Chapter 4 Higher Order Differential Equations

Highest differentiation: $\dfrac{d^n y}{dx^n}$, $n > 1$

Most of the methods in Chapter 4 are applied for the linear DE.

附錄四 DE 的分類

- DE
  - Linear
    - Constant coefficients
      - Homogeneous
      - Nonhomogeneous
    - Cauchy-Euler
      - Homogeneous
      - Nonhomogeneous
    - Others
      - Homogeneous
      - Nonhomogeneous
  - Nonlinear

附錄五 Higher Order DE 解法

- linear
  - homogeneous part
    - reduction of order
    - auxiliary function
    - Cauchy-Euler
  - particular solution
    - “guess” method
    - annihilator
    - variation of parameters
  - multiple linear DEs
    - elimination method
- nonlinear
  - reduction of order
  - Taylor series
  - numerical method
- both (but mainly linear)
  - series solution
  - transform
    - Laplace transform
    - Fourier series
    - Fourier cosine series
    - Fourier sine series
    - Fourier transform

4-1 Linear Differential Equations: Basic Theory

4.1.1 Initial-Value and Boundary Value Problems

4.1.1.1 The n^{th} Order Initial Value Problem

i.e., the n^{th} order linear DE with the constraints at the same point

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \qquad y'(x_0) = y_1, \qquad y''(x_0) = y_2, \qquad \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ y^{(n-1)}(x_0) = y_{n-1}$$

n initial conditions

Theorem 4.1.1

For an interval I that contains the point x_0

① If $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous at $x = x_0$

② $a_n(x_0) \neq 0$

(很像Section 2-3 當中沒有 singular point 的條件)

then for the problem on page 139, the solution $y(x)$ exists and is unique on the interval I that contains the point x_0

(Interval I 的範圍，取決於何時 $a_n(x) = 0$ 以及 何時 $a_k(x)$ ($k = 0 \sim n$) 不為continuous)

Otherwise, the solution is either non-unique or does not exist.

(infinite number of solutions) (no solution)

Example 1 (text page 119)

$$3y''' + 5y'' - y' + 7y = 0 \qquad y(1) = 0 \qquad y'(1) = 0 \qquad y''(1) = 0$$

Example 2 (text page 120)

$$y'' - 4y' = 12x \qquad y(0) = 4 \qquad y'(0) = 1$$

- $x^2 y'' - 2xy' + 2y = 6 \qquad y(0) = 3 \qquad y'(0) = 1$

有無限多組解

$y = cx^2 + x + 3$ $\qquad$ c 為任意之常數

- 比較：

$$x^2y'' - 2xy' + 2y = 6 \qquad y(1) = 3 \qquad y'(1) = 1$$

There is only one solution

$$y = x^2 - x + 3$$

$$x \in (0, \infty)$$

- Note:

The initial value can also be the form as:

$$\alpha y(x_0) + \beta y'(x_0) = 0$$

$$\sum_{n=0}^{N-1} \alpha_n y^{(n)}(x_0) = 0$$ (general initial condition)

4.1.1.2 n^{th} Order Boundary Value Problem

Boundary conditions are specified at different points

比較：Initial conditions are specified at the same points

例子：$a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $\quad y(a) = y_0, \quad y(b) = y_1$

或 $\quad y'(a) = y_0, \quad y(b) = y_1$

或 $$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$

An n^{th} order linear DE with n boundary conditions may have a <u>unique</u> solution, <u>no</u> solution, or <u>infinite</u> number of solutions.

