_{k+1}) - H_d(F_{k+1})]W(F_{k+1}) = (-1)^{k+2} e$$

(參考 page 54)

(Step 2): From page 46, $[R(F_m) - H_d(F_m)]W(F_m) = (-1)^{m+1}e$ (where m = 0, 1, 2, ….. , k+1) can be written as

$$\boxed{\sum_{n=0}^{k} s[n]\cos(2\pi F_m n) + (-1)^m W^{-1}(F_m)e = H_d(F_m)}$$

where m = 0, 1, 2, ….. , k+1. (k+2 equations)

Expressed by the matrix form:

k+2 unknowns s[n] (n=0~k), e

change with iteration

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \begin{array}{c} s[n] \\ n=0 \end{array} & n=1 & n=2 & & n=k & e \end{array} \\
\begin{array}{c} m=0 \\ m=1 \\ m=2 \\ \\ m=k \\ m=k+1 \end{array} &
\underbrace{\begin{bmatrix}
1 & \cos(2\pi F_0) & \cos(4\pi F_0) & \cdots & \cos(2\pi k F_0) & 1/W(F_0) \\
1 & \cos(2\pi F_1) & \cos(4\pi F_1) & \cdots & \cos(2\pi k F_1) & -1/W(F_1) \\
1 & \cos(2\pi F_2) & \cos(4\pi F_2) & \cdots & \cos(2\pi k F_2) & 1/W(F_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos(2\pi F_k) & \cos(4\pi F_k) & \cdots & \cos(2\pi k F_k) & (-1)^k/W(F_k) \\
1 & \cos(2\pi F_{k+1}) & \cos(4\pi F_{k+1}) & \cdots & \cos(2\pi k F_{k+1}) & (-1)^{k+1}/W(F_{k+1})
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ \vdots \\ s[k] \\ e \end{bmatrix}}_{S}
=
\underbrace{\begin{bmatrix} H_d[F_0] \\ H_d[F_1] \\ H_d[F_2] \\ \vdots \\ H_d[F_k] \\ H_d[F_{k+1}] \end{bmatrix}}_{H}
\end{array}
$$

Square matrix

<u>Solve $s[0]$, $s[1]$, $s[2]$, ….. , $s[k]$ from the above matrix</u>

(performing the matrix inversion).

$As = H$

$s = A^{-1}H$

(Step 3): Compute err(F) for $0 \le F \le 0.5$, exclude the transition band.

$$err(F) = [R(F) - H_d(F)]W(F) = \{\sum_{n=0}^{k} s[n]\cos(2\pi nF) - H_d(F)\}W(F)$$

Set $W(F) = 0$ at the transition band.

F= [0 : 0.0001 : 0.5] 5001 points

interval

(Step 4): Find k+2 **local maximal (or minimal) points** of err(F)

local maximal point: if $q(\tau) > q(\tau + \Delta_F)$ and $q(\tau) > q(\tau - \Delta_F)$,

then τ is a local maximal of $q(x)$.

local minimal point: if $q(\tau) < q(\tau + \Delta_F)$ and $q(\tau) < q(\tau - \Delta_F)$,

then τ is a local minimal of $q(x)$.

Other rules: Pages 63 and 64 0, 0.5

Denote the local maximal (or minimal) points by $F_0, F_1, \ldots\ldots, F_k, F_{k+1}$

(new extreme points)

These k+2 extreme points could include the boundary points of the transition band

(Step 5):

Set E_0 = Max(|err(F)|).

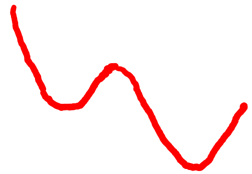
$$\begin{cases} E_0 : \text{現在的} Max\,|\,err(F)\,| \\ E_1 : \text{前一次iteration的} Max\,|\,err(F)| \end{cases}$$

(Case a) If $E_1 - E_0 > \Delta$, or $E_1 - E_0 < 0$ (or the first iteration) $\rightarrow$

set $F_n = P_n$ and $E_1 = E_0$, return to Step 2.

(Case b) If $0 \leq E_1 - E_0 \leq \Delta$ $\rightarrow$ continue to Step 6.

(Step 6):



Set $h[k] = s[0]$,

$h[k+n] = s[n]/2$, $h[k-n] = s[n]/2$ for $n = 1, 2, 3, \ldots, k$

(referred to page 45)

Then $h[n]$ is the impulse response of the designed filter.

◎ 2-G Mini-Max FIR Filter 設計時需注意的地方

(1) Extreme points 不要選在 transition band

Initial guess的extreme points只要注意別取在transition band裡，即能保證 converge，不同的guess會影響converge的速度但不影響結果

(2) E_1 (error of the previous iteration) < E_0 (present error) 時，亦不為收斂

(3) Remember to update $W(F_m)$ and $H_d(F_m)$ according to the locations of F_m.

