工程數學--微分方程

Differential Equations (DE)

授課者：丁建均

教學網頁：http://djj.ee.ntu.edu.tw/DE.htm
(請上課前來這個網站將講義印好)

歡迎大家來修課！

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上課時間： 星期三 第 3, 4 節 (AM 10:20~12:10)

上課地點： 明達205

課本： "Differential Equations-with Boundary-Value Problem", 9th edition, Dennis G. Zill and Michael R. Cullen, 2017. (metric version)

評分方式：四次作業一次小考 15%， 期中考 42.5%， 期末考 42.5%

教學網頁：http://djj.ee.ntu.edu.tw/DE.htm

共同教學網頁： http://cc.ee.ntu.edu.tw/~tomme/DE/DE.html

注意事項：

(1)請上課前，來這個網頁，將上課資料印好。

http://djj.ee.ntu.edu.tw/DE.htm

(2) 請各位同學踴躍出席 。

(3) 作業不可以抄襲。作業若寫錯但有用心寫仍可以有 40%~90% 的分數，但抄襲或借人抄襲不給分。

(4) 我週一至週五下午都在辦公室，有什麼問題 ，歡迎同學們來找我

上課日期

Week Number	Date (Wednesday)	Remark
1.	9/22	
2.	9/29	
3.	10/6	
4.	10/13: HW1	
5.	10/20	
6.	10/27	
7.	11/3: HW2	
8.	11/10:	
9.	11/17: Midterms	範圍： (Sections 2-2 ~ 4-5)
10.	11/24	
11.	12/1: HW3	
12.	12/8	
13.	12/15: Quiz	
14.	12/22	
15.	12/29: HW4	
16.	1/5: Finals	範圍： (Sections 4-6 ~ 12-4)

課程大綱

- Introduction (Chap. 1)
- First Order DE
  - 解法 (Chap. 2)
  - 應用 (Chap. 3)
  - 矩陣解 (Chap. 8，範圍外)
- Higher Order DE
  - 解法 (Chap. 4)
  - 應用 (Sec. 5-1)
  - 非線性 (Sections 4-10, 5-3, 工數特論)
  - 多項式解法 (Chap. 6)
- Partial DE
  - 解法 (Sections 12-1, 12-4)
  - 直角座標 (Chapter 12，工數特論)
  - 圓座標 (Chapter 13，工數特論)
- Transforms
  - Laplace Transform (Chap. 7 ，範圍外)
  - Fourier Series (Chap. 11)
  - Fourier Transform (Chap. 14，工數特論)

授課範圍

期中考範圍

Sections 1-1, 1-2, 1-3

Sections 2-1, 2-2, 2-3, 2-4, 2-5, 2-6

Sections 3-1, 3-2

Sections 4-1, 4-2, 4-3, 4-4, 4-5

期末考範圍

Sections 4-6, 4-7

Section 5-1

Sections 6-1, 6-2, 6-3

Sections 11-1, 11-2, 11-3

Sections 12-1, 12-4

blue colors: 要考的章節

Chapter 1 Introduction to Differential Equations

1.1 Definitions and Terminology (術語)

(1) Differential Equation (DE): any equation containing derivation
(text page 3, definition 1.1)

(1-1)

$$\frac{dy(x)}{dx} = 1$$

x: independent variable 自變數

$y(x)$: dependent variable 應變數

$$\int_0^x \sin(t) f(x-t)\,dt + \frac{d^3 f(x)}{dx^3} = \cos(x)$$

- Note: In the text book, $f(x)$ is often simplified as f

- notations of differentiation

$$\frac{df}{dx}, \quad \frac{d^2 f}{dx^2}, \quad \frac{d^3 f}{dx^3}, \quad \frac{d^4 f}{dx^4}, \quad \ldots\ldots\ldots \quad \text{Leibniz notation}$$

$$f', \quad f'', \quad f''', \quad f^{(4)}, \quad \ldots\ldots\ldots \quad \text{prime notation}$$

$$\dot{f}, \quad \ddot{f}, \quad \dddot{f}, \quad \ddddot{f}, \quad \ldots\ldots\ldots \quad \text{dot notation}$$

$$f_x, \quad f_{xx}, \quad f_{xxx}, \quad f_{xxxx}, \quad \ldots\ldots\ldots \quad \text{subscript notation}$$