Example 3 (text page 120)

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

(1) $y(0) = 0 \qquad y(\pi/2) = 0$

$y = c_2 \sin(4x) \qquad c_2$ is any constant (infinite number of solutions)

(2) $y(0) = 0 \qquad y(\pi/8) = 0$

$y = 0 \qquad$ (unique solution)

4.1.2 Homogeneous Equations

4.1.2.1 Definition

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x)=0 \longrightarrow$ homogeneous

$g(x)\neq 0 \longrightarrow$ nonhomogeneous

- 重要名詞：Associated homogeneous equation

The associated homogeneous equation of a nonhomogeneous DE:

Setting $g(x) = 0$

- Review: Section 2-3, pages 53, 55

4.1.2.2 New Notations

Notation: $D^n y = \dfrac{d^n y}{dx^n}$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y \xrightarrow{\text{可改寫成}} D^2y + 5Dy + 6y \xrightarrow{\text{可改寫成}} (D^2 + 5D + 6)y$$

可再改寫成

$$\boxed{\begin{array}{c} L(y) \\ L = D^2 + 5D + 6 \end{array}}$$

4.1.2.3 Solution of the Homogeneous Equation

[Theorem 4.1.5]

For an n^{th} order homogeneous linear DE $L(y) = 0$, if

① $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are the solutions of $L(y) = 0$

② $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are linearly independent

then any solution of the homogeneous linear DE can be expressed as:

$$\boxed{y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n}$$

可以和矩陣的概念相比較

From Theorem 4.1.5:

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

$y_1(t), y_2(t), \ldots\ldots, y_n(t)$: fundamental set of solutions

$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$: general solution of the homogenous linear DE

(又稱做 **complementary function**)

也是重要名詞

Definition 4.1 Linear Dependence / Independence

If there is no solution other than $c_1 = c_2 = \ldots\ldots = c_n = 0$ for the following equality

$$c_1 y_1(x) + c_2 y_2(x) + \cdots\cdots + c_n y_n(x) = 0$$

then $y_1(t)$, $y_2(t)$, ….., $y_n(t)$ are said to be linearly independent.

Otherwise, they are linearly dependent.

判斷是否為 linearly independent 的方法: **Wronskian**

Definition 4.2 Wronskian

$$W\left(y_1, y_2, \cdots\cdots, y_n\right) = \det\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

$$W\left(y_1, y_2, \cdots\cdots, y_n\right) \neq 0 \longrightarrow \text{linearly independent}$$

4.1.2.4 Examples

Example 9 (text page 127)

$$y''' - 6y'' + 11y' - 6y = 0$$

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$\det\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1, y_2, and y_3 are linear independent for any x

general solution:

$$\boxed{y = c_1e^x + c_2e^{2x} + c_3e^{3x}} \qquad x \in (-\infty, \infty)$$

4.1.3 Nonhomogeneous Equations (可和 page 55 相比較)

Nonhomogeneous linear DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

Part 1

Associated homogeneous DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)$$
$$+\cdots+a_1(x)y'(x)+a_0(x)y=0$$

find ***n* linearly independent solutions**

$$y_1(x),y_2(x),\cdots\cdots,y_n(x)$$

Part 2

particular solution y_p
(**any** solution of the nonhomogeneous linear DE)

$$g(x)=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$y_p(x)=y_{p_1}(x)+y_{p_2}(x)+\cdots\cdots+y_{p_k}(x)$$

general solution of the nonhomogeneous linear DE

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_ny_n(x)+y_p(x)$$

Theorem 4.1.6 general solution of a nonhomogeneous linear DE

$$y = y_c + y_p$$

general solution of the associated homogeneous function (complementary function)

particular solution (any solution) of the nonhomogeneous linear DE

$\oplus$

general solution of the nonhomogeneous linear DE

Example 10 (text page 128)

$$\boxed{y''' - 6y'' + 11y' - 6y = 3x}$$

$$y''' - 6y'' + 11y' - 6y = 0$$

Three linearly independent solution

$$e^x, \quad e^{2x}, \quad e^{3x}$$

Check by Wronskian (Example 9)

$$\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

Particular solution

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

General solution: $\boxed{y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x}$

Theorem 4.1.7 Superposition Principle

If $y_{p_1}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_1(x)$$

$y_{p_2}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_2(x)$$

$$\vdots$$

$y_{p_k}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_k(x)$$

then $\boxed{y_{p_1}(x)+y_{p_2}(x)+\cdots+y_{p_k}(x)}$ is the particular solution of

$$\boxed{\begin{aligned}&a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)\\&=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)\end{aligned}}$$