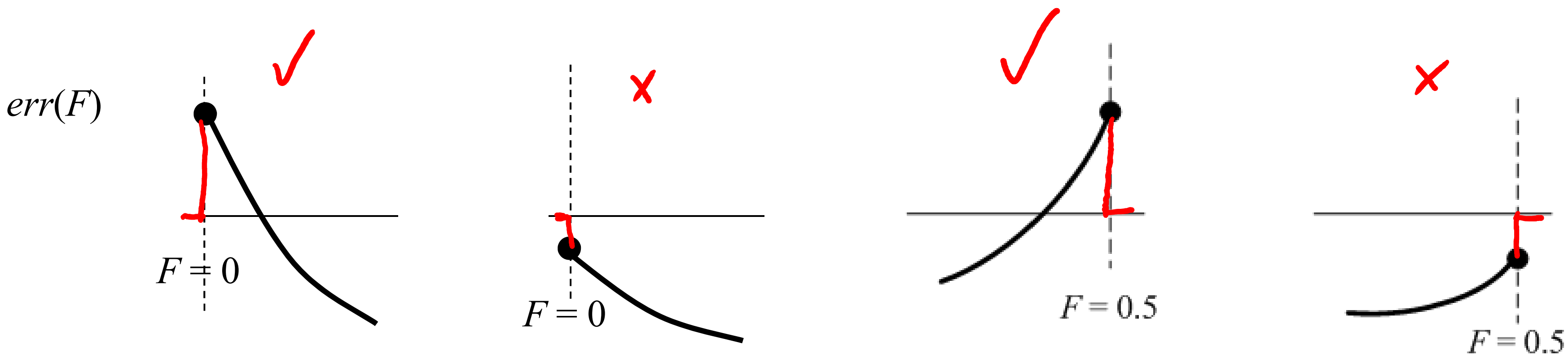
(4) Extreme points 判斷的規則：

(a) The local peaks or local dips that are not at boundaries must be extreme points.

Local peaks: $err(F) > err(F + \Delta_F)$ and $err(F) > err(F - \Delta_F)$

Local dips: $err(F) < err(F + \Delta_F)$ and $err(F) < err(F - \Delta_F)$

(b) For boundary points ($F = 0$, $F = 0.5$)



Add a zero to the outside and conclude whether the point is a local maximum or a local minimum.

(5) 有時，會找到多於 k+2 個 extreme points, 該如何選 $P_0, P_1, \ldots\ldots, P_k, P_{k+1}$

boundaries : 0, 0.5

(a) 優先選擇不在 boundaries 的 extreme points

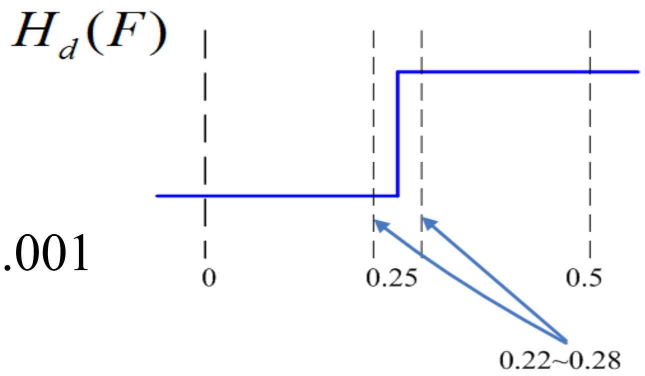
(b) 其次選擇 boundary extreme points 當中 |err(F)| 較大的，

直到湊足 k+2 個 extreme points 為止

(c) 找好 extreme points 之後，要記得重新依 F 值大小排序

◎ 2-H Examples for Mini-Max FIR Filter Design

- **Example 1:** Design a 9-length highpass filter in the mini-max sense



ideal filter: $H_d(F) = 0$ for $0 \le F < 0.25$,

$H_d(F) = 1$ for $0.25 < F \le 0.5$,

transition band: $0.22 < F < 0.28$ $\Delta = 0.001$

weighting function: $W(F) = 0.25$ for $0 \le F \le 0.22$,

$W(F) = 1$ for $0.28 \le F \le 0.5$,

(**Step 1**) Since $N = 9$, $k = (N\text{-}1)/2 = 4$, $k+2 = 6$,

$\rightarrow$ **Choose 6 extreme frequencies**

(e.g., $F_0 = 0$, $F_1 = 0.1$, $F_2 = 0.2$, $F_3 = 0.3$, $F_4 = 0.4$, $F_5 = 0.5$)

$$[R(F_n) - H_d(F_n)]W(F_n) = (-1)^{n+1}e, \qquad n = 0, 1, 2, 3, 4, 5.$$

(Step 2)