(2) Ordinary Differential Equation (ODE):

differentiation with respect to one independent variable

$$\frac{d^3u}{dx^3}+\frac{d^2u}{dx^2}+\frac{du}{dx}+\cos(6x)u=0 \qquad \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}=2xy+z$$

(3) Partial Differential Equation (PDE):

differentiation with respect to two or more independent variables

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0 \qquad \frac{\partial x}{\partial t}=\frac{\partial y}{\partial \tau}$$

(4) Order of a Differentiation Equation: the order of the highest derivative in the equation

$$\underline{\frac{d^7u}{dx^7}}+2\frac{d^6u}{dx^6}+2\frac{d^5u}{dx^5}+4\frac{d^4u}{dx^4}=0$$ 7th order

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}-5y=e^x$$ 2nd order

(5) Linear Differentiation Equation:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

(i) For y, only the terms $y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^n y}{dx^n}$ appear.

(ii) All of the coefficient terms $a_m(x)$ $m =$ 1, 2, …, n are independent of y.

Property of linear differentiation equations:

If $a_n(x)\frac{d^n y_1}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y_1}{dx^{n-1}}+\cdots+a_1(x)\frac{dy_1}{dx}+a_0(x)y_1=g_1(x)$

$$a_n(x)\frac{d^n y_2}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y_2}{dx^{n-1}}+\cdots+a_1(x)\frac{dy_2}{dx}+a_0(x)y_2=g_2(x)$$

and $y_3 = by_1 + cy_2$, then

$$a_n(x)\frac{d^n y_3}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y_3}{dx^{n-1}}+\cdots+a_1(x)\frac{dy_3}{dx}+a_0(x)y_3=bg_1(x)+cg_2(x)$$

(if $g(x)$ is treated as the input and $y(x)$ is the output)

(6) Non-Linear Differentiation Equation

$$(y+3)\frac{d^2y}{dx^2}+\frac{dy}{dx}+2y=x$$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y^2=e^x$$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+e^y=e^x$$

[Example 1.1.2] Linear and Nonlinear ODEs

(a) The equations

$$(y - x)dx + 4x dy = 0,\ y'' - 2y + y = 0,\ x^3 \frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

nonlinear term: coefficient depends on y

nonlinear term: nonlinear function of y

nonlinear term: power not 1

$$(1 - y)y' + 2y = e^x,\ \frac{d^2 y}{dx^2} + \sin y = 0,\ \text{and}\ \frac{d^4 y}{dx^4} + y^2 = 0$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

(7) Explicit Solution (text page 8)

The solution is expressed as $y = \phi(x)$

(8) Implicit Solution (text page 8)

Example: $\dfrac{dy^2}{dx} = -x$,

Solution: $\dfrac{1}{2}x^2 + y^2 = c$ (implicit solution)

or $$y = \sqrt{c - x^2/2}$$ $$y = -\sqrt{c - x^2/2}$$ (explicit solution)

1.2 Initial Value Problem (IVP)

A differentiation equation always has more than one solution.

for $\dfrac{dy}{dx} = 1$,

$y = x$, $y = x+1$, $y = x+2$ … are all the solutions of the above differentiation equation.

General form of the solution: $\boldsymbol{y = x + c}$, where c is any constant.

The initial value (未必在 $x = 0$) is helpful for obtain the unique solution.

$$\frac{dy}{dx} = 1 \text{ and } y(0) = 2 \longrightarrow y = x+2$$

$$\frac{dy}{dx} = 1 \text{ and } y(2) = 3.5 \longrightarrow y = x+1.5$$

The k^{th} order linear differential equation usually requires k independent initial conditions (or k independent boundary conditions) to obtain the unique solution.

$$\frac{d^2y}{dx^2} = 1$$

solution: $y = x^2/2 + bx + c$,

b and c can be any constant

$y(1) = 2$ and $y(2) = 3$ (boundary conditions，在不同點)

$y(0) = 1$ and $y'(0) = 5$ (initial conditions ，在相同點)

$y(0) = 1$ and $y'(3) = 2$ (boundary conditions，在不同點)

For the k^{th} order differential equation, the initial conditions can be $0^{\text{th}} \sim (k-1)^{\text{th}}$ derivatives at some points.

