Chapter 12 Boundary-Value Problem in Rectangular Coordinates

• Role of Chapter 12:

Discuss the boundary-value problem for the case of **two independent variables**.

(x-y 座標) (圓座標的問題在 Chapter 13 當中有討論但不在這學期的上課範圍之中)

Use the methods of (1) separation of variables or (2) the Fourier transform to solve the problem.

(1) separation of variables → Chapter 12

(2) the Fourier transform → Section 14.4

本章的架構

- 12.1 介紹解法 Separation of variables (很重要)
- 12.2 名詞和定義
- 12.4 Wave equation
- 12.5 Laplace's equation

12.4、12.5：可視為12.1 的應用題 (有點複雜要多練習)

兩大重點：

(1) 熟悉 separation of variables 解 PDE 的方法

(2) 名詞和定義

縮寫: boundary value problem (BVP)

initial value problem (IVP)

例：
$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

BVP: $u(0,t)=0$ $\quad$ $u(L,t)=0$

IVP: $u(x,0)=f(x)$ $\quad$ $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

partial differential equation (PDE)

ordinary differential equation (ODE)

Section 12.1 Separable Partial Differential Equations

12.1.1 Section 12.1 綱要

(1) linear second order partial differential equation for two independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

7 terms

$B^2-4AC>0$: hyperbolic, $B^2-4AC=0$: parabolic

$B^2-4AC<0$: elliptic

(2) Partial differential equation (PDE) 的第二種解法：

Separation of variables (see pages 724-726).

名詞： real separation constant (page 724)

12.1.2 Linear Second Order Partial Differential Equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

independent variables: x, y dependent variables: $u(x, y)$, 簡寫成 u

homogeneous : $G(x, y) = 0$, nonhomogeneous : $G(x, y) \neq 0$

particular solution, general solution 的定義一如往昔

【Theorem 12.1.1】 Superposition Principle

If u_1, u_2, …., u_k are solutions of a homogeneous linear partial differential equation, then

$$u = c_1 u_1 + c_2 u_2 + \cdots\cdots + c_k u_k$$

is also a solution of the homogeneous linear partial differential equation.

12.1.3 Method of Separation of Variables

解 PDE with BVP (or IVP) 的方法

(1) method of separation of variables

若 PDE 當中有對 x 及對 y 的偏微分，

假設解為 $u(x, y) = X(x)Y(y)$

(2) using the Fourier transform (or Fourier cosine transform, Fourier sine transform) (see Section 14.4)

共通的精神： PDE ⟶ ODE

Note: Laplace transform can also be used for solving the PDE (Section 14-2，期末考範圍外)

Method of Separation of Variables 的流程

(Step 1) 假設解為 $u(x, y) = X(x)Y(y)$

解法關鍵

(Step 2) 將 $u(x, y) = X(x)Y(y)$ 代入 PDE，把 PDE 變成

“ function of X ” = “ function of Y ” = $-\lambda$

的型態

λ 被稱為 real separation constant

Steps 3, 4, 5 要分成不同的 Cases 來解

除了trivial 的情形外，所有可能的 cases 都要考慮

(Step 3) 將 function of $X = -\lambda$ 的解算出，即為 $X(x)$

註：(a) 如果有等於零的 boundary (initial) conditions，也要在這一步考慮

(例如 pages 754, 770 的下方)

(b) 有時，先解 $Y(y)$ 會比較容易

(視 boundary (initial) conditions 而定)

(c) 在這一步中，有的時候，會得出 λ 的限制

(Step 4) 將 function of $Y = -\lambda$ 的解算出，即為 $Y(y)$

需注意的地方和 Step 3 相同

(Step 5) $u(x, y) = X(x)Y(y)$

(Step 6) 將所有可能的解全部加起來

(Step 7) 用非零的 boundary (initial) conditions 將 coefficients 求出

註： 這一步經常會用到 Fourier series, Fourier cosine series 或 Fourier sine series

※ 若沒有 boundary (initial) conditions，Steps 6, 7 可以省略

Rules:

x 的 BVP (IVP) 簡單 ⟶ 先算 $X(x)$

y 的 BVP (IVP) 簡單 ⟶ 先算 $Y(y)$

沒有 BVP (IVP) ⟶ 先算 $X(x)$ 或 $Y(y)$ 皆可

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0$$

$$u(0,y)=0 \qquad u(L,y)=0$$

$$u(x,0)=f(x) \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0}=g(x)$$

先算 $X(x)$

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}$$

$$u(0,y)=f(y) \qquad u(L,y)=0$$

$$\left.\frac{\partial}{\partial y}u(x,y)\right|_{y=0}=0 \qquad \left.\frac{\partial}{\partial y}u(x,y)\right|_{y=H}=0$$

先算 $Y(y)$

Note: Separation of variables 的方法其實未必可以得出 PDE 所有的解

有些解無法用 $X(x)Y(y)$ 來表示

Separation of variables 的主要好處是比其他方法簡單

Example 2 (text page 462)

$$\frac{\partial u^2}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

Step 1 假設解為 $u(x,y) = X(x)Y(y)$ (解法關鍵)

Step 2 將 $u(x,y) = X(x)Y(y)$ 代入 $\frac{\partial u^2}{\partial x^2} = 4\frac{\partial u}{\partial y}$

$$X''(x)Y(y) = 4X(x)Y'(y)$$

$$\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)}$$

real separation constant

令 $\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)} = -\lambda$ (解法關鍵)

$$X''(x) + 4\lambda X(x) = 0 \qquad Y'(y) + \lambda Y(y) = 0$$

$$X''(x) + 4\lambda X(x) = 0 \qquad Y'(y) + \lambda Y(y) = 0$$

Case 1 for Steps 3, 4, 5 $\lambda = 0$

Step 3-1 $X''(x) = 0$

auxiliary function $m^2 = 0$ roots : 0, 0

$$X(x) = c_1 + c_2 x$$

Step 4-1 $Y'(y) = 0$ $Y(y) = c_3$

Step 5-1 $u(x,y) = X(x)Y(y) = (c_1 + c_2 x)c_3 = A_1 + B_1 x$

$$A_1 = c_1 c_3 \qquad B_1 = c_2 c_3$$

Case 2 for Steps 3, 4, 5 $\lambda < 0$

為了方便起見，令 $\lambda = -\alpha^2$

Step 3-2 $X''(x) - 4\alpha^2 X(x) = 0$ roots of the auxiliary function: 2α, -2α

$$X(x) = d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$$

通常將解改寫成 $X(x) = c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

Step 4-2 $\dfrac{Y'(y)}{Y(y)} = \alpha^2$ $Y'(y) - \alpha^2 Y(y) = 0$

$$Y'(y) - \alpha^2 Y(y) = 0 \qquad Y(y) = c_6 e^{\alpha^2 y}$$

Step 5-2 $u(x,y) = X(x)Y(y) = A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x)$

$$A_2 = c_4 c_6 \qquad B_2 = c_5 c_6$$

Case 3 for Step 3 $\lambda > 0$

為了方便起見，令 $\lambda = \alpha^2$

Step 3-3 $X''(x) + 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $j2\alpha, -j2\alpha$

$$X(x) = c_7 \cos(2\alpha x) + c_8 \sin(2\alpha x)$$

Step 4-3 $\dfrac{Y'(y)}{Y(y)} = -\alpha^2$ $\quad Y'(y) + \alpha^2 Y(y) = 0$ $\quad Y(y) = c_9 e^{-\alpha^2 y}$