Example 11 (text page 129)

$y_{p_1}(x) = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2}(x) = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3}(x) = xe^{x}$ is a particular solution of $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

$\boxed{y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}}$ is a particular solution of

$$\boxed{y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}}$$

4.1.4 名詞

- initial conditions, boundary conditions (pages 139, 143)
  (重要名詞)
- associated homogeneous equation , complementary function (page 145)
  (重要名詞)
- fundamental set of solutions (page 148)
- Wronskian (page 150)
- particular solution (page 152)
- general solution of the homogenous linear DE (page 148)
- general solution of the nonhomogenous linear DE (page 152)

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_ny_n(x)+y_p(x)$$

4.1.5 本節要注意的地方

(1) Most of the theories in Section 4.1 are applied to the linear DE

(2) 注意 initial conditions 和 boundary conditions 之間的不同

(3) 快速判斷 linear independent

(補充 1) Theorem 4.1.1 的解釋

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0 \qquad y'(x_0)=y_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad y^{(n-1)}(x_0)=y_{n-1}$$

When $a_n(x_0) \neq 0$

$$y^{(n)}(x_0)+\frac{a_{n-1}(x_0)}{a_n(x_0)}y^{(n-1)}(x_0)+\cdots+\frac{a_1(x_0)}{a_n(x_0)}y'(x_0)+\frac{a_0(x)}{a_n(x_0)}y(x_0)=\frac{g(x_0)}{a_n(x_0)}$$

find $y^{(n)}(x_0)$

$$y^{(n-1)}(x_0+\Delta)=y^{(n-1)}(x_0)+y^{(n)}(x_0)\Delta$$

find $y^{(n-1)}(x_0+\Delta)$

(根據 $f'(t)=\dfrac{f(t+\Delta)-f(t)}{\Delta}$ ， $f(t+\Delta)=f(t)+f'(t)\Delta$)

以此類推

$$y^{(n-2)}(x_0+\Delta)=y^{(n-2)}(x_0)+y^{(n-1)}(x_0)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+\Delta)$$

$$y^{(n-3)}(x_0+\Delta)=y^{(n-3)}(x_0)+y^{(n-2)}(x_0)\Delta \longrightarrow \text{find } y^{(n-3)}(x_0+\Delta)$$

$$\vdots$$

$$y(x_0+\Delta)=y(x_0)+y'(x_0)\Delta \longrightarrow \boxed{\text{find } y(x_0+\Delta)}$$

$$\downarrow$$

$$y^{(n)}(x_0+\Delta)+\frac{a_{n-1}(x_0+\Delta)}{a_n(x_0+\Delta)}y^{(n-1)}(x_0+\Delta)+\cdots+\frac{a_1(x_0+\Delta)}{a_n(x_0+\Delta)}y'(x_0+\Delta)$$

$$+\frac{a_0(x+\Delta)}{a_n(x_0+\Delta)}y(x_0+\Delta)=\frac{g(x_0+\Delta)}{a_n(x_0+\Delta)} \longrightarrow \text{find } y^{(n)}(x_0+\Delta)$$

$$\downarrow$$

$$y^{(n-1)}(x_0+2\Delta)=y^{(n-1)}(x_0+\Delta)+y^{(n)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-1)}(x_0+2\Delta)$$

$$y^{(n-2)}(x_0+2\Delta)=y^{(n-2)}(x_0+\Delta)+y^{(n-1)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+2\Delta)$$

$$\vdots$$

$$y(x_0+2\Delta) = y(x_0+\Delta) + y'(x_0+\Delta)\Delta \longrightarrow \boxed{\text{find } y(x_0+2\Delta)}$$

