Chapter 4 Higher Order Differential Equations

Highest differentiation: $\dfrac{d^n y}{dx^n}, \ n > 1$

Most of the methods in Chapter 4 are applied for the linear DE.

附錄四 DE 的分類

- DE
  - Linear
    - Constant coefficients
      - Homogeneous
      - Nonhomogeneous
    - Cauchy-Euler
      - Homogeneous
      - Nonhomogeneous
    - Others
      - Homogeneous
      - Nonhomogeneous
  - Nonlinear

附錄五 Higher Order DE 解法

- linear
  - homogeneous part
    - reduction of order
    - auxiliary function
    - Cauchy-Euler
  - particular solution
    - "guess" method
    - annihilator
    - variation of parameters
  - multiple linear DEs
    - elimination method
- nonlinear
  - reduction of order
  - Taylor series
  - numerical method
- both (but mainly linear)
  - series solution
  - transform
    - Laplace transform
    - Fourier series
    - Fourier cosine /sine series
    - Fourier transform

4-1 Linear Differential Equations: Basic Theory

4.1.1 Initial-Value and Boundary Value Problems

4.1.1.1 The n^{th} Order Initial Value Problem

i.e., the n^{th} order linear DE with the constraints at the same point

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \quad \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; y^{(n-1)}(x_0) = y_{n-1}$$

n initial conditions

Theorem 4.1.1

For an interval I that contains the point x_0

① If $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous at $x = x_0$

② $a_n(x_0) \neq 0$

(很像Section 2-3 當中沒有 singular point 的條件)

then for the problem on page 144, the solution $y(x)$ exists and is unique on the interval I that contains the point x_0

(Interval I 的範圍，取決於何時 $a_n(x) = 0$ 以及 何時 $a_k(x)$ $(k = 0 \sim n)$ 不為continuous)

Otherwise, the solution is either non-unique or does not exist.

(infinite number of solutions) (no solution)

Example 1 (text page 119)

$$3y''' + 5y'' - y' + 7y = 0 \qquad y(1) = 0 \qquad y'(1) = 0 \qquad y''(1) = 0$$

Example 2 (text page 120)

$$y'' - 4y' = 12x \qquad y(0) = 4 \qquad y'(0) = 1$$

- $x^2y'' - 2xy' + 2y = 6 \qquad y(0) = 3 \qquad y'(0) = 1$

有無限多組解

$y = cx^2 + x + 3$ c 為任意之常數

- 比較：

$$x^2 y'' - 2xy' + 2y = 6 \qquad y(1) = 3 \qquad y'(1) = 1$$

There is only one solution

$$y = x^2 - x + 3$$

$$x \in (0, \infty)$$

- Note:

The initial value can also be the form as:

$$\alpha y(x_0) + \beta y'(x_0) = 0$$

$$\sum_{n=0}^{N-1} \alpha_n y^{(n)}(x_0) = 0$$ (general initial condition)

4.1.1.2 n^{th} Order Boundary Value Problem

Boundary conditions are specified at different points

比較：Initial conditions are specified at the same points

例子：$a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $\quad y(a) = y_0, \quad y(b) = y_1$

或 $\quad y'(a) = y_0, \quad y(b) = y_1$

或 $$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$

An n^{th} order linear DE with n boundary conditions may have a <u>unique</u> solution, <u>no</u> solution, or <u>infinite</u> number of solutions.

Example 3 (text page 120)

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

(1) $y(0) = 0 \qquad y(\pi/2) = 0$

$y = c_2 \sin(4x)$ $\quad c_2$ is any constant (infinite number of solutions)

(2) $y(0) = 0 \qquad y(\pi/8) = 0$

$y = 0$ $\quad$ (unique solution)

4.1.2 Homogeneous Equations

4.1.2.1 Definition

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x) = 0 \longrightarrow$ homogeneous

$g(x) \neq 0 \longrightarrow$ nonhomogeneous

- 重要名詞：Associated homogeneous equation

The associated homogeneous equation of a nonhomogeneous DE:

Setting $g(x) = 0$

- Review: Section 2-3, pages 57, 59

4.1.3 Nonhomogeneous Equations (可和 page 59 相比較)

Nonhomogeneous linear DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

Part 1

Associated homogeneous DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)$$
$$+\cdots+a_1(x)y'(x)+a_0(x)y=0$$

find ***n* linearly independent solutions**

$$y_1(x),y_2(x),\cdots\cdots,y_n(x)$$

Part 2

particular solution y_p
(**any** solution of the nonhomogeneous linear DE)

$$g(x)=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)$$

$$y_p(x)=y_{p_1}(x)+y_{p_2}(x)+\cdots\cdots+y_{p_k}(x)$$

general solution of the nonhomogeneous linear DE

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_ny_n(x)+y_p(x)$$

Theorem 4.1.6 general solution of a nonhomogeneous linear DE

$$y = \underline{y_c} + \underline{y_p}$$

<u>general solution</u> of the associated homogeneous function (complementary function)

particular solution (<u>any solution</u>) of the nonhomogeneous linear DE

$\oplus$

<u>general solution</u> of the nonhomogeneous linear DE

4.1.3.1 Solution of the Homogeneous Equation

[Important Theory]: An n^{th} order homogeneous linear DE has n linearly independent solutions.

[Theorem 4.1.5]

For an n^{th} order homogeneous linear DE, if

① $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are the solutions of this DE

② $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are linearly independent

then any solution of the homogeneous linear DE can be expressed as:

$$\boxed{y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n}$$

可以和矩陣的概念相比較

From Theorem 4.1.5:

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

$y_1(t)$, $y_2(t)$, ….., $y_n(t)$: fundamental set of solutions

$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$: general solution of the homogenous linear DE

(又稱做 **complementary function**)

也是重要名詞

[Definition 4.1] Linear Dependence / Independence

If there is no solution other than $c_1 = c_2 = \ldots\ldots = c_n = 0$ for the following equality

$$c_1 y_1(x) + c_2 y_2(x) + \cdots\cdots + c_n y_n(x) = 0$$

then $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are said to be linearly independent.

Otherwise, they are linearly dependent.

