Chapter 5 Modeling with Higher Order Differential Equations

Chapter 4 的應用題

自然界，有不少的系統可以用 linear DE 來表示

其中有不少的系統可以進一步簡化成 linear DE with constant coefficients

5-1 Linear Models: Initial Value Problem

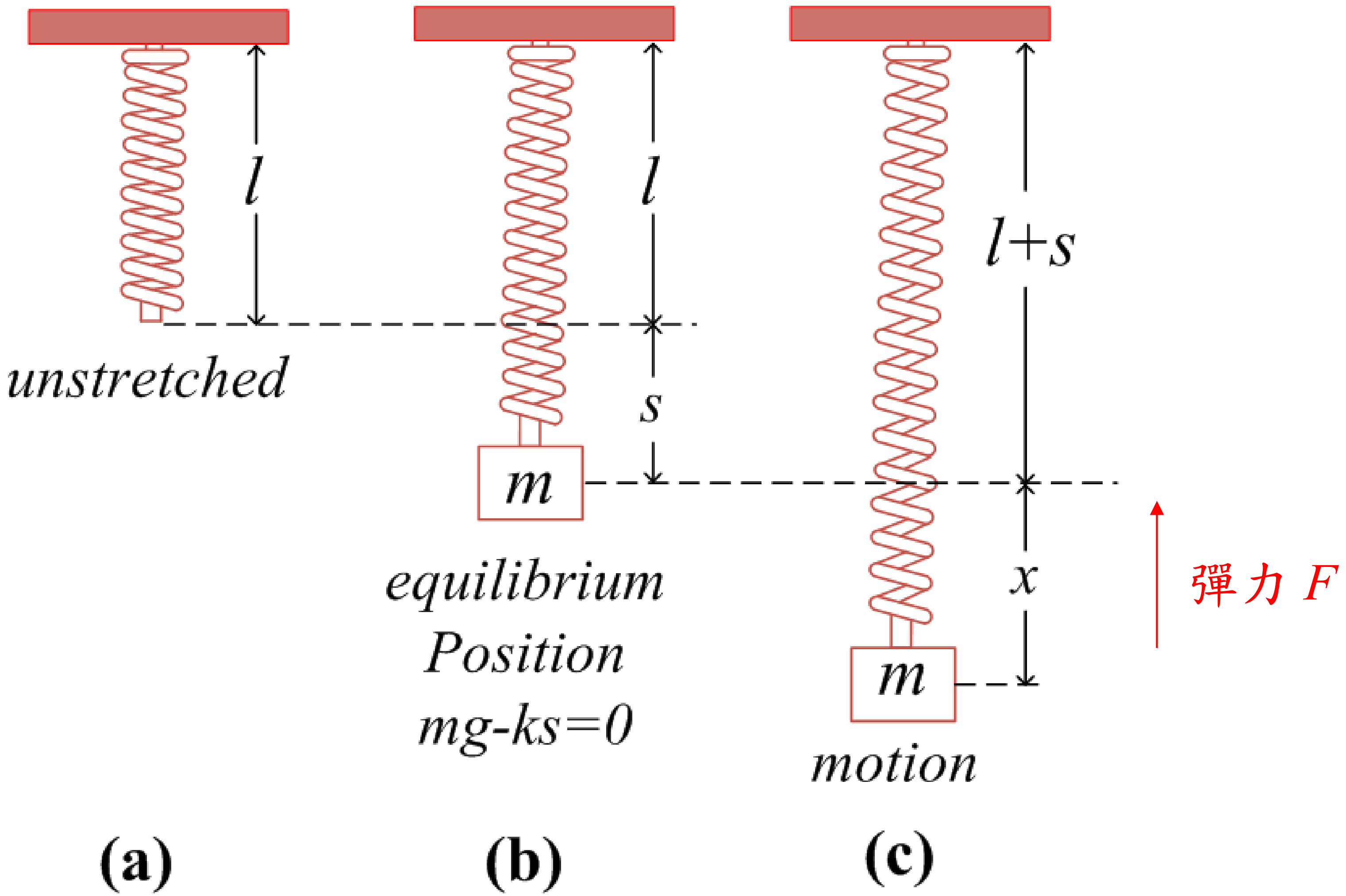
位置：x,　　速度：$\frac{d}{dt}x$　　加速度 $\frac{d^2}{dt^2}x$

$$F = ma \longrightarrow F = m\frac{d^2x}{dt^2}$$

$$F - \beta v = ma \longrightarrow F - \beta\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