$$\downarrow$$

$$y^{(n)}(x_0+2\Delta) + \frac{a_{n-1}(x_0+2\Delta)}{a_n(x_0+2\Delta)} y^{(n-1)}(x_0+2\Delta) + \cdots + \frac{a_1(x_0+2\Delta)}{a_n(x_0+2\Delta)} y'(x_0+2\Delta)$$
$$+ \frac{a_0(x+2\Delta)}{a_n(x_0+2\Delta)} y(x_0+2\Delta) = \frac{g(x_0+2\Delta)}{a_n(x_0+2\Delta)}$$

$$\downarrow$$

以此類推，可將 $y(x_0+3\Delta)$, $y(x_0+4\Delta)$, $y(x_0+5\Delta)$, ……………

以至於將 $y(x)$ 所有的值都找出來。

(求 $y(x)$ for $x > x_0$ 時, 用正的 Δ 值，

求 $y(x)$ for $x < x_0$ 時, 用負的 Δ 值)

Requirement 1: $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous 是為了讓 $a_k(x_0+m\Delta)$ 皆可以定義

Requirement 2: $a_n(x) \neq 0$ 是為了讓 $a_k(x_0+m\Delta)/a_n(x_0+m\Delta)$ 不為無限大

4-2 Reduction of Order

4.2.1 適用情形

(1) (2) (3)

Suitable for the 2nd order linear homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(4) One of the nontrivial solution $y_1(x)$ has been known.

4.2.2 解法

假設 $\boxed{y_2(x) = u(x)\,y_1(x)}$

先將DE 變成 Standard form

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

If $\boxed{y(x) = u(x)\,y_1(x)}$, (比較 Section 2-3)

$$y' = uy_1' + u'y_1 \qquad y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$uy_1'' + 2u'y_1' + u''y_1 + P(x)uy_1' + P(x)u'y_1 + Q(x)uy_1 = 0$$

$$u\underbrace{(y_1'' + P(x)\,y_1' + Q(x)\,y_1)}_{\text{zero}} + 2u'y_1' + u''y_1 + P(x)u'y_1 = 0$$

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\downarrow$$

$$\frac{dw}{dx}y_1 + w(2\frac{dy_1}{dx} + P(x)y_1) = 0$$

multiplied by $dx/(y_1 w)$

$$\downarrow$$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable
(with 3 variables)

$$\downarrow$$

$$\int\frac{dw}{w} + 2\int\frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\downarrow$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\downarrow$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$$\ln\left|wy_1^2\right| = -\int P(x)\,dx + c$$

$$wy_1^2 = \pm e^{-\int P(x)dx + c}$$

$$w = c_1 e^{-\int P(x)dx} / y_1^2$$

$$\boxed{u = \int w\,dx = c_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx + c_2}$$

$$\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx}$$

We can set $c_1 = 1$ and $c_2 = 0$

(因為我們算 $u(x)$ 的目的，只是為了要算出與 $y_1(x)$ 互相independent 的另一個解)

4.2.3 例子

Example 1 (text page 132)

$$\boxed{y'' - y = 0}$$

We have known that $y_1 = e^x$ is one of the solution

$P(x) = 0$ $\qquad y_2(x) = e^x \int ce^{-2x} dx = -\frac{1}{2} ce^{-x}$

Specially, set $c = -2$, ($y_2(x)$ 只要 independent of $y_1(x)$ 即可所以 c 的值可以任意設)

$$\boxed{y_2(x) = e^{-x}}$$

General solution: $\boxed{y(x) = c_1 e^x + c_2 e^{-x}}$

Example 2 (text page 133) (將課本 x 的範圍做更改)

$$\boxed{x^2 y'' - 3xy' + 4y = 0} \qquad \text{when } x \in (-\infty, 0)$$

We have known that $y_1 = x^2$ is one of the solution

Note: the interval of x

If $x \in (0, \infty)$ $(x > 0)$, $\int dx/x = \ln x$ 如課本

If $x < 0$, $\int dx/x = \ln(-x)$

$$y_2(x) = x^2 \int \frac{e^{3\ln(-x)}}{x^4} dx = x^2 \int \frac{(-x)^3}{(-x)^4} dx$$

$$= -x^2 \int \frac{1}{x} dx = -x^2 \ln|x|$$

$$\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln|x|}$$

4.2.4 本節需注意的地方

(1) 記住公式 $\boxed{y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx}$

(2) 若不背公式 (不建議)，在計算過程中別忘了對 $w(x)$ 做積分

(3) 別忘了 $P(x)$ 是 "standard form" 一次微分項的 coefficient term

(4) 同樣有 singular point 的問題

(5) 因為 $y_2(x)$ 是 homogeneous linear DE 的 "任意解"，所以計算時，常數的選擇以方便為原則

(6) 由於 $\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ 的計算較複雜且花時間，所以要多加練習

