Chapter 14 Integral Transform Method

Integral transform 可以表示成如下的積分式的 transform

$$F(s)=\int_a^b \underline{K(s,t)}\, f(t)\,dt$$

• kernel

Laplace transform is one of the integral transform

$$L\{f(t)\}=\int_0^\infty e^{-st} f(t)\,dt$$

本章討論的 integral transform: **Fourier transform**

$$\boxed{\Im\{f(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} f(t)\,dt}$$

Chapter 14 可看成是 Chapter 7 和 Chapter 11 的綜合

Fourier Transform:

(1) 可看成將 Laplace transform 的 s 換成 $-j\alpha$

並且將 $\int_0^\infty$ 換成 $\frac{1}{2\pi}\int_{-\infty}^\infty$

(2) 或者可看成 Fourier series 當 p 為無限大的情形

叮嚀： Chapter 14 的公式定義眾多，且非常相近，要注意彼此之間的差異以及適用情形，以免混淆

Section 14.3 Fourier Integral

14.3.1 綱要

(1) Fourier integral: (和 Fourier series 的定義比較)

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(\alpha)\cos(\alpha x)+B(\alpha)\sin(\alpha x)\right]d\alpha$$

$$A(\alpha)=\int_{-\infty}^{\infty}f(x)\cos(\alpha x)dx \qquad B(\alpha)=\int_{-\infty}^{\infty}f(x)\sin(\alpha x)dx$$

(2) complex form 或 exponential form of Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha) e^{-j\alpha x} d\alpha$$

$$C(\alpha) = \int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx$$

比較： Fourier integral 原本的定義

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \left[A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x) \right] d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x) dx \qquad B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x) dx$$

(3) Fourier cosine integral 或 cosine integral

$$f(x)=\frac{2}{\pi}\int_0^\infty A(\alpha)\cos(\alpha x)d\alpha \qquad A(\alpha)=\int_0^\infty f(x)\cos(\alpha x)dx$$

適用情形： (1) even 或 (2) interval: [0, ∞)

(4) Fourier sine integral 或 sine integral

$$f(x)=\frac{2}{\pi}\int_0^\infty B(\alpha)\sin(\alpha x)d\alpha \qquad B(\alpha)=\int_0^\infty f(x)\sin(\alpha x)dx$$

適用情形： (1) odd 或 (2) interval: [0, ∞)

(5) Others

名詞：absolutely integrable (page 671)
partial integral (page 683)

特殊公式： $\int_0^\infty \frac{\sin\alpha}{\alpha}d\alpha=\frac{\pi}{2}$

14.3.2 From Fourier Series to Fourier Integral

複習：Section 11-2 的 Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right) \qquad a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\,dt$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt$$

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \frac{1}{p}\sum_{n=1}^{\infty}\left(\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x$$
$$+\frac{1}{p}\sum_{n=1}^{\infty}\left(\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x$$

$$f(x)=\frac{1}{2p}\int_{-p}^{p}f(t)dt+\frac{1}{p}\sum_{n=1}^{\infty}\left(\int_{-p}^{p}f(t)\cos\frac{n\pi}{p}tdt\right)\cos\frac{n\pi}{p}x$$

$$+\frac{1}{p}\sum_{n=1}^{\infty}\left(\int_{-p}^{p}f(t)\sin\frac{n\pi}{p}tdt\right)\sin\frac{n\pi}{p}x$$

令 $\Delta\alpha=\frac{\pi}{p}$　　$\frac{1}{p}=\frac{\Delta\alpha}{\pi}$

$$f(x)=\frac{\Delta\alpha}{2\pi}\int_{-p}^{p}f(t)dt+\frac{\Delta\alpha}{\pi}\sum_{n=1}^{\infty}\left(\int_{-p}^{p}f(t)\cos(n\Delta\alpha\cdot t)dt\right)\cos(n\Delta\alpha\cdot x)$$

$$+\frac{\Delta\alpha}{\pi}\sum_{n=1}^{\infty}\left(\int_{-p}^{p}f(t)\sin(n\Delta\alpha\cdot t)dt\right)\sin(n\Delta\alpha\cdot x)$$

$$=\frac{1}{2\pi}\int_{-p}^{p}f(t)dt\cos(0\Delta\alpha\cdot x)\Delta\alpha+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-p}^{p}f(t)\cos(n\Delta\alpha\cdot t)dt\right)\cos(n\Delta\alpha\cdot x)\Delta\alpha$$

$$+\frac{1}{2\pi}\int_{-p}^{p}f(t)dt\sin(0\Delta\alpha\cdot x)\Delta\alpha+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-p}^{p}f(t)\sin(n\Delta\alpha\cdot t)dt\right)\sin(n\Delta\alpha\cdot x)\Delta\alpha$$

