

4-6 Variation of Parameters

4-6-1 方法的限制

The method can solve the particular solution for **any linear DE**

(1) May not have constant coefficients

(2) $g(x)$ may not be of the special forms

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

4-6-2 Case of the 2nd order linear DE

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y = g(x)$$

associated homogeneous equation: $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y = 0$

Suppose that the solution of the associated homogeneous equation is

$$c_1y_1(x) + c_2y_2(x)$$

Then the **particular solution** is assumed as:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

(方法的基本精神)

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

代入原式後，總是可以簡化

$$y'_p = u'_1y_1 + u_1y'_1 + u'_2y_2 + u_2y'_2$$

$$y''_p = u''_1y_1 + 2u'_1y'_1 + u_1y''_1 + u''_2y_2 + 2u'_2y'_2 + u_2y''_2$$

$$\text{代入 } y''(x) + P(x)y'(x) + Q(x)y = f(x)$$

$$P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)}, \quad f(x) = \frac{g(x)}{a_2(x)}$$

$$y''_p + P(x)y'_p + Q(x)y_p = u_1 \overset{\text{zero}}{[y''_1 + Py'_1 + Qy_1]} + u_2 \overset{\text{zero}}{[y''_2 + Py'_2 + Qy_2]} + y_1u''_1 + 2u'_1y'_1 + y_2u''_2 + 2u'_2y'_2 + P[y_1u'_1 + y_2u'_2]$$

$$y_p'' + P(x)y_p' + Q(x)y_p = f(x), \quad \text{代入 } y_p = u_1y_1 + u_2y_2$$

↓ 簡化

$$y_p'' + P(x)y_p' + Q(x)y_p = y_1u_1'' + 2u_1'y_1' + y_2u_2'' + 2u_2'y_2' + P[y_1u_1' + y_2u_2']$$

$$\frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x)$$

進一步簡化：

$$\text{假設 } y_1u_1' + y_2u_2' = 0$$

$$y_1'u_1' + y_2'u_2' = f(x)$$

聯立方程式

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases}$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1 + y_2' u_2 = f(x) \end{cases} \longrightarrow \begin{cases} u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \\ u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \end{cases} \longrightarrow \begin{cases} u_1 = \int u_1'(x) dx \\ u_2 = \int u_2'(x) dx \end{cases}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

| | : determinant

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

可以和 1st order case (page 62) 相比較

Determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Matrix Inverse

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{ab - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{ab - bc} \begin{bmatrix} dy_1 - by_2 \\ -cy_1 + ay_2 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{\begin{vmatrix} y_1 & b \\ y_2 & d \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{\begin{vmatrix} a & y_1 \\ b & y_2 \end{vmatrix}}$$

4-6-3 Process for the 2nd Order Case

Step 2-1 變成 standard form

$$y''(x) + P(x)y'(x) + Q(x)y = f(x)$$

Step 2-2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 2-3

$$u_1' = \frac{W_1}{W}$$

$$u_2' = \frac{W_2}{W}$$

Step 2-4

$$u_1 = \int u_1'(x) dx$$

$$u_2 = \int u_2'(x) dx$$

Step 2-5

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

4-6-4 Examples

Example 1 (text page 162)

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: solution of $y'' - 4y' + 4y = 0$:

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Step 2-2: $y_p = u_1 y_1 + u_2 y_2$, $y_1 = e^{2x}$, $y_2 = x e^{2x}$

$$W = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & 2x e^{2x} + e^{2x} \end{vmatrix} = e^{4x} \quad W_1 = \begin{vmatrix} 0 & x e^{2x} \\ (x+1)e^{2x} & 2x e^{2x} + e^{2x} \end{vmatrix} = -(x+1)x e^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 2-3: $u_1' = \frac{W_1}{W} = -x^2 - x$ $u_2' = \frac{W_2}{W} = x + 1$

Step 2-4: $u_1 = \int u_1' dx = \int (-x^2 - x) dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + \cancel{c_1}$

$$u_2 = \int u_2' dx = \int (x+1) dx = \frac{1}{2}x^2 + x + \cancel{c_2}$$

Step 2-5: $y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}$

Step 3: $y = c_1e^{2x} + c_2xe^{2x} + \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}$

Example 2 (text page 163) $4y'' + 36y = \csc 3x$

Step 1: solution of $4y'' + 36y = 0$: $y_c = c_1 \cos 3x + c_2 \sin 3x$

Step 2-1: standard form: $y'' + 9y = \csc 3x / 4$ $f(x) = \csc 3x / 4$

Step 2-2: $W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$ $W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -1/4$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4} \frac{\cos 3x}{\sin 3x}$$

Step 2-3: $u_1' = \frac{W_1}{W} = -\frac{1}{12}$ $u_2' = \frac{W_2}{W} = \frac{1}{12} \frac{\cos 3x}{\sin 3x}$

Step 2-4: $u_1 = -\frac{x}{12}$ $u_2 = \frac{1}{36} \ln |\sin 3x|$

(未完待續)

注意 $\frac{1}{12} \int \frac{\cos 3x}{\sin 3x} dx$ 算法

Step 2-5: $y_p = -\frac{x}{12} \cos 3x + \frac{1}{36} \sin 3x \ln |\sin 3x|$

Step 3: $y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12} \cos 3x + \frac{1}{36} \sin 3x \ln |\sin 3x|$

Note: 課本 Interval $(0, \pi/6)$ 應該改為 $(0, \pi/3)$

Example 3 (text page 164) $y'' - y = 1/x$

$$y_c = c_1 e^x + c_2 e^{-x} \quad f(x) = 1/x$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Note: $\int \frac{e^x}{x} dx$ 沒有 analytic 的解

所以直接表示成 $\int_{x_0}^x \frac{e^t}{t} dt$ (複習 page 48)

4-6-5 Case of the Higher Order Linear DE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

Solution of the associated homogeneous equation:

$$y_c = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + \cdots + c_ny_n(x)$$

The **particular solution** is assumed as:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x) + \cdots + u_n(x)y_n(x)$$

$$u'_k(x) = \frac{W_k}{W} \implies u_k(x) = \int u'_k(x) dx$$

$$u'_k(x) = \frac{W_k}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y'_1 & y'_2 & y'_3 & \cdots & y'_n \\ y''_1 & y''_2 & y''_3 & \cdots & y''_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W_k = \begin{vmatrix} y_1 & y_2 & \cdots & y_{k-1} & 0 & y_{k+1} & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_{k-1} & 0 & y'_{k+1} & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{k-1}^{(n-2)} & 0 & y_{k+1}^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{k-1}^{(n-1)} & f(x) & y_{k+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$f(x) = g(x) / a_n(x)$$

W_k : replace the k^{th} column of W by

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

$$f(x) = \frac{g(x)}{a_n(x)}$$

For example, when $n = 3$,

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

4-6-6 Process of the Higher Order Case

Step 2-1 變成 standard form

$$y^{(n)}(x) + \frac{a_{n-1}(x)}{a_n(x)} y^{(n-1)}(x) + \cdots + \frac{a_1(x)}{a_n(x)} y'(x) + \frac{a_0(x)}{a_n(x)} y = \frac{g(x)}{a_n(x)}$$

Step 2-2 Calculate W, W_1, W_2, \dots, W_n (see page 247)