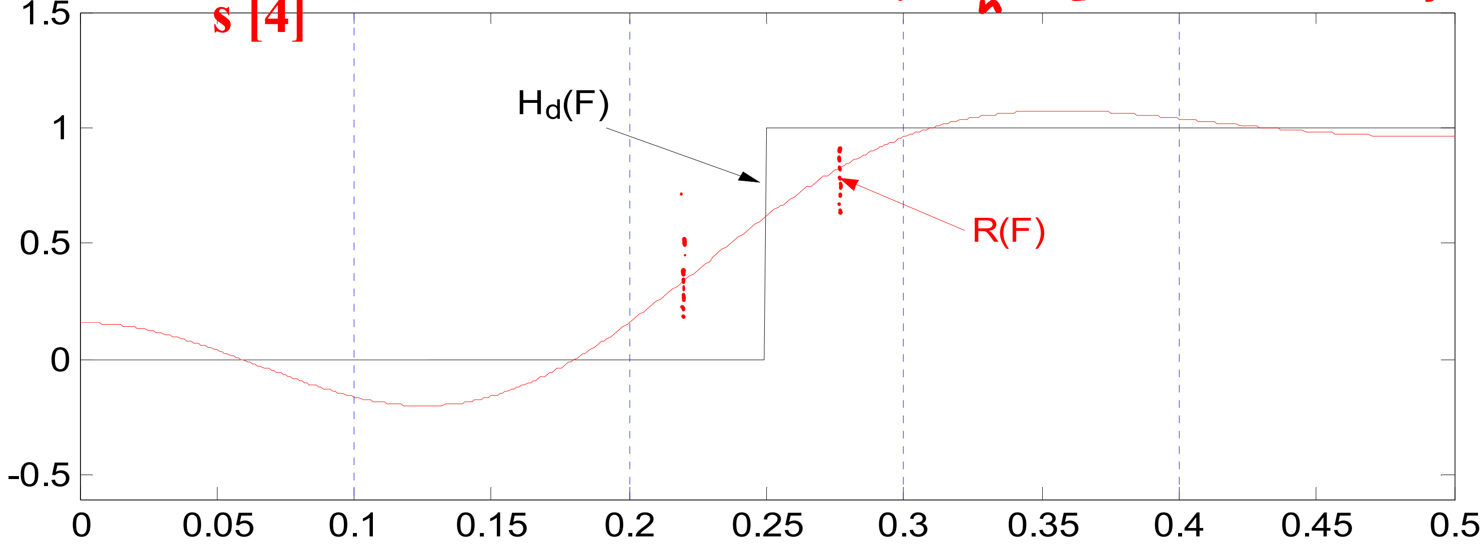
$\cos(2\pi F_m n)$

$$
\begin{array}{r l}
 & \begin{array}{cccccccc} n=0 & n=1 & & & n=4 & \dfrac{(-1)^m}{W(F_m)} & & H_d(F_m) \end{array} \\
\begin{array}{r} m=0 \\ m=1 \\ \\ \\ \\ m=5 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 4 \\
1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\
1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\
1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\
1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\
1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}
\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{array}
$$

$$
\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 4 \\
1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\
1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\
1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\
1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\
1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}^{-1}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 0.5120 \\ -0.6472 \\ -0.0297 \\ 0.2472 \\ 0.0777 \\ -0.040 \end{bmatrix}
$$

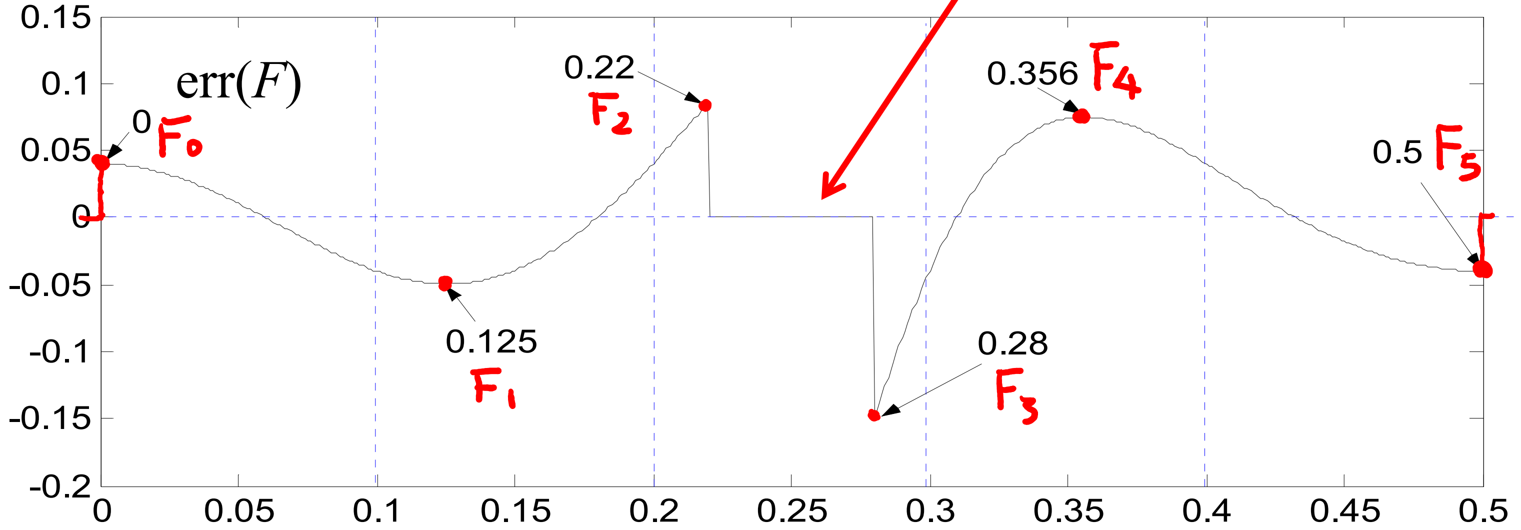
$$R(F) = \underbrace{0.5120}_{s[0]} - \underbrace{0.6472}_{s[1]}\cos(2\pi F) - \underbrace{0.0297}_{s[2]}\cos(4\pi F) + \underbrace{0.2472}_{s[3]}\cos(6\pi F) + \underbrace{0.0777}_{s[4]}\cos(8\pi F)$$

$R(F) = \sum_n s[n]\cos(2\pi n F)$



(Step 3) $\mathrm{err}(F) = [R(F) - H_d(F)]W(F)$

$W(F) = 0$ for $0.22 < F < 0.28$



添零

(Step 4) Extreme points:

$F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$

(Step 5) E_0 **=** Max[|err(F)|] = $\underline{0.1501}$, return to Step 2.