1.3 Differential Equations as Mathematical Model

Physical meaning of differentiation:

the variation at certain time or certain place

[Example 1]: $v(t)=\dfrac{dx(t)}{dt}$, $a(t)=\dfrac{dv(t)}{dt}=\dfrac{d^2x(t)}{dt^2}$

$$F-\beta v=ma \Rightarrow F-\beta\frac{dx(t)}{dt}=m\frac{d^2x(t)}{dt^2}$$

$x(t)$: location, $v(t)$: velocity, $a(t)$: acceleration
F: force, β: coefficient of friction, m: mass

[Example 2]: 人口隨著時間而增加的模型

$$\frac{dA(t)}{dt} = kA(t)$$

A: population

人口增加量和人口呈正比

[Example 3]: 開水溫度隨著時間會變冷的模型

$$\frac{dT}{dt} = k(T - T_m)$$

T: 熱開水溫度,

T_m: 環境溫度

t: 時間

大一微積分所學的：

$\int f(t)dt$ 的解　　例如：$\int \frac{1}{t}dt = \ln|t| + c$

$$\boxed{\frac{dA(t)}{dt} = f(t) \Rightarrow A(t) = \int f(t)dt + c}$$

Example: $\frac{dA(t)}{dt} = \frac{1}{t} \longrightarrow A(t) = \ln|t| + c$

$$\frac{dA(t)}{dt} = \frac{1}{t^2+4} \Rightarrow A(t) = \int \frac{1}{t^2+4}dt + c = ?$$

Problems

(1) 若等號兩邊都出現 dependent variable (如 pages 18, 19 的例子)

(2) 若 order of DE 大於 1 (如 page 17 的例子)

該如何解？

Review

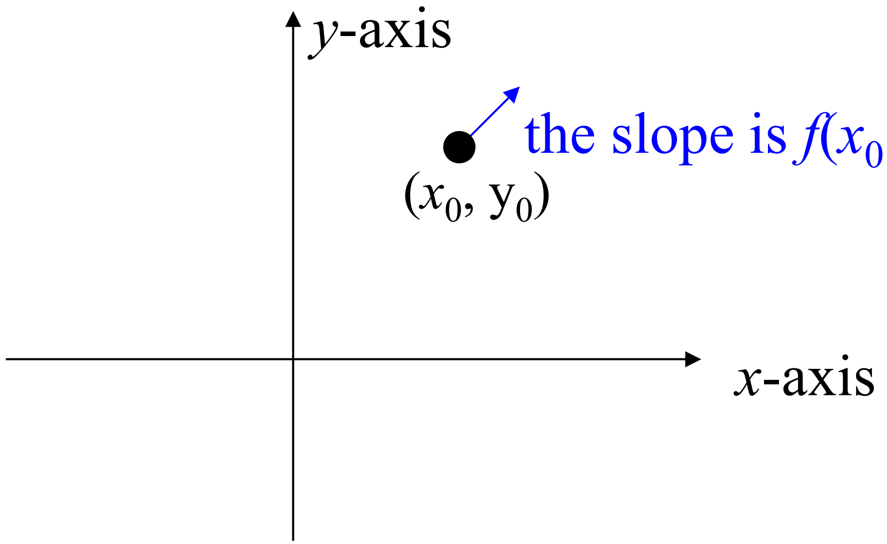
- dependent variable and independent variable
- DE
- PDE and ODE
- Order of DE
- linear DE and nonlinear DE
- explicit solution and implicit solution
- initial value; boundary value
- IVP

