Chapter 11 Orthogonal Functions and Fourier Series

複習： linear algebra 關於 orthogonal (正交) basis 的介紹

在 linear algebra 當中

(1) inner product $(\mathbf{f}_1, \mathbf{f}_2) = \sum_n \mathbf{f}_1[n]\mathbf{f}_2[n]$

(2) orthogonal $\sum_n \mathbf{f}_1[n]\mathbf{f}_2[n] = 0$

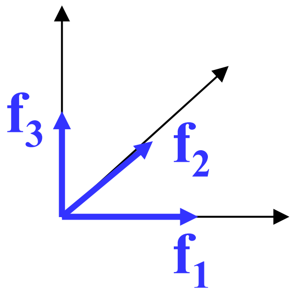
(3) 若 $\mathbf{f}_1$, $\mathbf{f}_2$, …., $\mathbf{f}_N$ 為 complete orthogonal set,

$$\mathbf{f}[n] = \sum_{m=1}^{N} a_m \mathbf{f}_\mathbf{m}[n] \quad \text{where} \quad a_m = \frac{\sum_{n=1}^{N} \mathbf{f}[n]\mathbf{f}_\mathbf{m}[n]}{\sum_{n=1}^{N} \mathbf{f}_\mathbf{m}[n]\mathbf{f}_\mathbf{m}[n]}$$

例如 在只有三個 entry 的情形下

$$\mathbf{f}_1 = [\underset{\mathbf{f}_1[1]}{1}, \underset{\mathbf{f}_1[2]}{0}, \underset{\mathbf{f}_1[3]}{0}] \qquad \mathbf{f}_2 = [\underset{\mathbf{f}_2[1]}{0}, \underset{\mathbf{f}_2[2]}{1}, \underset{\mathbf{f}_2[3]}{0}] \qquad \mathbf{f}_3 = [\underset{\mathbf{f}_3[1]}{0}, \underset{\mathbf{f}_3[2]}{0}, \underset{\mathbf{f}_3[3]}{1}]$$

是一組 complete orthogonal set



問題：在 continuous 當中該如何定義 orthogonal?

Section 11.1 Orthogonal Functions

11.1.1 綱要：熟悉幾個重要定義

(1) inner product (pp. 597)

(2) orthogonal (pp. 599)

(3) orthogonal set (pp. 600)

(4) square norm (pp. 602)

(5) norm (pp. 602)

(6) orthonormal set (pp. 602)

(7) normalize (pp. 604)

(8) complete (pp. 605)

(9) orthogonal series expansion (pp. 606)

(10) generalized Fourier series (pp. 606)

(11) weight function (pp. 608)

With weighting functions, many definition s are changed.

學習方式：(1) 可以多和 linear algebra 當中的定義多比較

(2) 複習三角函式的公式 (see pp. 611-612)

11.1.2 定義

(1) inner product on an interval $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\,dx \qquad (f_1, f_2 \text{ 為 real 時})$$

比較： discrete case $(\mathbf{f_1}, \mathbf{f_2}) = \sum_n \mathbf{f_1}[n]\mathbf{f_2}[n]$

補充：more standard definition for inner product

$$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x)\,dx$$

with conjugation

Inner product 性質

(a) $(\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{f}_2, \mathbf{f}_1)^*$ *: conjugation

(b) $(k\,\mathbf{f}_1, \mathbf{f}_2) = k\,(\mathbf{f}_1, \mathbf{f}_2)$, k 為 scalar (或稱為constant)

(c) $(\mathbf{f}, \mathbf{f}) = 0$ if and only if $\mathbf{f} = 0$, $(\mathbf{f}, \mathbf{f}) > 0$ if and only if $\mathbf{f} \neq 0$,

(d) $(\mathbf{f}_1 + \mathbf{f}_2\,, \mathbf{g}) = (\mathbf{f}_1, \mathbf{g}) + (\mathbf{f}_2\,, \mathbf{g})$

discrete case 亦有這些性質

(2) orthogonal on an interval [a, b]

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\,dx = 0$$ (f_1, f_2 為 real 時)

或 $$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x)\,dx = 0$$ (more standard definition)

比較： discrete case $\sum_n \mathbf{f_1}[n]\mathbf{f_2}[n] = 0$

例子: 當 [a, b] = [-1, 1],

1 和 x^k (k 為奇數) 互為 orthogonal

$$\int_{-1}^{1} 1 \cdot x^k dx = \left.\frac{x^{k+1}}{k+1}\right|_{-1}^{1} = \frac{1^{k+1} - (-1)^{k+1}}{k+1} = \frac{1-1}{k+1} = 0$$

注意：任何 even function 和任何 odd function 在 [-a, a] 之間必為 orthogonal,

包括 Example 1 (text page 426) 的 x^2 和 x^3 在 [-1, 1] 之間也是 orthogonal

(3) orthogonal set

有一組 functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ………..

若 $$\int_a^b \phi_m(x)\phi_n^*(x)\,dx = 0$$ for any $m \neq n$

則 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. 被稱作 orthogonal set on an interval $[a, b]$

Example 2 (text page 426)

Show that the set {1, cos$x$, cos2x, cos3x, …..} is an orthogonal set on the interval $[-\pi, \pi]$

when one of the functions is 1

$$\int_{-\pi}^{\pi} 1 \cdot \cos nx\,dx = \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = 0$$

when both the two functions are not 1

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)dx$$

$$= \frac{\sin(m+n)x}{2(m+n)}\Big|_{-\pi}^{\pi} + \frac{\sin(m-n)x}{2(m-n)}\Big|_{-\pi}^{\pi}$$

$$= \frac{\sin((m+n)\pi)}{2(m+n)} - \frac{\sin(-(m+n)\pi)}{2(m+n)} + \frac{\sin((m-n)\pi)}{2(m-n)} - \frac{\sin(-(m-n)\pi)}{2(m-n)} = 0$$

(4) square norm

$$\|f(x)\|^2 = (f(x), f(x)) = \int_a^b f(x) f^*(x)\, dx$$

比較： discrete case $\sum_n \mathbf{f}[n]\mathbf{f}^*[n]$

(5) norm

$$\|f(x)\| = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x) f^*(x)\, dx}$$

(6) orthonormal set

對一個 orthogonal set, 若更進一步的滿足

$$\int_a^b \phi_n(x)\phi_n^*(x)\, dx = 1 \qquad \text{for all } n$$

則被稱為 orthonormal set

Example 3 (text page 427)