Step 2-3 $u'_1 = \frac{W_1}{W}$ $u'_2 = \frac{W_2}{W}$ $u'_n = \frac{W_n}{W}$

Step 2-4 $u_1 = \int u'_1(x) dx$ $u_2 = \int u'_2(x) dx$ $u_n = \int u'_n(x) dx$

Step 2-5 $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$

Exercise 26 $y''' + 4y' = \sec 2x$

Complementary function: $y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2 \sin 2x & 2 \cos 2x \\ 0 & -4 \cos 2x & -4 \sin 2x \end{vmatrix} = 8$$

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2 \sin 2x & 2 \cos 2x \\ \sec 2x & -4 \cos 2x & -4 \sin 2x \end{vmatrix} = 2 \sec 2x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2 \cos 2x \\ 0 & \sec 2x & -4 \sin 2x \end{vmatrix} = -2 \quad W_3 = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2 \sin 2x & 0 \\ 0 & -4 \cos 2x & \sec 2x \end{vmatrix} = -2 \tan 2x$$

$$u_1' = \frac{W_1}{W} = \frac{\sec 2x}{4} \quad u_2' = \frac{W_2}{W} = \frac{-1}{4} \quad u_3' = \frac{W_3}{W} = \frac{-\tan 2x}{4}$$

$$u_1 = \frac{1}{8} \ln |\sec 2x + \tan 2x| \quad u_2 = \frac{-x}{4} \quad u_3 = \frac{1}{8} \ln |\cos 2x|$$

$$y(x) = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

$$+ \frac{1}{8} \ln |\sec 2x + \tan 2x| + \frac{-x}{4} \cos 2x + \frac{1}{8} (\ln |\cos 2x|) \sin 2x$$

for $-\pi/4 < x < \pi/4$

Note: $-\pi/4$, $\pi/4$ are singular points

4-6-7 本節需注意的地方

(1) 養成先解 associated homogeneous equation 的習慣

(2) 記熟幾個重要公式

(3) 這裡 $||$ 指的是 determinant

(4) 算出 $u_1'(x)$ 和 $u_2'(x)$ 後別忘了作積分

特別要小心

(5) $f(x) = g(x)/a_n(x)$ (和 1st order 的情形一樣，使用 standard form)

(6) 計算 $u_1'(x)$ 和 $u_2'(x)$ 的積分時， $+c$ 可忽略

因為我們的目的是算 particular solution y_p

y_p 是任何一個能滿足原式的解

(7) 這方法解的範圍，不包含 $a_n(x) = 0$ 的地方

4-7 Cauchy-Euler Equation

4-7-1 解法限制條件

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = g(x)$$

not constant coefficients

but the coefficients of $y^{(k)}(x)$ have the form of $a_k x^k$

a_k is some constant

associated homogeneous
equation

particular solution

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

4-7-2 解法

Associated homogeneous equation of the Cauchy-Euler equation

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

Guess the solution as $y(x) = x^m$, then

$$\begin{aligned} & a_n x^n m(m-1)(m-2)\cdots(m-n+1)x^{m-n} + \\ & a_{n-1} x^{n-1} m(m-1)(m-2)\cdots(m-n+2)x^{m-n+1} + \\ & a_{n-2} x^{n-2} m(m-1)(m-2)\cdots(m-n+3)x^{m-n+2} + \\ & \quad \vdots \\ & + a_1 x m x^{m-1} \\ & + a_0 x^m = 0 \end{aligned}$$

$$\begin{aligned}
 & a_n m(m-1)(m-2)\cdots(m-n+1) \\
 & + a_{n-1} m(m-1)(m-2)\cdots(m-n+2) \\
 & + a_{n-2} m(m-1)(m-2)\cdots(m-n+3) \\
 & \quad \vdots \\
 & + a_1 m \\
 & + a_0 = 0
 \end{aligned}$$

⇒ auxiliary function

比較: 和 constant coefficient
時有何不同?

規則: 把 $x^k \frac{d^k}{dx^k}$ 變成 $\frac{m!}{(m-k)!}$

4-7-3 For the 2nd Order Case

$$a_2x^2y''(x) + a_1xy'(x) + a_0y = 0$$

auxiliary function:

$$a_2m(m-1) + a_1m + a_0 = 0$$

$$a_2m^2 + (a_1 - a_2)m + a_0 = 0$$

roots

$$m_1 = \frac{a_2 - a_1 + \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2} \quad m_2 = \frac{a_2 - a_1 - \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2}$$

[Case 1]: $m_1 \neq m_2$ and m_1, m_2 are real

two independent solution of the homogeneous part:

$$x^{m_1} \quad \text{and} \quad x^{m_2}$$

$$y_c = c_1x^{m_1} + c_2x^{m_2}$$

[Case 2]: $m_1 = m_2$

Use the method of reduction of order

$$y_1 = x^{m_1}$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx = x^{m_1} \int \frac{e^{-\int \frac{a_1}{a_2 x} dx}}{x^{2m_1}} dx$$

Note 1: 原式 $\longrightarrow y''(x) + \frac{a_1}{a_2 x} y'(x) + \frac{a_0}{a_2 x^2} y = 0, \quad P(x) = \frac{a_1}{a_2 x}$

Note 2: 此時 $m_1 = m_2 = \frac{a_2 - a_1}{2a_2}$

$$\begin{aligned}
 y_2(x) &= x^{m_1} \int \frac{e^{-\int \frac{a_1}{a_2 x} dx}}{x^{2m_1}} dx = x^{m_1} \int \frac{e^{\frac{-a_1 \ln|x|}{a_2}}}{x^{2m_1}} dx = x^{m_1} \int \frac{|x|^{-\frac{a_1}{a_2}}}{x^{2m_1}} dx \\
 &= (\pm 1)^{\frac{-a_1}{a_2}} x^{m_1} \int x^{-\frac{a_1}{a_2} - 2m_1} dx = x^{m_1} \int x^{-1} dx = x^{m_1} \ln|x|
 \end{aligned}$$

If $y_2(x)$ is a solution of a homogeneous DE

then $c y_2(x)$ is also a solution of the homogeneous DE

If we constrain that $x > 0$, then $y_2 = x^{m_1} \ln x$

$$y_c = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

[Case 3]: $m_1 \neq m_2$ and m_1, m_2 are the form of

$$m_1 = \alpha + j\beta \quad m_2 = \alpha - j\beta$$

two independent solution of the homogeneous part:

$$x^{\alpha+j\beta} \quad \text{and} \quad x^{\alpha-j\beta}$$

$$y_c = C_1 x^{\alpha+j\beta} + C_2 x^{\alpha-j\beta}$$

$$x^{\alpha+j\beta} = (e^{\ln x})^{\alpha+j\beta} = e^{(\alpha+j\beta)\ln x} = e^{\alpha \ln x} e^{j\beta \ln x}$$

$$= x^\alpha (\cos(\beta \ln x) + j \sin(\beta \ln x))$$

$$\text{同理 } x^{\alpha-j\beta} = x^\alpha (\cos(\beta \ln x) - j \sin(\beta \ln x))$$

$$y_c = x^\alpha [(C_1 + C_2) \cos(\beta \ln x) + j(C_1 - C_2) \sin(\beta \ln x)]$$

$$y_c = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

Example 1 (text page 167)

$$x^2 y''(x) - 2xy'(x) - 4y = 0$$

Example 2 (text page 168)