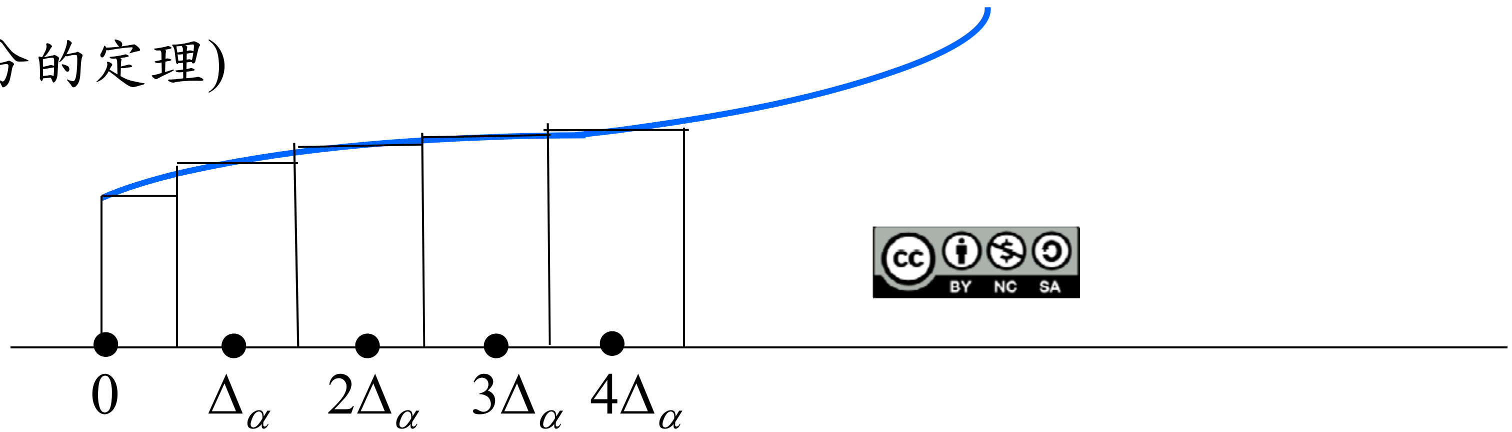
When $p\longrightarrow\infty$,　$\Delta\alpha\longrightarrow 0$

(即 週期 $\longrightarrow\infty$)

When $p \longrightarrow \infty$, $\Delta\alpha \longrightarrow 0$

$$\lim_{\Delta_\alpha \to 0}\left[S(0)\frac{\Delta_\alpha}{2}+\sum_{n=1}^{\infty}S(n\Delta_\alpha)\Delta_\alpha\right]=\int_0^{\infty}S(\alpha)d\alpha$$

(積分的定理)



$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left(\int_{-p}^{p}f(t)\cos(\alpha t)dt\right)\cos(\alpha x)d\alpha+\frac{1}{\pi}\int_0^{\infty}\left(\int_{-p}^{p}f(t)\sin(\alpha t)dt\right)\sin(\alpha x)d\alpha$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty}f(t)\cos(\alpha t)dt\right)\cos(\alpha x)d\alpha+\left(\int_{-\infty}^{\infty}f(t)\sin(\alpha t)dt\right)\sin(\alpha x)d\alpha\right]$$

14.3.3 Fourier Integral

Fourier Integral:

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(\alpha)\cos(\alpha x)+B(\alpha)\sin(\alpha x)\right]d\alpha$$

$$A(\alpha)=\int_{-\infty}^{\infty}f(x)\cos(\alpha x)dx \qquad B(\alpha)=\int_{-\infty}^{\infty}f(x)\sin(\alpha x)dx$$

Fourier integral 存在的 sufficient condition:

$$\int_{-\infty}^{\infty}\left|f(x)\right|dx \quad \text{converges}$$

若這個條件滿足，$f(x)$ 為 absolutely integrable

嚴格來說，當

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx \qquad B(\alpha)=\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx$$

$$f_1(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(\alpha)\cos(\alpha x)+B(\alpha)\sin(\alpha x)\right]d\alpha$$

$f_1(x)$ 和 $f(x)$ 未必相等

但一般還是寫成 $f(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(\alpha)\cos(\alpha x)d\alpha+B(\alpha)\sin(\alpha x)d\alpha\right]$

Theorem 14.3.1 Condition for convergence

When (1) $f(x)$ 為 piecewise continuous

(2) $f'(x)$ 為 piecewise continuous

(3) $f(x)$ 為 absolutely integrable

The Fourier integral of $f(x)$ (即上一頁的 $f_1(x)$) converges to $f(x)$ at a point of continuity.

At the point of discontinuity, $f_1(x)$ converges to

$$\frac{f(x+)+f(x-)}{2}$$

Example 1 (text page 531)

Find the Fourier integral representation of $f(x)$

$$f(x)=\begin{cases}0 & x<0\\ 1 & 0<x<2\\ 0 & x>2\end{cases}$$

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx=\int_{0}^{2}\cos(\alpha x)dx=\left.\frac{\sin(\alpha x)}{\alpha}\right|_{0}^{2}=\frac{\sin(2\alpha)}{\alpha}$$

$$B(\alpha)=\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx=\int_{0}^{2}\sin(\alpha x)dx=-\left.\frac{\cos(\alpha x)}{\alpha}\right|_{0}^{2}=\frac{1-\cos(2\alpha)}{\alpha}$$

$$\boxed{f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\sin(2\alpha)}{\alpha}\cos(\alpha x)+\frac{1-\cos(2\alpha)}{\alpha}\sin(\alpha x)\right]d\alpha}$$

Example 1 的解的另一種表示法

$$f(x)=\frac{1}{\pi}\int_0^\infty\left[\frac{\sin(2\alpha)}{\alpha}\cos(\alpha x)+\frac{1-\cos(2\alpha)}{\alpha}\sin(\alpha x)\right]d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\left[\frac{2\sin\alpha\cos\alpha}{\alpha}\cos(\alpha x)+\frac{2\sin^2\alpha}{\alpha}\sin(\alpha x)\right]d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\left[\frac{2\sin\alpha\{\cos\alpha\cos(\alpha x)+\sin\alpha\sin(\alpha x)\}}{\alpha}\right]d\alpha$$

$$=\frac{1}{\pi}\int_0^\infty\left[\frac{2\sin\alpha\cos(\alpha x-\alpha)}{\alpha}\right]d\alpha$$