Chapter 2 First Order Differential Equation

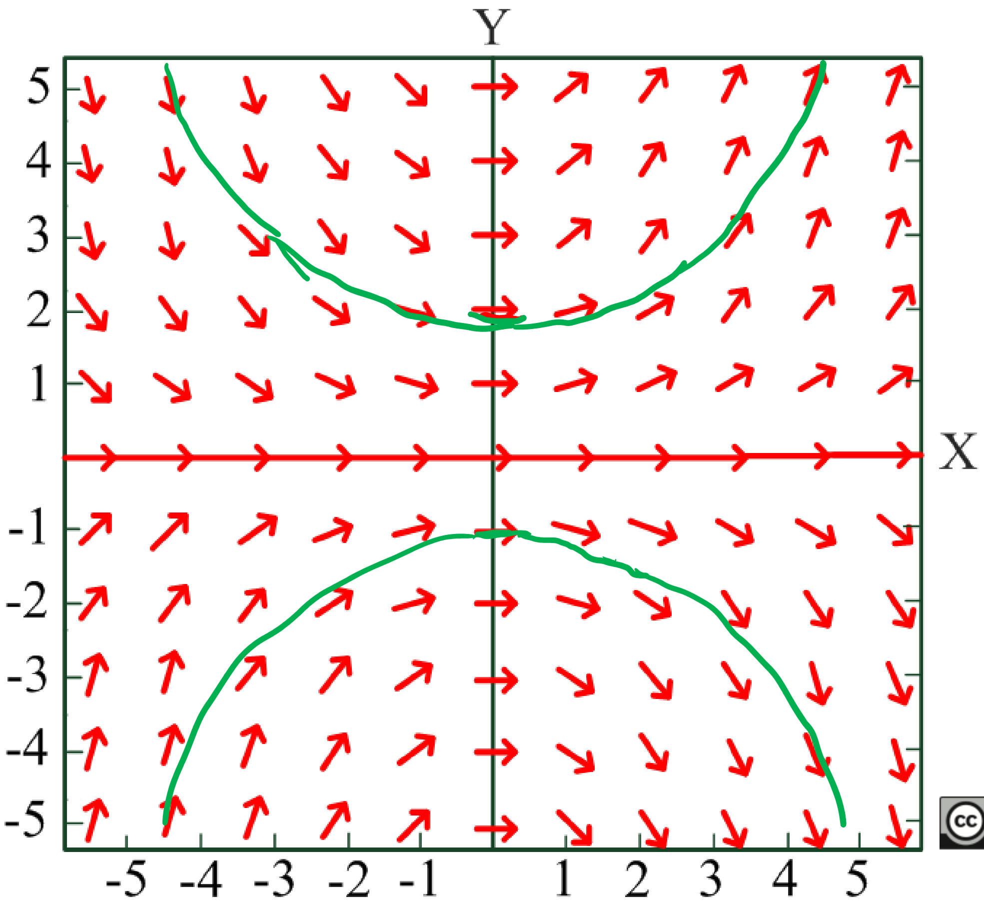
2-1 Solution Curves without a Solution

Instead of using analytic methods, the DE can be solved by graphs (圖解)

slopes and the field directions: $\dfrac{dy}{dx} = f\left(x, y\right)$

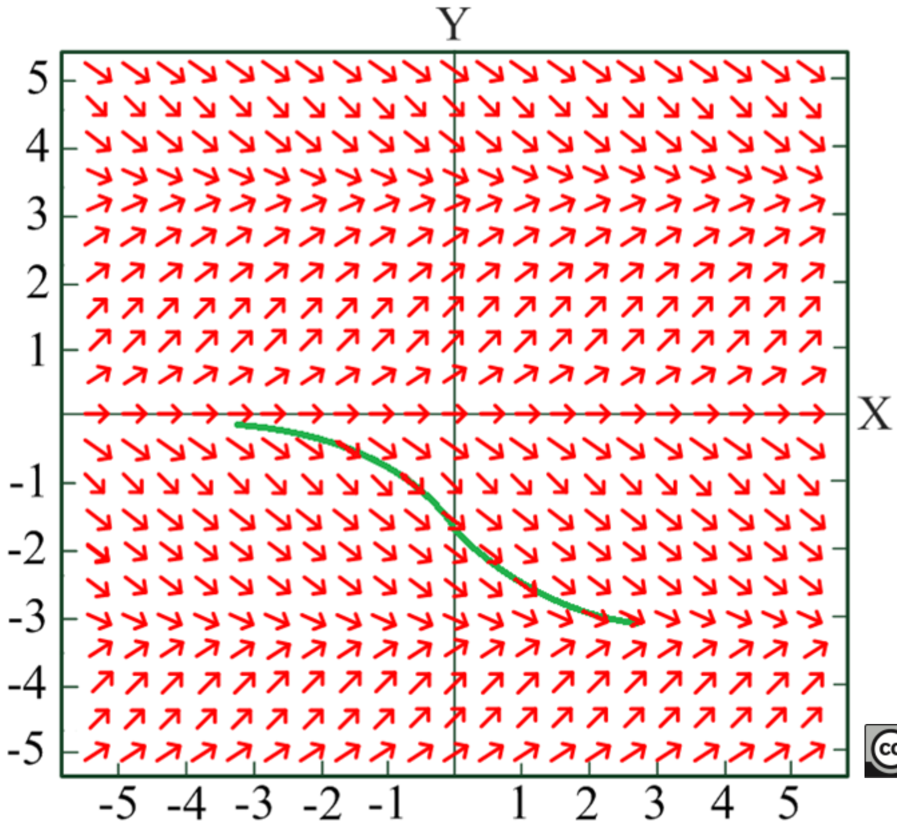


Example 1 $dy/dx = 0.2xy$



From： Fig. 2-1-3(a) in “Differential Equations-with Boundary-Value Problem”, 9th ed., Dennis G. Zill and Michael R. Cullen.