判斷是否為 linearly independent 的方法: **Wronskian**

[Definition 4.2] Wronskian

$$W\left(y_1, y_2, \cdots\cdots, y_n\right) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

$$W\left(y_1, y_2, \cdots\cdots, y_n\right) \neq 0 \longrightarrow \text{linearly independent}$$

4.1.3.2 Examples

Example 9 (text page 127)

$$y''' - 6y'' + 11y' - 6y = 0$$

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$\det\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1, y_2, and y_3 are linear independent for any x

general solution:

$$\boxed{y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}} \qquad x \in (-\infty, \infty)$$

Example 10 (text page 128)

$$\boxed{y''' - 6y'' + 11y' - 6y = 3x}$$

$$y''' - 6y'' + 11y' - 6y = 0$$

Three linearly independent solution

$$e^{x}, \quad e^{2x}, \quad e^{3x}$$

Check by Wronskian (Example 9)

$$\begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

Particular solution

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

General solution: $\boxed{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x}$

4.1.3.3 Superposition Principle for Particular Solutions

[Theorem 4.1.7] Superposition Principle

If $y_{p_1}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_1(x)$$

$y_{p_2}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_2(x)$$

$$\vdots$$

$y_{p_k}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_k(x)$$

then $\boxed{y_{p_1}(x)+y_{p_2}(x)+\cdots+y_{p_k}(x)}$ is the particular solution of

$$\boxed{\begin{aligned}&a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)\\&=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)\end{aligned}}$$

Example 11 (text page 129)

$y_{p_1}(x) = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2}(x) = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3}(x) = xe^{x}$ is a particular solution of $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

$\boxed{y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}}$ is a particular solution of

$$\boxed{y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}}$$

4.1.4 New Notations

Notation: $D^n y = \dfrac{d^n y}{dx^n}$

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y \xrightarrow{\text{可改寫成}} D^2 y + 5Dy + 6y \xrightarrow{\text{可改寫成}} (D^2 + 5D + 6)y$$

可再改寫成

$$\boxed{\begin{gathered} L(y) \\ L = D^2 + 5D + 6 \end{gathered}}$$

4.1.5 名詞

- initial conditions, boundary conditions (pages 144, 148)
  (重要名詞)
- associated homogeneous equation , (page 150)
  (重要名詞)
- fundamental set of solutions (page 154)
- Wronskian (page 156)
- complementary function (general solution of the homogenous linear DE)
  (重要名詞) (page 154)
- particular solution (page 151)
- general solution of the nonhomogenous linear DE (page 151)

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots\cdots + c_n y_n(x) + y_p(x)$$

4.1.6 本節要注意的地方

(1) Most of the theories in Section 4.1 are applied to the linear DE

(2) 注意 initial conditions 和 boundary conditions 之間的不同

(3) 快速判斷 linear independent

(補充 1) Theorem 4.1.1 的解釋

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0 \qquad y'(x_0)=y_1 \qquad \cdots\cdots\cdots\cdots \qquad y^{(n-1)}(x_0)=y_{n-1}$$

When $a_n(x_0) \neq 0$

$$y^{(n)}(x_0)+\frac{a_{n-1}(x_0)}{a_n(x_0)}y^{(n-1)}(x_0)+\cdots+\frac{a_1(x_0)}{a_n(x_0)}y'(x_0)+\frac{a_0(x)}{a_n(x_0)}y(x_0)=\frac{g(x_0)}{a_n(x_0)}$$

→ find $y^{(n)}(x_0)$

$$y^{(n-1)}(x_0+\Delta)=y^{(n-1)}(x_0)+y^{(n)}(x_0)\Delta$$

→ find $y^{(n-1)}(x_0+\Delta)$

(根據 $f'(t)=\dfrac{f(t+\Delta)-f(t)}{\Delta}$ ， $f(t+\Delta)=f(t)+f'(t)\Delta$)

以此類推

$$y^{(n-2)}(x_0+\Delta)=y^{(n-2)}(x_0)+y^{(n-1)}(x_0)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+\Delta)$$

$$y^{(n-3)}(x_0+\Delta)=y^{(n-3)}(x_0)+y^{(n-2)}(x_0)\Delta \longrightarrow \text{find } y^{(n-3)}(x_0+\Delta)$$

$$\vdots$$

$$y(x_0+\Delta)=y(x_0)+y'(x_0)\Delta \longrightarrow \boxed{\text{find } y(x_0+\Delta)}$$

$$\downarrow$$

$$y^{(n)}(x_0+\Delta)+\frac{a_{n-1}(x_0+\Delta)}{a_n(x_0+\Delta)}y^{(n-1)}(x_0+\Delta)+\cdots+\frac{a_1(x_0+\Delta)}{a_n(x_0+\Delta)}y'(x_0+\Delta)$$

$$+\frac{a_0(x+\Delta)}{a_n(x_0+\Delta)}y(x_0+\Delta)=\frac{g(x_0+\Delta)}{a_n(x_0+\Delta)} \longrightarrow \text{find } y^{(n)}(x_0+\Delta)$$

$$\downarrow$$

$$y^{(n-1)}(x_0+2\Delta)=y^{(n-1)}(x_0+\Delta)+y^{(n)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-1)}(x_0+2\Delta)$$

$$y^{(n-2)}(x_0+2\Delta)=y^{(n-2)}(x_0+\Delta)+y^{(n-1)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+2\Delta)$$

$$\vdots$$

$$y\left(x_0+2\Delta\right)=y\left(x_0+\Delta\right)+y'\left(x_0+\Delta\right)\Delta$$

$\boxed{\text{find } y(x_0+2\Delta)}$

$$y^{(n)}\left(x_0+2\Delta\right)+\frac{a_{n-1}\left(x_0+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}y^{(n-1)}\left(x_0+2\Delta\right)+\cdots+\frac{a_1\left(x_0+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}y'\left(x_0+2\Delta\right)$$

$$+\frac{a_0\left(x+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}y\left(x_0+2\Delta\right)=\frac{g\left(x_0+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}$$

以此類推，可將 $y(x_0+3\Delta)$, $y(x_0+4\Delta)$, $y(x_0+5\Delta)$, ……………

以至於將 $y(x)$ 所有的值都找出來。

(求 $y(x)$ for $x > x_0$ 時, 用正的 Δ 值，

求 $y(x)$ for $x < x_0$ 時, 用負的 Δ 值)

Requirement 1: $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous
是為了讓 $a_k(x_0+m\Delta)$ 皆可以定義

Requirement 2: $a_n(x) \neq 0$ 是為了讓 $a_k(x_0+m\Delta) / a_n(x_0+m\Delta)$ 不為無限大

4-2 Reduction of Order

4.2.1 適用情形

(1) (2) (3)

Suitable for the 2^{nd} order linear homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(4) One of the nontrivial solution $y_1(x)$ has been known.