v: 速度，βv: 磨擦力

5-1-1 ~ 5-1-3 Spring / Mass Systems

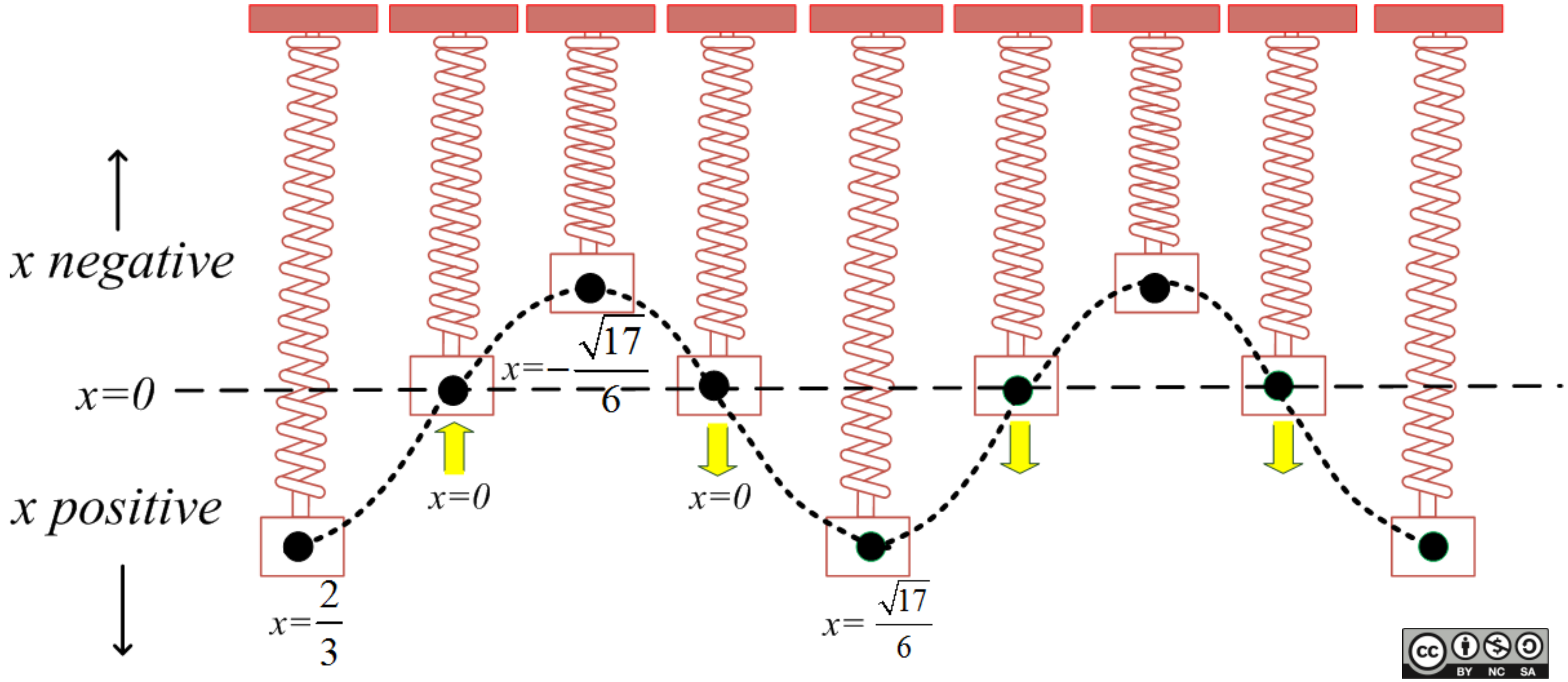


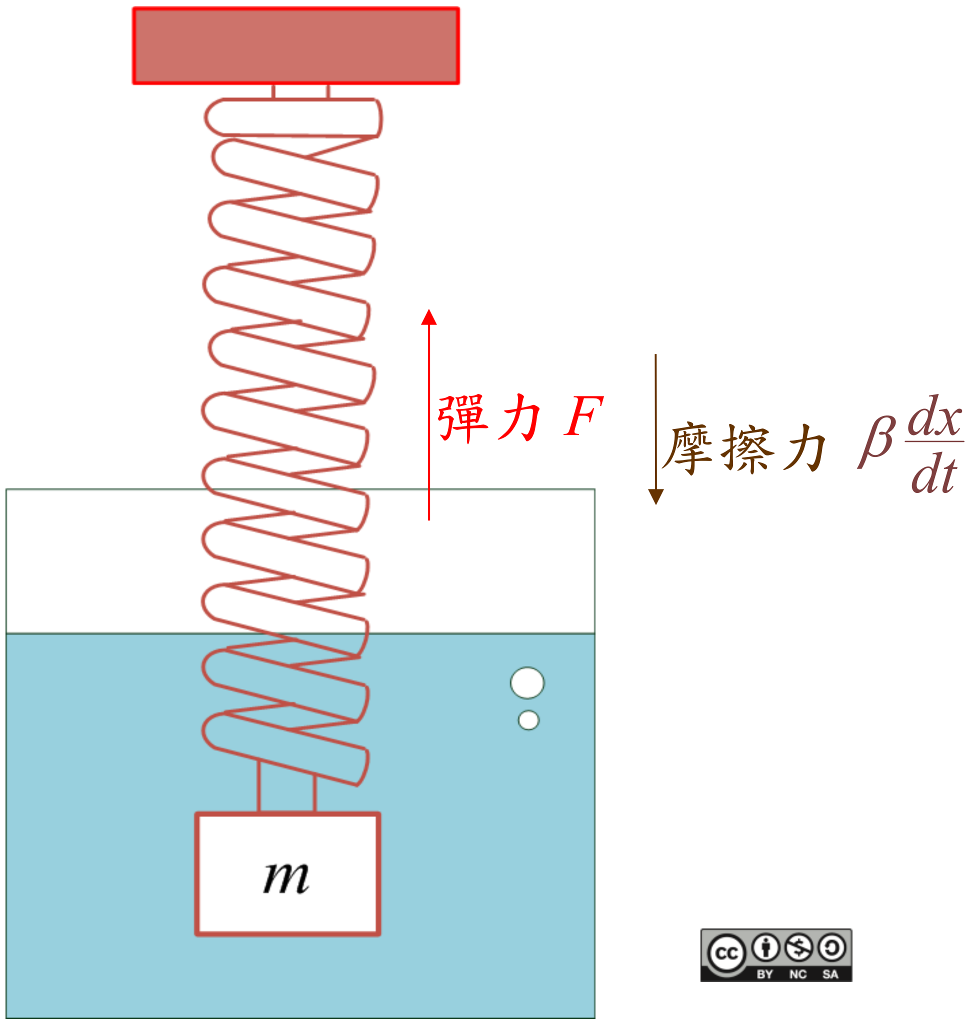
Spring/mass system

$$m\frac{d^2x}{dt^2} = F \longrightarrow m\frac{d^2x}{dt^2} = -kx$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Solution: $x(t) = c_1 \cos\omega t + c_2 \sin\omega t \quad \omega = \sqrt{\frac{k}{m}}$




彈力 F
摩擦力 $\beta\frac{dx}{dt}$
m

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}=F \longrightarrow m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}=-kx$$

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0$$

解有成三種情形 $\beta^2-4mk>0$

$$\beta^2-4mk=0$$

$$\beta^2-4mk<0$$

需要注意的概念

(1) 名詞一

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=g(t)$$

$g(t)$ 被稱作 input 或 deriving function 或 forcing function

$y(t)$ 被稱作 output 或 response

(2) 名詞二

對一個 2nd order linear DE with constant coefficients

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

auxiliary function $a_2 m^2 + a_1 m + a_0 = 0$

當 $a_1^2 - 4a_2 a_0 > 0$ 時，稱作 overdamped

當 $a_1^2 - 4a_2 a_0 = 0$ 時，稱作 critical damped

當 $a_1^2 - 4a_2 a_0 < 0$ 時，稱作 underdamped

當 $-4a_2 a_0 < 0$，$a_1 = 0$ 時，稱作 undamped

(3) $a_2 y''(x) + a_1 y'(x) + a_0 y(x) = g(x)$當中

a_1 的值將影響衰減速度

當 a_2, a_1, a_0 的值皆為正， a_1/a_2 的值越大，衰減的進度越快

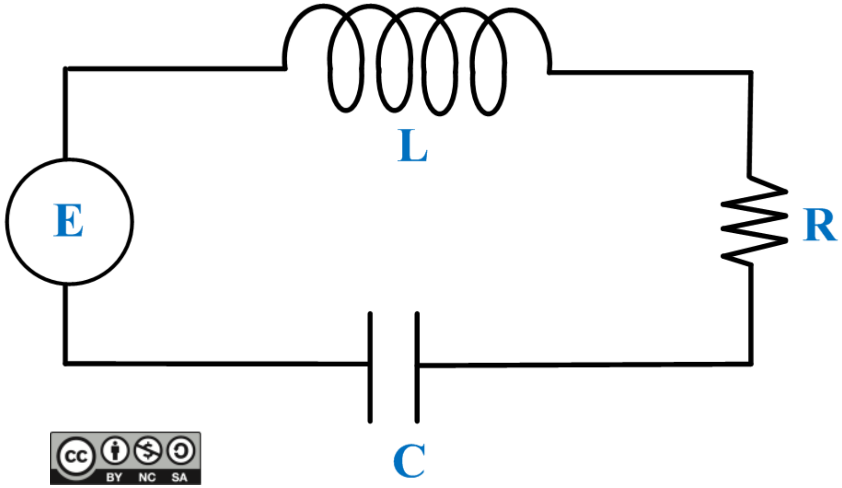
When $a_1^2 - 4a_2a_0 < 0$ $\quad y_c = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$

$$\alpha = -a_1/2a_2,$$

When $a_1^2 - 4a_2a_0 > 0$ $\quad y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} \qquad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

5-1-4 RLC circuit



inductance 的電壓 $L\dfrac{di}{dt}$

capacitor 的電壓 $\dfrac{q}{C}$

resistor 的電壓 Ri

$$\frac{q}{C}+Ri+L\frac{di}{dt}=E(t) \qquad \text{using} \quad i=\frac{dq}{dt}$$

$$\frac{q}{C}+R\frac{dq}{dt}+L\frac{d^2q}{dt^2}=E(t)$$

一定可以解

$$\frac{q}{C}+R\frac{dq}{dt}+L\frac{d^2q}{dt^2}=E(t)$$

auxiliary function $\quad Lm^2+Rm+1/C=0$

roots: $\quad m_1=\dfrac{-R+\sqrt{R^2-4L/C}}{2L} \qquad m_2=\dfrac{-R-\sqrt{R^2-4L/C}}{2L}$

Case 1: $R^2-4L/C>0$ **(overdamped)**

($m_1 \neq m_2$, m_1, m_2 are real)