Calculate the norms of $\{1, \cos x, \cos 2x, \cos 3x, \ldots..\}$

$$\int_{-\pi}^{\pi} 1 \cdot 1\, dx = x\Big|_{-\pi}^{\pi} = 2\pi$$

$$\int_{-\pi}^{\pi} \cos nx \cos nx dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos 2nx + 1) dx$$ 運用三角函式公式

$$= \frac{\sin 2nx}{4n} + \frac{x}{2}\Big|_{-\pi}^{\pi} = \frac{\sin 2n\pi}{4n} + \frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n} - \frac{(-\pi)}{2} = \pi$$

$$\|1\| = \sqrt{2\pi} \qquad \|\cos nx\| = \sqrt{\pi}$$

$\{1, \cos x, \cos 2x, \cos 3x, \ldots..\}$ normalization as a orthonormal set

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots\ldots\ldots\right\}$$

(7) normalize

將 norm 變為 1

$$\psi(x) \longrightarrow v(x) = \frac{\psi(x)}{\left\|\psi(x)\right\|}$$

注意，此時

$$(v(x), v(x)) = \left(\frac{\psi(x)}{\left\|\psi(x)\right\|}, \frac{\psi(x)}{\left\|\psi(x)\right\|}\right) = \frac{1}{\left\|\psi(x)\right\|^2}(\psi(x), \psi(x)) = 1$$

可藉由 normalization, 將 orthogonal set 變成 orthonormal set

(8) complete

若在 interval $[a, b]$ 之間，任何一個 function $f(x)$ 都可以表示成 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ………..的 linear combination

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots\cdots\cdots = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

則 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. 被稱作 complete

比較：在 linear algebra 當中，對 3-D vector 而言

$\mathbf{e}_1 = [1, 0, 0]$, $\mathbf{e}_2 = [0, 1, 0]$, $\mathbf{e}_3 = [0, 0, 1]$ 為 complete

Any 3-D vector $[a, b, c]$ can be expressed as $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$

(9)(10)

若 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ………..為complete

可將 $f(x)$ 表示成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

被稱作 (9) orthogonal series expansion

當 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. 不為 orthogonal, c_n 不易算

當 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. 為 orthogonal

$$\int_a^b f(x)\phi_n^*(x)dx = \sum_{m=0}^{\infty} c_m \int_a^b \phi_m(x)\phi_n^*(x)dx = c_n \int_a^b \phi_n(x)\phi_n^*(x)dx$$

$$c_n = \frac{\int_a^b f(x)\phi_n^*(x)dx}{\int_a^b \phi_n(x)\phi_n^*(x)dx}$$

c_n 被稱作 (10) generalized Fourier series

當 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. 為 orthonormal

$$c_n = \int_a^b f(x)\phi_n^*(x)\,dx$$

11.1.3 Orthogonal with Weight Function

(11) inner product with weight function

$$\left(f_1(x), f_2(x)\right) = \int_a^b w(x) f_1(x) f_2^*(x)\, dx$$

其中 $w(x)$ 被稱作 weight function

加上了 weight function 後

(11-1) orthogonal 的定義改成

$$\left(f_m, f_n\right) = \int_a^b w(x) f_m(x) f_n^*(x)\, dx = 0 \quad \text{for } m \neq n$$

(11-2) square norm 的定義改成

$$\left\| f(x) \right\|^2 = \int_a^b w(x) f(x) f^*(x)\, dx$$

(11-3) norm 的定義改成

$$\|f(x)\| = \sqrt{\int_a^b w(x) f(x) f^*(x) dx}$$

(11-4) orthonormal 的定義改成

$$\int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

$$\int_a^b w(x) f_n(x) f_n^*(x) dx = 1$$

(11-5) normalize 的算法改成

$$v(x) = \frac{\psi(x)}{\|\psi(x)\|} = \frac{\psi(x)}{\sqrt{\int_a^b w(x)\psi(x)\psi^*(x) dx}}$$

(11-6) orthogonal series expansion of $f(x)$ 以及 generalize Fourier series 的算法改成

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x)$$

$$\boxed{c_n=\frac{\int_a^b w(x)f(x)\phi_n^*(x)dx}{\int_a^b w(x)\phi_n(x)\phi_n^*(x)dx}}$$

11.1.4 三角函數表

(要複習)

$\cos(a + b)$	$\cos a \cos b - \sin a \sin b$
$\sin(a + b)$	$\sin a \cos b + \cos a \sin b$
$\cos(a - b)$	$\cos a \cos b + \sin a \sin b$
$\sin(a - b)$	$\sin a \cos b - \cos a \sin b$
$\cos a \cos b$	$[\cos(a + b) + \cos(a - b)]/2$
$\sin a \sin b$	$[\cos(a - b) - \cos(a + b)]/2$
$\sin a \cos b$	$[\sin(a + b) + \sin(a - b)]/2$

$\cos(2a)$	$\cos^2 a - \sin^2 a$ or $1 - 2\sin^2 a$ or $2\cos^2 a - 1$
$\sin(2a)$	$2\sin a \cos a$
$\cos^2 a$	$[\cos(2a) + 1]/2$
$\sin^2 a$	$[1 - \cos(2a)]/2$

11.1.5 Section 11.1 需要注意的地方

(1) Norm 和 square of norm 要分清楚

做 normalization 時，要除以 norm

(2) 熟悉三角函數的公式

(i) 記住幾個，其他的就不難推算出來

(ii) 許多公式可以由 $\cos(a)=\frac{e^{ja}+e^{-ja}}{2}$ 導出來

$$\sin(a)=\frac{e^{ja}-e^{-ja}}{2j}=\frac{je^{-ja}-je^{ja}}{2}$$

複習： Legendre polynomials 是一種 orthogonal set

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \text{if } m \neq n$$

其他常用的 orthogonal set

Hermite polynomials (with weight function) (補充)

Chebyshev polynomials (with weight function) (補充)

Cosine series

Sine series

Fourier series

Section 11.2 Fourier Series

11.2.1 綱要

trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots\ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\ldots\right\}$$

orthogonal set on the interval of $[-p, p]$

↑

be proved on pages 620~622

$\cos\frac{n\pi}{p}x$ 週期：$\frac{2p}{n}$ 頻率：$\frac{n}{2p}$

(2) Fourier Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)\,dx \qquad a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}x\,dx$$

(紅色部分特別注意，勿記錯公式)

(3) 名詞

trigonometric function (page 620)

Fourier series (trigonometric series) (page 624)

Fourier coefficients (page 624)

fundamental period (page 629)

period extension (page 629)

partial sum (page 631)

物理意義：

Fourier Series == 對信號做頻率分析

「頻率」(frequency) 是個常用字，以 Hz (每秒多少個週期)為單位

說話聲音: 100~1200 Hz

人耳可聽見的聲音： 20~20000Hz

廣播 (AM): 5×10^{5} ~ 1.6×10^{6} Hz

廣播 (FM): 8.8×10^{7} ~ 1.08×10^{8} Hz

無線電視： 7.6×10^{7} ~ 8.8×10^{7}, 1.74×10^{8} ~ 2.16×10^{8} Hz

行動通訊： 5.1×10^{8} Hz ~ 2.75×10^{11} Hz

可見光： 4×10^{14} Hz ~ 8×10^{14} Hz

測量頻率的方式： Fourier series
Fourier transform

11.2.2 Trigonometric Functions

trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots\ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\ldots\right\}$$