4.2.2 解法

假設 $\boxed{y_2(x) = u(x)\,y_1(x)}$

先將DE 變成 Standard form

$$y'' + P(x)y' + Q(x)y = 0$$

If $\boxed{y(x) = u(x)\,y_1(x)}$, (比較 Section 2-3)

$$y' = uy_1' + u'y_1 \qquad y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$uy_1'' + 2u'y_1' + u''y_1 + P(x)uy_1' + P(x)u'y_1 + Q(x)uy_1 = 0$$

$$u\underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{\text{zero}} + 2u'y_1' + u''y_1 + P(x)u'y_1 = 0$$

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\downarrow$$

$$\frac{dw}{dx}y_1 + w(2\frac{dy_1}{dx} + P(x)y_1) = 0$$

multiplied by $dx/(y_1 w)$

$$\downarrow$$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable
(with 3 variables)

$$\downarrow$$

$$\int \frac{dw}{w} + 2\int \frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\downarrow$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\downarrow$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$$\ln\left|wy_1^2\right| = -\int P(x)\,dx + c$$

$$wy_1^2 = \pm e^{-\int P(x)dx + c}$$

$$w = c_1 e^{-\int P(x)dx} / y_1^2$$

$$\boxed{u = \int w\,dx = c_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx + c_2}$$

$$\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx}$$

We can set $c_1 = 1$ and $c_2 = 0$

(因為我們算 $u(x)$ 的目的，只是為了要算出與 $y_1(x)$ 互相independent的另一個解)

4.2.3 例子

Example 1 (text page 132)

$$\boxed{y'' - y = 0}$$

We have known that $y_1 = e^x$ is one of the solution

$P(x) = 0$ $\quad y_2(x) = e^x \int ce^{-2x} dx = -\frac{1}{2} ce^{-x}$

Specially, set $c = -2$, ($y_2(x)$ 只要 independent of $y_1(x)$ 即可所以 c 的值可以任意設)

$$\boxed{y_2(x) = e^{-x}}$$

General solution: $\boxed{y(x) = c_1 e^x + c_2 e^{-x}}$

Example 2 (text page 133) (將課本 x 的範圍做更改)

$$\boxed{x^2 y'' - 3xy' + 4y = 0} \qquad \text{when } x \in (-\infty, 0)$$

We have known that $y_1 = x^2$ is one of the solution

Note: the interval of x

If $x \in (0, \infty)$ $(x > 0)$, $\int dx/x = \ln x$ 如課本

If $x < 0$, $\int dx/x = \ln(-x)$

$$y_2(x) = x^2 \int \frac{e^{3\ln(-x)}}{x^4} dx = x^2 \int \frac{(-x)^3}{(-x)^4} dx$$

$$= -x^2 \int \frac{1}{x} dx = -x^2 \ln|x|$$

$$\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln|x|}$$

4.2.4 本節需注意的地方

(1) 記住公式 $\boxed{y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx}$

(2) 若不背公式 (不建議)，在計算過程中別忘了對 $w(x)$ 做積分

(3) 別忘了 $P(x)$ 是 "standard form" 一次微分項的 coefficient term

(4) 同樣有 singular point 的問題

(5) 因為 $y_2(x)$ 是 homogeneous linear DE 的 "任意解" ，所以計算時，常數的選擇以方便為原則

(6) 由於 $\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ 的計算較複雜且花時間，所以要多加練習

多算習題

附錄六： Hyperbolic Function

$$\sinh(x)=\frac{e^{x}-e^{-x}}{2}$$

比較： $\sin(x)=\frac{e^{jx}-e^{-jx}}{2j}$

$$\cosh(x)=\frac{e^{x}+e^{-x}}{2}$$

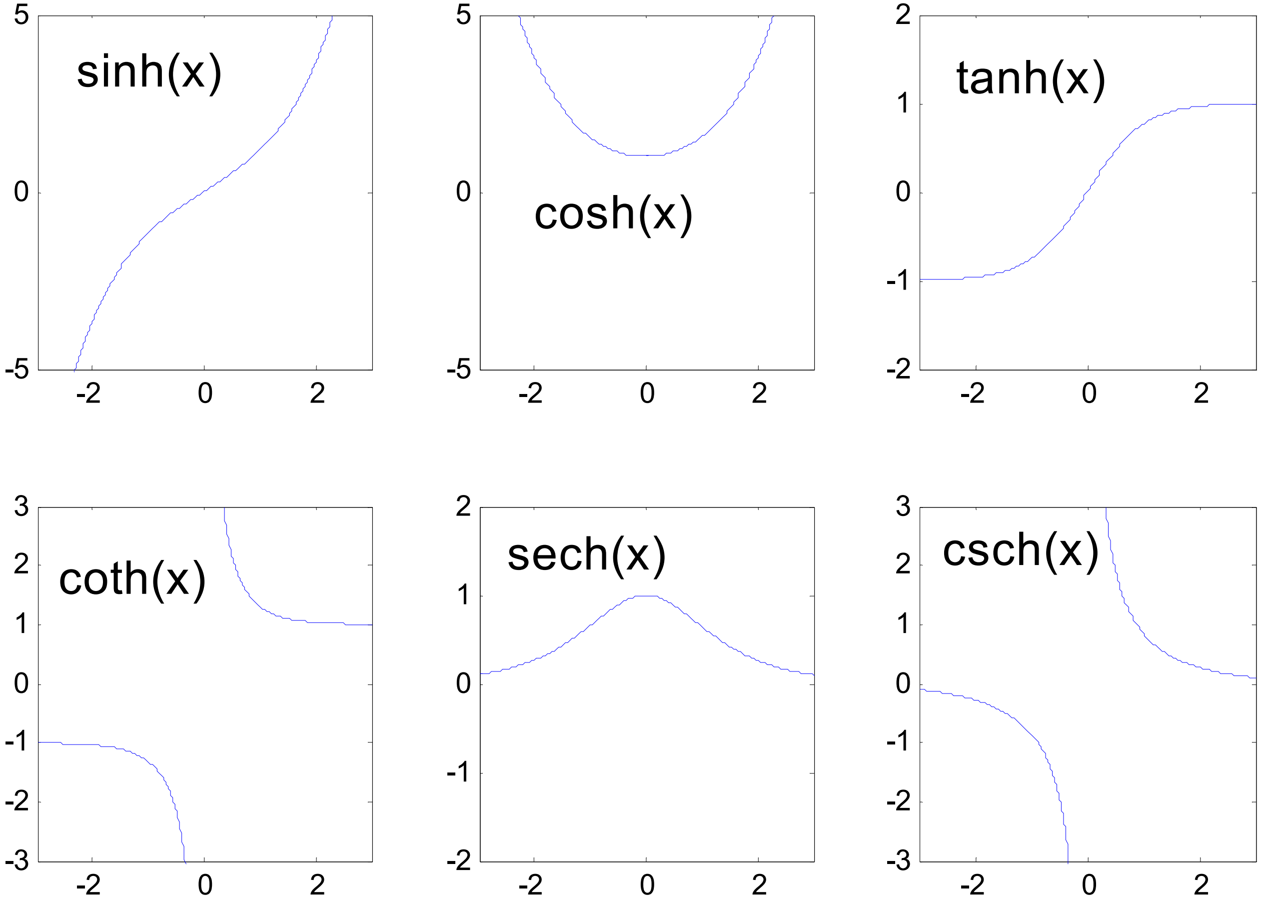
$$\cos(x)=\frac{e^{jx}+e^{-jx}}{2}$$

$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\coth(x)=\frac{\cosh(x)}{\sinh(x)}=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$$\operatorname{sech}(x)=\frac{1}{\cosh(x)}=\frac{2}{e^{x}+e^{-x}}$$