(別忘了複習三角函數的公式, pages 611 and 612)

14.3.4 Fourier Transform 意外的提供了一些方程式積分的算法

由 Example 1

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin(2\alpha)}{\alpha}\cos(\alpha x) + \frac{1-\cos(2\alpha)}{\alpha}\sin(\alpha x)\right]d\alpha$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\frac{2\sin\alpha\cos(\alpha x-\alpha)}{\alpha}\right]d\alpha$$

When $x = 1$, since $f(x) = 1$

$$\frac{2}{\pi}\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha = 1$$

$$\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha = \frac{\pi}{2}$$

補充： sinc function 的定義： $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

常用在 sampling theory, filter design, 及通訊上

14.3.5 Fourier Cosine and Sine Integrals

(A) Fourier cosine integral 或 cosine integral

$$f(x)=\frac{2}{\pi}\int_0^{\infty}A(\alpha)\cos(\alpha x)d\alpha$$

$$A(\alpha)=\int_0^{\infty}f(x)\cos(\alpha x)dx$$

類比於 cosine series

注意：有三個地方和 Fourier integral 不同

(1)

(2)

(3)

適用情形: (1) $f(x)$ is even, $f(x) = f(-x)$

(2) 只知道 $f(x)$ 當 $x > 0$ 的時候的值

(類似於Section 11.3 的 half-range expansion,

而且假設 $f(x) = f(-x)$)

(B) Fourier sine integral 或 sine integral

$$f(x)=\frac{2}{\pi}\int_0^{\infty}B(\alpha)\sin(\alpha x)d\alpha$$

$$B(\alpha)=\int_0^{\infty}f(x)\sin(\alpha x)dx$$

類比於 sine series

適用情形: (1) $f(x)$ is odd, $f(x) = -f(-x)$

(2) 只知道 $f(x)$ 當 $x > 0$ 的時候的值

(類比於Section 11.3 的 half-range expansion,

而且假設 $f(x) = -f(-x)$)

Example 3 (text page 533)

Represent $f(x) = e^{-x}, \qquad x > 0$

(a) by a cosine integral (b) by a sine integral

Solution:

(a) $A(\alpha) = \int_0^\infty e^{-x}\cos(\alpha x)dx$

Suppose that $\frac{d}{dx}\left[b_1 e^{-x}\cos(\alpha x) + b_2 e^{-x}\sin(\alpha x)\right] = e^{-x}\cos(\alpha x)$

$$-b_1 e^{-x}\cos(\alpha x) - b_1\alpha e^{-x}\sin(\alpha x) - b_2 e^{-x}\sin(\alpha x) + b_2\alpha e^{-x}\cos(\alpha x) = e^{-x}\cos(\alpha x)$$

$$\begin{cases} -b_1 + b_2\alpha = 1 \\ -b_1\alpha - b_2 = 0 \end{cases} \Longrightarrow \quad b_1 = -\frac{1}{1+\alpha^2}, \quad b_2 = \frac{\alpha}{1+\alpha^2}$$

$$A(\alpha) = -\frac{1}{1+\alpha^2}e^{-x}\cos(\alpha x) + \frac{\alpha}{1+\alpha^2}e^{-x}\sin(\alpha x)\Big|_0^\infty = \frac{1}{1+\alpha^2}$$

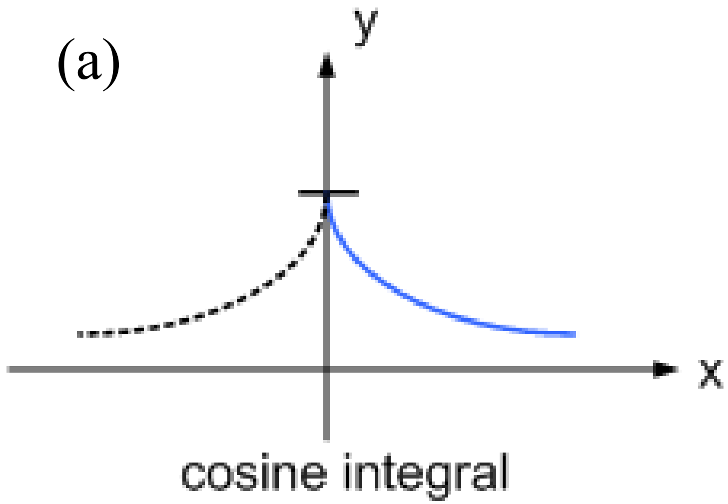
(其實，有一個取巧的快速算法，用 Laplace transform)

cosine integral:

$$f(x)=\frac{2}{\pi}\int_0^{\infty}A(\alpha)\cos(\alpha x)d\alpha=\frac{2}{\pi}\int_0^{\infty}\frac{\cos(\alpha x)}{1+\alpha^2}d\alpha$$

(b) $B(\alpha)=\int_0^{\infty}e^{-x}\sin(\alpha x)dx=\frac{\alpha}{1+\alpha^2}$

$$f(x)=\frac{2}{\pi}\int_0^{\infty}B(\alpha)\sin(\alpha x)d\alpha=\frac{2}{\pi}\int_0^{\infty}\frac{\alpha\sin(\alpha x)}{1+\alpha^2}d\alpha$$