Second iteration

(Step 2) Using $F_0 = 0, F_1 = 0.125, F_2 = 0.22, F_3 = 0.28, F_4 = 0.356, F_5 = 0.5$

$\rightarrow s[0] = 0.5018, s[1] = -0.6341, s[2] = -0.0194, s[3] = 0.3355,$
$s[4] = 0.1385$

(Step 3) err(F) = $[R(F) - H_d(F)]W(F)$,

(Step 4) extreme points : 0, 0.132, 0.22, 0.28, 0.336, 0.5

(Step 5) E_0 **=** Max[|err(F)|] = $\underline{0.0951}$, return to Step 2.

Third iteration

(Step 2), **(Step 3)**, **(Step 4)**, peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5) E_0 **=** $\underline{0.0821}$, return to Step 2.

Fourth iteration

(Step 2), **(Step 3)** , **(Step 4)**, peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5) E_0 **=** $\underline{0.0820}$, $E_1 - E_0 = 0.0001 \leq \Delta$, continues to Step 6.

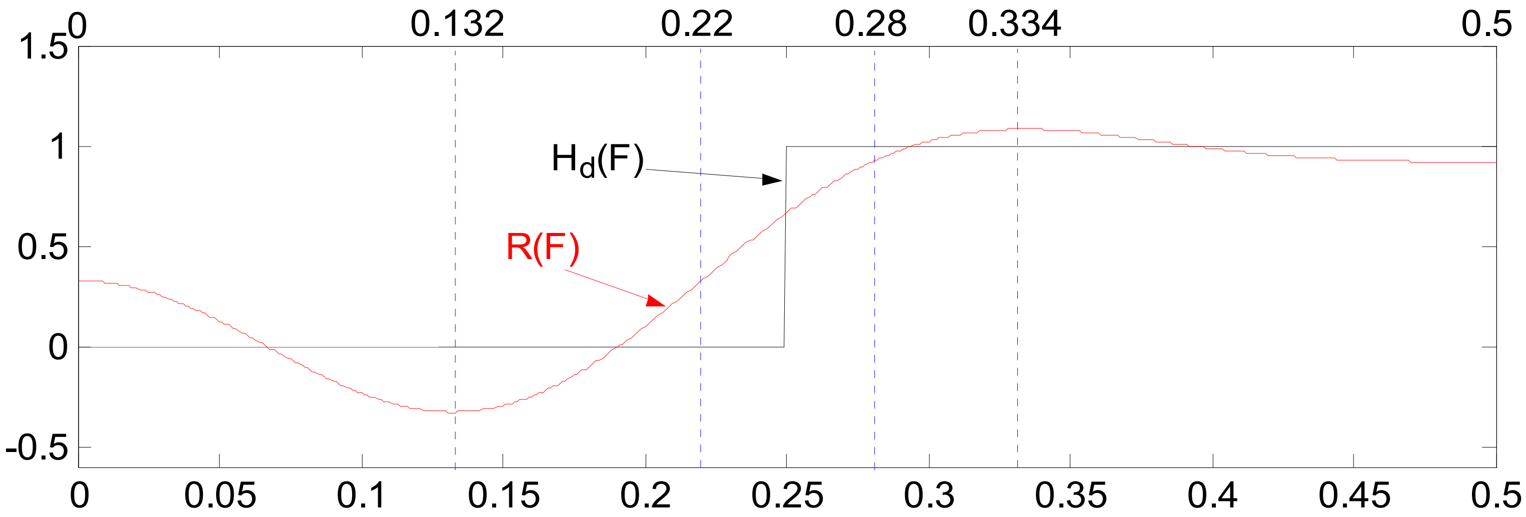
(Step 6) From $s[0] = 0.4990$, $s[1] = -0.6267$, $s[2] = -0.0203$, $s[3] = 0.3316$, $s[4] = 0.1442$

k:4

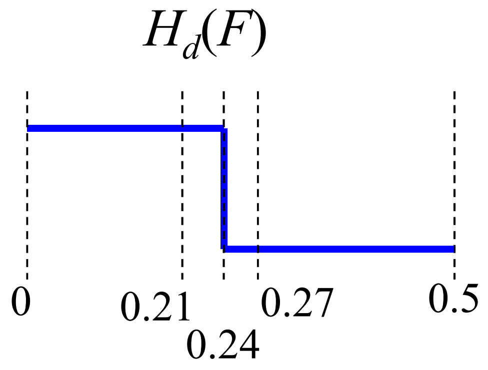
$h[4] = s[0] = 0.4990$, $h[3] = h[5] = s[1]/2 = -0.3134$,

$h[2] = h[6] = s[2]/2 = -0.0101$, $h[1] = h[7] = s[3]/2 = 0.1658$,

$h[0] = h[8] = s[4]/2 = 0.0721$.