$$\operatorname{csch}(x)=\frac{1}{\sinh(x)}=\frac{2}{e^{x}-e^{-x}}$$


sinh(x)
cosh(x)
tanh(x)
coth(x)
sech(x)
csch(x)
5
0
-5
-2
0
2
2
1
-1
-2
3
-3

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$

$$\sinh(0) = 0$$

$$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$

$$\cosh(0) = 1$$

$$\frac{d}{dx}\tanh(ax) = a\,\mathrm{sech}^2(ax)$$

$$\sinh'(0) = 1$$

$$\frac{d}{dx}\coth(ax) = -a\,\mathrm{csch}^2(ax)$$

$$\cosh'(0) = 0$$

$$\frac{d}{dx}\mathrm{sech}(ax) = -a\,\mathrm{sech}(ax)\tanh(ax)$$

$$\sin(ix) = i\sinh(x)$$

$$\frac{d}{dx}\mathrm{csch}(ax) = -a\,\mathrm{csch}(ax)\coth(ax)$$

$$\cos(ix) = \cosh(x)$$

$$\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a} + c$$

$$\int \cosh(ax)\,dx = \frac{\sinh(ax)}{a} + c$$

$$\int \tanh(ax)\,dx = \frac{\ln\left|\cosh(ax)\right|}{a} + c$$

$$\int \coth(ax)\,dx = \frac{\ln\left|\sinh(ax)\right|}{a} + c$$

$$\int \operatorname{sech}(ax)\,dx = \frac{2\tan^{-1}\left(\tanh(\frac{a}{2}x)\right)}{a} + c$$

$$\int \operatorname{csch}(ax)\,dx = \frac{\ln\left|\tanh(\frac{a}{2}x)\right|}{a} + c$$

附錄七 Linear DE 解法的步驟 (參照講義 page 151)

Step 1: Find the general solution (i.e., the complementary function) of the associated homogeneous DE

(Sections 4-2, 4-3, 4-7)

Step 2: Find the particular solution

(Sections 4-4, 4-5, 4-6)

Step 3: Combine the complementary function and the particular solution

Extra Step: Consider the initial (or boundary) conditions

4-3 Homogeneous Linear Equations with Constant Coefficients

本節使用 auxiliary equation 的方法來解 homogeneous DE

KK: [ɔgˋzɪljərɪ]

4-3-1 限制條件

限制條件: (1) homogeneous

(2) linear

(3) constant coefficients

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

a_0, a_1, a_2, …. , a_n are constants

(the simplest case of the higher order DEs)

4-3-2 解法

解法核心：

Suppose that the solutions has the form of e^{mx}

Example: $y''(x) - 3\,y'(x) + 2\,y(x) = 0$

Set $y(x) = e^{mx}$, $\quad m^2 e^{mx} - 3m\, e^{mx} + 2\, e^{mx} = 0$

$$m^2 - 3m + 2 = 0 \longrightarrow \text{solve } m$$

可以直接把 n 次微分用 m^n 取代，變成一個多項式

這個多項式被稱為 auxiliary equation

- **解法流程**

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

Step 1-1
auxiliary function

$$\boxed{a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0}$$

Step 1-1 Find n roots , m_1, m_2, m_3, …., m_n

(If m_1, m_2, m_3, …., m_n are <u>distinct</u>)

Step 1-2 n linearly independent solutions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots\ldots, e^{m_n x}$

(有三個 Cases)

Step 1-3 Complementary function

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}}$$

4-3-3 Three Cases for Roots (2nd Order DE)

$$\boxed{a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0}$$

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$

roots $m_1 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$ $\qquad m_2 = \dfrac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$

solutions

Case 1 $m_1 \neq m_2$, m_1, m_2 are real (其實 m_1, m_2 不必限制為 real)

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case 2 $m_1 = m_2$ (m_1 and m_2 are of course real)

First solution: $\boxed{y_1 = e^{m_1 x}}$

Second solution: using the method of "Reduction of Order"

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

$$= e^{m_1 x}\int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx$$

$$= e^{m_1 x}\int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \qquad m_1 = \frac{-a_1}{2a_2}$$

$$= e^{m_1 x}\int dx = e^{m_1 x}(x + c)$$

$$\boxed{y_2(x) = xe^{m_1 x}}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

$$\alpha = -a_1/2a_2, \qquad \beta = \sqrt{4a_2a_0 - a_1^2}/2a_2$$

Solution: $\boxed{y = C_1e^{\alpha x + j\beta x} + C_2e^{\alpha x - j\beta x}}$

Another form:

$$y = e^{\alpha x}\left(C_1e^{j\beta x} + C_2e^{-j\beta x}\right)$$

$$= e^{\alpha x}\left(C_1\cos\beta x + jC_1\sin\beta x + C_2\cos\beta x - jC_2\sin\beta x\right)$$

set $c_1 = C_1 + C_2$ and $c_2 = jC_1 - jC_2$

$\boxed{y = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)}$ c_1 and c_2 are some constant

Example 1 (text page 137)

(a) $\boxed{2y'' - 5y' - 3y = 0}$

$2m^2 - 5m - 3 = 0, \quad m_1 = -1/2, \; m_2 = 3$

$$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$$

(b) $\boxed{y'' - 10y' + 25y = 0}$

$m^2 - 10m + 25 = 0, \quad m_1 = 5, \; m_2 = 5$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

(c) $\boxed{y'' + 4y' + 7y = 0}$

$m^2 + 4m + 7 = 0, \quad m_1 = -2 + i\sqrt{3}, \quad m_2 = -2 - i\sqrt{3}$

$$\boxed{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}$$

4-3-4 Three + 1 Cases for Roots (Higher Order DE)

For higher order case $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$

auxiliary function: $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$

roots: $m_1, m_2, m_3, \ldots, m_n$

(1) If $m_p \neq m_q$ for $p = 1, 2, \ldots, n$ and $p \neq q$

(也就是這個多項式在 m_q 的地方只有一個根)

then $\boxed{e^{m_q x}}$ is a solution of the DE.

重覆次數

(2) If the multiplicities of m_q is k (當這個多項式在 m_q 的地方有 k 個根),

$$\boxed{e^{m_q x}, xe^{m_q x}, x^2 e^{m_q x}, \cdots\cdots, x^{k-1} e^{m_q x}}$$

are the solutions of the DE.

(3) If both $\alpha + j\beta$ and $\alpha - j\beta$ are the roots of the auxiliary function, then

$$\boxed{e^{\alpha x}\cos(\beta x),\, e^{\alpha x}\sin(\beta x)}$$

are the solutions of the DE.