Example 2 $dy/dx = \sin(y)$, $y(0) = -3/2$



From： Fig. 2-1-4 in “Differential Equations-with Boundary-Value Problem”, 9th ed., Dennis G. Zill and Michael R. Cullen.

With initial conditions, one curve can be obtained

Advantage:

It can solve some 1st order DEs that cannot be solved by mathematics.

Disadvantage:

It can only be used for the case of the 1st order DE.

It requires a lot of time

Section 2-6 A Numerical Method

- Another way to solve the DE without analytic methods
- independent variable $x \xrightarrow{\text{sampling(取樣)}} x_0,\ x_1, x_2, \ldots\ldots\ldots$
- Find the solution of $\dfrac{dy(x)}{dx} = f(x, y)$

Since $\dfrac{dy(x)}{dx} = f(x, y) \xrightarrow{\text{approximation}} \dfrac{y(x_{n+1}) - y(x_n)}{x_{n+1} - x_n} = f(x_n, y(x_n))$

$$y(x_{n+1}) = \underbrace{y(x_n)}_{\text{前一點的值}} + f(x_n, y(x_n))\underbrace{(x_{n+1} - x_n)}_{\text{取樣間格}}$$

$$\frac{dy(x)}{dx} = f(x, y) \qquad\qquad y(x_{n+1}) = y(x_n) + f(x_n, y(x_n))(x_{n+1} - x_n)$$

If $y(x_0)$ is known

$$y(x_1) = y(x_0) + f(x_0, y(x_0))(x_1 - x_0)$$

$$y(x_2) = y(x_1) + f(x_1, y(x_1))(x_2 - x_1)$$

$$y(x_3) = y(x_2) + f(x_2, y(x_2))(x_3 - x_2)$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\frac{dy(x)}{dx} = f(x,y) \qquad\qquad y(x_{n+1}) = y(x_n) + f(x_n, y(x_n))(x_{n+1} - x_n)$$

Example:

- $dy(x)/dx = 0.2xy \longrightarrow y(x_{n+1}) = y(x_n) + 0.2x_n y(x_n)*(x_{n+1} - x_n)$.
- $dy/dx = \sin(x) \longrightarrow y(x_{n+1}) = y(x_n) + \sin(x_n)*(x_{n+1} - x_n)$.