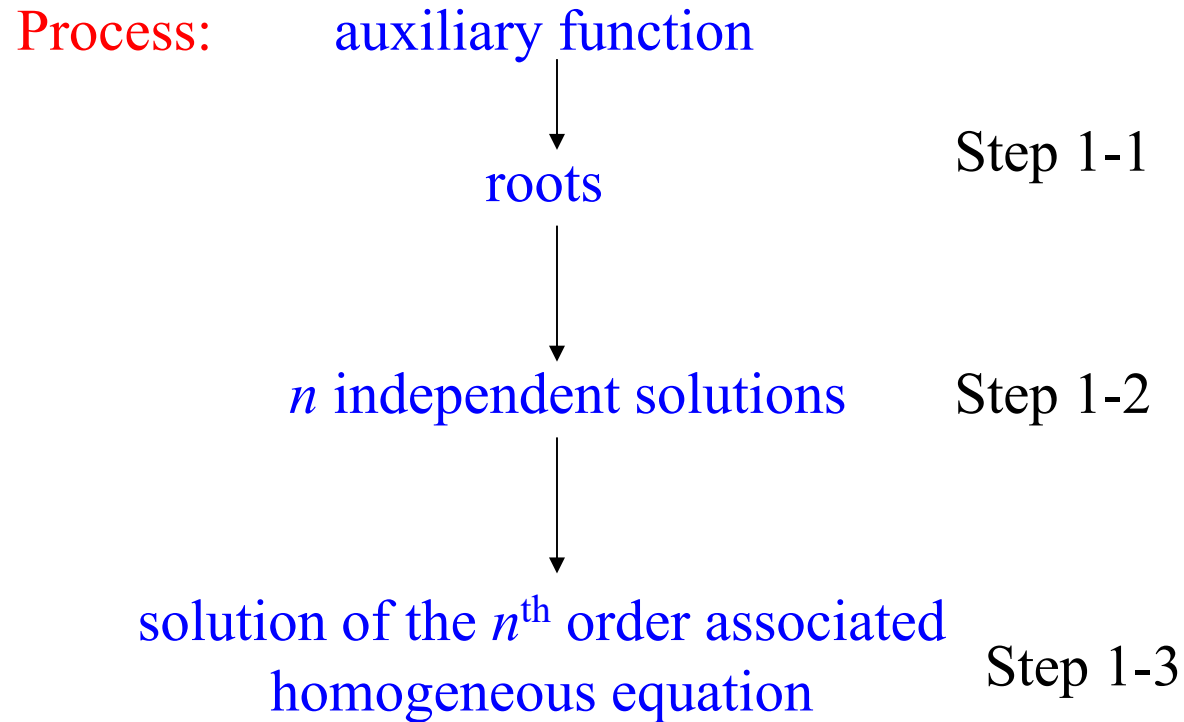
$$4x^2 y''(x) + 8xy'(x) + y = 0$$

Example 3 (text page 169)

$$4x^2 y''(x) + 17y = 0$$

$$y(1) = -1 \quad y'(1) = -\frac{1}{2}$$

4-7-4 For the Higher Order Case



(1) 若 auxiliary function 在 m_0 的地方只有一個根

$$x^{m_0}$$

是 associated homogeneous equation 的其中一個解

(2) 若 auxiliary function 在 m_0 的地方有 k 個重根

$$x^{m_0}, x^{m_0} \ln x, x^{m_0} (\ln x)^2, \dots, x^{m_0} (\ln x)^{k-1}$$

皆為 associated homogeneous equation 的解

(3) 若 auxiliary function 在 $\alpha + j\beta$ 和 $\alpha - j\beta$ 的地方各有一個根
(未出現重根)

$$x^\alpha \cos(\beta \ln x), \quad x^\alpha \sin(\beta \ln x)$$

是 associated homogeneous equation 的其中二個解

(4) 若 auxiliary function 在 $\alpha + j\beta$ 和 $\alpha - j\beta$ 的地方皆有 k 個重根

$$x^\alpha \cos(\beta \ln x), \quad x^\alpha \cos(\beta \ln x) \ln x, \quad x^\alpha \cos(\beta \ln x) (\ln x)^2, \quad \dots, \quad x^\alpha \cos(\beta \ln x) (\ln x)^{k-1}$$

$$x^\alpha \cos(\beta \ln x) (\ln x)^{k-1}$$

$$x^\alpha \sin(\beta \ln x), \quad x^\alpha \sin(\beta \ln x) \ln x, \quad x^\alpha \sin(\beta \ln x) (\ln x)^2, \quad \dots, \quad x^\alpha \sin(\beta \ln x) (\ln x)^{k-1}$$

$$x^\alpha \sin(\beta \ln x) (\ln x)^{k-1}$$

是 associated homogeneous equation 的其中 $2k$ 個解

Example 4 (text page 169)

$$x^3 y'''(x) + 5x^2 y''(x) + 7xy'(x) + 8y = 0$$

auxiliary function

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

$$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$$

$$m^3 + 2m^2 + 4m + 8 = 0$$

$$(m+2)(m^2+4) = 0$$

4-7-5 Nonhomogeneous Case

To solve the nonhomogeneous Cauchy-Euler equation:

Method 1: (See Example 5)

(1) Find the complementary function (general solutions of the associated homogeneous equation) from the rules on pages 255-258, 262-263.

(2) Use the method in Sec.4-6 (Variation of Parameters) to find the particular solution.

(3) Solution = complementary function + particular solution

Method 2: See Example 6 , 很重要

$$\text{Set } x = e^t, \quad t = \ln x$$

Example 5 (text page 169, illustration for method 1)

$$x^2 y''(x) - 3xy'(x) + 3y = 2x^4 e^x$$

Step 1 solution of the associated homogeneous equation

auxiliary function

$$m(m-1) - 3m + 3 = 0 \quad m^2 - 4m + 3 = 0 \quad m_1 = 1$$

$$m_2 = 3$$

$$y_c = c_1 x + c_2 x^3$$

Step 2-2 Particular solution $W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2 e^x & 3x^2 \end{vmatrix} = -2x^5 e^x \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2 e^x \end{vmatrix} = 2x^3 e^x$$

Step 2-3 $u'_1 = \frac{W_1}{W} = -x^2 e^x \quad u'_2 = \frac{W_2}{W} = e^x$

Step 2-4 $u_1 = \int u_1' dx = -x^2 e^x + 2x e^x - 2e^x$

$$u_2 = \int u_2' dx = e^x$$

Step 2-5 $y_p = u_1 y_1 + u_2 y_2 = 2x^2 e^x - 2x e^x$

Step 3 $y = c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x$

Example 6 (text page 170, illustration for method 2)

$$x^2 y''(x) - xy'(x) + y = \ln x$$

Set $x = e^t$, $t = \ln x$

$$\frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = \frac{1}{x} \frac{dy}{dt} \quad (\text{chain rule})$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dt}{dx} \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{1}{x} \frac{d}{dt} \left(\frac{1}{x} \frac{dy}{dt} \right) \\ &= \frac{1}{x^2} \frac{d^2 y}{dt^2} + \frac{1}{x} \left(\frac{d}{dt} \frac{1}{x} \right) \left(\frac{dy}{dt} \right) = \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \end{aligned}$$

Therefore, the original equation is changed into

$$\frac{d^2}{dt^2} y(t) - 2 \frac{d}{dt} y(t) + y(t) = t$$

$$\frac{d^2}{dt^2} y(t) - 2 \frac{d}{dt} y(t) + y(t) = t$$

$$\Rightarrow y(t) = c_1 e^t + c_2 t e^t + t + 2$$

$$\Rightarrow y(x) = c_1 x + c_2 x \ln x + \ln x + 2 \quad (\text{別忘了 } t = \ln x \text{ 要代回來})$$

Note 1: 以此類推

$$x^k \frac{d^k y}{dx^k} = (D_t - k + 1) \cdots (D_t - 1) D_t y$$

D_t means $\frac{d}{dt}$

Note 2: 簡化計算的小技巧: 使用 Cauchy-Euler equation 的 auxiliary function

4-7-6 本節要注意的地方

(1) 本節公式記憶的方法：

把 Section 4-3 的 e^x 改成 x ， x 改成 $\ln(x)$

把 auxiliary function 的 m^n 改成 $m(m-1)(m-2)\cdots(m-n+1)$

(2) 如何解 particular solution?