Step 5-3 $u(x,y) = A_3 e^{-\alpha^2 y} \cos(2\alpha x) + B_3 e^{-\alpha^2 y} \sin(2\alpha x)$

若要處理 boundary conditions，或著想得到 general solution，要將所有可能的解都加起來

Step 6

$$u(x,y) = A_1 + B_1 x + \sum_{\alpha>0} [A_{2,\alpha} e^{\alpha^2 y} \cosh(2\alpha x) + B_{2,\alpha} e^{\alpha^2 y} \sinh(2\alpha x)]$$

$$+ \sum_{\alpha>0} [A_{3,\alpha} e^{-\alpha^2 y} \cos(2\alpha x) + B_{3,\alpha} e^{-\alpha^2 y} \sin(2\alpha x)]$$

α 是任意實數

(註：nonseparable 的解在這一步得到)

12.1.4 Classification

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

$B^2-4AC>0$ ⟶ The PDE is said to be hyperbolic (雙曲線)

$B^2-4AC=0$ ⟶ The PDE is said to be parabolic (拋物線)

$B^2-4AC<0$ ⟶ The PDE is said to be elliptic (橢圓形)

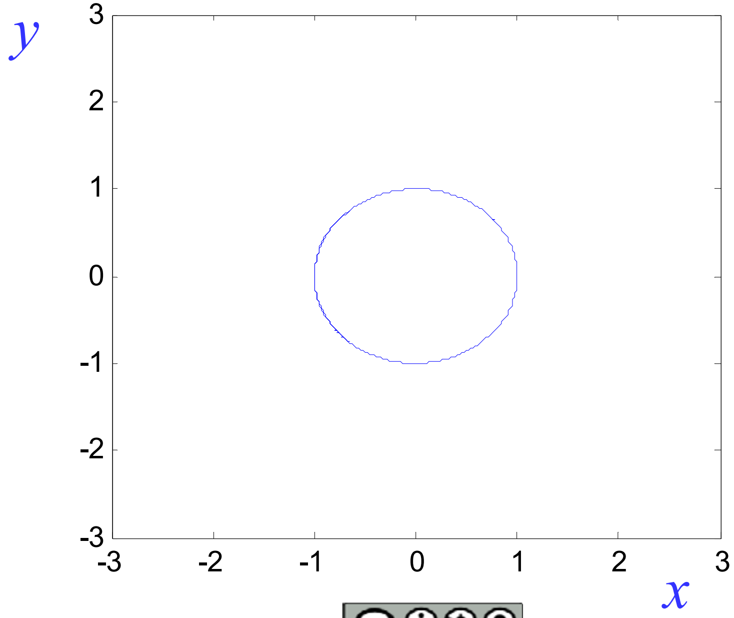
這些命名方式，是根據 2 次多項式在 x-y 平面上的軌跡

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

當 $x^2 + y^2 - 1 = 0$

$$x^2 + y^2 = 1$$

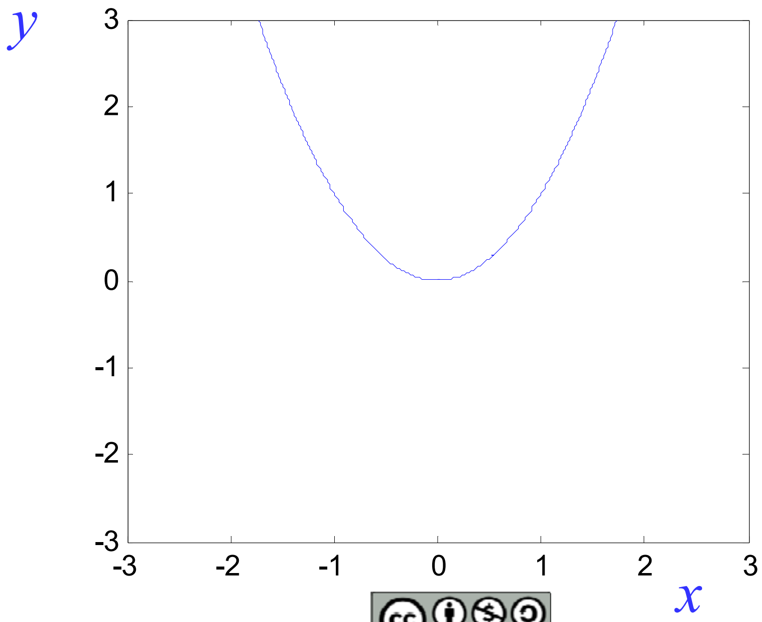
$$B^2 - 4AC = -4 < 0$$



當 $x^2 - y = 0$

$$y = x^2$$

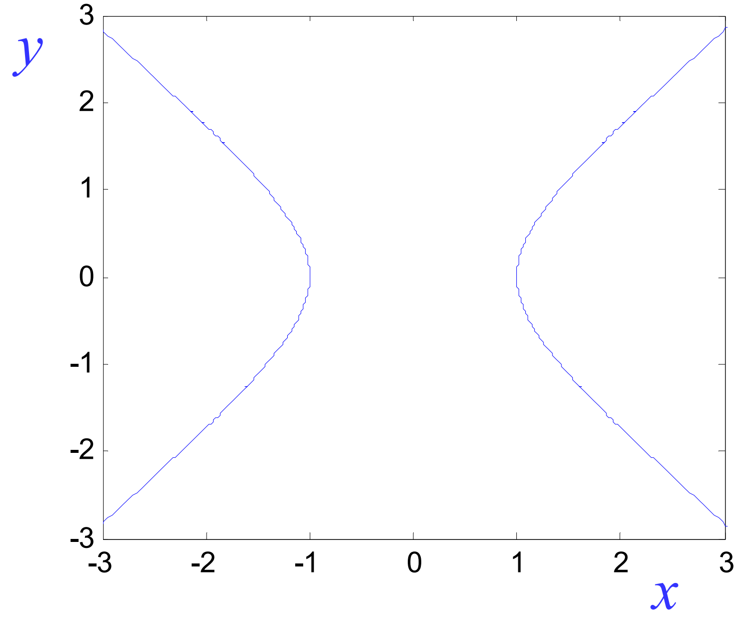
$$B^2 - 4AC = 0$$



當 $x^2 - y^2 - 1 = 0$

$x^2 - y^2 = 1$

$B^2 - 4AC = 4 > 0$



記憶秘訣：只要清楚幾個「特例」，就可以記住當

$$B^2 - 4AC < 0, \quad B^2 - 4AC = 0, \quad B^2 - 4AC > 0$$

的時候，應該是什麼圖形

Example 3 (text page 463)

$$3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

12.1.5 本節需要注意的地方

(1) 本節除了定義以外，只有兩個重點： classification of equations 以及 method of separation of variables.