Complementary function:

$$q_c(t)=c_1e^{m_1t}+c_2e^{m_2t}$$

註：由於 R, L, C 的值都是正的, $\sqrt{R^2-4L/C}<R$ 必定可以滿足

所以 m_1, m_2 都是負的

$q_c(t)=0 \quad$ when $t\to\infty$

Particular solution (1) $E(t)$ 有的時候可用” guess” 的方法來解

(2) $E(t)$ 用 variation of parameters 的方法一定解得出來(但較耗時)

$$W = \begin{vmatrix} e^{m_1 t} & e^{m_2 t} \\ m_1 e^{m_1 t} & m_2 e^{m_2 t} \end{vmatrix} = (m_2 - m_1) e^{(m_1 + m_2)t}$$

$$W_1 = \begin{vmatrix} 0 & e^{m_2 t} \\ ? & m_2 e^{m_2 t} \end{vmatrix} = ? \qquad W_2 = \begin{vmatrix} e^{m_1 t} & 0 \\ m_1 e^{m_1 t} & ? \end{vmatrix} = ?$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{m_2 t} E(t)/L}{(m_2 - m_1) e^{(m_1 + m_2)t}} \qquad u_2' = \frac{W_2}{W} = \frac{e^{m_1 t} E(t)/L}{(m_2 - m_1) e^{(m_1 + m_2)t}}$$

$$u_1 = \frac{-\int E(t) e^{-m_1 t} dt}{L(m_2 - m_1)} \qquad u_2 = \frac{\int e^{-m_2 t} E(t) dt}{L(m_2 - m_1)}$$

$$q_p(t) = -\frac{e^{m_1 t}\int E(t)e^{-m_1 t}dt}{L(m_2 - m_1)} + \frac{e^{m_2 t}\int E(t)e^{-m_2 t}dt}{L(m_2 - m_1)}$$

$$\boxed{q(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} - \frac{e^{m_1 t}\int E(t)e^{-m_1 t}dt}{L(m_2 - m_1)} + \frac{e^{m_2 t}\int E(t)e^{-m_2 t}dt}{L(m_2 - m_1)}}$$

Specially, when $E(t) = E_0$ where E_0 is some constant

$$q_p(t) = -\frac{E_0 e^{m_1 t}\int e^{-m_1 t}dt}{L(m_2 - m_1)} + \frac{E_0 e^{m_2 t}\int e^{-m_2 t}dt}{L(m_2 - m_1)} = \frac{E_0}{L(m_2 - m_1)}\left(\frac{1}{m_1} - \frac{1}{m_2}\right)$$

$$= \frac{E_0}{Lm_1 m_2} = E_0 C \qquad (m_1 m_2 = 1/LC)$$

$$\boxed{q(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + E_0 C}$$

Case 2: $R^2 - 4L/C = 0$ **(critically damped)**

$(m_1 = m_2 = -R/2L)$

$$q_c(t) = c_1 e^{-Rt/2L} + c_2 t\, e^{-Rt/2L}$$

$q_c(t) = 0$ when $t \to \infty$

Particular solution

$$q_p(t) = \frac{e^{-Rt/2L}}{L}\left[t\int E(t)e^{Rt/2L}dt - \int E(t)te^{Rt/2L}dt\right]$$

$$\boxed{q(t) = c_1 e^{-Rt/2L} + c_2 t\, e^{-Rt/2L} + \frac{e^{-Rt/2L}}{L}\left[t\int E(t)e^{Rt/2L}dt - \int E(t)te^{Rt/2L}dt\right]}$$

When $E(t) = E_0$, $q_p(t) = E_0 C$

Case 3: $R^2 - 4L/C < 0$ **(underdamped)**

$$m_1 = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

$$\alpha = \frac{-R}{2L} \quad , \qquad \beta = \frac{\sqrt{4L/C - R^2}}{2L}$$

$$q_c(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right) \qquad q_c(t) = 0 \quad \text{when } t \to \infty$$

Particular solution

$$q_p(t) = \frac{e^{\alpha t}}{L\beta}\left[\sin\beta t \int E(t)e^{-\alpha t}\cos\beta t dt - \cos\beta t \int E(t)e^{-\alpha t}\sin\beta t dt\right]$$

General solution

$$\boxed{\begin{aligned} q(t) &= e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right) \\ &\quad + \frac{e^{\alpha t}}{L\beta}\left[\sin\beta t \int E(t)e^{-\alpha t}\cos\beta t dt - \cos\beta t \int E(t)e^{-\alpha t}\sin\beta t dt\right] \end{aligned}}$$

When $E(t) = E_0$ where E_0 is some constant

$$\boxed{q(t) = e^{\alpha t}\left(c_1 \cos\beta t + c_2 \sin\beta t\right) + E_0 C}$$

When $R = 0$, then $\alpha = 0$

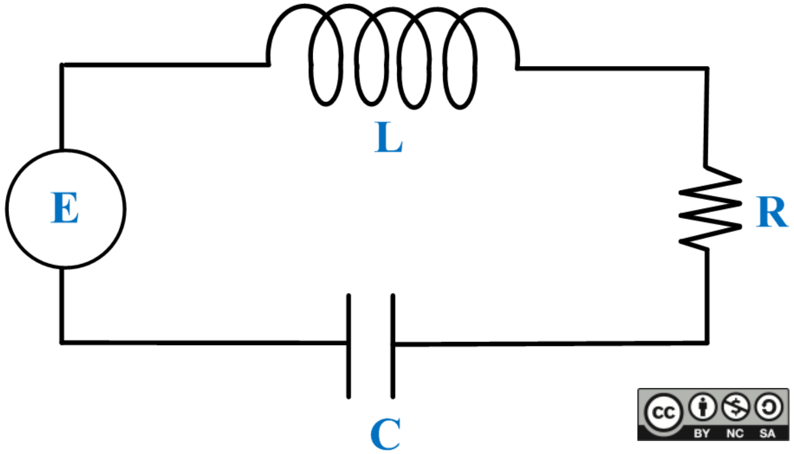
$$q(t) = c_1 \cos\beta t + c_2 \sin\beta t + \frac{1}{L\beta}\left[\sin\beta t \int E(t)\cos\beta t dt - \cos\beta t \int E(t)\sin\beta t dt\right]$$

When $R = 0$, $E(t) = E_0$

$$q(t) = c_1 \cos\beta t + c_2 \sin\beta t + E_0 C$$

$$i(t) = \frac{d}{dt}q(t) = d_1 \cos\beta t + d_2 \sin\beta t \qquad d_1 = \beta c_2 \qquad d_2 = -\beta c_1$$