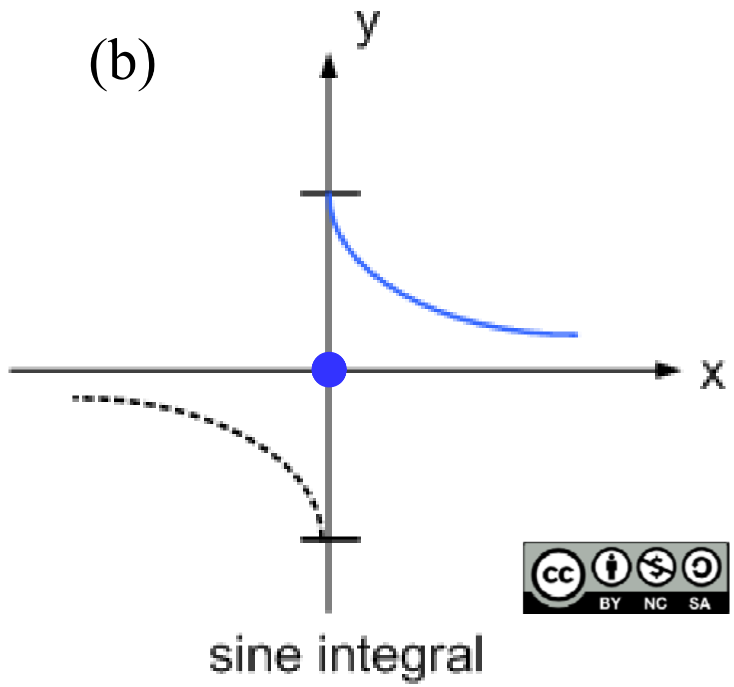


Fig. 14.3.4

14.3.6 Partial Integral

partial integral for Fourier integral

$$F_b(x) = \frac{1}{\pi}\int_0^b [A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)]d\alpha$$

partial integral for cosine integral

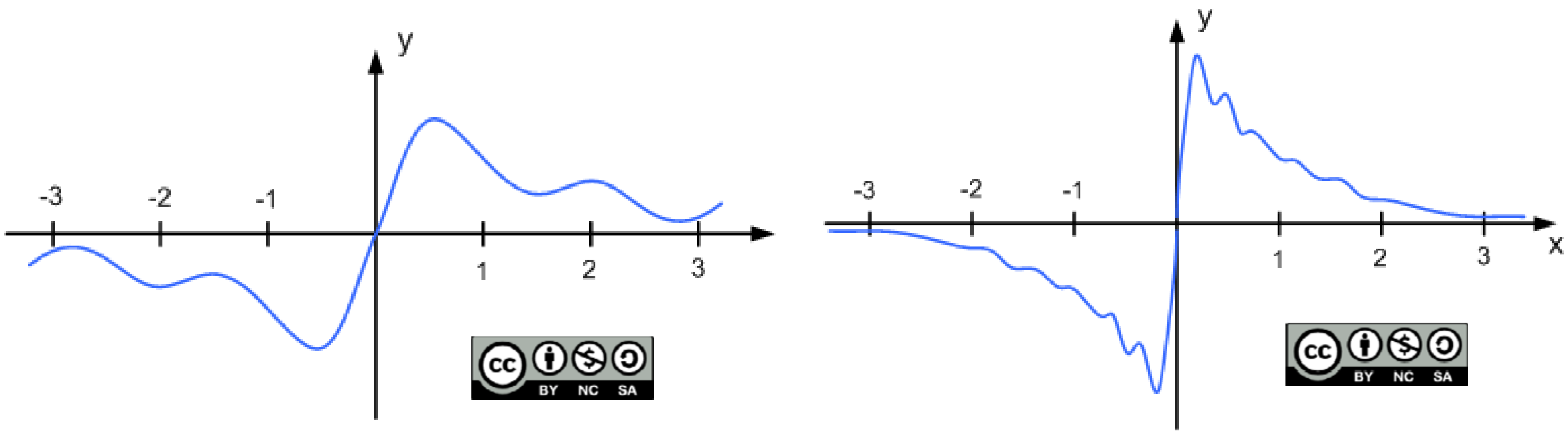
$$F_b(x) = \frac{2}{\pi}\int_0^b A(\alpha)\cos(\alpha x)d\alpha$$

partial integral for sine integral

$$F_b(x) = \frac{2}{\pi}\int_0^b A(\alpha)\sin(\alpha x)d\alpha$$

(用 b 取代 ∞)

For Example 3



(a) $F_5(x)$

($b = 5$)

(b) $F_{20}(x)$

($b = 20$)

Fig. 14.3.5

14.3.7 Complex Form

complex form or exponential form of Fourier integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha) e^{-j\alpha x} d\alpha$$

$$C(\alpha) = \int_{-\infty}^{\infty} f(x) e^{j\alpha x} d\alpha$$

remember: $e^{j\alpha x} = \cos\alpha x + j\sin\alpha x$

Proof:

由講義 page 671 Fourier integral 的定義

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty}f(t)\cos(\alpha t)dt\right)\cos(\alpha x)+\left(\int_{-\infty}^{\infty}f(t)\sin(\alpha t)dt\right)\sin(\alpha x)\right]d\alpha$$

$$=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(t)[\cos(\alpha t)\cos(\alpha x)+\sin(\alpha t)\sin(\alpha x)]dtd\alpha$$

$$=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(t)\cos(\alpha(t-x))dtd\alpha$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)\cos(\alpha(t-x))dtd\alpha$$

注意： $f(t)\cos(\alpha(t-x))$ 對 α 而言是 even function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos(\alpha(t-x))\,dt\,d\alpha$$

From $\int_{-\infty}^{\infty} f(t)\sin(\alpha(t-x))\,d\alpha = 0$

(因為 $f(t)\sin(\alpha(t-x))$ 對 α 而言是 odd function)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos(\alpha(t-x)) + j\sin(\alpha(t-x))]\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{j\alpha(t-x)}\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)e^{j\alpha t}\,dt\right]e^{-j\alpha x}\,d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{-j\alpha x}\,d\alpha \qquad C(\alpha) = \int_{-\infty}^{\infty} f(x)e^{j\alpha x}\,dx$$

14.3.8 Section 14.3 需要注意的地方

(1) 公式積分的外面，要乘 $\dfrac{1}{\pi}$ (Fourier integral)

或 $\dfrac{1}{2\pi}$ (Complex form of Fourier integral)

或 $\dfrac{2}{\pi}$ (cosine integral, sine integral)

(2) 一些積分的計算會常常用到

$\int x\cos(\alpha x)dx$ $\int x\sin(\alpha x)dx$ 算法： $\int uv' = uv - \int u'v$

$\int e^{-x}\cos(\alpha x)dx$ $\int e^{-x}\sin(\alpha x)dx$

算法：假設解為 $b_1e^{-x}\cos(\alpha x)+b_2e^{-x}\sin(\alpha x)$

或者用 Laplace transform 的公式, $s = 1$

Section 14.4 Fourier Transforms

14.4.1 綱要

Fourier transform，其實就是 complex form of Fourier integral

公式：$\Im[f(x)]=\int_{-\infty}^{\infty} f(x)e^{j\alpha x}dx=F(\alpha)$

$\Im$ 代表 Fourier transform

$$\Im^{-1}[F(\alpha)]=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-j\alpha x}d\alpha=f(x)$$

本節著重於 (1) 定義

(2) 性質 ⟵ 學習方式：多和 Laplace transform 比較

(3) Solving the boundary value problem (pages 703-713)

↑

有一點複雜，且常考，要勤於練習

(A) 六大定義

(1) Fourier transform

$$\Im[f(x)] = \int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx = F(\alpha)$$

(2) inverse Fourier transform

$$\Im^{-1}[F(\alpha)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{-j\alpha x} d\alpha = f(x)$$

(3) Fourier sine transform

$$\Im_s[f(x)] = \int_{0}^{\infty} f(x)\sin(\alpha x) dx = F(\alpha)$$

(4) inverse Fourier sine transform

$$\Im_s^{-1}[F(\alpha)] = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\sin(\alpha x) d\alpha = f(x)$$

(5) Fourier cosine transform

$$\Im_c[f(x)] = \int_{0}^{\infty} f(x)\cos(\alpha x) dx = F(\alpha)$$

(6) inverse Fourier cosine transform

$$\Im_c^{-1}[F(\alpha)] = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\cos(\alpha x) d\alpha = f(x)$$

注意：除了 $e^{j\alpha x}$ 變成 $\cos(\alpha x)$ 以外

還有三個地方和 Fourier transform 不同

(B) 微分性質

(7) for Fourier transform $\Im\left[f'(x)\right] = -j\alpha F(\alpha)$

(8) $\Im\left[f^{(n)}(x)\right] = (-j\alpha)^{(n)} F(\alpha)$

(9) for Fourier sine transform $\Im_s\left[f'(x)\right] = -\alpha\,\Im_c\left[f(x)\right]$

不同

(10) $\Im_s\left[f''(x)\right] = -\alpha^2\Im_s\left[f(x)\right] + \alpha f(0)$

(11) for Fourier cosine transform $\Im_c\left[f'(x)\right] = \alpha\,\Im_s\left[f(x)\right] - f(0)$

不同

(12) $\Im_c\left[f''(x)\right] = -\alpha^2\,\Im_c\left[f(x)\right] - f'(0)$

(C) Problems with boundary conditions (多練習)

(13) 可考慮用 Fourier transform 的情形 $-\infty < x < \infty$

(14) 可考慮用 Fourier sine transform 的情形 $0 < x < \infty$ $U(x, y) = 0$ when $x = 0$

(15) 可考慮用 Fourier cosine transform 的情形 $0 < x < \infty$ $\left.\frac{\partial}{\partial x} U(x, y)\right|_{x=0} = 0$

另外，要熟悉 page 704 的計算流程

(D) 名詞

transform pair (page 693)

heat equation $k\frac{\partial u^2}{\partial x^2} = \frac{\partial u}{\partial t}$ (page 705)

14.4.2 Transform Pair

Transform pair 的定義：

若　甲 $\xrightarrow{A}$ 乙　　乙 $\xrightarrow{B}$ 甲

則 A 和 B 形成一個 transform pair

甲 和 乙

14.4.3 Fourier Transform

Fourier transform pair

$$\Im[f(x)] = \int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx = F(\alpha)$$

$$\Im^{-1}[F(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-j\alpha x} d\alpha = f(x)$$

和之前 complex form of Fourier integral 相比較

只不過把 $C(\alpha)$ 換成 $F(\alpha)$

為何要取兩個名字???