(2) 然而， method of separation of variables 解法的流程，稍有些複雜，需要熟悉 (Sections 12-4, 12-5 都將用這個方法)

關鍵：記住第一步 $u(x, y) = X(x)Y(y)$

第二步 function of X = function of $Y = -\lambda$

(3) Method of separation of variables 在計算時，會分成很多個 cases.

(4) Separation of variables要解 BVP 和 IVP時，需要將每個 cases 得出來的解都加起來 (Step 6)

(5) 為了方便解決 BVP 或 IVP，經常將 $d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$

改寫成 $c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

(6) Hyperbolic, parabolic, elliptic 的條件，可以用幾個 special cases 來記

(7) "等於零" 的 BVP 或IVP 可以先於 Steps 3, 4 當中考慮

(例如 pages 754, 770 的下方)

"不等於零" 的 BVP 或IVP 則在 Step 7 當中處理

Section 12.2 Classical PDEs and Boundary-Value Problems

12.2.1 本節綱要

(1) one-dimensional heat equation (或簡稱為 heat equation)

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \qquad k>0$$

(2) one-dimensional wave equation (或簡稱為 wave equation)

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

(3) two-dimensional form of Laplace's equation (或簡稱為 Laplace's equation)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

名詞：

heat equation, (page 741)

wave equation, (page 743)

Laplace's equation, (page 746)

Laplacian, (page 747)

Dirichlet condition, (page 750)

Neumann condition, (page 750)

Robin condition (page 750)

本節的重點：熟悉這七大名詞，和它們所對應的公式

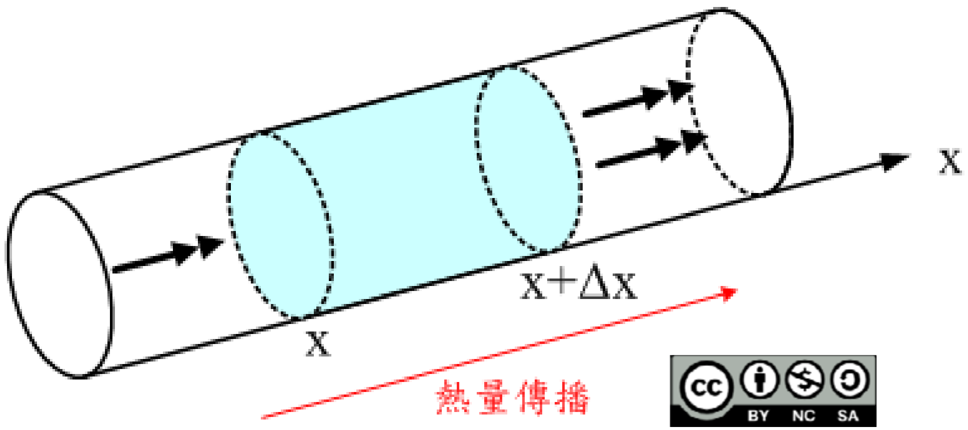
12.2.2 One-Dimensional Heat Equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

由來：熱傳導的理論 (物理上的說明見 text page 466)

$u(x, t)$: temperature, t: time, x: location

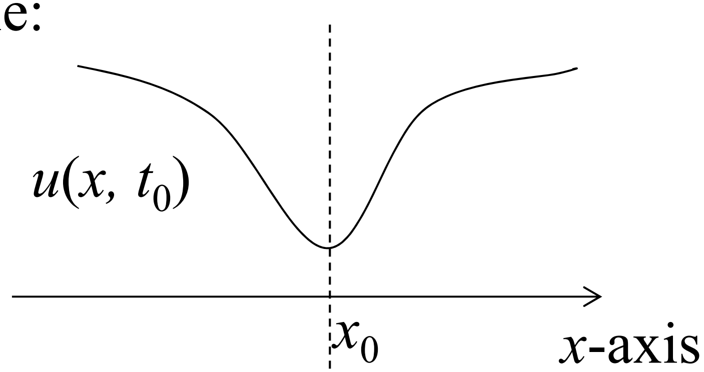
Fig. 12.2.1



heat equation 別名：diffusion equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Example:



$u(x, t)$: temperature,
t: time, x: location

x_0 的溫度將上升 $\left.\frac{\partial u(x_0, t)}{\partial t}\right|_{t=t_0} > 0$

12.2.3 One-Dimensional Wave Equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

「拉像皮筋」的模型 (物理上的說明見 text page 468)