以 DE 的觀點來解釋 RLC 電路的問題



(1) $R^2 < 4L/C \longrightarrow$ 產生弦波

(2) R 越小，弦波衰減得越慢

例子

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E(t)$$

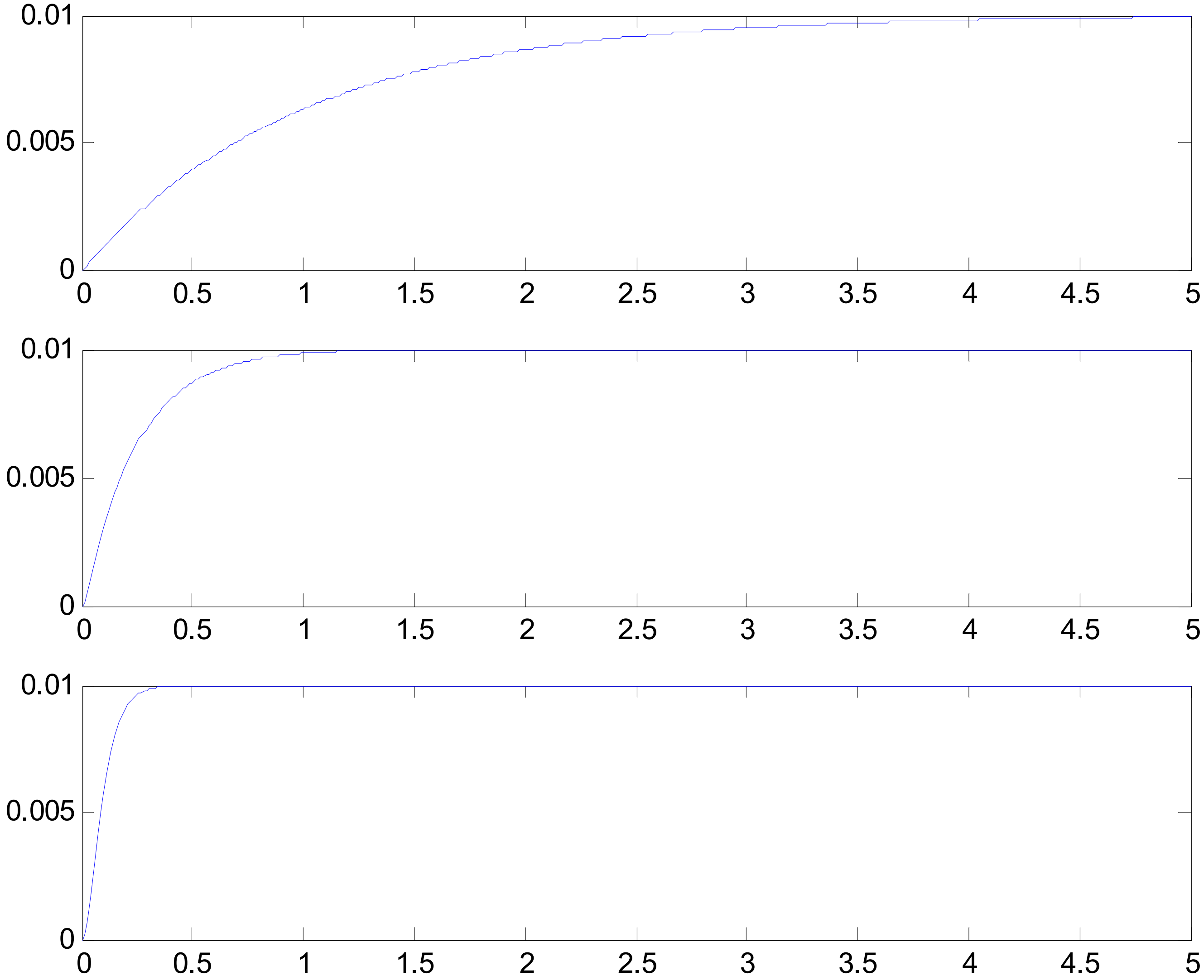
$$E(\mathrm{t}) = 1,\ L = 0.25,\ C = 0.01$$

$$a_1^2-4a_2a_0=R^2-100$$

$R = 100$

$R = 25$

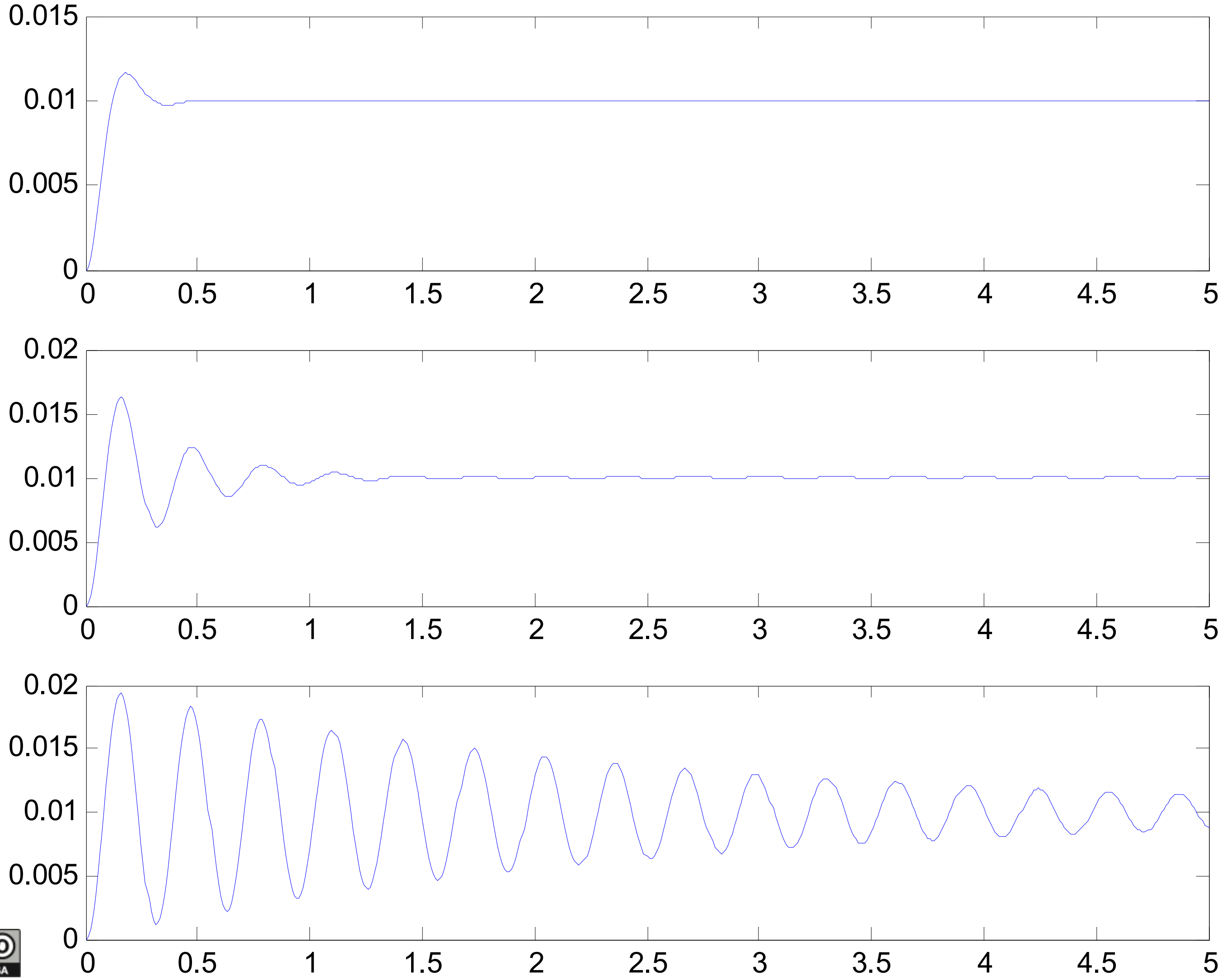
$R = 10$



$R = 5$

$R = 1.5$

$R = 0.2$


0.015
0.01
0.005
0
0
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1
1.5
2
2.5
3
3.5
4
4.5
5
0.02
0.015
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0.015
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2
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3
3.5
4
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5

5-1-5 Express the Solutions by Other Forms

(1) Express the Solution by the Form of Amplitude and Phase

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

當 $a_1^2 - 4a_2 a_0 < 0$ 時，solution 為 $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$$\alpha = -a_1 / 2a_2, \quad \beta = \sqrt{4a_2 a_0 - a_1^2} / 2a_2$$