Trigonometric functions is orthogonal on the interval of $[-p\ ,\ p]$

要用 $C_3^2+2=5$ 次的inner products 來證明

(1) 1 VS. Cosine

$$\int_{-p}^{p} 1\cdot\cos\frac{\pi k}{p}x\,dx=\frac{p}{\pi k}\sin\frac{\pi k}{p}x\Big|_{-p}^{p}=\frac{p}{\pi k}\sin\pi k-\frac{p}{\pi k}\sin(-\pi k)=0-0=0$$

(2) 1 VS. Sine

$$\int_{-p}^{p} 1\cdot\sin\frac{\pi k}{p}x\,dx=-\frac{p}{\pi k}\cos\frac{\pi k}{p}x\Big|_{-p}^{p}=-\frac{p}{\pi k}\cos\pi k+\frac{p}{\pi k}\cos(-\pi k)=0$$

(3) Cosine VS. Sine

$$\int_{-p}^{p} \cos\frac{\pi k}{p}x \cdot \sin\frac{\pi h}{p}x dx = \int_{-p}^{p} \frac{1}{2}\left[\sin\frac{\pi(h+k)}{p}x - \sin\frac{\pi(h-k)}{p}x\right]dx$$

$$= \frac{p}{2\pi}\left[-\frac{1}{h+k}\cos(\frac{\pi(h+k)x}{p}) + \frac{1}{h-k}\cos(\frac{\pi(h-k)x}{p})\right]\Bigg|_{-p}^{p}$$

$$= \frac{p}{2\pi}\Bigg[-\frac{1}{h+k}[\cos(\pi(h+k)) - \cos(-\pi(h+k))]$$

$$+\frac{1}{h-k}[\cos(\pi(h-k)) - \cos(-\pi(h-k))]\Bigg] = 0 \qquad (\text{when } h \neq k)$$

when $(h = k)$

$$\int_{-p}^{p} \cos\frac{\pi k}{p}x \cdot \sin\frac{\pi h}{p}x dx = \int_{-p}^{p} \frac{1}{2}\sin\frac{2\pi k}{p}x dx = -\frac{p}{4\pi k}\cos\frac{2\pi k}{p}\Bigg|_{-p}^{p} = 0$$

(4) Cosine VS. Cosine, $k \neq h$

$$\int_{-p}^{p} \cos\frac{\pi k}{p}x \cdot \cos\frac{\pi h}{p}x dx = \int_{-p}^{p} \frac{1}{2}\left[\cos\frac{\pi(h+k)}{p}x + \cos\frac{\pi(h-k)}{p}x\right]dx$$

$$= \frac{p}{2\pi}\left[\frac{1}{h+k}\sin(\frac{\pi(h+k)x}{p}) + \frac{1}{h-k}\sin(\frac{\pi(h-k)x}{p})\right]\Bigg|_{-p}^{p}$$

$$= \frac{p}{2\pi}\left[\frac{1}{h+k}[\sin(\pi(h+k)) - \sin(-\pi(h+k))]\right.$$

$$\left. + \frac{1}{h-k}[\sin(\pi(h-k)) - \sin(-\pi(h-k))]\right] = 0 \qquad \text{when } h \neq k$$

(5) Sine VS. Sine, $k \neq h$

$$\int_{-p}^{p} \sin\frac{\pi k}{p}x \cdot \sin\frac{\pi h}{p}x dx = \int_{-p}^{p} \frac{1}{2}\left[\cos\frac{\pi(h-k)}{p}x - \cos\frac{\pi(h+k)}{p}x\right]dx$$

$$= \frac{p}{2\pi}\left[\frac{1}{h-k}\sin(\frac{\pi(h-k)x}{p}) - \frac{1}{h+k}\sin(\frac{\pi(h+k)x}{p})\right]\Bigg|_{-p}^{p} = 0 \quad \text{when } h \neq k$$

Square norms of trigonometric functions

$$\|1\|^2 = \int_{-p}^{p} 1 \cdot 1 dx = x\Big|_{-p}^{p} = 2p$$

$$\left\|\cos\frac{\pi k}{p}x\right\|^2 = \int_{-p}^{p}\cos^2\frac{\pi k}{p}x dx = \frac{1}{2}\int_{-p}^{p}(1+\cos\frac{2\pi k}{p}x)dx = \frac{1}{2}\left(x+\frac{p}{2\pi k}\sin\frac{2\pi k}{p}x\right)\Bigg|_{-p}^{p} = p$$

$$\left\|\sin\frac{\pi kx}{p}\right\|^2 = \int_{-p}^{p}\sin^2\frac{\pi kx}{p}dx = \frac{1}{2}\int_{-p}^{p}(1-\cos\frac{2\pi kx}{p})dx = \frac{1}{2}\left(x-\frac{p}{2\pi k}\sin\frac{2\pi kx}{p}\right)\Bigg|_{-p}^{p} = p$$

11.2.3 Fourier Series

The Fourier series is the orthogonal series expansion (see page 606) by trigonometric functions

(Fourier series又被稱作 trigonometric series)

The Fourier Series of a function $f(x)$ defined on the interval $[-p, p]$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx$$

a_0, a_n, b_n 被稱作 Fourier coefficients

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$

Example 1 (text page 433)

$$f(x)=\begin{cases} 0 & \text{for} \quad -\pi < x < 0 \\ \pi - x & \text{for} \quad 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{\pi - x}{\pi}\frac{1}{n}\sin nx\Big|_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}(-1)\frac{1}{n}\sin nx\,dx$$

$$= -\frac{1}{n^2\pi}\cos nx\Big|_0^{\pi} = \frac{1-\cos n\pi}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx dx$$

$$= -\frac{\pi - x}{\pi}\frac{1}{n}\cos nx\Big|_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}(-1)(-\frac{1}{n}\cos nx)dx$$

$$= \frac{1}{n} - \frac{1}{n^2\pi}\sin nx\Big|_0^{\pi} = \frac{1}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos\frac{n\pi}{p}x + \frac{1}{n}\sin\frac{n\pi}{p}x\right)$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right)$$

$$p = \pi$$

11.2.4 Conditions for Convergence

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$ 其實未必成立

If $a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx \quad a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$

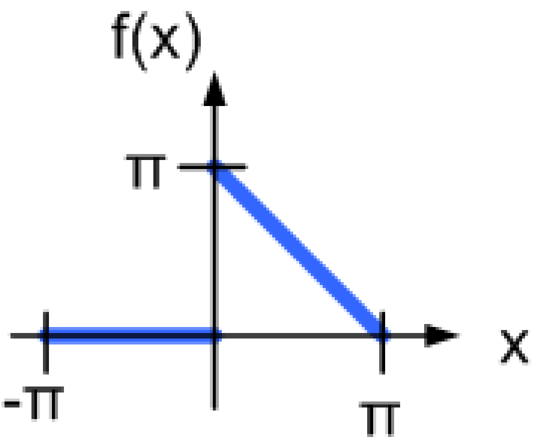
$$f_1(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