Fourier transform 存在的條件

(1) $\int_{-\infty}^{\infty} |f(x)| d\alpha < \infty$ (absolutely integrable)

(2) $f(x)$ and $f'(x)$ 為 piecewise continuous

- Fourier transform 和 Laplace transform 之間的關係：

把 s 換成 $-j\alpha$

Laplace: $$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

14.4.4 Fourier Sine Transform and Fourier Cosine Transform

Fourier sine transform pair

$$\Im_s\left[f(x)\right] = \int_0^\infty f(x)\sin(\alpha x)dx = F_s(\alpha)$$

$$\Im_s^{-1}\left[F(\alpha)\right] = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin(\alpha x)d\alpha = f(x)$$

Fourier cosine transform pair

$$\Im_c\left[f(x)\right] = \int_0^\infty f(x)\cos(\alpha x)dx = F_c(\alpha)$$

$$\Im_c^{-1}\left[F(\alpha)\right] = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos(\alpha x)d\alpha = f(x)$$

Fourier sine / cosine transform 存在的條件

(1) $\int_0^\infty \left|f(x)\right| d\alpha < \infty$ (absolutely integrable)

(2) $f(x)$ and $f'(x)$ 為 piecewise continuous

Fourier sine / cosine transform 的意義：

(1) 當 $f(x)$ 為 even

Fourier transform ⟶ Fourier cosine transform

$$\int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx = F(\alpha)$$

$$\int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx = \int_{-\infty}^{\infty} f(x)(\cos\alpha x + j\sin\alpha x) dx$$

$$= \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + \underbrace{j\int_{-\infty}^{\infty} f(x)\sin\alpha x dx}_{\text{等於 } 0}$$

$$= \int_{-\infty}^{\infty} f(x)\cos\alpha x dx$$

$$= 2\int_{0}^{\infty} f(x)\cos\alpha x dx$$

for Fourier cosine transform

$$F_c(\alpha) = F(\alpha)/2$$

(2) Inverse Fourier transform ⟶ inverse Fourier cosine transform

If $f(x)$ is even

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-j\alpha x}d\alpha = f(x)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} F_c(\alpha)e^{-j\alpha x}d\alpha = f(x) \quad \text{(由前頁)}$$

由於對 Fourier cosine transform 而言

$$F_c(\alpha) = F_c(-\alpha) \quad \text{(even function)}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} F_c(\alpha)e^{-j\alpha x}d\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty} F_c(\alpha)\cos(\alpha x)d\alpha - \frac{j}{\pi}\underbrace{\int_{-\infty}^{\infty} F_c(\alpha)\sin(\alpha x)d\alpha}_{\text{等於 } 0} = f(x)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} F_c(\alpha)\cos(\alpha x)d\alpha = f(x)$$

$$\frac{2}{\pi}\int_{0}^{\infty} F_c(\alpha)\cos(\alpha x)d\alpha = f(x)$$

Fourier cosine transform 等同於 Fourier transform 當 input $f(x)$ 為 even 的情形。當 $f(x)$ 為 even，

$$F_c(\alpha) = F(\alpha)/2$$

Fourier sine transform 等同於 Fourier transform 當 input $f(x)$ 為 odd 的情形。當 $f(x)$ 為 odd，

$$F_s(\alpha) = F(\alpha)/j2$$

- 然而，若 $f(x)$ 只有在 $x \in [0, \infty)$ 之間有定義，也可以用 Fourier cosine / sine transform

(類似於Section 11.3 的 half-range expansion)

14.4.5 微分性質

(1) Fourier transform 的微分性質

$$\Im[f'(x)] = \int_{-\infty}^{\infty} f'(x) e^{j\alpha x} dx = f(x) e^{j\alpha x}\Big|_{-\infty}^{\infty} - j\alpha \int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx$$

$$= -j\alpha \Im[f(x)]$$

微分性質做了一些假設：$f(x) = 0$ when $x \to \infty$ and $x \to -\infty$

以此類推 $\Im\left[f^{(n)}(x)\right] = (-j\alpha)^{(n)} F(\alpha)$

比較：對 Laplace transform

$$L\{f'(x)\} = sL\{f(x)\} - f(0) \qquad \int_{0}^{\infty} f(x) e^{-sx} dx$$

對 Fourier transform

$s \to -j\alpha$, without initial conditions

(2) Fourier sine transform 的微分性質

$$\Im_s[f'(x)] = \int_0^\infty f'(x)\sin(\alpha x)\,dx = f(x)\sin(\alpha x)\Big|_0^\infty - \alpha\int_0^\infty f(x)\cos(\alpha x)\,dx$$

$$= -\alpha\,\Im_c[f(x)]$$

(3) Fourier cosine transform 的微分性質

$$\Im_c[f'(x)] = \int_0^\infty f'(x)\cos(\alpha x)\,dx = f(x)\cos(\alpha x)\Big|_0^\infty + \alpha\int_0^\infty f(x)\sin(\alpha x)\,dx$$

$$= \alpha\,\Im_s[f(x)] - f(0)$$

注意： (1) Fourier sine, cosine transforms 互換

(2) α 正負號不同

(3) Fourier cosine transform 要考慮 initial condition

$$\Im_s\left[f''(x)\right] = -\alpha\,\Im_c\left[f'(x)\right] = -\alpha^2\,\Im_s\left[f(x)\right] + \alpha f(0)$$

$$\Im_c\left[f''(x)\right] = \alpha\,\Im_s\left[f'(x)\right] - f'(0) = -\alpha^2\,\Im_c\left[f(x)\right] - f'(0)$$