Solution 可改寫成

$$y = Ae^{\alpha x}\sin(\beta x + \phi) \qquad A = \sqrt{c_1^2 + c_2^2}$$

$$\phi = \sin^{-1}(c_1 / A) = \cos^{-1}(c_2 / A)$$

$$y = Ae^{\alpha x}\sin\left(\beta x + \phi\right)$$

$Ae^{\alpha x}$: damped amplitude

$\dfrac{\beta}{2\pi}$: damped frequency

ϕ: phase angle

(2) Express the Solution by Hyperbolic Functions

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

當 $a_1 = 0$

且 $a_2 > 0,\ a_0 < 0$ (或 $a_0 > 0,\ a_2 < 0$)

$$y = c_1 e^{m_1 x} + c_2 e^{-m_1 x} \qquad m_1 = \sqrt{-\frac{a_0}{a_2}}$$

$$y = c_3 \cosh(m_1 x) + c_4 \sinh(m_1 x)$$

$$\cosh(m_1 x) = \frac{e^{m_1 x} + e^{-m_1 x}}{2} \qquad c_3 = c_1 + c_2$$

$$\sinh(m_1 x) = \frac{e^{m_1 x} - e^{-m_1 x}}{2} \qquad c_4 = c_1 - c_2$$

5-1-6 本節要注意的地方

(1) 將力學現象寫成 DE 時，要注意正負號 (根據力的方向)

(2) 注意 page 274 的四個專有名詞

(3) 注意 linear DE with constant coefficients 的解，有其他的寫法

(see pages 287 and 289)

5-2 Linear Models: Boundary-Value Problem

(不在考試範圍)

Section 5-2 的問題，和 Section 5-1 類似

(都是 Linear DE)

只是將 initial value problems 變成 boundary value problems

複習： 將 IVP 改成 boundary value problems，

對 solution 有什麼影響？

Section 5-1 的例子

(1) 牛頓運動定律

(2) 彈簧運動 (subsection 5-1-1~5-1-3)

(3) RLC Circuit (subsection 5-1-4)

Section 5-2 的例子

(1) 樑彎曲 (a) 橫放 (subsection 5-2-1)

(b) 上方施力 (subsection 5-2-2)

(2) 跳繩 (subsection 5-2-3)

4-9 Solving Systems of Linear Equations by Elimination

4-9-1 方法適用的情形和限制

處理有 2 個以上 dependent variables 的問題

例如：Section 3-3 電路學上 "並聯" 的例子

限制：必需是 linear and constant coefficients

4-9-2 方法

(Step 1) 先將 $\dfrac{d^n}{dt^n}$ 寫成 D^n

(Step 2) 再用聯立方程(或線性代數)的方式

將各個 dependent variable 所對應的 DE 寫出

(Step 3) 再運用 Sections 4-3, 4-4 的方法，

得出各 dependent variables 的解

(Step 4) 代回原式，求出 unknowns 之間的關係

(別忘了這一步，可以參考講義 page 298)

4-9-3 範例

Figure 3.3.4 的例子 (See Pages 96, 97)

$$L_1\frac{di_2(t)}{dt}+(R_1+R_2)i_2(t)+R_1i_3(t)=E(t)$$

$$R_1i_2(t)+L_2\frac{di_3(t)}{dt}+R_1i_3(t)=E(t)$$

令 $L_1=L_2=1,\ R_1=4,\ R_2=6,\ E(t)=10$

$$\boxed{\begin{aligned}&\frac{di_2(t)}{dt}+10i_2(t)+4i_3(t)=10\\&4i_2(t)+\frac{di_3(t)}{dt}+4i_3(t)=10\end{aligned}}$$

求解

Step 1 $(D+10)i_2 + 4i_3 = 10$ …….. (式 1)

$4i_2 + (D+4)i_3 = 10$ …….. (式 2)

Step 2-1 $(D + 4) \times$ (式 1) $- 4 \times$ (式 2)

$$(D+4)(D+10)i_2 - 16i_2 = (D+4)10 - 4\times 10$$

Note: $(D+4)10 = \frac{d}{dt}10 + 4\times 10 = 40$

$$(D^2 + 14D + 24)i_2 = 0$$

Step 3-1 auxiliary function: $m^2 + 14m + 24 = 0$

roots $m_1 = -2$ $m_2 = -12$

$$\boxed{i_2(t) = c_1 e^{-2t} + c_2 e^{-12t}}$$

Step 2-2 $4 \times$ (式 1) − (D+10) $\times$ (式 2)

$$16i_3-(D+4)(D+10)i_3=4\times10-(D+10)10$$

$$-(D^2+14D+24)i_3=-60$$

$$(D^2+14D+24)i_3=60$$

Step 3-2 auxiliary function: $m^2+14m+24=0$

roots $m_1=-2 \qquad m_2=-12$

complementary function for $i_{3,c}(t)=c_3e^{-2t}+c_4e^{-12t}$

Particular solution: $i_{3,p}(t)=A$

$$24A=60 \quad , \quad A=5/2$$

$$\boxed{i_3\left(t\right)=c_3e^{-2t}+c_4e^{-12t}+5/2}$$

Step 4

將

$$i_2(t) = c_1 e^{-2t} + c_2 e^{-12t}$$
$$i_3(t) = c_3 e^{-2t} + c_4 e^{-12t} + 5/2$$

代回(式 1)

$$(8c_1 + 4c_3)e^{-2t} + (-2c_2 + 4c_4)e^{-12t} + 10 = 10$$

$$\begin{aligned} 8c_1 + 4c_3 &= 0 \\ -2c_2 + 4c_4 &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} c_3 &= -2c_1 \\ c_4 &= c_2/2 \end{aligned}$$

$$\boxed{i_2(t) = c_1 e^{-2t} + c_2 e^{-12t}}$$

$$\boxed{i_3(t) = -2c_1 e^{-2t} + c_2 e^{-12t}/2 + 5/2}$$

此即 $i_2(t)$ 和 $i_3(t)$ 的解

問題：需要代回另一式嗎？

將 $i_2(t) = c_1 e^{-2t} + c_2 e^{-12t}$

$i_3(t) = -2c_1 e^{-2t} + c_2 e^{-12t}/2 + 5/2$ 代回(式 2)

$$(4+4-8)c_1 e^{-2t} + (4-6+2)c_2 e^{-12t} + 10 = 10$$

無論 c_1 和 c_2 的值為多少，等號皆成立

●● 較快速的解法

Step 2-2 將 $i_2(t)$ 解出來以後

直接將 $i_2(t)$ 代回 (式1)

$$(D+10)(c_1e^{-2t}+c_2e^{-12t})+4i_3=10$$

$$-2c_1e^{-2t}+10c_1e^{-2t}-12c_2e^{-12t}+10e^{-2t}+4i_3=10$$

$$4i_3=-8c_1e^{-2t}+2c_2e^{-12t}+10$$

$$i_3=-2c_1e^{-2t}+c_2e^{-12t}/2+5/2$$

但這種簡化的解法不是任何情形都適用

(式子當中沒有對第二個 dependent variable $i_3(t)$ 作微分時才適用)

Example on text page 184

$$Dx - 3y = 0$$
$$2x - Dy = 0$$

Example 1 (text page 185)

$$Dx + (D+2)y = 0$$
$$(D-3)x - 2y = 0$$

Example 3 (text page 186)

$$\frac{d}{dt}x_1 = -\frac{2}{25}x_1 + \frac{1}{50}x_2$$
$$\frac{d}{dt}x_2 = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

Example 2 (text page 185)

$$x' - 4x + y'' = t^2$$
$$x' + x + y' = 0$$

Step 1 $(D-4)x + D^2 y = t^2$ (式1)

$(D+1)x + Dy = 0$ (式2)