u(*x*, *t*): height, *t*: time, *x*: location

(1) 剛開始時

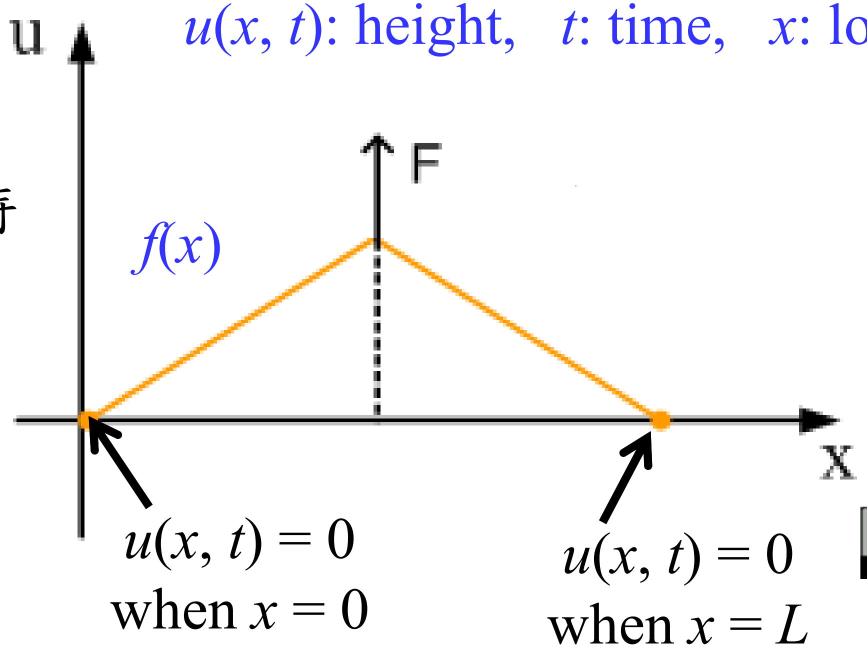


Fig. 12.2.4

wave equation 別名： telegraph equation

(2) 手放開之後產生振動

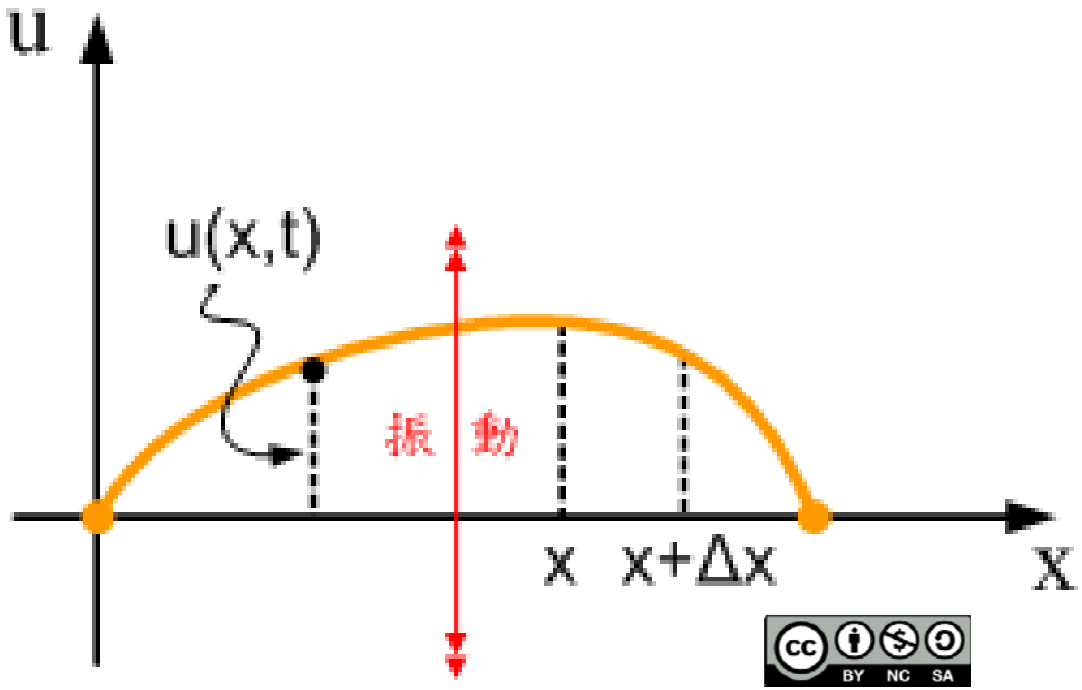


Fig. 12.2.2

- Wave equation 其他的應用：

  Theory of high-frequency transmission line

  Fluid mechanics (流體力學)

  Acoustics (聲學)

  Elasticity (彈力學)

  Microwave engineering (電波工程)

12.2.4 Two-Dimensional Form of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

如課本 Fig. 12.2.3，溫度隨著位置而變化的模型

$u(x, y)$: temperature,

x, y: location

Laplace's Equation 亦可用 Laplacian 表示, $\nabla^2 u(x, y) = 0$

Laplacian: ∇^2

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

- Laplace's Equation 的其他應用

  Static displacement of membranes

  Edge detection (邊緣偵測)

  Microwave engineering (電波工程)

12.2.5 Modification

加上外力，或與外界的交互作用

例：heat equation 的 modification

$$k\frac{\partial^2 u}{\partial x^2}-h\left(u-u_m\right)=\frac{\partial u}{\partial t}$$

例：wave equation 的 modification

$$a^2\frac{\partial^2 u}{\partial x^2}+F\left(x,t,u,u_t\right)=\frac{\partial^2 u}{\partial t^2}$$

12.2.6 Boundary Conditions 或 Initial Conditions

Dirichlet condition $u = \ldots\ldots..$ (沒微分)

Neumann condition $\frac{\partial u}{\partial n} = \cdots\cdots$ (有微分)

Robin condition $\frac{\partial u}{\partial n} + hu = \cdots\cdots$ (混合)

h is a constant

Section 12.4 Wave Equation

12.4.1 本節綱要

要解決的問題 (one-dimensional wave equation)

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L \qquad t > 0$$

BVP and IVP

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad \text{for } t > 0$$

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \qquad \text{for } 0 < x < L$$

解法見 page 753-761
例子見 page 762

實際上，Sections 12.4 和 12.5 可看成是 Section 12.1 的 method of separation of variables 的練習題

(可見得 method of separation of variables 有多重要)

名詞：

standing waves (page 763)	normal modes (page 763)
first standing wave (page 764)	fundamental frequency (page 764)
nodes (page 766)	overtones (page 766)

12.4.2 Solutions for Wave Equations (自己挑戰解解看)

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \qquad 0<x<L \qquad t>0$$

四大條件 $u(0,t)=0$ $\qquad u(L,t)=0$ $\qquad$ for $t>0$

$u(x,0)=f(x)$ $\qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$ $\qquad$ for $0<x<L$

求解 (使用method of separation of variables)

Step 1 假設解為 $u(x,t)=X(x)T(t)$

Step 2 將 $u(x,y)=X(x)T(t)$ 代入 $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$

$$a^2X''(x)T(t)=X(x)T''(t) \qquad \frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}$$

令 $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{a^2T(t)}=-\lambda$

得出2 個 ODEs $X''(x)+\lambda X(x)=0$ $T''(t)+a^2\lambda T(t)=0$

Steps 3, 4, 5 的前處理

(1) 因為 x 的 boundary condition 較簡單，所以先解 $X(x)$

(2) 分成 $\lambda=0$, $\lambda<0$, $\lambda>0$ 三個 cases

(3) 由於 $u(0,t)=0$ for all $t>0$ $u(0, t) = X(0)T(t) = 0$

$T(t)$ 不可為 0 (否則 $u(x, t) = X(x)T(t) = 0$ for any x, t)

所以 $X(0) = 0$

同理，由 $u(L,t)=0$ 可以立即判斷 $X(L) = 0$

$X''(x)+\lambda X(x)=0$ subject to $X(0) = 0$ and $X(L) = 0$

$X''(x)+\lambda X(x)=0$ subject to $X(0)=0$ and $X(L)=0$

$T''(t)+a^2\lambda T(t)=0$

Case 1 for Steps 3, 4, 5 $\lambda=0$

Step 3-1 $X''(x)=0$ $X(x)=d_1x+d_0$

根據 boundary conditions

$$d_0=0 \qquad\qquad d_0=0$$
$$d_1L+d_0=0 \quad\Longrightarrow\quad d_1=0 \qquad X(x)=0$$