多算習題

附錄六： Hyperbolic Function

$$\sinh(x)=\frac{e^{x}-e^{-x}}{2}$$

比較： $\sin(x)=\dfrac{e^{jx}-e^{-jx}}{2j}$

$$\cosh(x)=\frac{e^{x}+e^{-x}}{2}$$

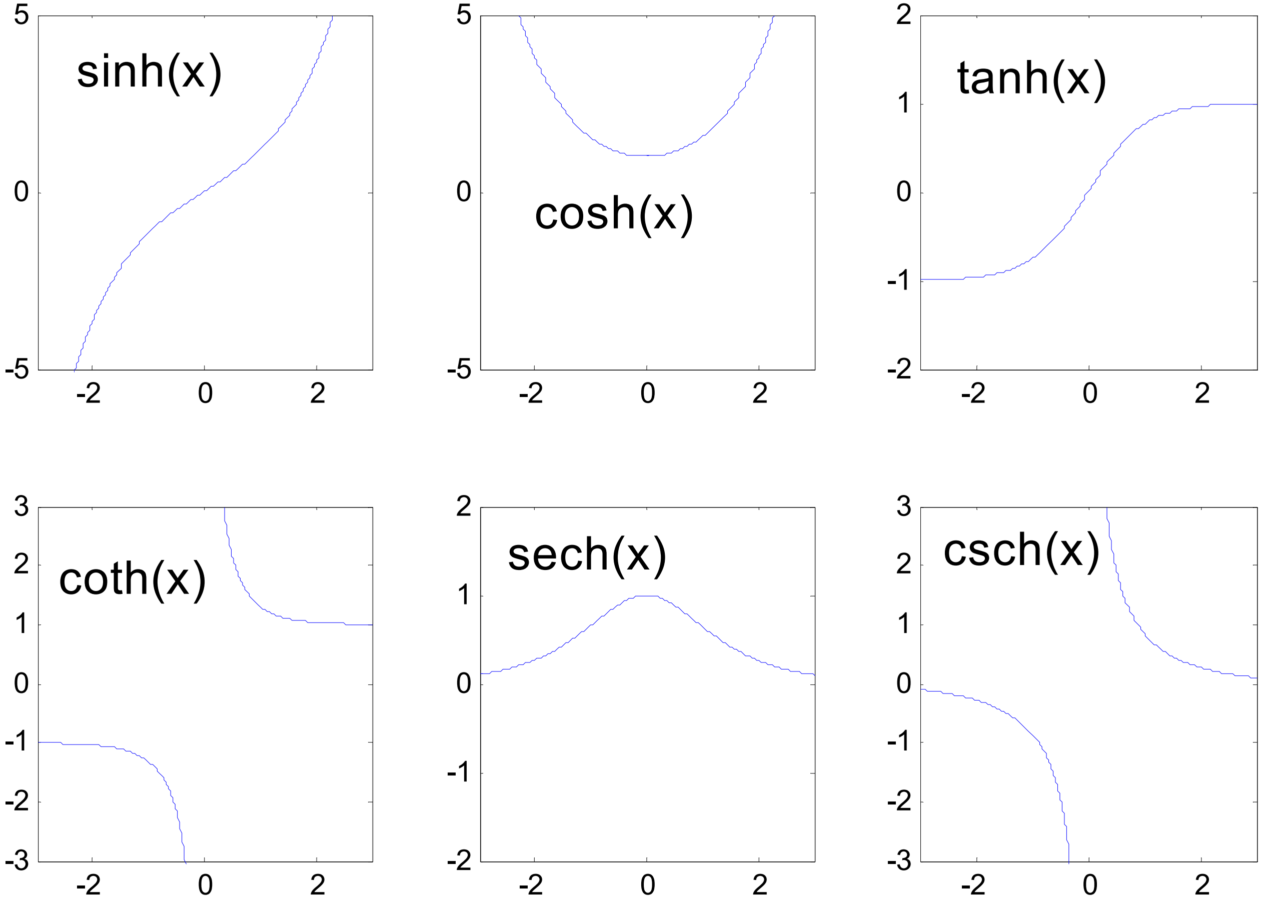
$$\cos(x)=\frac{e^{jx}+e^{-jx}}{2}$$

$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\coth(x)=\frac{\cosh(x)}{\sinh(x)}=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$$\operatorname{sech}(x)=\frac{1}{\cosh(x)}=\frac{2}{e^{x}+e^{-x}}$$

$$\operatorname{csch}(x)=\frac{1}{\sinh(x)}=\frac{2}{e^{x}-e^{-x}}$$


sinh(x)
cosh(x)
tanh(x)
coth(x)
sech(x)
csch(x)

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$

$$\sinh(0) = 0$$

$$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$

$$\cosh(0) = 1$$

$$\frac{d}{dx}\tanh(ax) = a\,\mathrm{sech}^2(ax)$$

$$\sinh'(0) = 1$$

$$\frac{d}{dx}\coth(ax) = -a\,\mathrm{csch}^2(ax)$$

$$\cosh'(0) = 0$$

$$\frac{d}{dx}\mathrm{sech}(ax) = -a\,\mathrm{sech}(ax)\tanh(ax)$$

$$\sin(ix) = i\sinh(x)$$

$$\frac{d}{dx}\mathrm{csch}(ax) = -a\,\mathrm{csch}(ax)\coth(ax)$$

$$\cos(ix) = \cosh(x)$$

$$\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a} + c$$

$$\int \cosh(ax)\,dx = \frac{\sinh(ax)}{a} + c$$

$$\int \tanh(ax)\,dx = \frac{\ln\left|\cosh(ax)\right|}{a} + c$$

$$\int \coth(ax)\,dx = \frac{\ln\left|\sinh(ax)\right|}{a} + c$$

$$\int \operatorname{sech}(ax)\,dx = \frac{2\tan^{-1}\left(\tanh(\frac{a}{2}x)\right)}{a} + c$$