Step 2-1 (式2) $\times D$ − (式1)

$$(D^2+4)x = -t^2$$

Step 3-1 complementary function: $x_c = c_1 \cos(2t) + c_2 \sin(2t)$

particular solution: $x_p = At^2 + Bt + C$

$$4At^2 + 4Bt + 4C + 2A = -t^2 \qquad A = -1/4, \quad B = 0, \quad C = 1/8$$

$$x_p = -\frac{1}{4}t^2 + \frac{1}{8}$$

$$x = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}t^2 + \frac{1}{8}$$

Step 2-2 (式1) ×(D+1) − (式2) × (D−4)

$$(D^3+4D)y=(D+1)t^2=t^2+2t$$

Step 3-2 complementary function:

$$y_c=c_3+c_4\cos 2t+c_5\sin 2t$$

particular solution

$$y_p=At^3+Bt^2+Ct$$

注意，不可設為 $y_p=At^2+Bt+C$

$$12At^2+8Bt^2+6A+4C=t^2+2t$$

$$A=1/12,\quad B=1/4,\quad C=-1/8$$

$$y_p=\frac{1}{12}t^3+\frac{1}{4}t^2-\frac{1}{8}t$$

$$y=c_3+c_4\cos 2t+c_5\sin 2t+\frac{1}{12}t^3+\frac{1}{4}t^2-\frac{1}{8}t$$

Step 4 (代入式2) (因為式2比式1容易)

$$(c_1 + 2c_2 + 2c_5)\cos(2t) + (c_2 - 2c_1 - 2c_4)\sin(2t) = 0$$

$$c_5 = -c_2 - \frac{1}{2}c_1, \quad c_4 = \frac{1}{2}c_2 - c_1$$

解

$$\boxed{x = c_1\cos(2t) + c_2\sin(2t) - \frac{1}{4}t^2 + \frac{1}{8}}$$

$$\boxed{y = c_3 + (\frac{1}{2}c_2 - c_1)\cos 2t - (c_2 + \frac{1}{2}c_1)\sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t}$$

4-9-4 多個 Dependent Variables

Exercise 19

$$\frac{d}{dt}x = 6y \qquad Dx - 6y = 0 \qquad (式1)$$

$$\frac{d}{dt}y = x + z \longrightarrow x - Dy + z = 0 \qquad (式2)$$

$$\frac{d}{dt}z = x + y \qquad x + y - Dz = 0 \qquad (式3)$$

Steps 2, 3： 分別簡化成只包含 x, y, z 的 DE

$$(式2) \times D + (式3) \qquad (D+1)x + (1-D^2)y = 0 \qquad (式4)$$

$$(式4) \times 6 + (式1) \times (1-D^2)$$

$$(-D^3 + 7D + 6)x = 0 \qquad -m^3 + 7m + 6 = 0$$

$$m = -1, -2, 3$$

$$\boxed{x = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{3t}}$$

(式4) × D − (式1) × (1+D)

$$(-D^3+7D+6)y=0 \qquad \boxed{y=c_4e^{-t}+c_5e^{-2t}+c_6e^{3t}}$$

(式3) × 6 + (式1) $\quad (D+6)x-6Dz \quad$ (式5)

(式1) × D − (式2) × 6 $\quad (D^2-6)x-6z=0 \quad$ (式6)

(式5) × (D^2 −6) + (式6) × (D+6)

$$(6D^3-42D-36)z=0 \qquad \boxed{z=c_7e^{-t}+c_8e^{-2t}+c_9e^{3t}}$$

Step 4：把 $c_4, c_5, c_6, c_7, c_8, c_9$ 用 c_1, c_2, c_3 表示

將 $\quad x=c_1x^{-t}+c_2x^{-2t}+c_3x^{3t} \quad$ 代回 (式1)

$$y=c_4x^{-t}+c_5x^{-2t}+c_6x^{3t}$$

$$6c_4=-c_1\ ,\ 6c_5=-2c_2\ ,\ 6c_6=3c_3$$

$$\boxed{y=-\frac{1}{6}c_1e^{-t}-\frac{1}{3}c_2e^{-2t}+\frac{1}{2}c_3e^{3t}}$$

將

$$x = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{3t}$$

$$y = -\frac{1}{6} c_1 e^{-t} - \frac{1}{3} c_2 e^{-2t} + \frac{1}{2} c_3 e^{3t}$$

代回 (式2)

$$z = c_7 e^{-t} + c_8 e^{-2t} + c_9 e^{3t}$$

$$c_7 = -\frac{5}{6} c_1 \,,\, c_8 = -\frac{1}{3} c_2 \,,\, c_9 = \frac{1}{2} c_3 \quad \boxed{z = -\frac{5}{6} c_1 e^{-t} - \frac{1}{3} c_2 e^{-2t} + \frac{1}{2} c_3 e^{3t}}$$

思考：y 和 z 有無快速解法？

Higher order 規則

(1) 假設有 N 個 dependent variables,

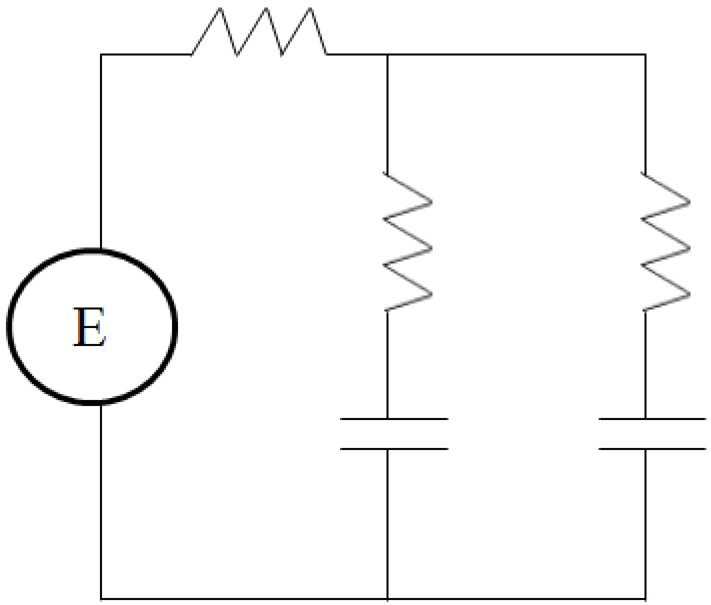
則至少需要有 N 個 DE 才可以得出 solutions

(2) 若每一個 DE 的 orders 分別為 $k_1, k_2, k_3, \ldots\ldots\ldots, k_N$

最後將「可能」得出 order 為 $k_1 + k_2 + k_3 + \ldots\ldots\ldots + k_N$ 的 DE

(3) 要代回其中 $N-1$ 個式，來求 unknowns 之間的關係

A circuit that can be modeled by a 2nd order polynomial.