Variation of Parameters 的方法

(3) 解的範圍將不包括 $x=0$ 的地方 (Why?)

還有很多 linear DE 沒有辦法解，怎麼辦

- (1) numerical approach (Section 4-9-3)
- (2) using special function (Chap. 6)
- (3) Laplace transform and Fourier transform (Chaps. 7, 11, 14)
- (4) 查表 (table lookup)

- (1) 即使用了 Section 4-7 的方法，大部分的 DE 還是沒有辦法解
- (2) 所幸，自然界真的有不少的例子是 linear DE
甚至是 constant coefficient linear DE

Chapter 5 Modeling with Higher Order Differential Equations

Chapter 4 的應用題

自然界，有不少的系統可以用 linear DE 來表示

其中有不少的系統可以進一步簡化成 linear DE with constant coefficients

5-1 Linear Models: Initial Value Problem

位置： x ， 速度： $\frac{d}{dt}x$ 加速度 $\frac{d^2}{dt^2}x$

$$F = ma \longrightarrow F = m \frac{d^2 x}{dt^2}$$

$$F - \beta v = ma \longrightarrow F - \beta \frac{dx}{dt} = m \frac{d^2 x}{dt^2}$$

v : 速度， βv : 磨擦力

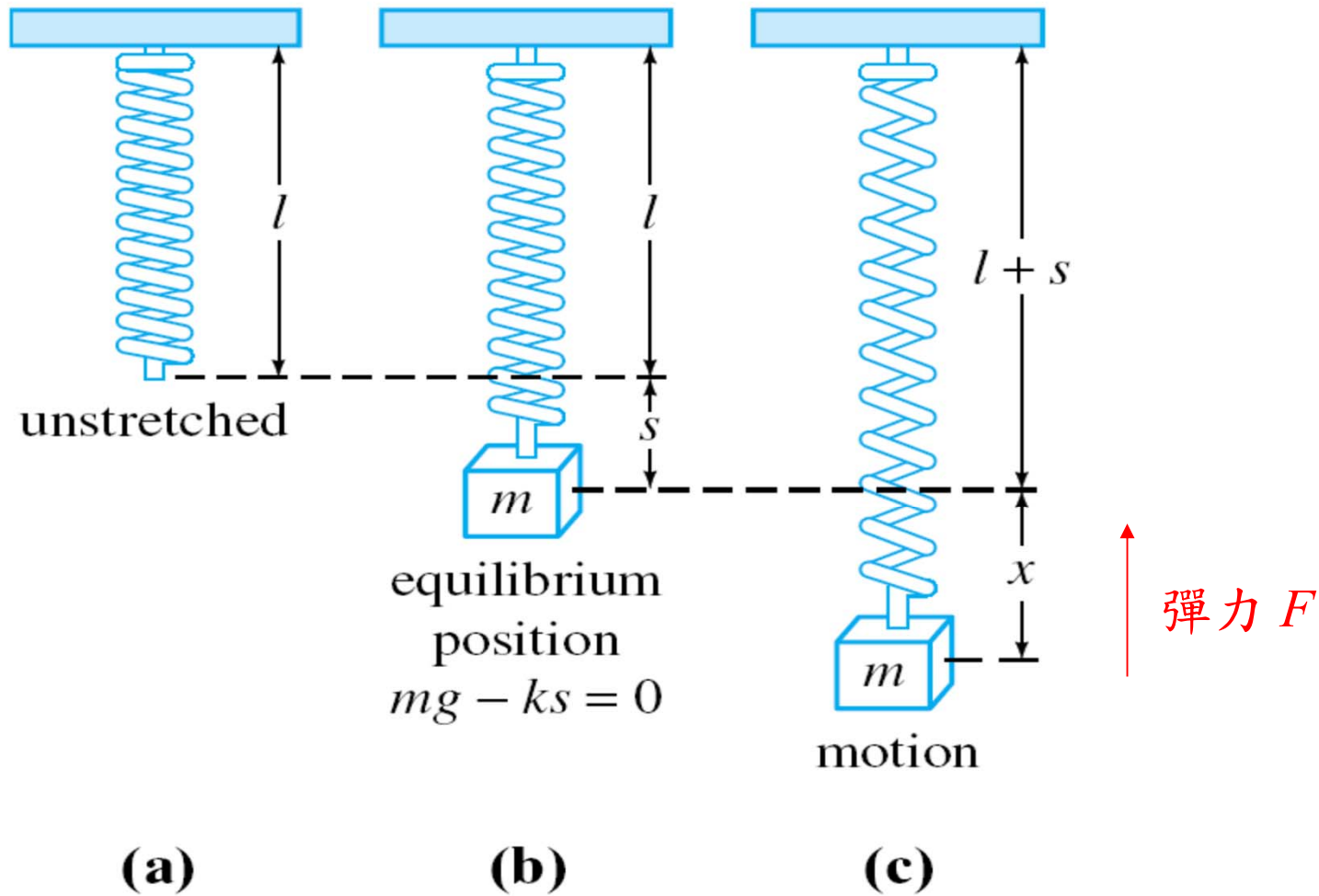


Figure 5.1.1 Spring/mass system

$$m \frac{d^2 x}{dt^2} = F \longrightarrow m \frac{d^2 x}{dt^2} = -kx$$

$$m \frac{d^2 x}{dt^2} + kx = 0$$

$$\text{Solution: } x(t) = c_1 \cos \omega t + c_2 \sin \omega t \quad \omega = \sqrt{\frac{k}{m}}$$