$$\int \operatorname{csch}(ax)\,dx = \frac{\ln\left|\tanh(\frac{a}{2}x)\right|}{a} + c$$

附錄七 Linear DE 解法的步驟 (參照講義 page 152)

Step 1: Find the general solution (i.e., the complementary function) of the associated homogeneous DE

(Sections 4-2, 4-3, 4-7)

Step 2: Find the particular solution

(Sections 4-4, 4-5, 4-6)

Step 3: Combine the complementary function and the particular solution

Extra Step: Consider the initial (or boundary) conditions

4-3 Homogeneous Linear Equations with Constant Coefficients

本節使用 auxiliary equation 的方法來解 homogeneous DE

KK: [ɔg`zɪ lj ər ɪ]

4-3-1 限制條件

限制條件: (1) homogeneous

(2) linear

(3) constant coefficients

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

a_0, a_1, a_2, ... , a_n are constants

(the simplest case of the higher order DEs)

4-3-2 解法

解法核心：

Suppose that the solutions has the form of e^{mx}

Example: $y''(x) - 3y'(x) + 2y(x) = 0$

Set $y(x) = e^{mx}$, $m^2 e^{mx} - 3m e^{mx} + 2e^{mx} = 0$

$$m^2 - 3m + 2 = 0 \longrightarrow \text{solve } m$$

可以直接把 <u>n 次微分</u>用 m^n 取代，變成一個多項式

這個多項式被稱為 auxiliary equation

- **解法流程**

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

Step 1-1
auxiliary function

$$\boxed{a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0}$$

Step 1-1 Find n roots , m_1, m_2, m_3, …., m_n

(If m_1, m_2, m_3, …., m_n are distinct)