4-9-5 本節需注意的地方

(1) Section 4.9 的方法只適用於 constant coefficients 的情形

(2) 每一個 dependent variable 的解，

homogeneous 的部分通常會有相同的型態

(3) 概念不難，但計算繁雜

(自我訓練解題速度和解題正確度是必要的)

(4) 驗算

(5) 別忘了 Step 4 計算 unknowns 之間的關係

(6) 何時可以用較快速的解法？

4-10 Nonlinear Differential Equations

Method 1: Reduction of Order

Method 2: Taylor Series

Method 3: Numerical Approach

4-10-1 Method 1: Reduction of Order

精神：變成 1st order DE

再用 1st order DE 的方法求解

(這方法的名字和 Section 4-2 一樣，但是不限於 linear，

而且不必知道其中一個解)

限制：The DE should have the form of

Case 1, page 313

$$F\left(x\,,\frac{d}{dx}y\,,\frac{d^2}{dx^2}y\right)=0$$

(Without the term y)

or

Case 2, page 315

$$F\left(y\,,\frac{d}{dx}y\,,\frac{d^2}{dx^2}y\right)=0$$

(Without the term x)

Case 1: The 2nd order DE has the form of $F\left(x, \frac{d}{dx}y, \frac{d^2}{dx^2}y\right) = 0$

(Without the term y)

$$F\left(x, \underbrace{\frac{d}{dx}y}_{u}, \underbrace{\frac{d^2}{dx^2}y}_{\frac{d}{dx}u}\right) = 0$$

解法： (Step 1) Set $u = \frac{d}{dx}y$

此時DE 變成 $F\left(x, u, \frac{d}{dx}u\right) = 0$ (對 u 而言，是 1st order DE)

(Step 2) 將 u 解出來 (用 Section 2 的方法)

(Step 3) 對 u 作積分，即解出 y

Example 1 (text page 189)

$$y'' = 2x(y')^2$$

(Step 1) $u = y'$

$$\frac{d}{dx}u = 2xu^2$$

問題： u 要用什麼方法解？

(Step 2) $u = -\dfrac{1}{x^2 + c}$

(Step 3) $y = -\displaystyle\int \frac{1}{x^2 + c}dx = ?$

Case 2: The 2nd order DE has the form of $F\left(y\,,\frac{d}{dx}y\,,\frac{d^2}{dx^2}y\right)=0$

(Without the term x)

$u \qquad u\frac{d}{dy}u$

解法：(Step 1) Set $u=\frac{d}{dx}y$

$$\frac{d^2}{dx^2}y=\frac{d}{dx}u=\frac{dy}{dx}\frac{d}{dy}u=u\frac{d}{dy}u \qquad \text{(Chain rule)}$$

此時DE 變成 $F\left(y\,,u\,,u\frac{d}{dy}u\right)=0$

(對 u 而言，是1st order DE, independent variable 為 y)

(Step 2) 將 u 解出來 (用 Section 2 的方法)

得出的解, u 是 y 的函數 $u = F_1(y)$

(Step 3) $\frac{dy}{dx} = F_1(y)$ $\frac{dy}{F_1(y)} = dx$

用 separable variable 的方法即可將解得出

Example 2 (text page 190)

$$yy'' = (y')^2$$

(Step 1) Set $u = \frac{d}{dx}y$

$$y \cdot u \frac{d}{dy}u = u^2$$

(Step 2) $\frac{du}{u} = \frac{dy}{y}$ $\qquad \ln|u| = \ln|y| + c_1$ $\qquad |u| = |y|e^{c_1}$

$$u = c_2 y \qquad (c_2 = \pm e^{c_1})$$

(Step 3) $\frac{dy}{dx} = c_2 y$ $\qquad \frac{dy}{y} = c_2 dx$ $\qquad \ln|y| = c_2 x + c_3$ $\qquad |y| = e^{c_2 x}e^{c_3}$

$$y = c_4 e^{c_2 x} \quad (c_4 = \pm e^{c_3})$$

4-10-2 Method 2: Taylor Series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots\cdots$$

更一般化的型態

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \frac{y^{(4)}(x_0)}{4!}(x - x_0)^4 + \cdots\cdots$$

Step 1 算出 $y(x_0)$, $y'(x_0)$, $y''(x_0)$, $y'''(x_0)$, $y^{(4)}(x_0)$, $\cdots\cdots$

Step 2 代回 Taylor series

Example 3 (text page 190)

$$y'' = x + y - y^2 \qquad y(0) = -1 \qquad y'(0) = 1$$

$$y'' = x + y - y^2 \qquad y''(0) = 0 + (-1) - 1^2 = -2$$

$$y''' = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2y'y \qquad y'''(0) = 4$$

$$y^{(4)} = \frac{d}{dx}(1 + y' - 2y'y) = y'' - 2y''y - 2(y')^2 \qquad y^{(4)}(0) = -8$$

$$y^{(5)} = \frac{d}{dx}(y'' - 2y''y - 2(y')^2) = y''' - 2y'''y - 6y'y'' \qquad y^{(5)}(0) = 24$$

$$\vdots$$

代回 Taylor series

$$\boxed{y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 \cdots\cdots}$$

限制：(1) $y(x)$ 在 x_0 的地方必需為 analytic,

($x = x_0$ 不為 singular point)

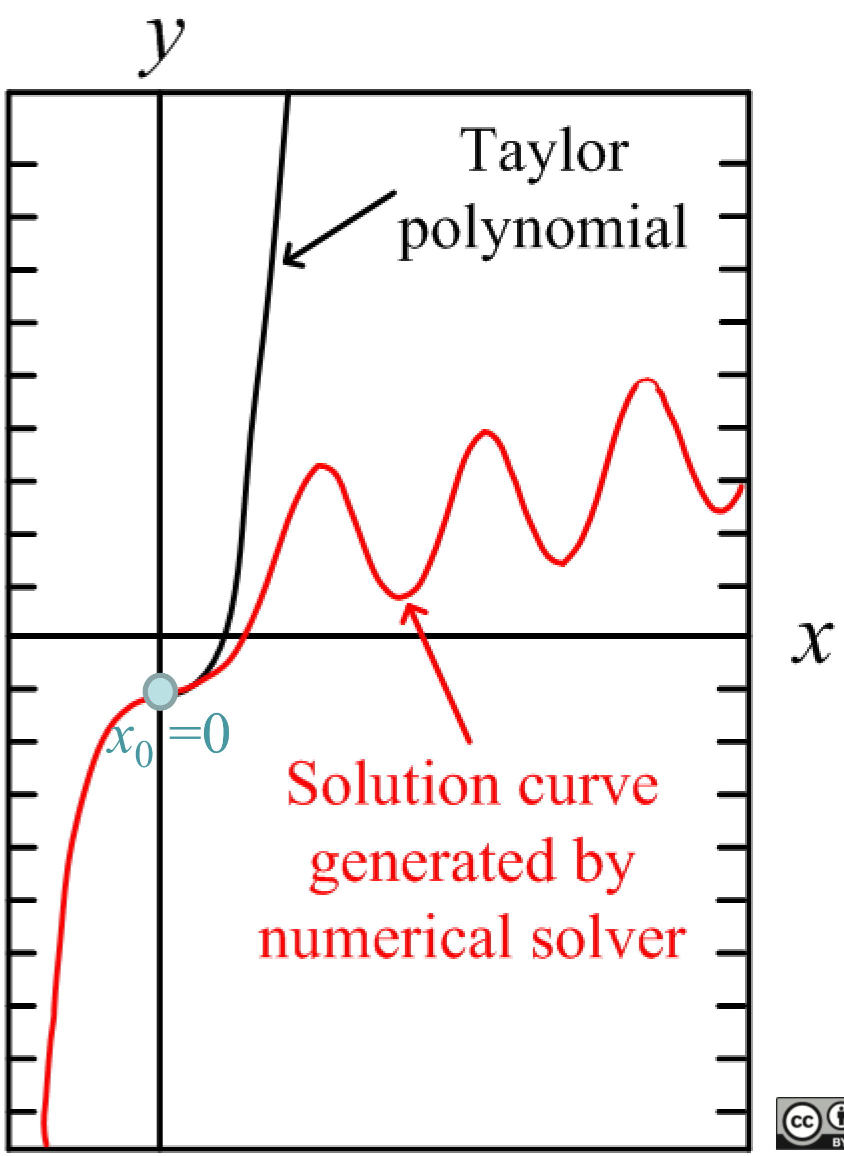
(2) 在解 n^{th} order DE 時，$y(x_0)$， $y'(x_0)$， $y''(x_0)$， …..