- **Example 2:** Design a 7-length digital filter in the mini-max sense



ideal filter: $H_d(F) = 1$ for $0 \le F < 0.24$,

$H_d(F) = 0$ for $0.24 < F \le 0.5$,

transition band: $0.21 < F < 0.27$

weighting function: $W(F) = 1$ for $0 \le F \le 0.21$,

$W(F) = 0.5$ for $0.27 \le F \le 0.5$,

$\Delta = 0.001$

(**Step 1**) Since $N = 7$, $k = (N\text{-}1)/2 = 3$, $k+2 = 5$,

$\rightarrow$ Choose 5 extreme frequencies

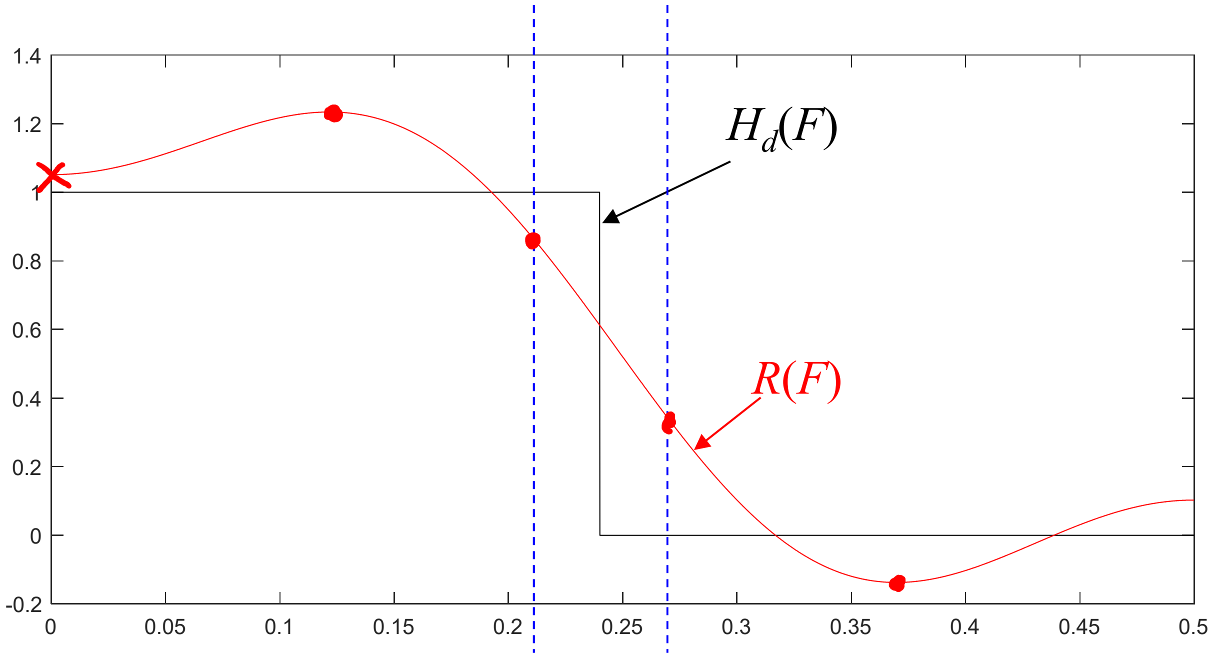
(e.g., $F_0 = 0$, $F_1 = 0.2$, $F_2 = 0.3$, $F_3 = 0.4$, $F_4 = 0.5$)

$n=0 \quad n=1 \quad \cos(2\pi F_m n) \quad n=2 \quad n=3 \quad \frac{(-1)^m}{W(F_m)}$

(Step 2)

$$
\begin{array}{c} m=0 \\ m=1 \\ \\ \\ m=4 \end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 0.309 & -0.809 & -0.809 & -1 \\
1 & -0.309 & -0.809 & 0.809 & 2 \\
1 & -0.809 & 0.309 & 0.309 & -2 \\
1 & -1 & 1 & -1 & 2
\end{bmatrix}
\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ e \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$\rightarrow\ s[0] = 0.5486,\ \ s[1] = 0.7215,\ \ s[2] = 0.0284,\ \ s[3] = \text{-}0.2472,\ \ e = \text{-}0.0514$



After Step 2,

(Step 3) $\mathrm{err}(F) = [0.5486 + 0.7215\cos(2\pi F) + 0.0284\cos(4\pi F)$
$- 0.2472\cos(6\pi F) - H_d(F)]W(F)$

(Step 4) extreme points: 0.1217, 0.21, 0.27, 0.3698, 0.5.

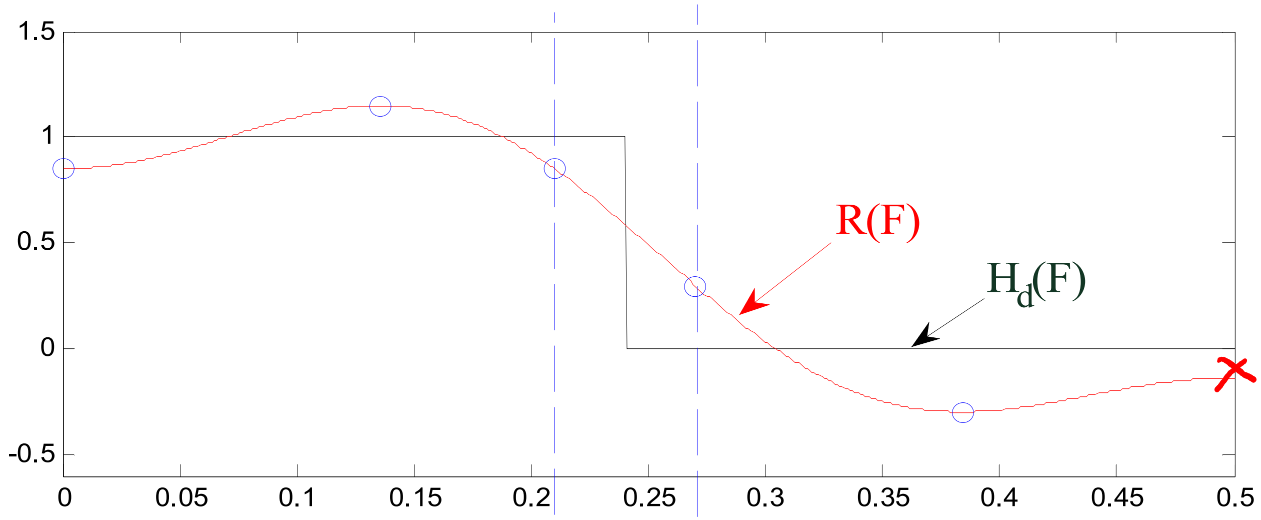
(Step 5) E_0 = Max[|err(F)|] = 0.2341, return to Step 2.