Step 1-2 n linearly independent solutions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots\ldots, e^{m_n x}$

(有三個 Cases)

Step 1-3 Complementary function

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}}$$

4-3-3 Three Cases for Roots (2nd Order DE)

$$\boxed{a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0}$$

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$

roots $m_1 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$ $\qquad m_2 = \dfrac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$

solutions

Case 1 $m_1 \neq m_2$, m_1, m_2 are real (其實 m_1, m_2 不必限制為 real)

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case 2 $m_1 = m_2$ (m_1 and m_2 are of course real)

First solution: $\boxed{y_1 = e^{m_1 x}}$

Second solution: using the method of "Reduction of Order"

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

$$= e^{m_1 x}\int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx$$

$$= e^{m_1 x}\int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \qquad m_1 = \frac{-a_1}{2a_2}$$

$$= e^{m_1 x}\int dx = e^{m_1 x}(x + c)$$

$$\boxed{y_2(x) = xe^{m_1 x}}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

$$\alpha = -a_1 / 2a_2, \qquad \beta = \sqrt{4a_2 a_0 - a_1^2} / 2a_2$$

Solution: $\boxed{y = C_1 e^{\alpha x + j\beta x} + C_2 e^{\alpha x - j\beta x}}$

Another form:

$$y = e^{\alpha x}\left(C_1 e^{j\beta x} + C_2 e^{-j\beta x}\right)$$

$$= e^{\alpha x}\left(C_1 \cos\beta x + jC_1 \sin\beta x + C_2 \cos\beta x - jC_2 \sin\beta x\right)$$

set $c_1 = C_1 + C_2$ and $c_2 = jC_1 - jC_2$

$\boxed{y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)}$ c_1 and c_2 are some constant

Example 1 (text page 137)

(a) $\boxed{2y'' - 5y' - 3y = 0}$

$$2m^2 - 5m - 3 = 0, \quad m_1 = -1/2, \; m_2 = 3$$

$$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$$

(b) $\boxed{y'' - 10y' + 25y = 0}$

$$m^2 - 10m + 25 = 0, \quad m_1 = 5, \; m_2 = 5$$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

(c) $\boxed{y'' + 4y' + 7y = 0}$

$$m^2 + 4m + 7 = 0, \quad m_1 = -2 + i\sqrt{3}, \quad m_2 = -2 - i\sqrt{3}$$

$$\boxed{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}$$

4-3-4 Three + 1 Cases for Roots (Higher Order DE)

For higher order case $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$

auxiliary function: $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$

roots: $m_1, m_2, m_3, \ldots, m_n$

(1) If $m_p \neq m_q$ for $p = 1, 2, \ldots, n$ and $p \neq q$

(也就是這個多項式在 m_q 的地方只有一個根)

then $\boxed{e^{m_q x}}$ is a solution of the DE.

重覆次數

(2) If the multiplicities of m_q is k (當這個多項式在 m_q 的地方有 k 個根),

$$\boxed{e^{m_q x}, xe^{m_q x}, x^2 e^{m_q x}, \cdots\cdots, x^{k-1} e^{m_q x}}$$

are the solutions of the DE.

(3) If both $\alpha + j\beta$ and $\alpha - j\beta$ are the roots of the auxiliary function,

then

$$\boxed{e^{\alpha x}\cos(\beta x),\, e^{\alpha x}\sin(\beta x)}$$

are the solutions of the DE.