(1) $f_1(x_0)=f(x_0)$ if $f(x)$ is continuous at x_0

(2) $f_1(x_0) = \dfrac{f(x_0+) + f(x_0-)}{2}$ if $f(x)$ is not continuous at x_0

$$f(x_0+) = \lim_{h \to 0} f(x_0 + h) \qquad f(x_0-) = \lim_{h \to 0} f(x_0 - h)$$

Example 1 的例子



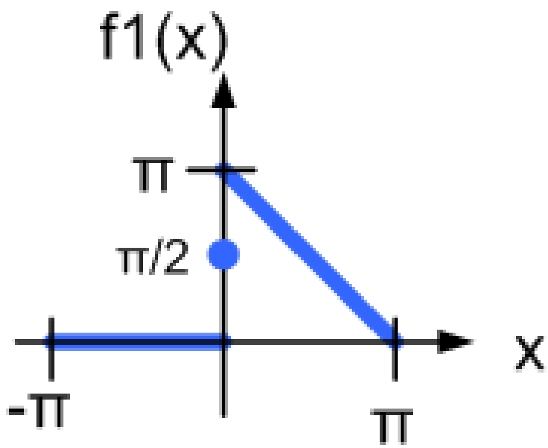


Fig. 11-2-1

11.2.5 Period Extension

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

fundamental period: $2p$

在 interval $x \in [-p, p]$ 以外的地方

$$f_1(x+2p) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(\frac{n\pi}{p}x + 2n\pi) + b_n \sin(\frac{n\pi}{p}x + 2n\pi)\right)$$

$$f_1(x+2p) = f_1(x)$$ (period Extension)