這個 case 得出 trivial solution $u(x,t)=X(x)T(t)=0$

$u(x,0)=f(x)$ 將無法滿足 $\lambda=0$ 時無解

無需再解 Step 4-1, Step 5-1

Case 2 of Steps 3, 4, 5: $\lambda < 0$

Step 3-2 令 $\lambda = -\alpha^2$

$$X''(x) - \alpha^2 X(x) = 0$$

Solution: $X(x) = d_2 e^{\alpha x} + d_3 e^{-\alpha x}$

可改寫成 $X(x) = d_4 \cosh(\alpha x) + d_5 \sinh(\alpha x)$ → 較易處理 boundary conditions

根據 boundary conditions $X(0) = 0$ and $X(L) = 0$

$$\begin{cases} d_4 = 0 \\ d_4 \cosh(\alpha L) + d_5 \sinh(\alpha L) = 0 \end{cases} \Longrightarrow \begin{cases} d_4 = 0 \\ d_5 = 0 \end{cases} \quad X(x) = 0$$

這個 case 得出 trivial solution $u(x, t) = X(x)T(t) = 0$

$u(x,0) = f(x)$ 將無法滿足 $\lambda < 0$ 時無解

無需再解 Step 4-2, Step 5-2

Case 3 of Steps 3, 4, 5: $\lambda > 0$

Step 3-3 令 $\lambda = \alpha^2$

$$X''(x) + \alpha^2 X(x) = 0$$

Solution: $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

根據 boundary conditions $X(0) = 0$ and $X(L) = 0$

$$\begin{cases} c_1 = 0 \\ c_1 \cos \alpha L + c_2 \sin \alpha L = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = 0 \\ \alpha = \dfrac{n\pi}{L} \\ c_2 = \text{any nonzero constant} \end{cases}$$

n 是任意整數

特別注意：

不可直接由 $\begin{cases} c_1 = 0 \\ c_1 \cos \alpha L + c_2 \sin \alpha L = 0 \end{cases}$ 就斷言 $\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$

應該看看是否有適當的 α, 讓第二個式子等於零

$$X(x) = c_2 \sin \frac{n\pi}{L} x$$ n 是任意正整數, c_2 是任意數

$$\alpha = \frac{n\pi}{L} \qquad \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}$$

Step 4-3 $T''(t) + a^2 \lambda T(t) = 0$

$$T''(t) + \frac{a^2 n^2 \pi^2}{L^2} T(t) = 0$$

Solution: $T(t) = c_3 \cos\left(\frac{na\pi}{L} t\right) + c_4 \sin\left(\frac{na\pi}{L} t\right)$ n 是任意整數

Step 5-3

$$u_n(x,t) = X(x)T(t) = c_2 \sin\left(\frac{n\pi}{L} x\right)\left[c_3 \cos\left(\frac{na\pi}{L} t\right) + c_4 \sin\left(\frac{na\pi}{L} t\right)\right]$$

$$= \sin\left(\frac{n\pi}{L} x\right)\left[A_n \cos\left(\frac{na\pi}{L} t\right) + B_n \sin\left(\frac{na\pi}{L} t\right)\right]$$ n 是任意整數

$$A_n = c_2 c_3, \quad B_n = c_2 c_4,$$

注意：$u_n(x,t)=\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$

只是其中一個解，因為 n 是任意整數

Step 6

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

討論：既然 n 是任意整數，那為什麼 n 是從1 加到 ∞，

而非由 $-\infty$ 加到 ∞？

因為 $$\sin\left(\frac{n\pi}{L}x\right)=-\sin\left(\frac{-n\pi}{L}x\right),\qquad \cos\left(\frac{na\pi}{L}t\right)=\cos\left(\frac{-na\pi}{L}t\right),$$

$$\sin\left(\frac{na\pi}{L}t\right)=-\sin\left(\frac{-na\pi}{L}t\right),\qquad \sin(0)=0$$

可證明 $$\sum_{n=-\infty}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[C_n\cos\left(\frac{na\pi}{L}t\right)+D_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$A_n=C_n-C_{-n}\qquad\qquad B_n=D_n+D_{-n}$$

Step 7

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

由 initial conditions

$$u(x,0)=f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$$

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right) \qquad g(x)=\sum_{n=1}^{\infty}B_n\frac{na\pi}{L}\sin\left(\frac{n\pi}{L}x\right)$$

也就是說，A_n 是 $f(x)$ 的 Fourier sine series,

$B_n\frac{na\pi}{L}$ 是 $g(x)$ 的 Fourier sine series

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$B_n\frac{na\pi}{L}=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \qquad B_n=\frac{2}{na\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

12.4.3 物理意義

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(0,t) = 0 \qquad u(L,t) = 0$$

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

u ：高度

$\frac{\partial u}{\partial t}$ ：速度

$\frac{\partial^2 u}{\partial t^2}$ ：加速度

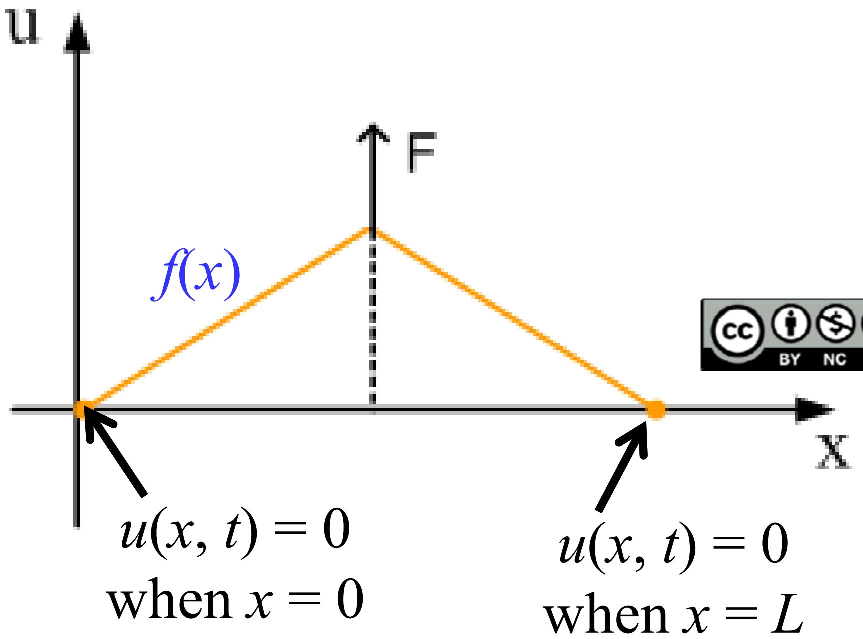


Fig. 12.2.4

12.4.4 名詞

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$u(x,t)=u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots\cdots\cdots$$

其中 $$u_n(x,t)=\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$=C_n\sin\left(\frac{n\pi}{L}x\right)\left[\sin\left(\frac{na\pi}{L}t+\phi_n\right)\right]$$

$$C_n=\sqrt{A_n^2+B_n^2}\qquad \cos\phi_n=\frac{B_n}{C_n}\qquad \sin\phi_n=\frac{A_n}{C_n}$$