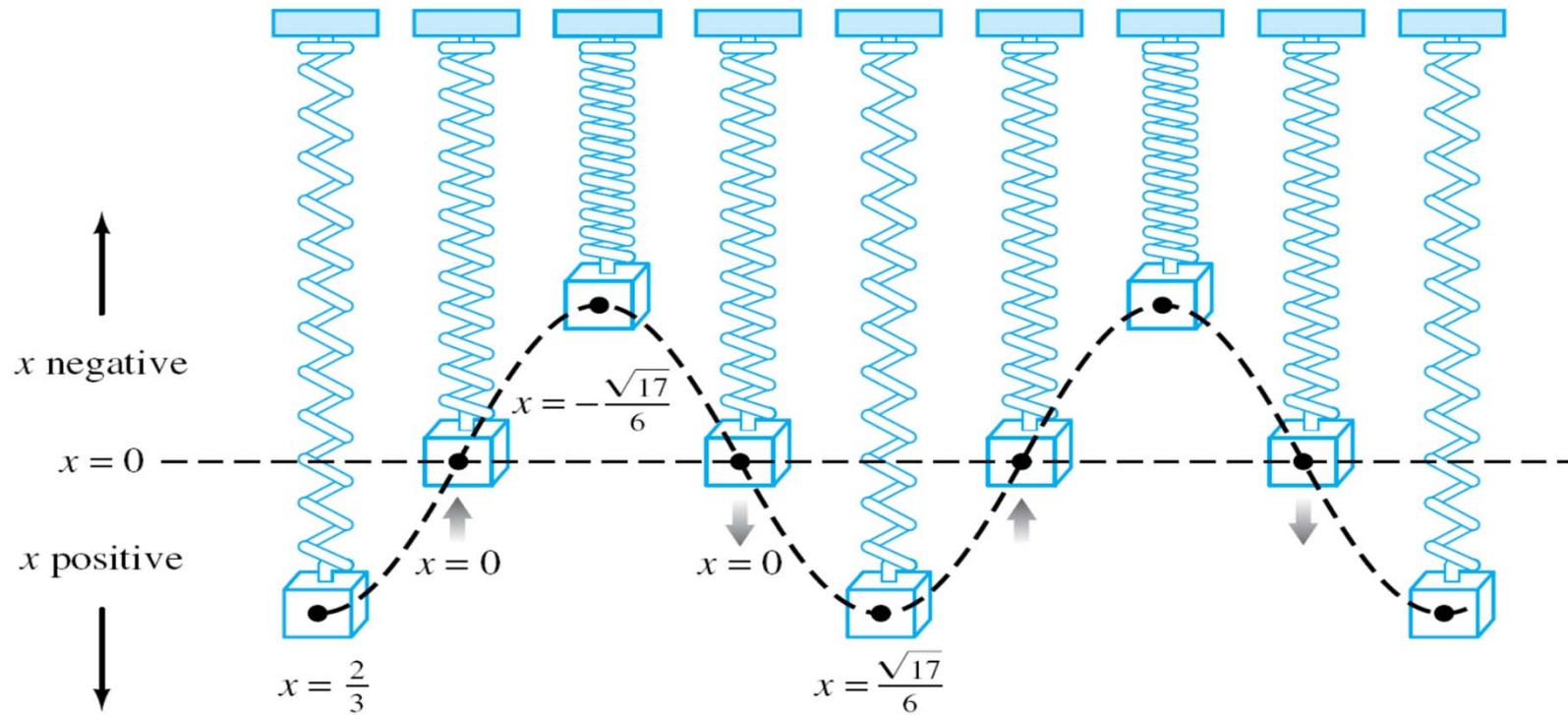


Figure 5.1.4

(a)

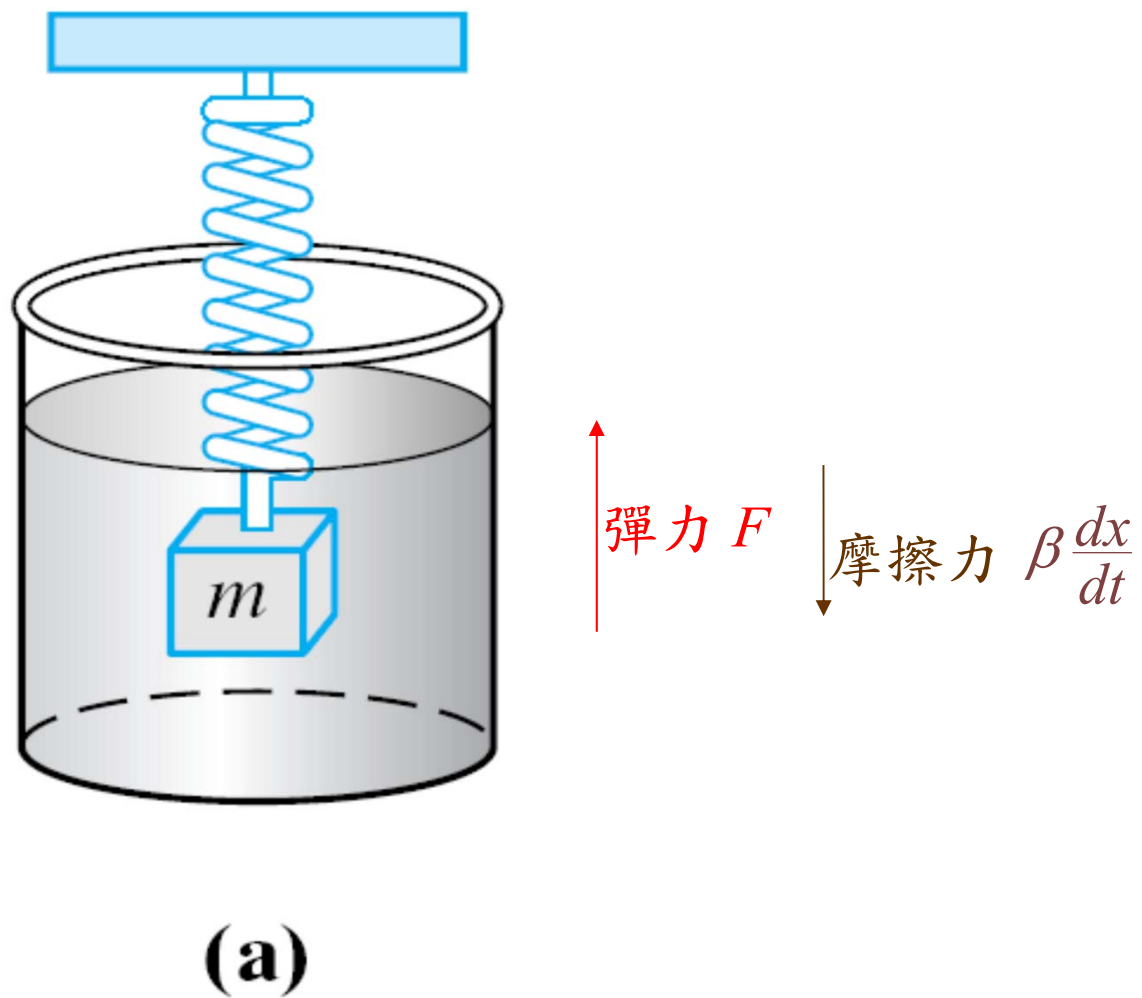


Figure 5.1.5

$$m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} = F \longrightarrow m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} = -kx$$

$$m \frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + kx = 0$$

解有成三種情形

$$(1) \quad \beta^2 - 4mk > 0 \quad x = c_1 e^{n_1 t} + c_2 e^{n_2 t} \quad n_1, n_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

$$(2) \quad \beta^2 - 4mk = 0 \quad x = c_1 e^{nt} + c_2 t e^{nt} \quad n = -\beta / 2m$$