14.4.6 Solving the Boundary Value Problem (BVP)

※ 概念複雜，要特別加強練習

(Condition 1) interval 為 $-\infty < v < \infty$ 時:

用 Fourier transform

(Condition 2) interval 為 $0 < v < \infty$，

有 "$u(v, \ldots..) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier sine transform

(Condition 3) interval 為 $0 < v < \infty$，

有 "$\frac{\partial}{\partial v}u(v,\cdots\cdots)=0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier cosine transform

使用 Fourier transform, Fourier cosine transform, Fourier sine transform 來解 partial differential equation (PDE) 的 BVP 或IVP 的解法流程

(Step 1) 以 page 703 的規則，來決定要針對 **哪一個** independent variable ，做**什麼** transform (Fourier, Fourier cosine, 或Fourier sine transform)

(Step 2) 對 PDE 做 Step 1 所決定的 transform, 則原本的 PDE 變成針對另外一個 independent variable 的 ordinary differential equation (ODE)

(Step 3) 將 Step 2 所得出的 ODE 的解算出來

(Step 4) Step 3 所得出來的解會有一些 constants，可以對 initial conditions (或 boundary conditions) 做 transform 將 constants 解出

(※ 和 Step 1 所做的 transform 一樣，只是 transform 的對象變成是 initial 或 boundary conditions，見 pages 705, 708 的例子)

(Step 5) 最後，別忘了做 inverse transform (畫龍點睛)

Example 1 (text page 538)

heat equation: $k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$ $\quad -\infty < x < \infty \quad t>0$

subject to $u(x,0)=f(x)$ where $f(x)=\begin{cases} u_0, & |x|<1 \\ 0, & |x|>1 \end{cases}$

Step 1 決定針對 x 做 Fourier transform

$$\Im_{x\to\alpha}\{u(x,t)\}=\int_{-\infty}^{\infty}u(x,t)e^{i\alpha x}dx=U(\alpha,t)$$

Step 2 $\Im_{x\to\alpha}\left\{k\frac{\partial^2 u}{\partial x^2}\right\}=\Im_{x\to\alpha}\left\{\frac{\partial u}{\partial t}\right\}$ 原本對 x, t 兩個變數做偏微分

$$-k\alpha^2 U(\alpha,t)=\frac{\partial U(\alpha,t)}{\partial t}$$

經過 Fourier transform 之後，
只剩下對 t 做偏微分

$$\frac{dU(\alpha,t)}{dt}+k\alpha^2U(\alpha,t)=0$$ 對於 t 而言，是 1st order ODE

Step 3 $U(\alpha,t)=c\,e^{-k\alpha^2t}$ 這邊的 c 值，對 t 而言是 constant ，

但是可能會 **dependent on α**(特別注意)

Step 4 根據 $u(x, 0) = f(x)$ 將 c 解出

和 Step 1 一樣，也是針對 x 做 Fourier transform

只是對象改成 initial condition

$$\Im_{x\to\alpha}\{u(x,0)\}=\int_{-\infty}^{\infty}f(x)e^{i\alpha x}dx=\int_{-1}^{1}u_0e^{i\alpha x}dx$$

$$=u_0\frac{e^{i\alpha}-e^{-i\alpha}}{i\alpha}=2u_0\frac{\sin\alpha}{\alpha}$$

因為 $\Im_{x\to\alpha}\{u(x,0)\}=U(\alpha,0)$

$$U(\alpha,0)=2u_0\frac{\sin\alpha}{\alpha}$$

$$U(\alpha,t)=c\,e^{-k\alpha^2 t} \xleftrightarrow{\text{比較係數}} U(\alpha,0)=2u_0\frac{\sin\alpha}{\alpha}$$

解出 $c=2u_0\dfrac{\sin\alpha}{\alpha}$

$$U(\alpha,t)=2u_0\frac{\sin\alpha}{\alpha}e^{-k\alpha^2 t}$$

Step 5 未完待續，別忘了最後要做 inverse Fourier transform

$$u(x,t)=\mathfrak{I}^{-1}_{\alpha\to x}\left[U(\alpha,t)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}2u_0\frac{\sin\alpha}{\alpha}e^{-k\alpha^2 t}e^{-j\alpha x}d\alpha$$

不易化簡，課本僅依據 $\dfrac{\sin\alpha}{\alpha}e^{-k\alpha^2 t}$ 對 α 而言是 even function 將 $u(x, t)$ 化簡為

$$u(x,t)=\frac{u_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}e^{-k\alpha^2 t}(\cos\alpha x-j\sin\alpha x)d\alpha$$

$$=\frac{u_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}e^{-k\alpha^2 t}d\alpha$$

Example 3 Laplace's equation (text page 540)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<\pi \qquad y>0$$

$$u(0,y)=0 \qquad u(\pi,y)=e^{-y} \qquad y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0 \qquad 0<x<\pi$$

Step 1 決定針對 y 做 Fourier cosine transform

$$\mathfrak{I}_{c,y\to\alpha}\{u(x,y)\}=\int_0^\infty u(x,y)\cos\alpha y\,dy=U(x,\alpha)$$