$u_n(x,t)$ 被稱作 standing waves (駐波) 或 normal modes

$n = 1$ 時， $u_1(x, t)$ 被稱作 first standing wave 或
first normal mode 或 fundamental mode of vibration

$$u_1(x,t) = C_1 \sin\left(\frac{\pi}{L}x\right)\left[\sin\left(\frac{a\pi}{L}t + \phi_1\right)\right]$$

$$u_1\left(x, t + \frac{2L}{a}\right) = C_1 \sin\left(\frac{\pi}{L}x\right)\left[\sin\left(\frac{a\pi}{L}t + 2\pi + \phi_1\right)\right] = u_1(x,t)$$

對於 t 而言，週期 $= \dfrac{2L}{a}$ 頻率 =1/週期 $= \dfrac{a}{2L}$

$f_1 = \dfrac{a}{2L}$ 被稱作 fundamental frequency (基頻) 或 first harmonic

以此類推， $u_2(x, t)$ 被稱作 second standing wave

$u_3(x, t)$ 被稱作 third standing wave

First standing wave

Second standing wave

Third standing wave

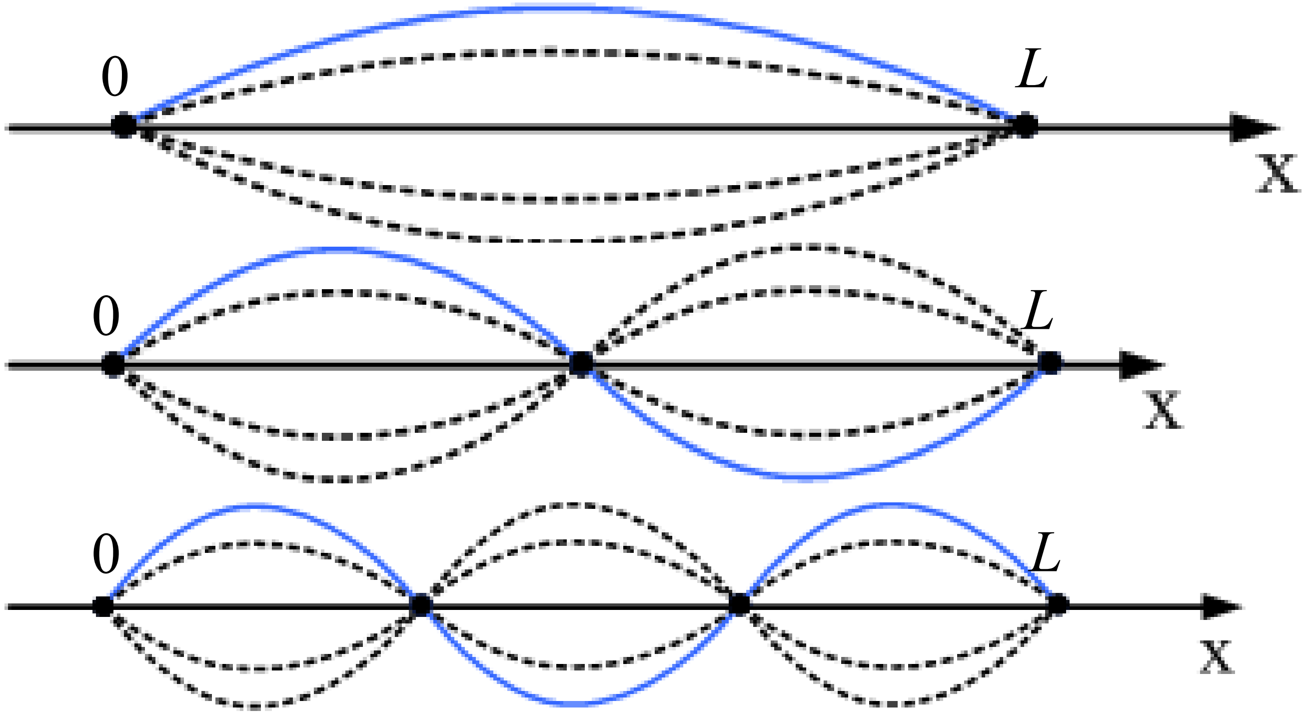


Fig. 12.4.2

$$u_n(x,t) = C_n \sin\left(\frac{n\pi}{L}x\right)\left[\sin\left(\frac{na\pi}{L}t + \phi_n\right)\right]$$

$x = \dfrac{L}{n}$ 時，無論 t 等於多少，$u_n\left(\dfrac{L}{n}, t\right) = 0$

$x = \dfrac{L}{n}$ 是 n^{th} standing wave 的 node (節點)

$$u_n(x,t) = u_n\left(x, t + \frac{2L}{an}\right)$$

$u_n(x, t)$ 的頻率 =1/週期 = $n\dfrac{a}{2L}$

$f_n = n\dfrac{a}{2L} = nf_1$ 被稱作 overtones (泛音)

Section 12.5 Laplace's Equation

12.5.1 Section 12.5 綱要

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

(使用method of separation of variables 來解)

「問題 1」 $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0 \quad \left.\frac{\partial u}{\partial x}\right|_{x=a}=0$ for $0<y<b,$

$u(x,0)=0 \quad u(x,b)=f(x)$ for $0<x<a$

「問題 2」 $u(0,y)=0 \quad u(a,y)=0$ for $0<y<b,$

$u(x,0)=0 \quad u(x,b)=f(x)$ for $0<x<a$

「問題 3」 $u(0,y)=F(y) \quad u(a,y)=G(y)$ for $0<x<a$

$u(x,0)=f(x) \quad u(x,b)=g(x)$ for $0<y<b,$

※ 特別注意 **"superposition principle"**

Sections 12.4, 12.5 的重點，只有二個

(1) Method of separation of variables 的練習

(a) Wave equation 和 (b) Laplace's equation

(2) Superposition principle

12.5.2 Solutions for Laplace's Equations (自己挑戰解解看)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad 0<x<a, \qquad 0<y<b,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \qquad \text{for } 0<y<b,$$

$$u(x,0)=0 \qquad u(x,b)=f(x) \quad \text{for } 0<x<a$$

Step 1　假設解為　$u(x, y)=X(x)Y(y)$

Step 2　代入　$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$　得出

$$X''(x)Y(y)+X(x)Y''(y)=0 \qquad \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}$$

令　$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$$

得出2 個 ODEs　$X''(x)+\lambda X(x)=0 \quad Y''(y)-\lambda Y(y)=0$

Steps 3, 4, 5 的前處理

(1) 因為 x 的 boundary condition 較簡單，所以先解 $X(x)$

(2) 分成 $\lambda = 0$, $\lambda < 0$, $\lambda > 0$ 三個 cases

(3) 由於 $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ for all $0 < y < b$,

$$\left.\frac{\partial X(x)Y(y)}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0$$