(4) If the multiplicities of $\alpha + j\beta$ is k and the multiplicities of $\alpha - j\beta$ is also k， then

$$\boxed{\begin{array}{l} e^{\alpha x}\cos(\beta x),\, xe^{\alpha x}\cos(\beta x),\, x^2 e^{\alpha x}\cos(\beta x),\, \cdots\cdots,\, x^{k-1}e^{\alpha x}\cos(\beta x) \\ e^{\alpha x}\sin(\beta x),\, xe^{\alpha x}\sin(\beta x),\, x^2 e^{\alpha x}\sin(\beta x),\, \cdots\cdots,\, x^{k-1}e^{\alpha x}\sin(\beta x) \end{array}}$$

are the solutions of the DE.

Note: If $\alpha + j\beta$ is a root of a real coefficient polynomial,

then $\alpha - j\beta$ is also a root of the polynomial.

$$a_n(\alpha + j\beta)^n + a_{n-1}(\alpha + j\beta)^{n-1} + \cdots + a_1(\alpha + j\beta) + a_0 = 0$$

a_0, a_1, a_2, …. , a_n are real

Example 3 (text page 138)

Solve $\boxed{y''' + 3y'' - 4y = 0}$

Step 1-1 $m^3 + 3m^2 - 4 = 0$

$$(m-1)\left(m^2 + 4m + 4\right) = 0 \qquad \boxed{m_1 = 1, \quad m_2 = m_3 = -2}$$

Step 1-2 3 independent solutions: e^x , e^{-2x} , xe^{-2x}

Step 1-3 general solution: $\boxed{y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}}$

Example 4 (text page 138)

Solve $\boxed{y^{(4)}(x) + 2y''(x) + y(x) = 0}$

Step 1-1 $m^4 + 2m^2 + 1 = 0$

$(m^2 + 1)^2 = 0$ $\quad$ $\boxed{\text{four roots: } \ \boldsymbol{i},\ \boldsymbol{i},\ -\boldsymbol{i},\ -\boldsymbol{i}}$

Step 1-2 4 independent solutions: $\cos x$, $x\cos x$, $\sin x$, $x\sin x$

Step 1-3 general solution: $\boxed{y = c_1 \cos x + c_2 x\cos x + c_3 \sin x + c_4 x \sin x}$

4-3-5 How to Find the Roots

(1) Formulas

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0} \Rightarrow m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \qquad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

$$\boxed{a_3 m^3 + a_2 m^2 + a_1 m + a_0}$$

$$\Rightarrow m_1 = S + T - \frac{a_2}{3a_3}$$

Solutions:

$$m_2 = -\frac{1}{2}(S+T) - \frac{a_2}{3a_3} + i\frac{1}{2}\sqrt{3}(S-T)$$

$$m_3 = -\frac{1}{2}(S+T) - \frac{a_2}{3a_3} - i\frac{1}{2}\sqrt{3}(S-T)$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^3}} \quad , \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^3}}$$

$$Q = \frac{1a_1}{3a_3} - \frac{a_2^2}{9a_3^2} \quad , \quad R = \frac{9a_1 a_2 - 27a_3 a_0 - 2a_2^3}{54a_3^2}$$

(太複雜了)

(2) Observing

例如：1 是否為 root $\longrightarrow$ 看係數和是否為 0

又如：

$$3m^3 + 5m^2 + 10m - 4$$

factor: 1,3 factor: 1,2,4

possible roots: ±1, ±2, ±4, ±1/3, ±2/3, ±4/3

test for each possible root $\longrightarrow$ find that 1/3 is indeed a root

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \frac{1}{3}\right)\left(3m^2 + 6m + 12\right)$$

(3) Solving the roots of a polynomial by software

Maple

Mathematica (by the commands of **Nsolve** and **FindRoot**)

Matlab (by the command of **roots**)

4-3-6 本節需注意的地方

(1) 注意重根和 conjugate complex roots 的情形

(2) 寫解答時，要將 "General solution" 寫出來

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

(3) 因式分解要熟練

(4) 本節的方法，也適用於 1st order 的情形

4-4 Undetermined Coefficients – Superposition Approach

This section introduces some method of "guessing" the particular solution.

4-4-1 方法適用條件

Suitable for linear (1) and constant coefficient (2) DE.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = g(x)$$

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ...contain finite number of terms.

4-4-2 方法

把握一個原則：

$g(x)$ 長什麼樣子，particular solution 就應該是什麼樣子.

記熟下一頁的規則

(計算時要把 A, B, C, … 這些 unknowns 解出來)

Trial Particular Solutions (from text page 146)

$g(x)$	Form of y_p
1 (any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
$x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

It comes from the "form rule". See page 203.

$$g(x) = e^{2x} + xe^{3x} \qquad y_p = ?$$

$$g(x) = \cos(x) + x^2 \sin(2x) \qquad y_p = ?$$

$$g(x) = \cosh(2x) \qquad y_p = ?$$

4-4-3 Examples

Example 2 $y'' - y' + y = \underline{2\sin 3x}$ (text page 144)

Step 1: find the solution of the associated homogeneous equation

Guess

Step 2: particular solution

$$\boxed{y_p = A\cos 3x + B\sin 3x}$$

$$y'_p = -3A\sin 3x + 3B\cos 3x$$

$$y''_p = -9A\cos 3x - 9B\sin 3x$$

$$y''_p - y'_p + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

$$\begin{cases} -8A - 3B = 0 \\ 3A - 8B = 2 \end{cases} \Longrightarrow A = 6/73,\ B = -16/73$$

$$\boxed{y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$$

Step 3: General solution:

$$\boxed{y = e^{x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$$

Example 3 $y''-2y'-3y=\underline{4x-5}+\underline{6xe^{2x}}$ (text page 145)

Step 1: Find the solution of

$$y''-2y'-3y=0.$$

$$y_c=c_1e^{3x}+c_2e^{-x}$$

Step 2: Particular solution

$$y''-2y'-3y=4x-5 \qquad\qquad y''-2y'-3y=6xe^{2x}$$

guess

$$\boxed{y_{p_1}=Ax+B}$$

$$y'_{p_1}=A$$

$$y''_{p_1}=0$$

$$-3Ax-2A-3B=4x-5$$

$$A=-\frac{4}{3},\quad B=\frac{23}{9}$$

$$\boxed{y_{p_1}=-\frac{4}{3}x+\frac{23}{9}}$$

guess

$$\boxed{y_{p_2}=Cxe^{2x}+Ee^{2x}}$$

$$y'_{p_2}=2Cxe^{2x}+Ce^{2x}+2Ee^{2x}$$

$$y''_{p_2}=4Cxe^{2x}+4Ce^{2x}+4Ee^{2x}$$

$$-3Cxe^{2x}+\left(2C-3E\right)e^{2x}=6xe^{2x}$$

$$C=-2,\quad E=-\frac{4}{3}$$

$$\boxed{y_{p_2}=-(2x+\frac{4}{3})e^{2x}}$$

Particular solution

$$y_p = y_{p_1} + y_{p_2} = -\frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^{x}$$

Step 3: General solution

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x} - \frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^{2x}}$$

4-4-4 方法的解釋：Form Rule

Form Rule: y_p should be a linear combination of $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ……………..