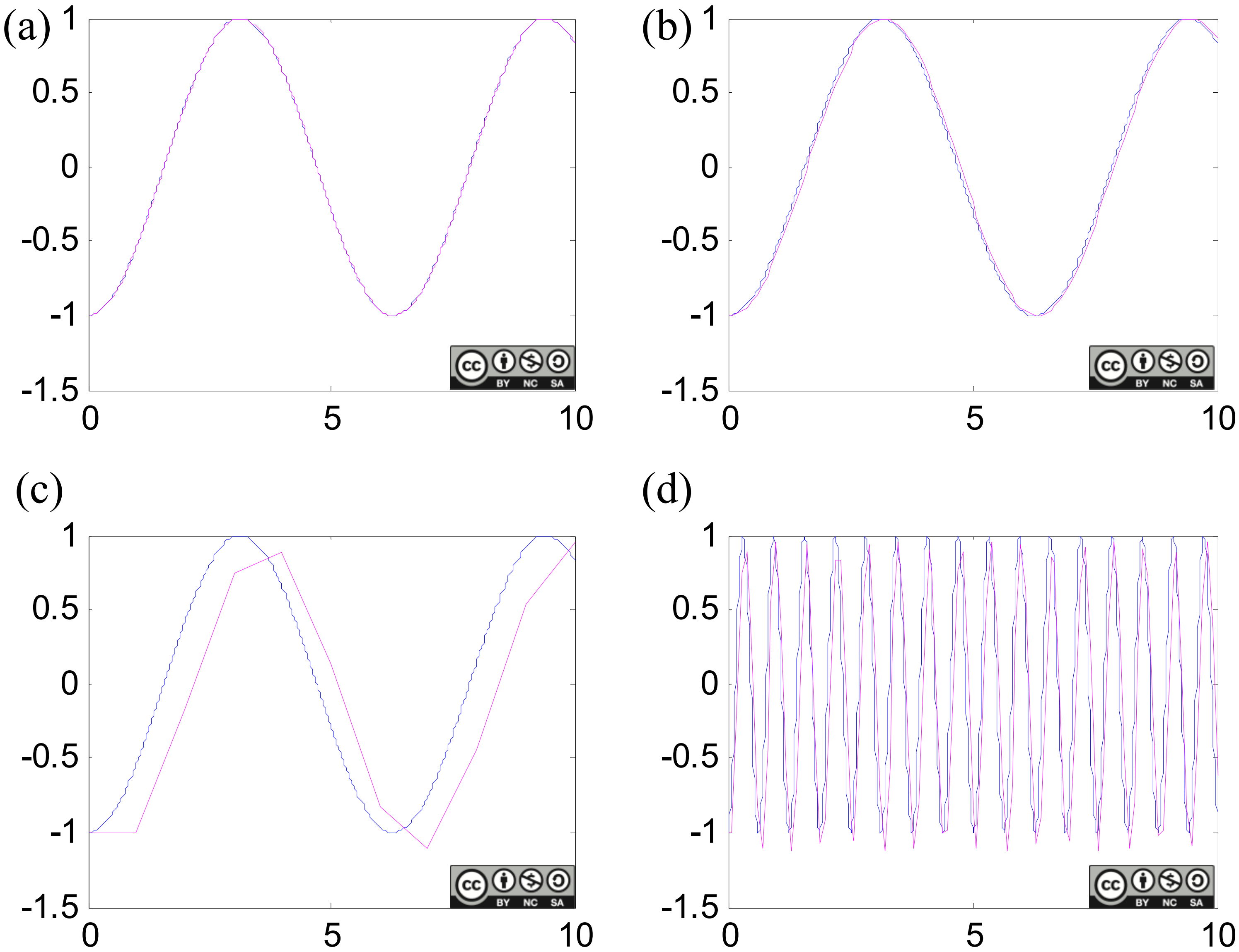
後頁為 $dy/dx = \sin(x)$, $y(0) = -1$,

(a) $x_{n+1} - x_n = 0.01$, (b) $x_{n+1} - x_n = 0.1$,

(c) $x_{n+1} - x_n = 1$, (d) $x_{n+1} - x_n = 0.1$, $dy/dx = 10\sin(10x)$ 的例子

Constraint for obtaining accurate results:

(1) small sampling interval (2) small variation of $f(x, y)$


(a)
(b)
(c)
(d)
1
0.5
0
-0.5
-1
-1.5
0
5
10
BY NC SA

Advantages

-- It can solve some 1st order DEs that cannot be solved by mathematics.

-- can be used for solving a complicated DE (not constrained for the 1st order case)

-- suitable for computer simulation

Disadvantages

-- numerical error (數值方法的課程對此有詳細探討)

附錄一　Table of Integration

$1/x$	$\ln\lvert x\rvert + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$
$\tan(x)$	$-\ln\lvert\cos(x)\rvert + c$
$\cot(x)$	$\ln\lvert\sin(x)\rvert + c$
a^x	$a^x/\ln(a) + c$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$
$1/\sqrt{a^2-x^2}$	$\sin^{-1}(x/a) + c$
$-1/\sqrt{a^2-x^2}$	$\cos^{-1}(x/a) + c$
$x\,e^{ax}$	$\dfrac{e^{ax}}{a}\left(x-\dfrac{1}{a}\right)+c$
$x^2\,e^{ax}$	$\dfrac{e^{ax}}{a}\left(x^2-\dfrac{2x}{a}+\dfrac{2}{a^2}\right)+c$

Exercises for Practicing

(not homework, but are encouraged to practice)

1-1: 1, 13, 19, 23, 37

1-2: 3, 13, 21, 33

1-3: 2, 7, 28

2-1: 1, 13, 20, 25, 33

2-6: 1, 3