$Y(y)$ 不可為 0 (否則 $u(x, y) = X(x)Y(y) = 0$)

所以 $\boxed{X'(0) = 0}$

同理，由 $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \longrightarrow \boxed{X'(a) = 0}$

同理，由 $u(x,0) = 0 \longrightarrow \boxed{Y(0) = 0}$

$X''(x)+\lambda X(x)=0$ $X'(0)=0$ $X'(a)=0$

$Y''(y)-\lambda Y(y)=0$ $Y(0)=0$

Case 1 of Steps 3, 4, 5: $\lambda=0$

Step 3-1 $X''(x)=0$ solution: $X(x)=c_1+c_2x$

由 boundary conditions $X'(0)=0$ $X'(a)=0$

$c_2=0$

$X(x)=c_1$

Step 4-1 $Y''(y)=0$ $Y(0)=0$

solution: $Y(y)=c_3+c_4y$

根據 boundary condition $Y(0)=0$, $c_3=0$

$Y(y)=c_4y$

Step 5-1

$$u(x,y)=X(x)Y(y)=c_1c_4y=A_0y \qquad A_0=c_1c_4$$

Case 2 of Steps 3, 4, 5: $\lambda<0$

令 $\lambda=-\alpha^2$

Step 3-2 $\quad X''(x)-\alpha^2X(x)=0 \qquad X'(0)=0 \qquad X'(a)=0$

solution: $\quad X(x)=d_2e^{\alpha x}+d_3e^{-\alpha x}$

可改寫成 $\quad X(x)=d_4\cosh(\alpha x)+d_5\sinh(\alpha x)$

由 boundary conditions $\quad X'(0)=0 \quad X'(a)=0$

以及 $\dfrac{d}{dx}\cosh(\alpha x)=\alpha\sinh(\alpha x), \qquad \dfrac{d}{dx}\sinh(\alpha x)=\alpha\cosh(\alpha x)$

$$\begin{cases} d_5\alpha=0 \\ d_4\alpha\sinh(\alpha a)+d_5\alpha\cosh(\alpha a)=0 \end{cases} \Longrightarrow \begin{cases} d_5=0 \\ d_4=0 \end{cases} \Longrightarrow X(x)=0$$

因此， case 2 得出 trivial solution $u(x, y) = X(x)Y(y) = 0$

$u(x,b) = f(x)$ 將無法滿足　　　$\lambda < 0$ 時無解

(不需再算 Steps 4-2, 5-2)

Case 3 of Steps 3, 4, 5: $\lambda > 0$

令 $\lambda = \alpha^2$

Step 3-3　$X''(x) + \alpha^2 X(x) = 0$　　$X'(0) = 0$　　$X'(a) = 0$

solution:　$X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

由 boundary conditions　$X'(0) = 0$　$X'(a) = 0$

$$\begin{cases} c_2\alpha = 0 \\ -c_1\alpha\sin(\alpha a) + c_2\alpha\cos(\alpha a) = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = \text{any nonzero constant} \\ \alpha = \dfrac{n\pi}{a} \quad n \text{ 是任意整數} \\ c_2 = 0 \end{cases}$$

再次注意：不可直接判斷成 $c_1 = 0$ and $c_2 = 0$

應該看看是否有適當的 α, 讓第二個式子等於零

$$X_n(x) = c_1 \cos\frac{n\pi}{a}x$$

n 是任意整數 $\lambda = \alpha^2 = \dfrac{n^2\pi^2}{a^2}$

Step 4-3 $Y''(y) - \dfrac{n^2\pi^2}{a^2}Y(y) = 0$ since $\lambda = \dfrac{n^2\pi^2}{a^2}$

$Y(0) = 0$

solution: $Y_n(y) = d_3 e^{\frac{n\pi}{a}y} + d_4 e^{-\frac{n\pi}{a}y}$

經常改寫為 $Y_n(y) = c_3 \cosh\left(\dfrac{n\pi}{a}y\right) + c_4 \sinh\left(\dfrac{n\pi}{a}y\right)$

根據 boundary condition $Y(0) = 0$ $c_3 = 0$

$$Y_n(y) = c_4 \sinh\left(\frac{n\pi}{a}y\right)$$

Step 5-3

$$u(x,y) = X(x)Y(y) = c_1 \cos\left(\frac{n\pi}{a}x\right) c_4 \sinh\left(\frac{n\pi}{a}y\right) = A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$A_n = c_1 c_4$

n 是任意整數

Step 6 把所有可能的解，全部加起來

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

為什麼 n 是從 1 加到 ∞，而非由 $-\infty$ 加到 ∞？

道理同講義 page 760

Step 7 $$\boxed{u(x,y)=A_0 y+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)}$$

nonzero boundary condition: $u(x,b)=f(x)$

$$f(x)=A_0 b+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

也就是說，$2A_0 b$ 和 $A_n\sinh\left(\frac{n\pi}{a}b\right)$ $(n = 1, 2, \ldots., \infty)$

是 $f(x)$ 的 Fourier cosine series 的 coefficients

remember： Section 11-3 的 Fourier cosine series 為

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x$$

$$a_0=\frac{2}{p}\int_0^p f(x)dx \qquad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$

$$2A_0 b = \frac{2}{a}\int_0^a f(x)dx$$

$$A_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi}{a}x dx$$

$$\boxed{A_0 = \frac{1}{ab}\int_0^a f(x)dx}$$

$$\boxed{A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\cos\frac{n\pi}{a}x dx}$$

12.5.3 Laplace's Equations with Dirichlet Problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad 0<x<a, \quad 0<y<b,$$

$$u(0,y)=0 \qquad u(a,y)=0 \qquad 0<y<b,$$

$$u(x,0)=0 \qquad u(x,b)=f(x) \qquad 0<x<a,$$

用 method of separation of variables，經過計算得出

$$u(x,y)=\sum_{n=1}^{\infty} A_n \sinh\frac{n\pi}{a}y\sin\frac{n\pi}{a}x$$

$$A_n=\frac{2}{a\sinh\frac{n\pi}{a}b}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

可自行練習解解看

12.5.4 Superposition Principle

Dirichlet Problem 可分解成兩個子問題

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$$u(0,y)=F(y) \qquad u(a,y)=G(y) \qquad \text{for } 0<y<b,$$

$$u(x,0)=f(x) \qquad u(x,b)=g(x) \qquad \text{for } 0<x<a,$$

當四個邊界都不為零時，很難直接用 separation of variable 的方法解出來

子問題 1 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ $\quad 0<x<a, \quad 0<y<b,$

$u(0,y)=0 \quad u(a,y)=0 \quad$ for $0<y<b,$

$u(x,0)=f(x) \quad u(x,b)=g(x) \quad$ for $0<x<a,$

子問題 2 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ $\quad 0<x<a, \quad 0<y<b,$