Why? 如此一來，在比較係數時才不會出現多餘的項

When $g(x) = x^n$

$$x^n \rightarrow x^{n-1} \rightarrow x^{n-2} \rightarrow x^{n-3} \rightarrow \cdots\cdots \rightarrow 1 \rightarrow 0$$

$$y_p = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots\ldots + A_0$$

When $g(x) = \cos kx$

$$\cos kx \rightarrow \sin kx$$

$$y_p = A_1 \cos kx + A_2 \sin kx$$

When $g(x) = \exp(kx)$

$$e^{kx}$$

$$y_p = A \exp(kx)$$

When $g(x) = x^n \exp(kx)$

$$g'(x) = nx^{n-1}e^{kx} + kx^n e^{kx}$$

$$g''(x) = n(n-1)x^{n-2}e^{kx} + 2nkx^{n-1}e^{kx} + k^2 x^n e^{kx}$$

$$g'''(x) = n(n-1)(n-2)x^{n-3}e^{kx} + 3kn(n-1)x^{n-2}e^{kx}$$
$$+3k^2 nx^{n-1}e^{kx} + k^3 x^n e^{kx}$$

$$\vdots$$

$$\vdots$$

會發現 $g(x)$ 不管多少次微分，永遠只出現

$$x^n e^{kx},\ x^{n-1}e^{kx},\ x^{n-2}e^{kx},\ x^{n-3}e^{kx},\ \cdots\cdots,\ e^{kx}$$

$$\boxed{y_p = c_n x^n e^{kx} + c_{n-1}x^{n-1}e^{kx} + c_{n-2}x^{n-2}e^{kx} + \cdots\cdots + c_0 e^{kx}}$$

4-4-5 Glitch of the method:

Example 4 $\quad y'' - 5y' + 4y = 8e^x \qquad$ (text page 146)

Particular solution guessed by Form Rule:

$$y_p = Ae^x$$

$$y_p'' - 5y_p' + 4y_p = Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

$$0 = 8e^x$$ (no solution)

Why?

Glitch condition 1: The **particular solution** we guess belongs to the complementary function.

For Example 4 $\quad y'' - 5y' + 4y = 8e^x$

Complementary function $\quad y_c = c_1 e^x + c_2 e^{4x} \qquad Ae^x \in y_c$

解決方法：再乘一個 x

$\boxed{y_p = Axe^x} \qquad y_p' = Axe^x + Ae^x$

$$y_p'' = Axe^x + 2Ae^x$$

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x \implies A = -8/3$$

$$\boxed{y_p = -\frac{8}{3}xe^x}$$

$$\boxed{y = c_1 e^x + c_2 e^{4x} - \frac{8}{3}xe^x}$$

Example 7 $\quad y'' - 2y' + y = e^x \qquad$ (text page 148)

$$y_c = c_1 e^x + c_2 x e^x$$

From Form Rule, the particular solution is Ae^x

$$Ae^x \in y_c$$

$$Axe^x \in y_c$$

如果乘一個 x 不夠，則再乘一個 x

$$\boxed{y_p = Ax^2 e^x} \qquad y_p' = (Ax^2 + 2Ax)e^x$$

$$y_p'' = (Ax^2 + 4Ax + 2A)e^x$$

$$y_p'' - 2y_p' + y_p = 2Ae^x = e^x \quad \Longrightarrow \quad A = 1/2$$

$$\boxed{y_p = x^2 e^x / 2}$$

$$\boxed{y = c_1 e^x + c_2 x e^x + x^2 e^x / 2}$$

Example 8 (text page 148)

$$y'' + y = \underbrace{4x}_{\text{子問題1}} + \underbrace{10\sin x}_{\text{子問題2}} \qquad y(\pi) = 0 \qquad y'(\pi) = 2$$

Step 1 $\quad y_c = c_1 \cos x + c_2 \sin x$

Step 2 $\quad y_p = Ax + B + C\sin x + E\cos x$

修正 $\quad y_p = \boxed{Ax + B} + \boxed{Cx\sin x + Ex\cos x}$

注意： sin$x$, cos$x$ 都要乘上 x

$$y_p = 4x - 5x\cos x$$

Step 3 $\quad \boxed{y = c_1 \cos x + c_2 \sin x + 4x - 5x\cos x}$

Step 4 $\quad$ Solving c_1 and c_2 by initial conditions (最後才解 IVP)

$$y(\pi) = -c_1 + 4\pi + 5\pi = 0 \Longrightarrow c_1 = 9\pi$$

$$y' = -c_1 \sin x + c_2 \cos x + 4 - 5\cos x + 5x\sin x$$

$$y'(\pi) = -c_2 + 9 = 2 \Longrightarrow c_2 = 7$$

$$\boxed{y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x}$$

Example 11 (text page 149)

子問題1

$$y_{p,1} = A$$

$$y_{p,2} = Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}$$

$$y^{(4)} + y''' = 1 - x^2e^{-x}$$

子問題2

$$y_c = \boxed{c_1} + c_2x + c_3x^2 + \boxed{c_4e^{-x}}$$

From Form Rule

$$y_p = \boxed{A} + Bx^2e^{-x} + Cxe^{-x} + \boxed{Ee^{-x}}$$

y_p 只要有一部分和 y_c 相同就作修正

修正

乘上 x^3　　乘上 x

$$\boxed{y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}}$$

來自同一個子問題的項，乘的要相同

If we choose $y_p = A + Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}$

$$y_p^{(4)} + y_{(p)}''' = \underline{-2Bxe^{-x} + (6B - C)e^{-x}} = 1 - x^2e^{-x}$$

沒有 $1, x^2e^{-x}$ 兩項，不能比較係數，無解

If we choose $\boxed{y_p = Ax^3 + Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}}$

$$y_p^{(4)} + y_{(p)}''' = \underline{6A - 2Bxe^{-x} + (6B - C)e^{-x}} = 1 - x^2e^{-x}$$

沒有 x^2e^{-x} 這一項，不能比較係數，無解

If we choose $\boxed{y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}}$

$$y_p^{(4)} + y_{(p)}'''$$

$$= 6A - 3Bx^2e^{-x} + \left(18B - 2C\right)xe^{-x} + (-18B + 6C - E)e^{-x}$$

$$= 1 - x^2e^{-x}$$

$A = 1/6$, $B = 1/3$, $C = 3$, $E = 12$

$$\boxed{y_p = \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}}$$

$$\boxed{y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}}$$

Glitch condition 2: $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ……………
contain **infinite number of terms**.