♦ $f_1(x)$ 是個週期為 $2p$ 的 函式 (這是 $f_1(x)$ 和 $f(x)$ 第二個不同的地方

Example 1 的例子

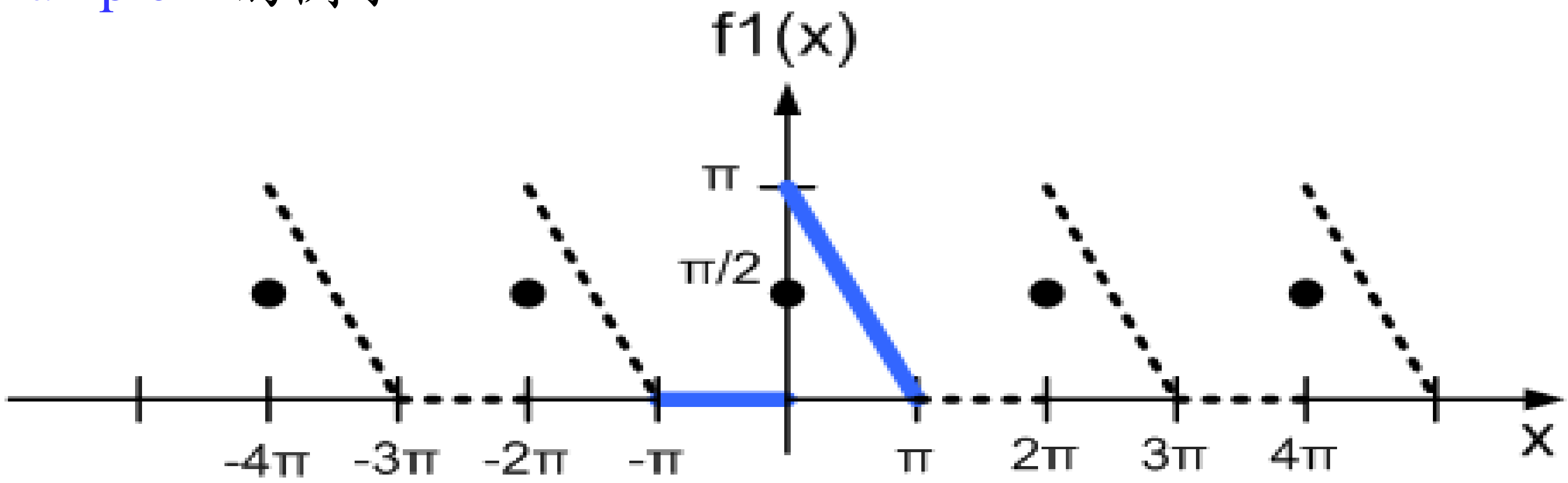


Fig. 11.2.2

對一個非週期的函式，Fourier series expansion 的結果不適用於 $x \notin [-p, p]$ 的區域

但是週期函式則可

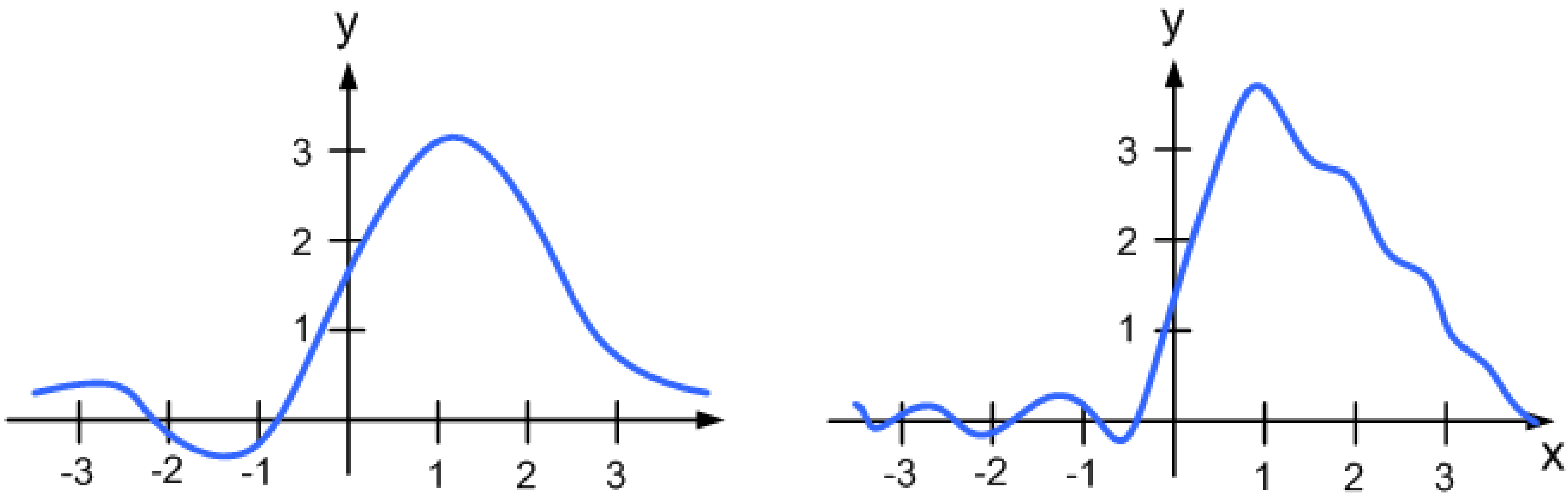
11.2.6 Sequence of Partial Sums

Sequence of Partial Sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

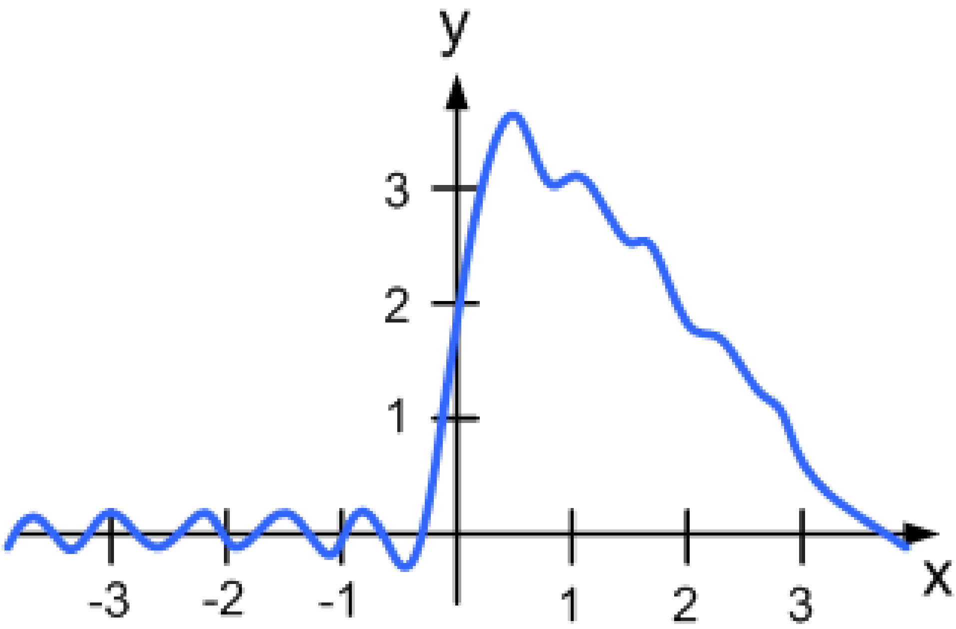
$$f_1(x) = \lim_{N \to \infty} S_N(x)$$

N 越大，越能逼近原來的 function



(a) $S_3(x)$ (b) $S_8(x)$

Fig. 11.2.3



(c) $S_{15}(x)$

Fig. 11.2.3

$N = 15$

11.2.7 Section 11.2 需要注意的地方

(1) Fourier series 的公式 (常背錯)

(a) 第一項是 $a_0/2$，而非 a_0

(b) 算 a_0, a_n, b_n 時，積分後別忘了除以 p

(p 是 interval width 的一半)

(2) 背熟三角函式公式

(3) 熟悉 $\int_a^b u(t)v'(t)dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)dt$

(在計算 Fourier coefficients 會常用到，如 Example 1)

(4) 當 n 為整數時, $\cos n\pi = (-1)^n$ 習慣這種表示法

(5) 正確而言， $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$

近似於

因為當 $f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$

$f_1(x)$ 和 $f(x)$ 之間有二個不同的地方

(a) 在 discontinuous 的地方 $f_1(x_0) = [f(x_0+) + f(x_0-)]/2$

(b) $f_1(x)$ 為 periodic, $f_1(x) = f_1(x+p)$

然而，習慣上，還是寫成 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$

數學史上最美麗的詩篇 --- 傅立葉級數

Clerk Maxwell

悲傷的傅立葉

Section 11.3 Fourier Cosine and Sine Series

11.3.1 綱要

(1)

Fourier Series

$f(x)$ is even — Fourier cosine series (或 cosine series)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x$$

$$a_0=\frac{2}{p}\int_0^p f(x)dx \qquad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$

$f(x)$ is odd — Fourier sine series (或 sine series)

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x$$

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}xdx$$

比較 page 637
和 page 616

(2) 重要名詞： Fourier cosine series, cosine series (page 643)

Fourier sine series, sine series (page 644)

Gibb's Phenomenon (page 647)

(3) Half-range extension: $[0, L]$

(a) cosine series: $f(x) = f(-x)$, interval is changed into $[-L, L]$, set $p = L$

(b) sine series: $f(x) = -f(-x)$, interval is changed into $[-L, L]$, set $p = L$

(c) Fourier series: (i) interval $[-p, p]$ is replaced by $[0, L]$,

(ii) p is replaced by $L/2$

(4) One of the applications: **Solving particular solution** (See page 657)

11.3.2 Even and Odd Functions

even function: $f(x) = f(-x)$

odd function: $f(x) = -f(-x)$

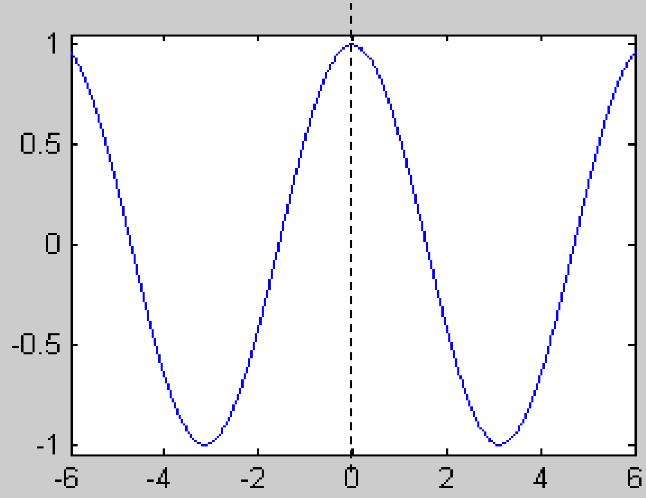
Example

$1, x^2, x^4, x^6, x^8$ ….. are even

x, x^3, x^5, x^7, x^9….. are odd

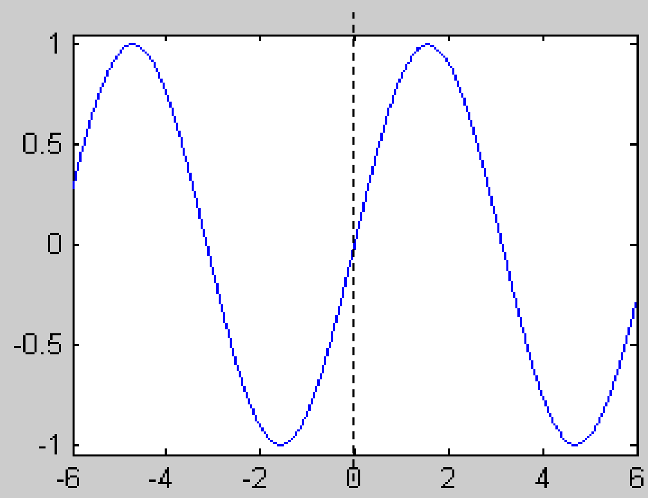
Cosine functions are even

$\cos(t)$



Sine functions are odd

$\sin(t)$



Several properties about even and odd functions

(a) The product of two even functions is even

$$例：x^2 \cdot x^4 = x^6$$

(b) The product of two odd functions is even

$$例：x \cdot x = x^2$$

(c) The product of an even function and an odd function is odd

$$例：x \cdot x^2 = x^3$$

(d) The sum (or difference) of two even function is still even

(e) The sum (or difference) of two odd function is still odd

(f) If $f(x)$ is even, then $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(g) If $f(x)$ is odd, then $$\int_{-a}^{a} f(x)\,dx = 0$$