$$(3) \quad \beta^2 - 4mk < 0 \quad x = e^{at} (c_1 \cos bt + c_2 \sin bt)$$

$$a = -\beta / 2m \quad b = \sqrt{4mk - \beta^2} / 2m$$

需要注意的概念

(1) 名詞一

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

$g(t)$ 被稱作 **input** 或 deriving function 或 forcing function

$y(t)$ 被稱作 **output** 或 **response**

(2) 名詞二

對一個 2nd order linear DE with constant coefficients

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

auxiliary function $a_2 m^2 + a_1 m + a_0 = 0$

當 $a_1^2 - 4a_2 a_0 > 0$ 時，稱作 overdamped

當 $a_1^2 - 4a_2 a_0 = 0$ 時，稱作 critical damped

當 $a_1^2 - 4a_2 a_0 < 0$ 時，稱作 underdamped

當 $-4a_2 a_0 < 0, a_1 = 0$ 時，稱作 undamped

(3) $a_2 y''(x) + a_1 y'(x) + a_0 y(x) = g(x)$ 當中

a_1 的值將影藉衰減速度

當 a_2, a_1, a_0 的值皆為正， a_1/a_2 的值越大，衰減的進度越快

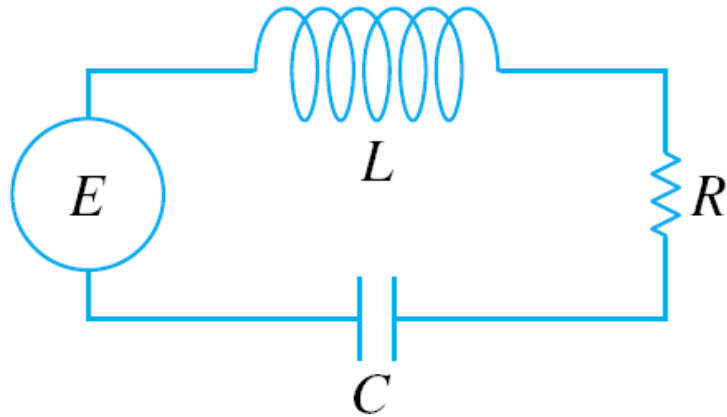
When $a_1^2 - 4a_2a_0 < 0$ $y_c = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$

$$\alpha = -a_1 / 2a_2,$$

When $a_1^2 - 4a_2a_0 > 0$ $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} \quad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

5-1-4 RLC circuit



inductance 的電壓 $L \frac{di}{dt}$

capacitor 的電壓 $\frac{q}{C}$

resistor 的電壓 Ri

$$\frac{q}{C} + Ri + L \frac{di}{dt} = E(t) \quad \text{using } i = \frac{dq}{dt}$$

$$\frac{q}{C} + R \frac{dq}{dt} + L \frac{d^2q}{dt^2} = E(t) \quad \text{一定可以解}$$

$$\frac{q}{C} + R \frac{dq}{dt} + L \frac{d^2q}{dt^2} = E(t)$$

auxiliary function $Lm^2 + Rm + 1/C = 0$

$$\text{roots: } m_1 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L} \quad m_2 = \frac{-R - \sqrt{R^2 - 4L/C}}{2L}$$

Case 1: $R^2 - 4L/C > 0$ (overdamped)

$(m_1 \neq m_2, m_1, m_2 \text{ are real})$

Complementary function:

$$q_c(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

註：由於 R, L, C 的值都是正的, $\sqrt{R^2 - 4L/C} < R$ 必定可以滿足

所以 m_1, m_2 都是負的

$$q_c(t) = 0 \quad \text{when } t \rightarrow \infty$$

Particular solution (1) $E(t)$ 有的時候可用” guess” 的方法來解

(2) $E(t)$ 用 variation of parameters 的方法一定解得出來(但較耗時)

$$W = \begin{vmatrix} e^{m_1 t} & e^{m_2 t} \\ m_1 e^{m_1 t} & m_2 e^{m_2 t} \end{vmatrix} = (m_2 - m_1) e^{(m_1 + m_2)t}$$

$$W_1 = \begin{vmatrix} 0 & e^{m_2 t} \\ ? & m_2 e^{m_2 t} \end{vmatrix} = ?$$

$$W_2 = \begin{vmatrix} e^{m_1 t} & 0 \\ m_1 e^{m_1 t} & ? \end{vmatrix} = ?$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{m_2 t} E(t) / L}{(m_2 - m_1) e^{(m_1 + m_2)t}}$$

$$u_2' = \frac{W_2}{W} = \frac{e^{m_1 t} E(t) / L}{(m_2 - m_1) e^{(m_1 + m_2)t}}$$

$$u_1 = \frac{-\int E(t) e^{-m_1 t} dt}{L(m_2 - m_1)}$$

$$u_2 = \frac{\int e^{-m_2 t} E(t) dt}{L(m_2 - m_1)}$$

$$q_p(t) = -\frac{e^{m_1 t} \int E(t) e^{-m_1 t} dt}{L(m_2 - m_1)} + \frac{e^{m_2 t} \int E(t) e^{-m_2 t} dt}{L(m_2 - m_1)}$$

$$q(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} - \frac{e^{m_1 t} \int E(t) e^{-m_1 t} dt}{L(m_2 - m_1)} + \frac{e^{m_2 t} \int E(t) e^{-m_2 t} dt}{L(m_2 - m_1)}$$

Specially, when $E(t) = E_0$ where E_0 is some constant

$$\begin{aligned} q_p(t) &= -\frac{E_0 e^{m_1 t} \int e^{-m_1 t} dt}{L(m_2 - m_1)} + \frac{E_0 e^{m_2 t} \int e^{-m_2 t} dt}{L(m_2 - m_1)} = \frac{E_0}{L(m_2 - m_1)} \left(\frac{1}{m_1} - \frac{1}{m_2} \right) \\ &= \frac{E_0}{L m_1 m_2} = E_0 C \quad (m_1 m_2 = 1/LC) \end{aligned}$$

$$q(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + E_0 C$$

Case 2: $R^2 - 4L/C = 0$ (critically damped)

$$(m_1 = m_2 = -R/2L)$$

$$q_c(t) = c_1 e^{-Rt/2L} + c_2 t e^{-Rt/2L}$$

$$q_c(t) = 0 \quad \text{when } t \rightarrow \infty$$

Particular solution

$$q_p(t) = \frac{e^{-Rt/2L}}{L} \left[t \int E(t) e^{Rt/2L} dt - \int E(t) t e^{Rt/2L} dt \right]$$

$$q(t) = c_1 e^{-Rt/2L} + c_2 t e^{-Rt/2L} + \frac{e^{-Rt/2L}}{L} \left[t \int E(t) e^{Rt/2L} dt - \int E(t) t e^{Rt/2L} dt \right]$$

When $E(t) = E_0$, $q_p(t) = E_0 C$

Case 3: $R^2 - 4L/C < 0$ (underdamped)

$$m_1 = \alpha + j\beta \quad m_2 = \alpha - j\beta$$

$$\alpha = \frac{-R}{2L}, \quad \beta = \frac{\sqrt{4L/C - R^2}}{2L}$$

$$q_c(t) = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t) \quad q_c(t) = 0 \quad \text{when } t \rightarrow \infty$$

Particular solution

$$q_p(t) = \frac{e^{\alpha t}}{L\beta} \left[\sin \beta t \int E(t) e^{-\alpha t} \cos \beta t dt - \cos \beta t \int E(t) e^{-\alpha t} \sin \beta t dt \right]$$

General solution

$$q(t) = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t) + \frac{e^{\alpha t}}{L\beta} \left[\sin \beta t \int E(t) e^{-\alpha t} \cos \beta t dt - \cos \beta t \int E(t) e^{-\alpha t} \sin \beta t dt \right]$$