If $g(x) = \ln x$

$$\ln x \to \frac{1}{x} \to \frac{1}{x^2} \to \frac{1}{x^3} \to \cdots\cdots$$

If $g(x) = \exp(x^2)$

$$g'(x) \to 2xe^{x^2}$$
$$g''(x) \to (4x^2 + 2)e^{x^2}$$
$$g'''(x) \to (8x^3 + 12x)e^{x^2}$$

$$\vdots$$

$$\vdots$$

4-4-6 本節需要注意的地方

(1) 記住 Table 4.1 的 particular solution 的假設方法
(其實和 "form rule" 有相密切的關聯)

(2) 注意 "glitch condition"

另外，"同一類" 的 term 要乘上相同的東西 (參考 Example 11)

(3) 所以要先算 complementary function，再算 particular solution

(4) 同樣的方法，也可以用在 1st order 的情形

(5) 本方法只適用於 linear, constant coefficient DE

4-5 Undetermined Coefficients – Annihilator Approach

For a linear DE:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

Annihilator Operator:

能夠「殲滅」 $g(x)$ 的 operator

4-5-1 方法適用條件

(1) Linear， (2) Constant coefficients

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ………contain finite number of terms.

4-5-2 Find the Annihilator

Example 1: (text page 153)

$g(x) = 1 - 5x^2 + 8x^3 \longrightarrow$ annihilator: D^4 $\quad D^k g(x) = \dfrac{d^k}{dx^k} g(x)$

$g(x) = e^{-3x} \longrightarrow$ annihilator: $D + 3$

$$\frac{d}{dx} g(x) + 3g(x) = 0$$

$g(x) = 4e^{2x} - 10xe^{2x} \longrightarrow$ annihilator: $(D-2)^2$

$$(D-2)^2 = D^2 - 4D + 4$$

$$\frac{d^2}{dx^2}g(x) - 4\frac{d}{dx}g(x) + 4g(x) = 0$$

註：當各個微分項的 coefficients 皆為 constants 時，function of D 的計算方式和 function of x 的計算方式相同

$$(x-2)^2 = x^2 - 4x + 4$$

$$\Rightarrow (D-2)^2 = D^2 - 4D + 4$$

General rule 1:

If $g(x)=\left(a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_0\right)e^{\alpha x}$

then the annihilator is $[D-\alpha]^{n+1}$

注意： annihilator 和 a_0, a_1, …… , a_n 無關

只和 α, n 有關

The annihilator is <u>independent of the constant multiplied in the front</u> of each term.

General rule 2:

If $g(x)=\left(a_n x^n+a_{n-1}x^{n-1}+\cdots\cdots+a_0\right)e^{\alpha x}\left(b_1\cos\beta x+b_2\sin\beta x\right)$

$b_1 \neq 0$ or $b_2 \neq 0$

then the annihilator is $\left[D^2-2\alpha D+\left(\alpha^2+\beta^2\right)\right]^{n+1}$

Example 2: (text page 154) $g(x)=5e^{-x}\cos 2x-9e^{-x}\sin 2x$

annihilator D^2+2D+5

Example 5: (text page 156) $g(x)=x\cos x-\cos x$

annihilator $\left[D^2+1\right]^2$

Example 6: (text page 157) $g(x)=10e^{-2x}\cos x$

annihilator D^2+4D+5

General rule 3:

If $g(x) = g_1(x) + g_2(x) + \ldots\ldots + g_k(x)$

$L_h[g_h(x)] = 0$ but $L_h[g_m(x)] \neq 0$ if $m \neq h$,

then the annihilator of $g(x)$ is the product of L_h $(h = 1 \sim k)$

$$\boxed{L_k L_{k-1} \cdots L_2 L_1}$$

Proof:
$$L_k L_{k-1} \cdots L_3 L_2 L_1 \left[g_1 + g_2 + g_3 + \cdots\cdots + g_k \right]$$
$$= L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 + L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 +$$
$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_3 + \cdots\cdots + L_k L_{k-1} \cdots L_3 L_2 L_1 g_k$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 = L_k L_{k-1} \cdots L_3 L_2 \left[L_1 g_1 \right] = 0$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 = L_k L_{k-1} \cdots L_3 L_1 \left[L_2 g_2 \right] = 0$$

(因為 L_1, L_2 為 linear DE with constant coefficient,

$L_1 L_2 = L_2 L_1$)

Similarly,

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_3 = L_k L_{k-1} \cdots L_4 L_2 L_1 \left[L_3 g_3 \right] = 0$$

$$\vdots$$

$$\vdots$$

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_3 = L_{k-1} \cdots L_4 L_3 L_2 L_1 \left[L_k g_k \right] = 0$$

Therefore,

$$\begin{aligned} & L_k L_{k-1} \cdots L_3 L_2 L_1 \left[g_1 + g_2 + g_3 + \cdots\cdots + g_k \right] \\ & = 0 + 0 + 0 + \cdots\cdots + 0 \\ & = 0 \end{aligned}$$

Example 7 (text page 157)

$$g(x) = \underline{5x^2 - 6x} + \underline{4x^2 e^{2x}} + \underline{3e^{5x}}$$

annihilator: D^3

annihilator: $(D-2)^3$

annihilator: $D-5$

annihilator of $g(x)$: $D^3(D-2)^3(D-5)$

4-5-3 Using the Annihilator to Find the Particular Solution

Step 2-1 Find the annihilator L_1 of $g(x)$

Step 2-2 如果原來的 linear & constant coefficient DE 是

$$L(y) = g(x)$$

那麼將 DE 變成如下的型態：

$$\boxed{L_1[L(y)] = L_1[g(x)] = 0}$$

(homogeneous linear & constant coefficient DE)

註： If $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$

then $L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

Step 2-3 Use the method in Section 4-3 to find the solution of

$$\boxed{L_1\left[L(y)\right]=0}$$

Step 2-4 Find the particular solution.