(Proof):

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{a}^{0} f(-x_1)\,dx_1 + \int_{0}^{a} f(x)\,dx \quad (令\ x_1 = -x,\ dx_1 = -dx)$$

$$= \int_{0}^{a} f(-x_1)\,dx_1 + \int_{0}^{a} f(x)\,dx$$

When $f(x) = f(-x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x_1)\,dx_1 + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

When $f(x) = -f(-x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} -f(x_1)\,dx_1 + \int_{0}^{a} f(x)\,dx = 0$$

11.3.3 Fourier Cosine and Sine Series

(1) The Fourier series of an **even function** on the interval $(-p, p)$ is the **cosine series** (或稱作 **Fourier cosine series**)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$

和之前 Fourier series 不一樣的地方有三個

適用情形： (1) $f(x)$ is even

(2) Half range extension (page 649)

(2) The Fourier series of an **odd function** on the interval $(-p, p)$ is the **sine series** (或稱作 **Fourier sine series**)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

和之前 Fourier series 不一樣的地方有三個

(是哪三個)

適用情形： (1) $f(x)$ is odd

(2) Half range extension (page 649)

Example 1 (text page 438)

Expand $f(x) = x,\ -2 < x < 2$ in a Fourier series

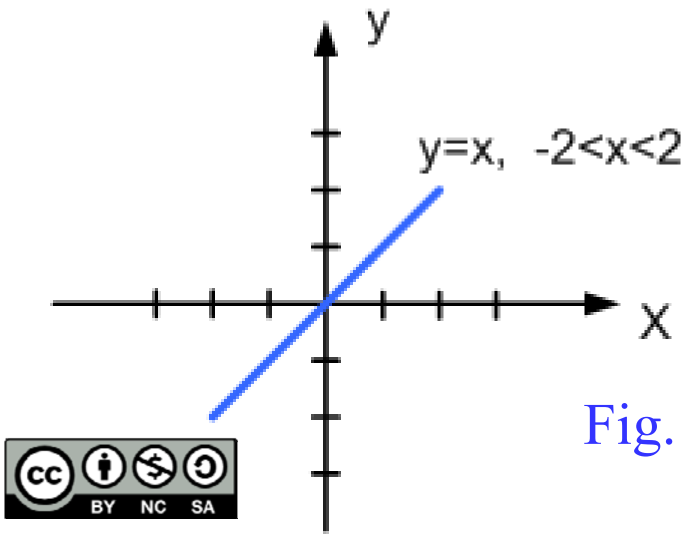


Fig. 11.3.3

$f(x)$ is odd

$\therefore$ expand $f(x)$ by a Fourier sine series

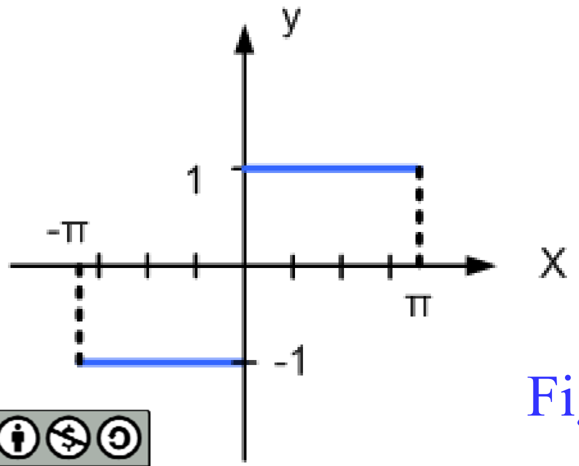
$$b_n = \frac{2}{2}\int_0^2 x\sin\frac{n\pi}{2}x\,dx = -\frac{2}{n\pi}x\cos\frac{n\pi}{2}x\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi}{2}x\,dx$$

$$= -\frac{2}{n\pi}2\cos n\pi + 0 + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}x\Big|_0^2 = -\frac{4}{n\pi}(-1)^n + 0 - 0 = \frac{4}{n\pi}(-1)^{n+1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi}{2}x}$$

Example 2 (text page 438)

$$f(x)=\begin{cases}-1 & -\pi<x<0\\ 1 & 0\le x<\pi\end{cases}$$



odd function, 使用 sine series

Fig. 11.3.5

$$b_n=\frac{2}{\pi}\int_0^{\pi}1\cdot\sin\frac{n\pi}{\pi}x\,dx=-\frac{2}{\pi}\frac{\cos nx}{n}\bigg|_0^{\pi}=\frac{2}{\pi}\frac{1-(-1)^n}{n}$$