$u(0,y)=F(y) \quad u(a,y)=G(y) \quad$ for $0<y<b,$

$u(x,0)=0 \quad u(x,b)=0 \quad$ for $0<x<a,$

假設 $u_1(x, y)$, $u_2(x, y)$ 分別是子問題 1, 子問題 2 的解

則 $\boldsymbol{u(x, y)=u_1(x, y)+u_2(x, y)}$ 是原來問題的解

(類似於 Section 4-1 的 superposition principle)

當 $u(x, y) = u_1(x, y) + u_2(x, y)$

$$u(0, y) = u_1(0, y) + u_2(0, y) = 0 + F(y) = F(y)$$

$$u(a, y) = u_1(a, y) + u_2(a, y) = 0 + G(y) = G(y)$$

$$u(x, 0) = u_1(x, 0) + u_2(x, 0) = f(x) + 0 = f(x)$$

$$u(x, b) = u_1(x, b) + u_2(x, b) = g(x) + 0 = g(x)$$

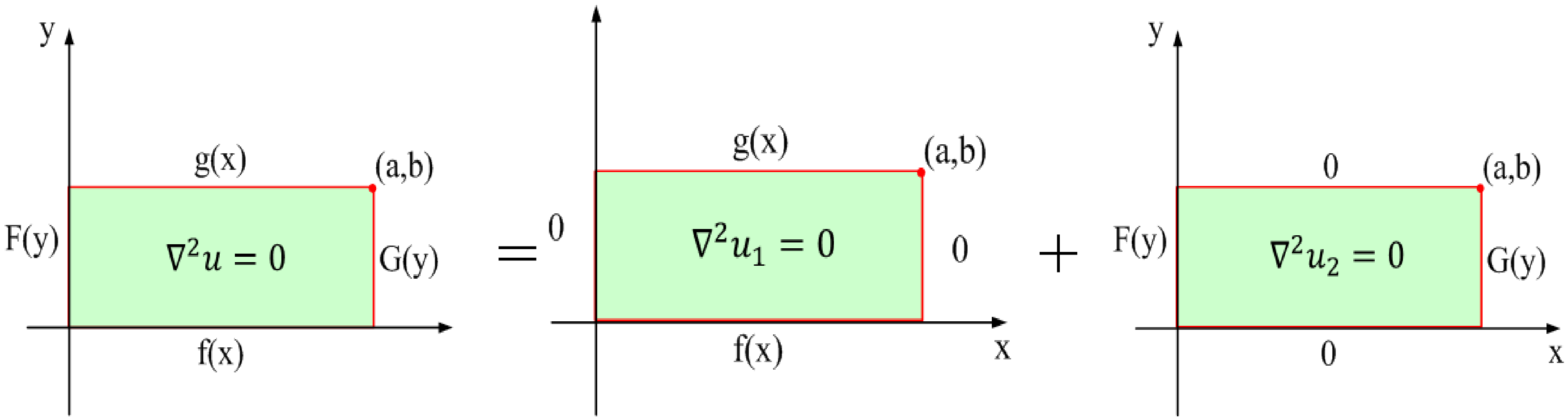


Fig. 12.5.3

子問題 1 的解 $u_1(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{a}y+B_n\sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x$

$$A_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$B_n=\frac{1}{\sinh\left(\frac{n\pi}{a}b\right)}\left[\frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx-A_n\cosh\left(\frac{n\pi}{a}b\right)\right]$$

子問題 2 的解 $u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{b}y+B_n\sinh\frac{n\pi}{b}y\right\}\sin\frac{n\pi}{b}x$

$$A_n=\frac{2}{b}\int_0^b F(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

$$B_n=\frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\left[\frac{2}{b}\int_0^b g(x)\sin\left(\frac{n\pi}{b}x\right)dx-A_n\cosh\left(\frac{n\pi}{b}a\right)\right]$$

原來問題的解 $u_1(x,y)+u_2(x,y)$

12.5.5 Sections 12.4 及 12.5 需要注意的地方

(1) Method of separation of variables 解 PDE 的過程雖然長，但是把握住講義 pages 724-726 的 7 個 steps，並練習幾次，就可以熟悉。

(這些對大二下和大三上的電磁學很重要)

(2) 雖然概念不難，但是計算過程很長且繁雜

所以一定要多研究簡化運算、快速判斷的方法

(3) 有沒有注意到，

若 boundary conditions 出現 $u(0, y) = 0$, $u(L, y) = 0$,

最後的解總是和 sine 有關 $X(x) = c_2 \sin \frac{n\pi}{L} x$ 週期為 $2L/n$

若 boundary conditions 出現 $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

最後的解總是和 cosine 或 constant 有關

$X(x) = c_1$ or $X_n(x) = c_1 \cos \frac{n\pi}{L} x$ 週期也為 $2L/n$

(4) 經驗足夠後，看到 $u(x, y)$ 的 boundary conditions

出現 $u(a, y) = 0$ $\longrightarrow$ 就知道 $X(a) = 0$ ，

看到 $u(x, b) = 0$ $\longrightarrow$ 就知道 $Y(b) = 0$ 。

看到 $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \longrightarrow$ 就知道 $X'(a) = 0$ ，

看到 $\left.\frac{\partial u}{\partial y}\right|_{y=b} = 0 \longrightarrow$ 就知道 $Y'(b) = 0$

(5) 對於 wave equations 而言，$X(x)$ 和 $T(t)$ 的解有相同的型態

如果 $X(x)$ 為 sine & cosine, $T(t)$ 也為 sine & cosine

對於 Laplace's equations而言，$X(x)$ 和 $Y(y)$ 的解型態不同

如果$X(x)$ 為 sine & cosine, $Y(y)$ 為 sinh & cosh

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}=-\lambda \qquad \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$$

(6) 要熟悉 $\cosh(x)$, $\sinh(x)$ 的性質

(7) Method of separation of variables 在計算上容易出錯的地方

(以 講義 pages 753-761 wave equations 為例)

(a) $\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{a^2T(t)} = -\lambda$

(b) Steps 3, 4, 5 要考慮所有 cases

(c) 不可直接由 $c_1 = 0$ 及 $c_1 \cos \alpha L + c_2 \sin \alpha L = 0$ 判斷 $c_1 = c_2 = 0$

因為 α 可以是 $\pi n/L$, 如講義 page 757 所述

(d) 在 Step 6，要將所有可能的解加起來，才是 $u(x, t)$ 的一般解

如講義 page 759 所述

Exercise for Practice

Section 12-1	3, 6, 9, 10, 12, 14, 16, 18, 22, 23, 27, 30, 32
Section 12-2	3, 4, 8, 10, 11
Section 12-4	1, 4, 7, 9, 11, 15, 17, 19, 20
Section 12-5	2, 5, 6, 9, 10, 11, 12, 14, 16, 17, 18
Review 12	1, 2, 5, 13, 14

Happy New Year!

祝各位期末考順利，寒假愉快！