Step 2 $\mathfrak{I}_{c,y\to\alpha}\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\mathfrak{I}_{c,y\to\alpha}\left\{\frac{\partial^2 u}{\partial y^2}\right\}=\mathfrak{I}_{c,y\to\alpha}\{0\}$

from $\mathfrak{I}_c\left[f''(y)\right]=-\alpha^2\,\mathfrak{I}_c\left[f(y)\right]-f'(0)$

$$\frac{d^2U(x,\alpha)}{dx^2}-\alpha^2U(x,\alpha)=0$$ 對於 x 的 2nd order ODE

Step 3 $$\frac{d^2U(x,\alpha)}{dx^2}-\alpha^2U(x,\alpha)=0$$

⇒ $$U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x$$

注意：雖然也可將解表示成 $U(x,\alpha)=c_3e^{\alpha x}+c_4e^{-\alpha x}$

$$c_1=\frac{c_3+c_4}{2}\qquad c_2=\frac{c_3-c_4}{2}$$

但是表示成 $U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x$

較容易處理 boundary value condition

$$\sinh x=\frac{e^x-e^{-x}}{2}\qquad \sinh 0=0$$

$$\cosh x=\frac{e^x+e^{-x}}{2}\qquad \frac{d}{dx}\cosh x=\sinh x\qquad \left.\frac{d}{dx}\cosh x\right|_{x=0}=0$$

$$U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x$$

Step 4 由 $u(0,y)=0$ $u(\pi,y)=e^{-y}$ 來解 c_1, c_2

和 Step 1 一樣，也是針對 y 做 Fourier cosine transform

只是對象改成 boundary conditions

$$(1)\ U(0,\alpha)=\Im_{c,y\to\alpha}\{u(0,y)\}=\int_0^\infty 0\cdot\cos\alpha y\,dy=0$$

$$(2)\ U(\pi,\alpha)=\Im_{c,y\to\alpha}\{u(\pi,y)\}=\int_0^\infty e^{-y}\cdot\cos\alpha y\,dy=\frac{1}{1+\alpha^2}$$

↑

(可以用Laplace transform 的「取巧法」)

分別代入 $U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x$

(1) $c_1=0$

(2) $c_1\cosh\pi\alpha+c_2\sinh\pi\alpha=\dfrac{1}{1+\alpha^2}$

$\Rightarrow$

$$c_1=0$$

$$c_2=\frac{1}{(1+\alpha^2)\sinh\pi\alpha}$$

$$U(x,\alpha)=\frac{\sinh\alpha x}{(1+\alpha^2)\sinh\pi\alpha}$$

Step 5 inverse cosine transform

$$u(x,y)=\mathfrak{I}_{c,\alpha\to y}^{-1}\left[U(x,\alpha)\right]=\frac{2}{\pi}\int_0^\infty\frac{\sinh\alpha x}{(1+\alpha^2)\sinh\pi\alpha}\cos\alpha y\,d\alpha$$

(算到這裡即可，難以繼續化簡)

14.4.7 Section 14.4 需要注意的地方

(1) 微分公式當中，Fourier cosine transform 和 Fourier sine transform 會有互換的情形。(See pages 701, 702)

(2) 公式會有很多小地方會背錯 (特別注意綱要的公式中紅色的地方)

(3) 在解 boundary value problem 時，要了解

何時用 Fourier transform，

何時用Fourier cosine transform,

何時用 Fourier sine transform (see page 703)

(4) 解 boundary value problem 流程雖複雜，但只要記住，

方法的精神，在於：

運用 transform，

將原本針對兩個以上 independent variables 做微分的 PDE，

變成只有針對一個 independent variable 做微分的 ODE

(5) 在解 partial differential equation 時，往往只針對一個 independent variable 做 Fourier transform, 另一個 independent variable 不受影響，如 Examples 1 and 2, pages 705 and 708 的例子

計算過程中，自己要清楚是對哪一個 independent variable 做Fourier transform

※ 本人習慣用下標做記號 $\mathfrak{I}_{x\to\alpha}\{u(x,t)\}$ (建議同學們使用)

(6) Step 4 和 Step 1 必需是針對同一個 independent variable 來做同一種 transform，只是處理的對象改成了 initial (or boundary) conditions (see pages 706, 710)

(7) 注意 page 709, 有時我們會用 $c_1\cosh\alpha x + c_2\sinh\alpha x$ 來取代 $c_3e^{\alpha x} + c_4e^{-\alpha x}$，以方便計算

附錄八：其他書籍常見的 Fourier Transform 的定義

在其他書上，常常把 Fourier transform 的定義寫成

$$\Im[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}[G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\Im[g(x)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}[G(\omega)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\Im[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx = G(f)$$

$$\Im^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} d\omega = g(x)$$

考試時還是用課本上的定義 (見 page 694)

期末考

(1) 由於內容眾多，各位對於所學的東西，一定要有系統化的整理與比較。

(2) 公式、定理、名詞、解法甚多，若要背公式就早一點背公式

(3) 保持最佳狀態，腦筋多轉彎

祝同學們期末考順利！

Exercise for Practice

Section 14-3 3, 4, 7, 12, 15, 16, 17, 19, 20

Section 14-4 1, 2, 3, 9, 12, 15, 16, 18, 19, 20, 21, 27

Review 14 2, 7, 8, 11, 15, 16