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

11.3.4 Gibbs Phenomenon

Example 2 的結果 $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

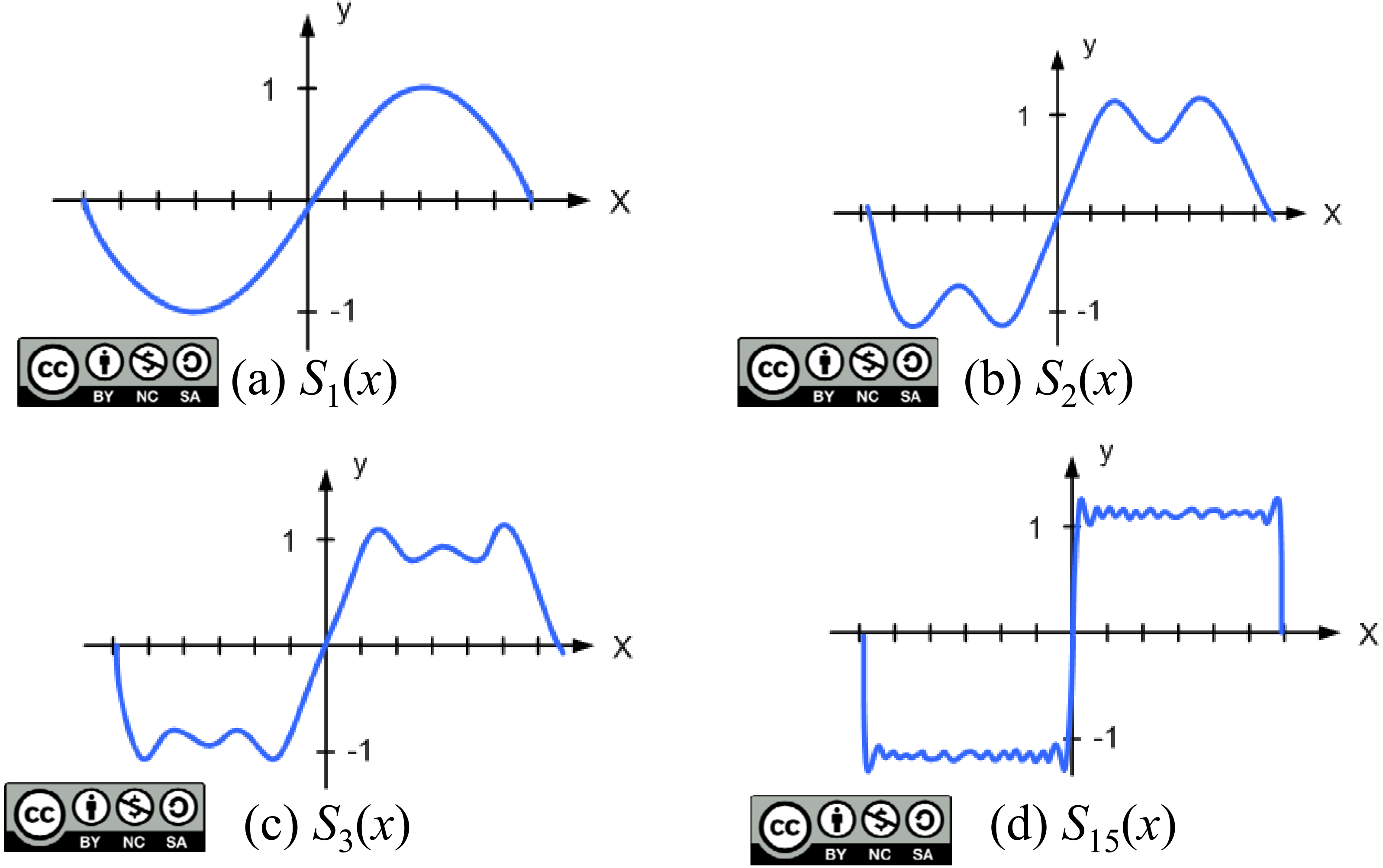
partial sum $S_N(x)=\frac{2}{\pi}\sum_{n=1}^{N}\frac{1-(-1)^n}{n}\sin nx$

當 N 不為無限大，在 discontinuities 附近會有 "overshooting"

"overshooting" 的大小不會隨著 N 而變小

但寬度會越來越窄，越來越靠近 discontinuities 的地方

這種現象，稱作 Gibb's phenomenon



(a) $S_1(x)$ (b) $S_2(x)$

(c) $S_3(x)$ (d) $S_{15}(x)$

Fig. 11.3.6

11.3.5 Half Range Extension

之前的例子：$f(x)$ is defined in the interval of $-p < x < p$

若問題改成

Expand $f(x)$, $0 < x < L$ ($f(x)$ 只有在 $0 < x < L$ 當中有定義)

(a) In a cosine series

(i) Interval: $[-L, L]$, (ii) 所有公式的 p 由 L 取代, (iii) 結果是 even

(b) in a sine series

(i) Interval: $[-L, L]$, (ii) 所有公式的 p 由 L 取代，(iii) 結果是 odd

(c) in a Fourier series

(i) Interval: $[0, L]$, (ii) 所有公式的 p 由 $L/2$ 取代

如 Example 3 (text page 440),

$f(x) = x^2, \quad 0 < x < L$

(a) in a cosine series

假設 $f(x) = f(-x)$ for $-L < x < 0$, (假設 $f(x)$ 是一個 even function)

interval變為 $(-L, L)$

原本 cosine series 公式

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \qquad a_0 = \frac{2}{p}\int_0^p f(x)dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$

現在只不過將 p 改成 L

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx$$

(b) in a sine series

假設 $f(x) = -f(-x)$ for $-L < x < 0$, (假設 $f(x)$ 是一個 odd function)

interval 變為 $(-L, L)$

原本 sine series 公式

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

現在只不過將 p 改成 L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

(c) in a Fourier series　　interval 仍為 $(0, L)$

原本 Fourier series 公式

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)\qquad a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx\qquad b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$

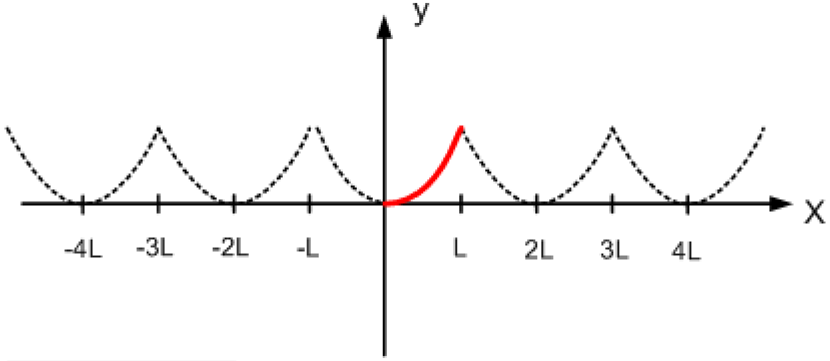
現在 (1) 將 interval $[-p, p]$ 換為 $[0, L]$,　(2) 將 p 換為 $L/2$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi}{L}x+b_n\sin\frac{2n\pi}{L}x\right)\qquad a_0=\frac{2}{L}\int_{0}^{L}f(x)dx$$

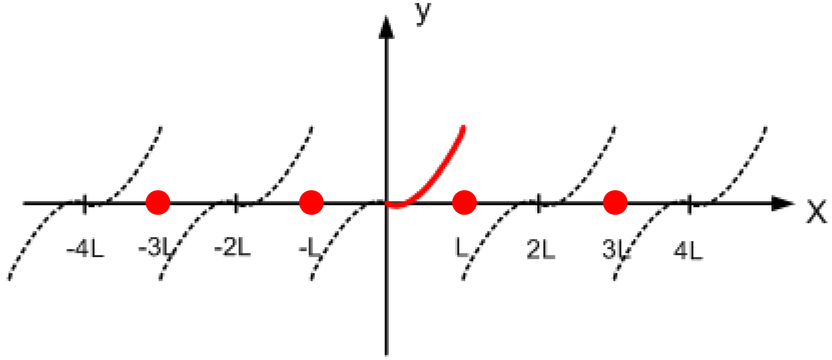
$$a_n=\frac{2}{L}\int_{0}^{L}f(x)\cos\frac{2n\pi}{L}xdx\qquad b_n=\frac{2}{L}\int_{0}^{L}f(x)\sin\frac{2n\pi}{L}xdx$$