(4) If the multiplicities of $\alpha + j\beta$ is k and the multiplicities of $\alpha - j\beta$ is also k， then

$$\boxed{\begin{array}{l} e^{\alpha x}\cos(\beta x),\, xe^{\alpha x}\cos(\beta x),\, x^2e^{\alpha x}\cos(\beta x),\cdots\cdots,\, x^{k-1}e^{\alpha x}\cos(\beta x) \\ e^{\alpha x}\sin(\beta x),\, xe^{\alpha x}\sin(\beta x),\, x^2e^{\alpha x}\sin(\beta x),\cdots\cdots,\, x^{k-1}e^{\alpha x}\sin(\beta x) \end{array}}$$

are the solutions of the DE.

Note: If $\alpha + j\beta$ is a root of a real coefficient polynomial,

then $\alpha - j\beta$ is also a root of the polynomial.

$$a_n(\alpha + j\beta)^n + a_{n-1}(\alpha + j\beta)^{n-1} + \cdots + a_1(\alpha + j\beta) + a_0 = 0$$

$a_0, a_1, a_2, \ldots, a_n$ are real

Example 3 (text page 138)

Solve $\boxed{y''' + 3y'' - 4y = 0}$

$$\downarrow$$

Step 1-1 $m^3 + 3m^2 - 4 = 0$

$$(m-1)(m^2+4m+4) = 0 \qquad \boxed{m_1 = 1, \quad m_2 = m_3 = -2}$$

Step 1-2 3 independent solutions: e^x , e^{-2x} , xe^{-2x}

Step 1-3 general solution: $\boxed{y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}}$

Example 4 (text page 138)

Solve $\boxed{y^{(4)}(x)+2y''(x)+y(x)=0}$

Step 1-1 $m^4+2m^2+1=0$

$(m^2+1)^2=0$ $\boxed{\text{four roots: } \boldsymbol{i},\ \boldsymbol{i},\ -\boldsymbol{i},\ -\boldsymbol{i}}$

Step 1-2 4 independent solutions: $\cos x$, $x\cos x$, $\sin x$, $x\sin x$

Step 1-3 general solution: $\boxed{y=c_1\cos x+c_2x\cos x+c_3\sin x+c_4x\sin x}$

4-3-5 How to Find the Roots

(1) Formulas

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0} \Longrightarrow m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \qquad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

$$\boxed{a_3 m^3 + a_2 m^2 + a_1 m + a_0}$$

$$\Longrightarrow m_1 = S + T - \frac{a_2}{3a_3}$$

Solutions:

$$m_2 = -\frac{1}{2}(S+T) - \frac{a_2}{3a_3} + i\frac{1}{2}\sqrt{3}(S-T)$$

$$m_3 = -\frac{1}{2}(S+T) - \frac{a_2}{3a_3} - i\frac{1}{2}\sqrt{3}(S-T)$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^3}} \quad , \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^3}}$$

$$Q = \frac{1a_1}{3a_3} - \frac{a_2^2}{9a_3^2} \quad , \quad R = \frac{9a_1 a_2 - 27a_3 a_0 - 2a_2^3}{54a_3^2}$$

(太複雜了)

(2) Observing

例如：1 是否為 root $\longrightarrow$ 看係數和是否為 0

又如：

$$3m^3 + 5m^2 + 10m - 4$$

factor: 1,3 factor: 1,2,4

possible roots: ±1, ±2, ±4, ±1/3, ±2/3, ±4/3

test for each possible root $\longrightarrow$ find that 1/3 is indeed a root

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \frac{1}{3}\right)\left(3m^2 + 6m + 12\right)$$

(3) Solving the roots of a polynomial by software

Maple

Mathematica (by the commands of **Nsolve** and **FindRoot**)

Matlab (by the command of **roots**)

4-3-6 本節需注意的地方

(1) 注意重根和 conjugate complex roots 的情形

(2) 寫解答時，要將 "General solution" 寫出來

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

(3) 因式分解要熟練

(4) 本節的方法，也適用於 1^{st} order 的情形

練習題

Sec. 4-1: 3, 7, 8, 10, 13, 20, 24, 29, 33, 36

Sec. 4-2: 2, 4, 9, 13, 14, 16, 18, 19

Sec. 4-3: 7, 16, 20, 22, 24, 28, 33, 39, 41, 52, 54, 56, 59, 61, 63