Iteration	1	2	3	4	5	6
Max[|err(F)|]	0.2341	0.3848	0.1685	0.1496	0.1493	0.1493

$E_1 < E_0$

It has to be continued.

After ~~7~~ 6 times of iteration



$s[0] = 0.4243$, $s[1] = 0.7559$, $s[2] = -0.0676$, $s[3] = -0.2619$, $e = 0.1493$

(Step 6):

$h[3] = 0.4243$, $h[2] = h[4] = s[1]/2 = 0.3780$,

$h[1] = h[5] = s[2]/2 = -0.0338$,

$h[0] = h[6] = s[3]/2 = -0.1309$, $h[n] = 0$ for $n < 0$ and $n > 6$

附錄二：Spectrum Analysis for Sampled Signals

(學信號處理的人一定要會的基本常識)

已知 $x[n]$ 是由一個 continuous signal $y(t)$ 取樣而得

$$x[n] = y(n\Delta_t)$$

DFT: $X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N}$ FT: $Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t} y(t)\, dt$

$t = n\Delta_t$

$dt = \Delta_t$

Q: $x[n]$ 的 DFT 和 $y(t)$ 的 Fourier transform 之間有什麼關係？

Basic rule：把間隔由 1 換成 f_s/N where $f_s = 1/\Delta_t$

$$f = m\frac{f_s}{N}$$

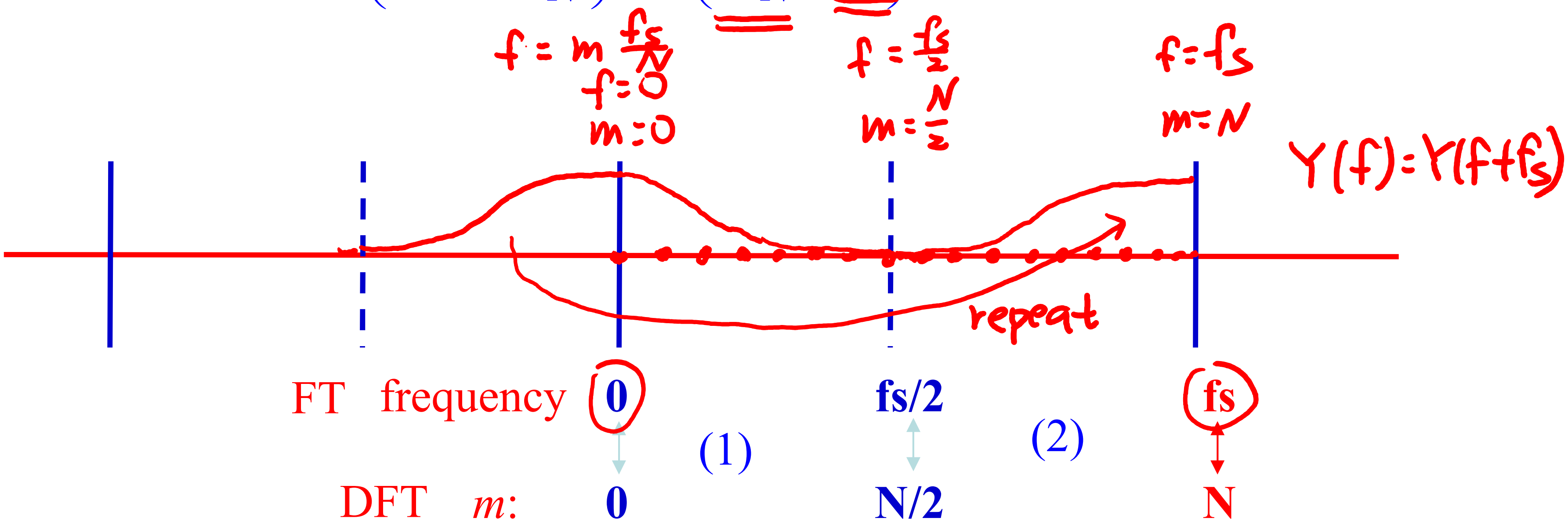
$f = m\Delta_f = \frac{m}{\Delta_t N} = m\frac{f_s}{N}$

(Very important)

ex: voice
20~20000Hz
$f_s > 2B = 40000$

(1) $X[m]\Delta_t \cong Y\left(m\frac{f_s}{N}\right)$ $f_s = 1/\Delta_t$ for $m \le N/2$

(2) $X[m]\Delta_t \cong Y\left((m-N)\frac{f_s}{N}\right) = Y\left(\underline{m\frac{f_s}{N}} - \underline{f_s}\right)$ for $m > N/2$