Example 3, $f(x) = x^2$, $0 < x < L$

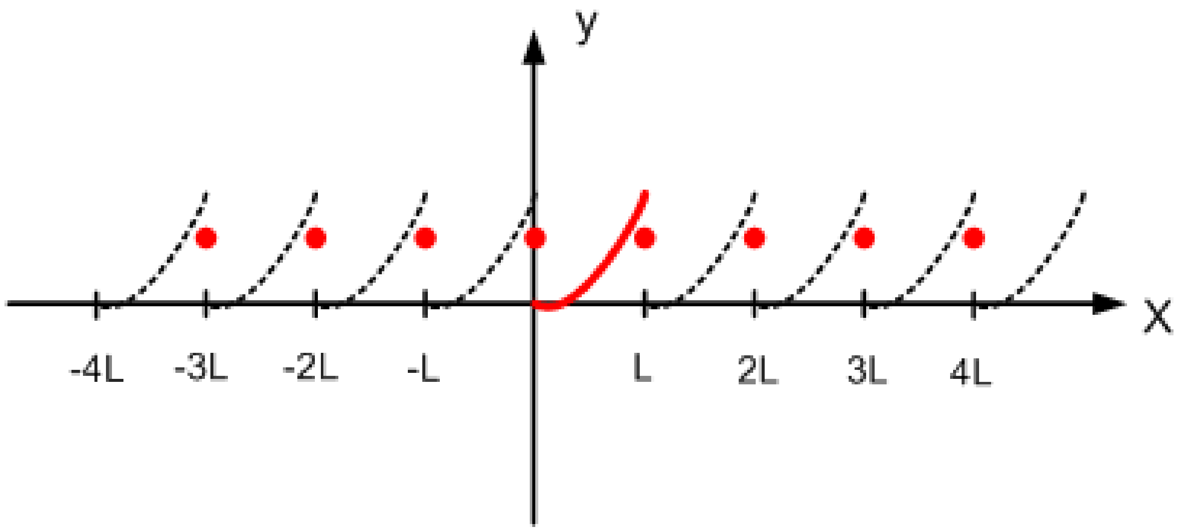
將三個方法的結果畫成圖形



cosine series



sine series



Fourier series

11.3.6 Solving Particular Solutions (第四個方法)

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = f(t)$$

$$f(t) = f(t + 2p)$$

方法的限制

(註：以下的步驟不包含解 homogeneous solution
homogeneous solution 還是需要用 Section 4-3 的方法來解)

(Step 1) 將 $f(t)$ 表示成 Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}t + b_n \sin\frac{n\pi}{p}t\right)$$

或 cosine series (當 $f(t)$ 為 even)

或 sine series (當 $f(t)$ 為 odd)

(Step 2) 假設 particular solution 的型態為

$$y_p(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{p}t + B_n \sin\frac{n\pi}{p}t \right)$$

(Step 3) 代回原式，比較係數，將 A_0, A_n, B_n 解出來

若所假設的 particular solution 和 homogeneous solution
有相同的地方，則要乘上 t

Example 4 (text page 441)

(相關物理定理請複習 Section 5.1)

$$\frac{1}{16}\frac{d^2x}{dt^2}+4x=f(t) \qquad f(t)=\pi t \quad \text{for } -1<t<1$$

$$f(t)=f(t-2)$$

Step 1 假設 $f(t)=\sum_{n=1}^{\infty} b_n \sin n\pi t$ (因為 $f(t)$ 是 odd)

$$b_n = 2\int_0^1 \pi t \sin(n\pi t)\,dt$$

$$= -2\frac{t}{n}\cos(n\pi t)\Big|_0^1 + \int_0^1 \frac{2}{n}\cos(n\pi t)\,dt$$

$$= -2\frac{1}{n}(-1)^n - 0 + \frac{2}{n^2\pi}\sin(n\pi t)\Big|_0^1 = \frac{2}{n}(-1)^{n+1}$$

$$f(t)=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin n\pi t$$

Step 2　　假設 particular solution 為

$$x_p(t)=\sum_{n=1}^{\infty}\left(A_n\cos n\pi t+B_n\sin n\pi t\right)$$

$$(p=1)$$

思考：為什麼這裡可以沒有常數項 A_0?

Step 3 將 $x_p(t)$ 和 Step 1 的結果代入 $\frac{1}{16}\frac{d^2x}{dt^2}+4x=f(t)$

$$\sum_{n=1}^{\infty}\left(-\frac{1}{16}A_n\frac{n^2\pi^2}{4}\cos n\pi t-\frac{1}{16}B_n n^2\pi^2\sin n\pi t\right)$$

$$+\sum_{n=1}^{\infty}\left(4A_n\cos n\pi t+4B_n\sin n\pi t\right)=\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin n\pi t$$

$$-\frac{1}{16}A_n\frac{n^2\pi^2}{4}+4A_n=0 \quad\Longrightarrow\quad A_n=0$$

$$-\frac{1}{16}B_n n^2\pi^2+4B_n=\frac{2}{n}(-1)^{n+1} \quad\Longrightarrow\quad B_n=\frac{32(-1)^{n+1}}{n(64-n^2\pi^2)}$$

Therefore, the particular solution is:

$$x_p(t)=\sum_{n=1}^{\infty}\frac{32(-1)^{n-1}}{n(64-n^2\pi^2)}\sin n\pi t$$

General solution:

$$x(t)=c_1\cos(8t)+c_2\sin(8t)+\sum_{n=1}^{\infty}\frac{32(-1)^{n-1}}{n(64-n^2\pi^2)}\sin n\pi t$$

注意：由於 $\frac{1}{16}\frac{d^2x}{dt^2}+4x=f(t)$ 當中並沒有一次，三次，五次....微分項，所以 particular solution 不可能會有 cosine terms

所以，在 Step 2 當中，可以直接假設

$$x_p(t)=\sum_{n=1}^{\infty}B_n\sin n\pi t$$

11.3.7 Section 11.3 需要注意的地方

(1) 公式一些地方易記錯

for cosine series and sine series,

$$a_0 = \frac{2}{p}\int_0^p f(x)dx \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}xdx$$

(2) Fourier series 的 half-range extension 和 cosine series 及 sine series 不同

p is replaced by $L/2$, $[-p, p]$ is replaced by $[0, L]$

(3) Half range extension 和 solving particular solution 這兩個部分較複雜，需要特別注意，並且多練習例題

Exercise for Practice

Section 11-1 3, 5, 6, 8, 13, 14, 17, 19, 20, 21, 22, 23

Section 11-2 2, 5, 9, 10, 12, 16, 19, 22, 23, 24

Section 11-3 14, 16, 18, 21, 22, 23, 28, 29, 33, 36, 37, 43, 46, 47a, 48a, 49, 52

Review 11 6, 12, 13, 14, 15, 17, 18