When $E(t) = E_0$ where E_0 is some constant

$$q(t) = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t) + E_0 C$$

When $R = 0$, then $\alpha = 0$

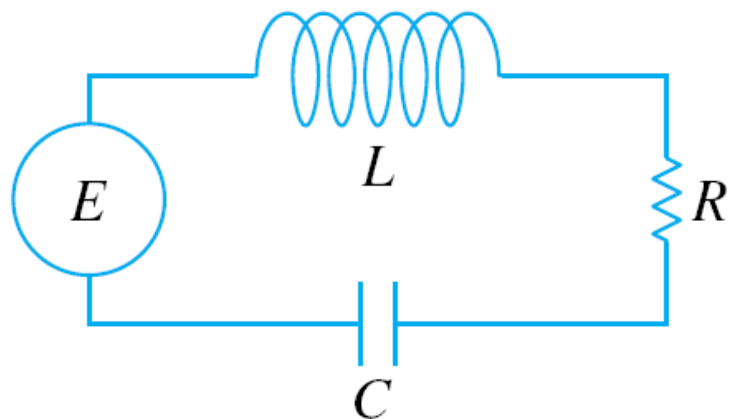
$$q(t) = c_1 \cos \beta t + c_2 \sin \beta t + \frac{1}{L\beta} \left[\sin \beta t \int E(t) \cos \beta t dt - \cos \beta t \int E(t) \sin \beta t dt \right]$$

When $R = 0$, $E(t) = E_0$

$$q(t) = c_1 \cos \beta t + c_2 \sin \beta t + E_0 C$$

$$i(t) = \frac{d}{dt} q(t) = d_1 \cos \beta t + d_2 \sin \beta t \quad d_1 = \beta c_2 \quad d_2 = -\beta c_1$$

以 DE 的觀點來解釋 RLC 電路的問題

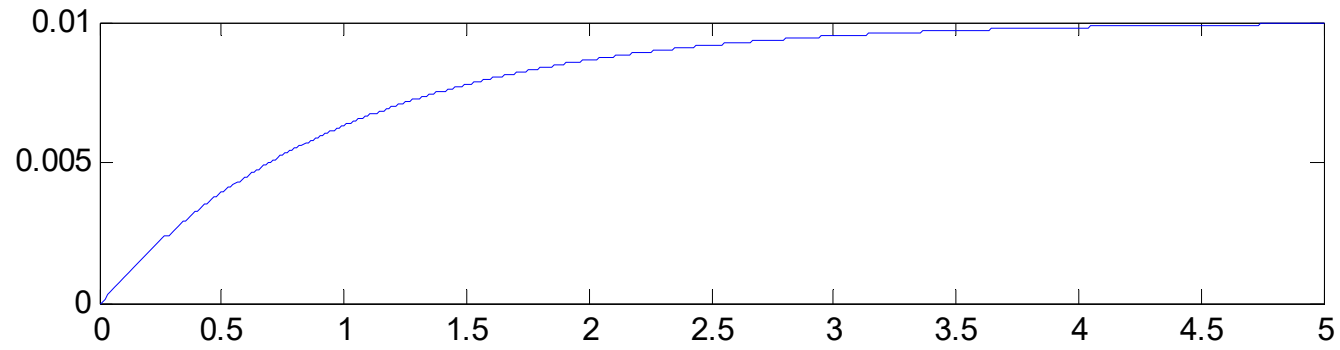
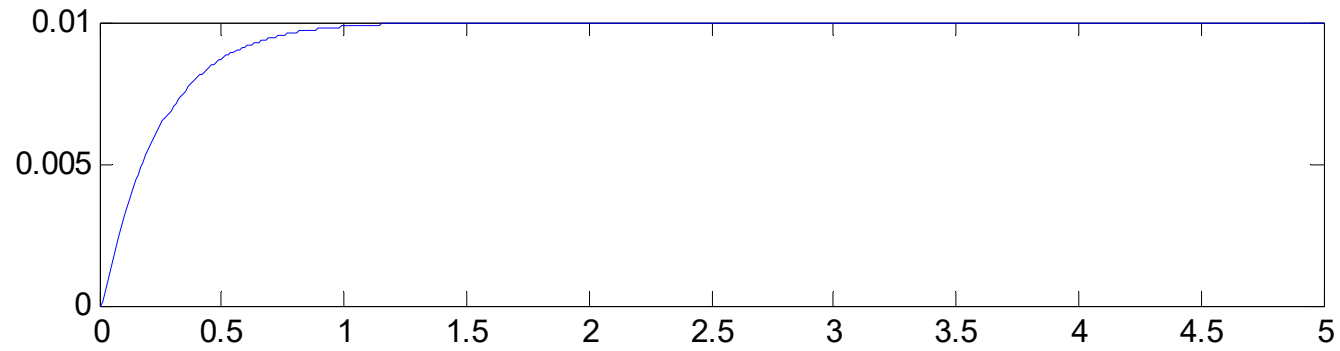
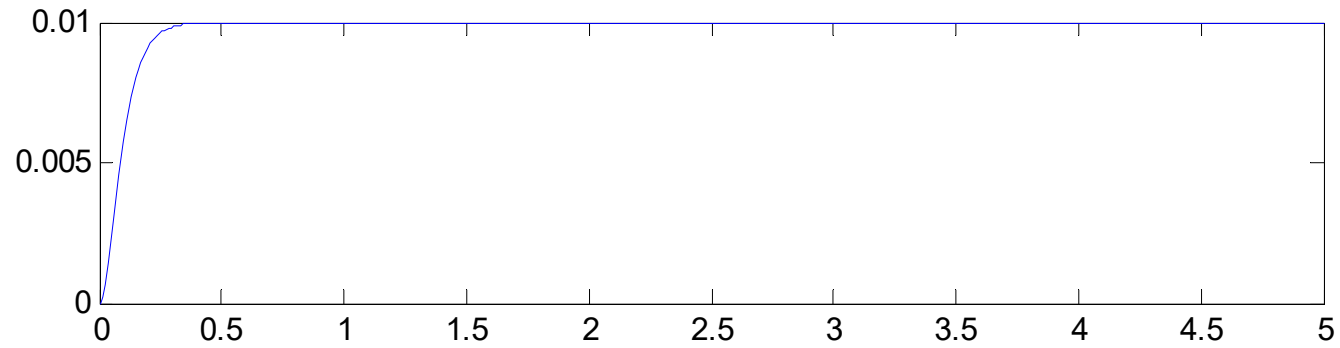