If the sampling frequency is f_s, the FT output has the period of f_s

The DFT output has the period of N

Ex 1: $f_s = 40000$ $N = 200,000$
if $f = 262$ $m = ?$
$m\frac{40000}{200000} = 262$, $m = 1310$

Proof： $Y(f)=\int_{-\infty}^{\infty} e^{-j2\pi f t} y(t)\,dt$

用 $t=n\Delta_t$, $f=m\Delta_f$ 代入

$$Y(m\Delta_f) \cong \sum_n e^{-j2\pi m\Delta_f n\Delta_t} y(n\Delta_t)\Delta_t = \Delta_t \sum_n e^{-j2\pi m\Delta_f n\Delta_t} x[n]$$

當 $\Delta_t\Delta_f=\frac{1}{N}$ i.e., $\Delta_f=\frac{1}{N\Delta_t}=\frac{f_s}{N}$

$$Y\left(m\frac{f_s}{N}\right) \cong \Delta_t \sum_n e^{-j2\pi\frac{mn}{N}} x[n]$$

$$=\Delta_t DFT\{x[n]\}$$

Ex 3 : $f_s=40{,}000$ $N=600{,}000$

① If $f=500$ Hz , $m=?$

② If $m=6000$, $f=?$ Hz

Ex2: $f_s=40000$, $N=200{,}000$

$X(10{,}000)$ corresponds to

$f=?$ $10000\times\frac{40000}{200000}=2000$ (Hz)

$X(160{,}000)$ corresponds to

$f=?$ $160000\times\frac{40000}{200000}=32000$

$32000-40000=-8000$ (Hz)

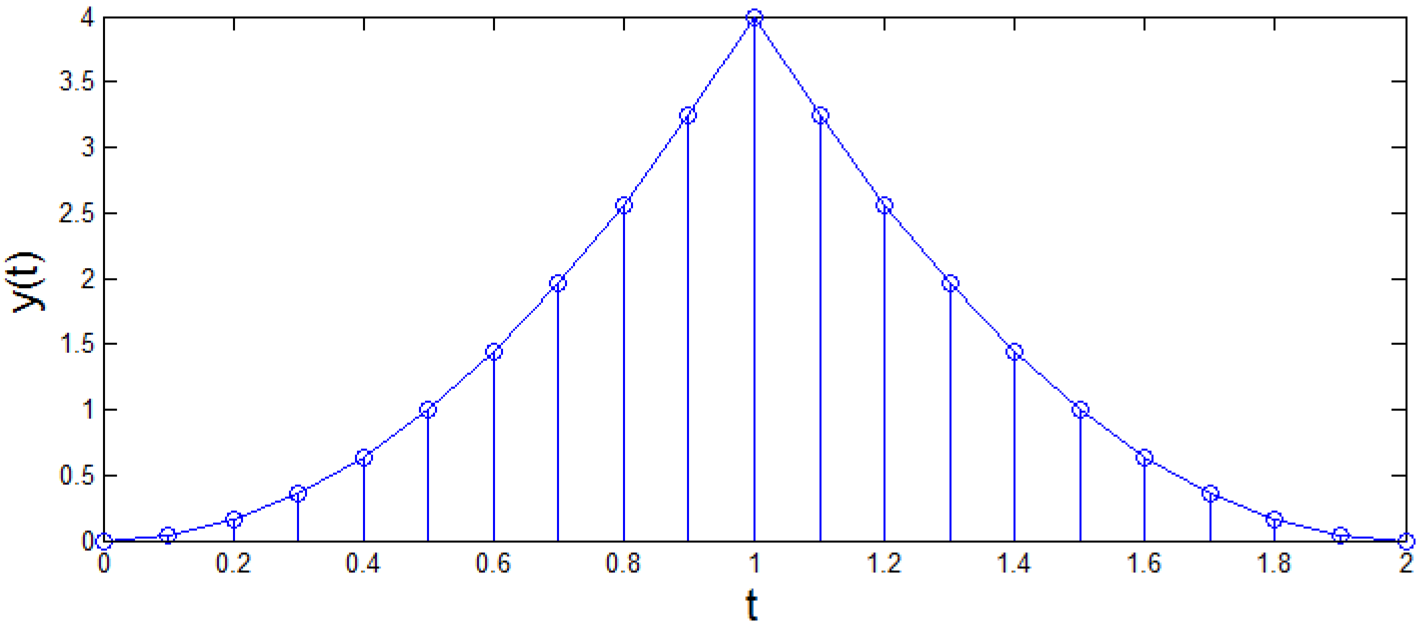
Example：已知

$$y(t) = (2t)^2 \quad \text{for } 0 \le t \le 1 \qquad\qquad y(t) = (4-2t)^2 \quad \text{for } 1 \le t \le 2$$