The particular solution y_p is a solution of

$$L_1\left[L(y)\right]=0$$

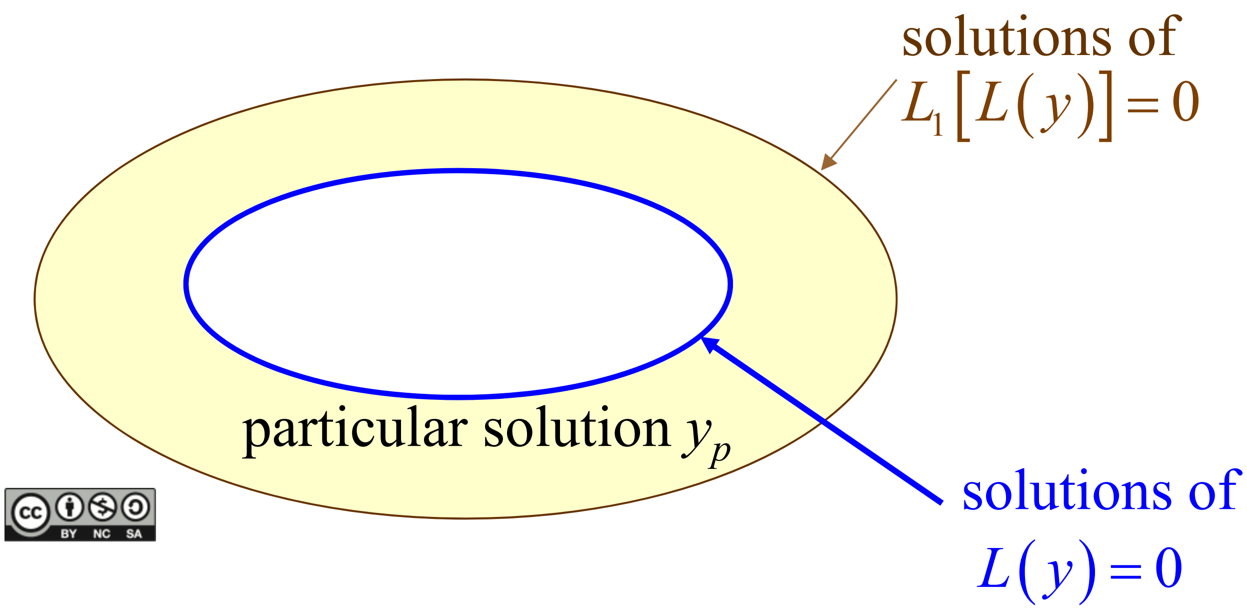
but not a solution of

$$L(y)=0$$

(Proof): Since $L(y_p)=g(x)$, if $g(x)\neq 0$, $L(y_p)$ should be nonzero.

Moreover, $L_1\left[L(y_p)\right]=L_1\left[g(x)\right]=0.$

Step 2-5 Solve the unknowns



particular solution y_p $\in$ solutions of $L_1[L(y)]=0$

$\notin$ solutions of $L(y)=0$

本節核心概念

4-5-4 Examples

Example 3 (text page 155)

$$y'' + 3y' + 2y = 4x^2$$

Step 1: Complementary function

(solution of the associated homogeneous function)

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2-1: Annihilation: D^3

$$L_1\left[L(y)\right] = L_1\left[g(x)\right] = 0$$

Step 2-2: $$D^3(D^2 + 3D + 2)y = 0$$

Step 2-3: auxiliary function $m^3(m^2 + 3m + 2) = 0$

roots: $m_1 = m_2 = m_3 = 0,\ m_4 = -1,\ m_5 = -2$

Solution for $L_1\left[L(\tilde{y})\right] = 0$:

移除和 complementary function 相同的部分

$$\tilde{y} = \boxed{d_1 + d_2 x + d_3 x^2} + d_4 e^{-x} + d_5 e^{-2x}$$

Step 2-4: particular solution $y_p = A + Bx + Cx^2$ $\quad y_p' = B + 2Cx$

$$y_p'' = 2C$$

Step 2-5: $y_p'' + 3y_p' + 2y = 2Cx^2 + (2B + 6C)x + (2A + 3B + 2C) = 4x^2$

$$\begin{cases} 2C = 4 \\ 2B + 6C = 0 \\ 2A + 3B + 2C = 0 \end{cases} \Longrightarrow \begin{aligned} C &= 2 \\ B &= -6 \\ A &= 7 \end{aligned}$$

$$\boxed{y_p = 7 - 6x + 2x^2}$$

Step 3: $\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2}$

Example 4 (text page 156)

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

Step 1: Complementary function

From auxiliary function, $m^2 - 3m = 0$, roots: 0, 3

$$y_c = c_1 + c_2 e^{3x}$$

Step 2-1: Find the annihilator

$D - 3$ annihilate $8e^{3x}$ but cannot annihilate $4\sin x$

$(D^2 + 1)$ annihilate $4\sin x$ but cannot annihilate $8e^{3x}$

$$\Downarrow$$

$(D - 3)(D^2 + 1)$ is the annihilator of $8e^{3x} + 4\sin x$

Step 2-2: $\boxed{(D-3)(D^2+1)\left(D^2 - 3D\right)y = 0}$

Step 2-3: auxiliary function: $\left(m-3\right)(m^2+1)\left(m^2-3m\right)$ 易犯錯的地方

$$= m\left(m-3\right)^2(m^2+1) = 0$$

solution of $\left(D-3\right)(D^2+1)\left(D^2-3D\right)\tilde{y}=0$:

$$\tilde{y} = \cancel{d_1} + \cancel{d_2 e^{3x}} + \boxed{d_3 x e^{3x} + d_4 \cos x + d_5 \sin x}$$

Step 2-4: particular solution

$$y_p = d_3 x e^{3x} + d_4 \cos x + d_5 \sin x$$

$\Downarrow$ 代回原式
並比較係數

Step 2-5: $y_p = \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x$

Step 3: general solution $\boxed{y = c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x}$

4-5-5 本節要注意的地方

(1) 所以要先算 complementary function，再算 particular solution

(2) 若有兩個以上的 annihilator，選其中較簡單的即可

(3) 計算 auxiliary function 時有時容易犯錯

(4) $L_1\left[L\left(\tilde{y}\right)\right]=0$ 的解和 $L\left(y\right)=0$ 的解不一樣。

(5) 這方法，只適用於 constant coefficient linear DE

(因為，還需借助 auxiliary function)

The thing that can be done by the annihilator approach can always be done by the "guessing" method in Section 4-4, too.

附錄八：Reviews for Higher Order DE

(A) Linear DE Complementary Function 3 大解法的前 2個

(1) Reduction of Order (Section 4-2)

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

適用情形：

(2) Auxiliary Function (Section 4-3)

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

$$\downarrow$$

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

4 Cases (See pages 183-185, 187-188)

適用情形：

(B) Linear DE Particular solution 3 大解法的前 2個

(1) Guess (Section 4-4) (熟悉講義 page 198 的表)

要訣： y_p should be a linear combination of $g(x)$, $g'(x)$,

$g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ……………..

適用情形：

遇到重覆，乘 x 或 $\ln x$

(2) Annihilator (Section 4-5)

若原本的 DE 為 $L[y(x)] = g(x)$ Annihilator: $L_1[\,g(x)] = 0$

Particular solution 為 $L_1\{L[y(x)]\} = 0$ 的解

(扣去和 $L[y(x)] = 0$ 的解重複的部分)

$y = y_c + y_p$

適用情形：

Annihilator 算法三大規則： Pages 217-219

練習題

Sec. 4-1: 3, 7, 8, 10, 13, 20, 24, 29, 33, 36

Sec. 4-2: 2, 4, 9, 13, 14, 16, 18, 19

Sec. 4-3: 7, 16, 20, 22, 24, 28, 33, 39, 41, 52, 54, 56, 59, 61, 63

Section 4-4 5, 6, 14, 17, 18, 24, 26, 32, 33, 39, 42

Section 4-5 2, 7, 8, 13, 18, 31, 45, 60, 62, 69, 70

Review 4 2, 21, 22, 25, 33, 34, 37