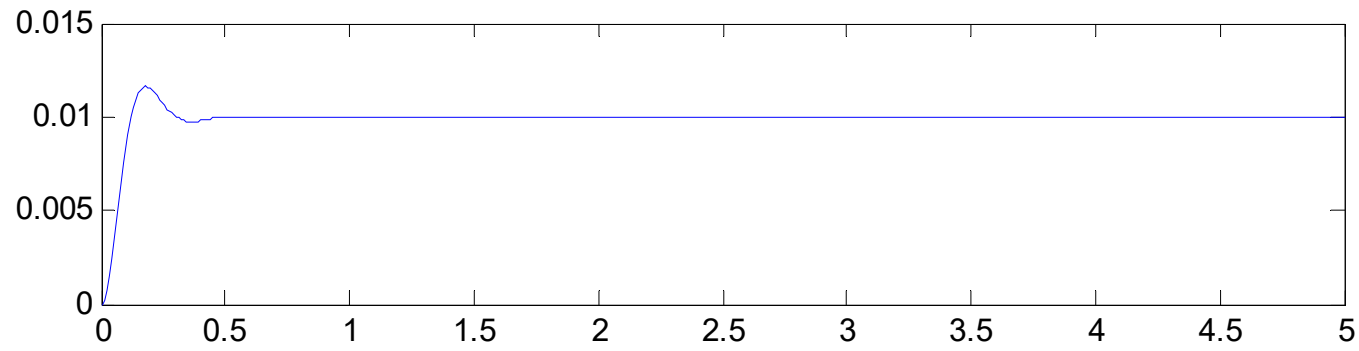
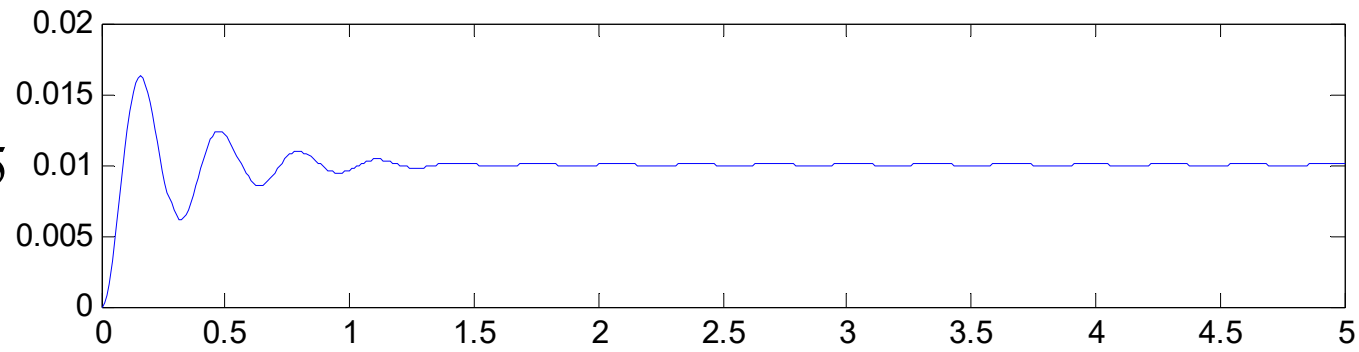
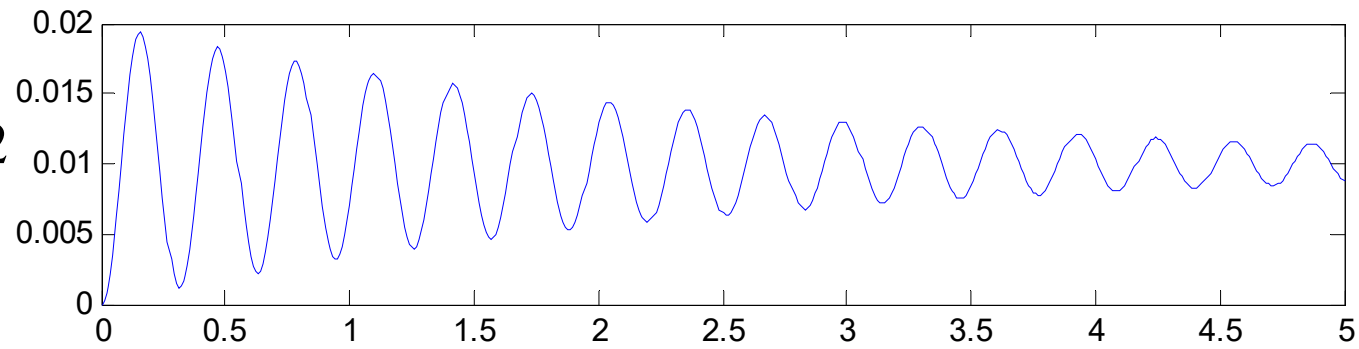
- (1) $R^2 < 4L/C$ \longrightarrow 產生弦波
- (2) R 越小，弦波衰減得越慢

例子

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E(t) \quad E(t) = 1, L = 0.25, C = 0.01$$

$$a_1^2 - 4a_2a_0 = R^2 - 100$$

$R = 100$  $R = 25$  $R = 10$ 

$R = 5$  $R = 1.5$  $R = 0.2$ 

5-1-5 Express the Solutions by Other Forms

(1) Express the Solution by the Form of Amplitude and Phase

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

當 $a_1^2 - 4a_2 a_0 < 0$ 時，solution 為 $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$

$$\alpha = -a_1 / 2a_2, \quad \beta = \sqrt{4a_2 a_0 - a_1^2} / 2a_2$$

Solution 可改寫成

$$y = A e^{\alpha x} \sin(\beta x + \phi)$$

$$A = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \sin^{-1}(c_1 / A) = \cos^{-1}(c_2 / A)$$

$$y = Ae^{\alpha x} \sin(\beta x + \phi)$$

$Ae^{\alpha x}$: damped amplitude

$\frac{\beta}{2\pi}$: damped frequency

ϕ : phase angle

(2) Express the Solution by Hyperbolic Functions

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

$$\text{當 } a_1 = 0$$

$$\text{且 } a_2 > 0, a_0 < 0 \text{ (或 } a_0 > 0, a_2 < 0)$$

$$y = c_1 e^{m_1 x} + c_2 e^{-m_1 x} \quad m_1 = \sqrt{-\frac{a_0}{a_2}}$$

$$y = c_3 \cosh(m_1 x) + c_4 \sinh(m_1 x)$$

$$\cosh(m_1 x) = \frac{e^{m_1 x} + e^{-m_1 x}}{2} \quad c_3 = c_1 + c_2$$

$$\sinh(m_1 x) = \frac{e^{m_1 x} - e^{-m_1 x}}{2} \quad c_4 = c_1 - c_2$$

5-1-6 本節要注意的地方

- (1) 將力學現象寫成 DE 時，要注意正負號 (根據力的方向)
- (2) 注意 page 280 的四個專有名詞
- (3) 注意 linear DE with constant coefficients 的解，有其他的寫法
(see pages 293 and 295)

附錄八(續) : Reviews for Higher Order DE

(接續 page 230)

(A) Linear DE Complementary Function 3 大解法

(3) Cauchy-Euler Equation (Section 4-7)

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$



$$a_n \frac{m!}{(m-n)!} + a_{n-1} \frac{m!}{(m-n+1)!} + \cdots + a_1 x \frac{m!}{(m-1)!} + a_0 = 0$$

適用情形：

(B) Linear DE Particular solution 3 大解法

(3) Variation of parameters (Section 4-6)

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

$$u'_k(x) = \frac{W_k}{W} \quad W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y'_1 & y'_2 & y'_3 & \cdots & y'_n \\ y''_1 & y''_2 & y''_3 & \cdots & y''_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

W_k : replace the k^{th} column of W by

適用情形：

$$f(x) = \frac{g(x)}{a_n(x)} \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

(4) For Cauchy-Euler Equation (Section 4-7)

可採用二種方法

(1) 先用

$$a_n \frac{m!}{(m-n)!} + a_{n-1} \frac{m!}{(m-n+1)!} + \cdots + a_1 x \frac{m!}{(m-1)!} + a_0 = 0$$

解 complementary function

再用 Variation of parameters 解 particular solution

(2) Use the method on pages 268, 269

Set $x = e^t$, $t = \ln x$

Exercises for practicing

Section 4-6 4, 5, 8, 13, 14, 17, 18, 21, 28, 29, 34

Section 4-7 11, 17, 18, 20, 21, 24, 32, 34, 36, 37, 40, 42

Review 4 27, 28, 29, 30, 32, 42

Section 5-1 1, 11, 29, 43, 44, 49, 52, 56, 60

Review 5 12, 21, 22