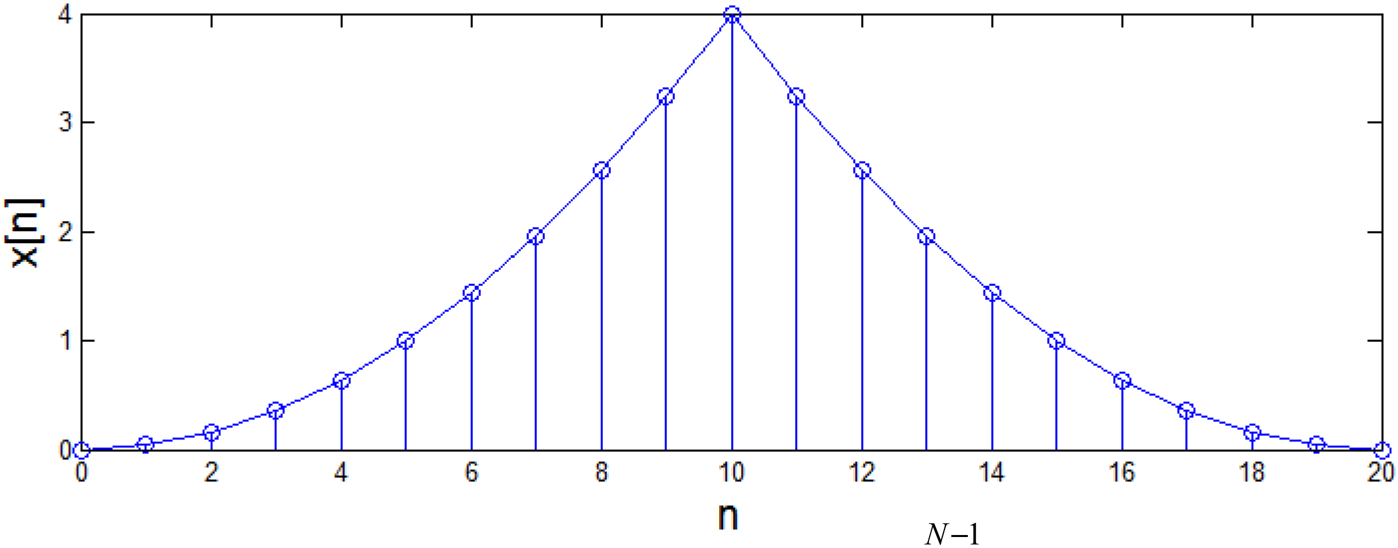
取樣間隔：$\Delta_t = 0.1$

$x[n] = y(n\,\Delta_t)$ for $0 \le n \le 20$ $N=21$

如何用 DFT 來正確的畫出 $y(t)$ 的頻譜？



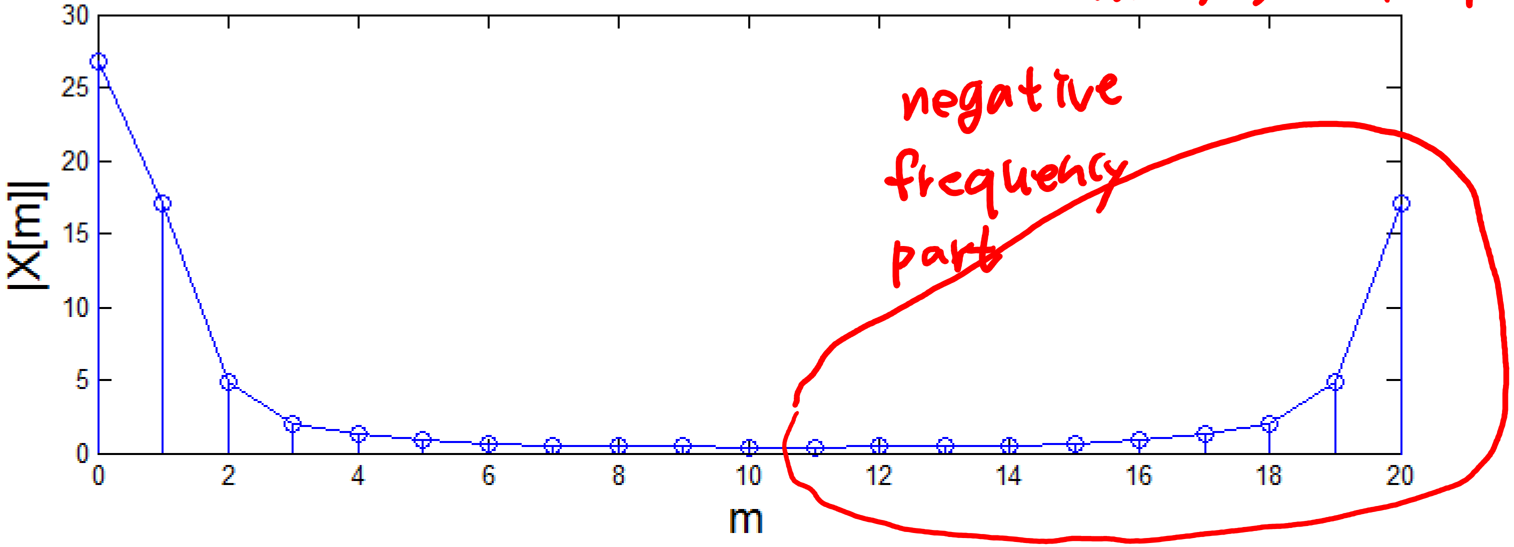
$x[n] = y(n\,\Delta_t)$ for $0 \le n \le 20$



(Step 1) Perform the DFT for $x[n]$

$$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N} \qquad N = 21$$

$m = 0, 1, \cdots N-1$



$f = m\frac{f_s}{N}$

(Step 2-1) $Y\left(m\frac{f_s}{N}\right) \cong X[m]\Delta_t$ for $m \le N/2$

(Step 2-2) $Y\left((m-N)\frac{f_s}{N}\right) \cong X[m]\Delta_t$ for $m > N/2$

In this example, $\frac{f_s}{N} = \frac{1}{N\Delta_t} = \frac{1}{21 \cdot 0.1} = 0.4762$