$y^{(n-1)}(x_0)$ 的值必需皆為已知

(3) 得出的解只有在 x_0 附近較為正確

問題：(1) Taylor series 應該取多少項？

(2) $|x - x_0|$ 的範圍？


y
Taylor
polynomial
x
$x_0=0$
Solution curve
generated by
numerical solver

4-10-3 Method 3: Numerical Method

$$\frac{d^2 y}{dx^2} = f(x, y, y') \qquad y(x_0) = y_0 \qquad y'(x_0) = u_0$$

解法 $\begin{cases} y' = u \\ u' = f(x, y, u) \end{cases}$ subject to $y(x_0) = y_0$, $u(x_0) = u_0$

使用Section 2-6 的 Euler's Method

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n) y'(x_n)$$

$$\begin{cases} y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n) u(x_n) \\ u(x_{n+1}) = u(x_n) + (x_{n+1} - x_n) u'(x_n) \\ \qquad\qquad = u(x_n) + (x_{n+1} - x_n) f(x_n, y(x_n), u(x_n)) \end{cases}$$

$$\begin{cases} y' = u \\ u' = f(x, y, u) \end{cases} \qquad y(x_0) = y_0\ , \quad u(x_0) = u_0$$

Recursive 的解法

Initial: $y(x_0) = y_0\ , \quad u(x_0) = u_0 \qquad n = 0$

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n)u(x_n)$$

$n = n + 1$

$$u(x_{n+1}) = u(x_n) + (x_{n+1} - x_n)f(x_n, y(x_n), u(x_n))$$

更一般化的情形

$$\frac{d^k y}{dx^k} = f\left(x, y, y', y'', \ldots\ldots, y^{(k-1)}\right)$$

$$y(x_0) = y_{0,0} \quad y'(x_0) = y_{0,1} \quad y''(x_0) = y_{0,2} \quad \ldots\ldots\ldots\ldots\ldots$$

$$y^{(k-1)}(x_0) = y_{0,k-1}$$

改變為

$$\begin{cases} y' = u_1 \\ u_1' = y'' = u_2 \\ u_2' = y''' = u_3 \\ \vdots \\ u_{k-2}' = y^{(k-1)} = u_{k-1} \\ u_{k-1}' = f\left(x, y, u_1, u_2, \ldots\ldots, u_{k-1}\right) \end{cases} \quad \text{subject to} \quad \begin{aligned} y(x_0) &= y_{0,0} \\ u_1(x_0) &= y_{0,1} \\ u_2(x_0) &= y_{0,2} \\ &\vdots \\ u_{k-1}(x_0) &= y_{0,k-1} \end{aligned}$$

Recursive 的解法

$$\text{Initial}: \quad y(x_0) = y_{0,0}\,, \quad u_1(x_0) = y_{0,1}\,, \quad u_2(x_0) = y_{0,2}\,, \quad \ldots\ldots\,, u_{k-1}(x_0) = y_{0,k-1}\,, \qquad n = 0$$

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n)u_1(x_n)$$

$$u_1(x_{n+1}) = u_1(x_n) + (x_{n+1} - x_n)u_2(x_n)$$

$$u_2(x_{n+1}) = u_2(x_n) + (x_{n+1} - x_n)u_3(x_n)$$

$$\vdots$$

$$u_{k-2}(x_{n+1}) = u_{k-2}(x_n) + (x_{n+1} - x_n)u_{k-1}(x_n)$$

$$u_{k-1}(x_{n+1}) = u_{k-1}(x_n) + (x_{n+1} - x_n)\, f\big(x_n, y(x_n), u_1(x_n), u_2(x_n), \ldots\ldots, u_{k-1}(x_n)\big)$$

$$n = n + 1$$

解法的限制：

$$\frac{d^k y}{dx^k} = f\left(x, y, y', y'', \ldots\ldots, y^{(k-1)}\right)$$

(1) 當 $f\left(x, y, y', y'', \ldots\ldots, y^{(k-1)}\right)$ 為無窮大 (例如 singular point)

或者 $f\left(x, y, y', y'', \ldots\ldots, y^{(k-1)}\right)$ 雖然不是無窮大，但是值相當大

用以上的方法會產生問題

(2) 必需有 k 個在同一點的initial conditions

4-10-4 本節需注意的地方

(1) Section 4.10 的方法並非任何情形都可以解

每一種方法都有一些限制

(2) Section 4.1 的定理不適用於本節 (例如 exercises 4.10 的第 1, 2 題)

(3) Method 1 比較有挑戰性，要多加變通

(4) Method 1 別忘了將 u 用 dy/dx 代回

5-3 Nonlinear Models

非線性彈簧的例子 (text pages 222, 223)

鍾擺的例子 (text pages 223, 224, 225)

電話線的例子 (text pages 225, 226)

火箭的例子 (text page 226)

拿鏈子的例子 (text pages 227, 228)

5-3-1 火箭的例子

$$F = ma \qquad F = m\frac{d^2y(t)}{dt^2}$$

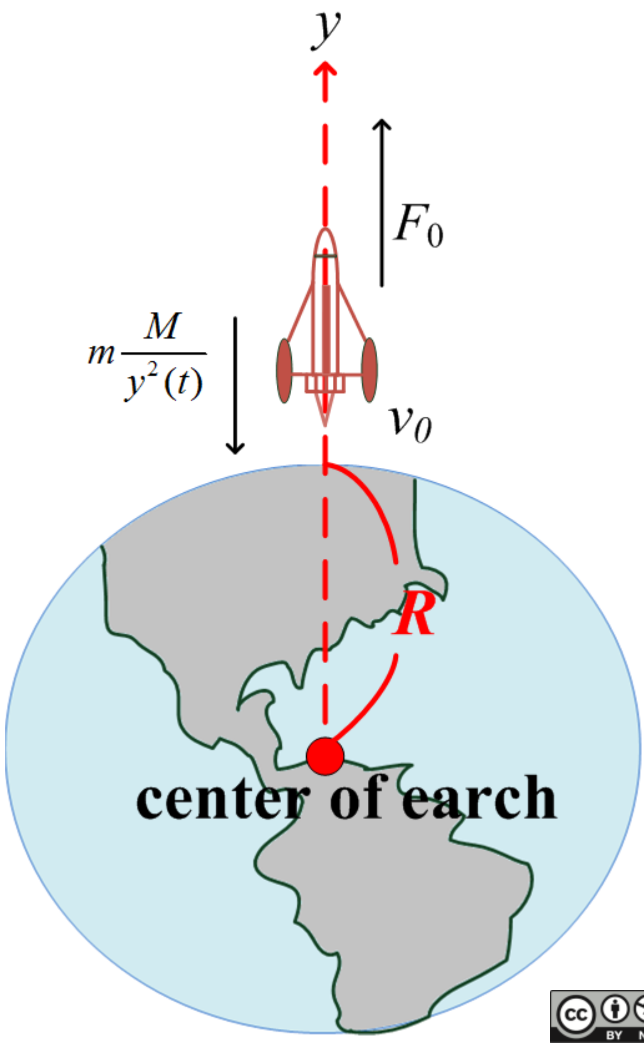
F 會隨著 y 而改變 (萬有引力定律)

$$\underbrace{-k\frac{mM}{y^2(t)}}_{\text{萬有引力}} = m\frac{d^2y(t)}{dt^2}$$

M: 地球的質量　　m: 火箭的質量

- 修正：

$$\underbrace{F_0}_{\text{推進力}} - k\frac{mM}{y^2(t)} = m\frac{d^2y(t)}{dt^2}$$



5-3-2 拿鏈子的例子

$$F = \frac{d}{dt} mv$$

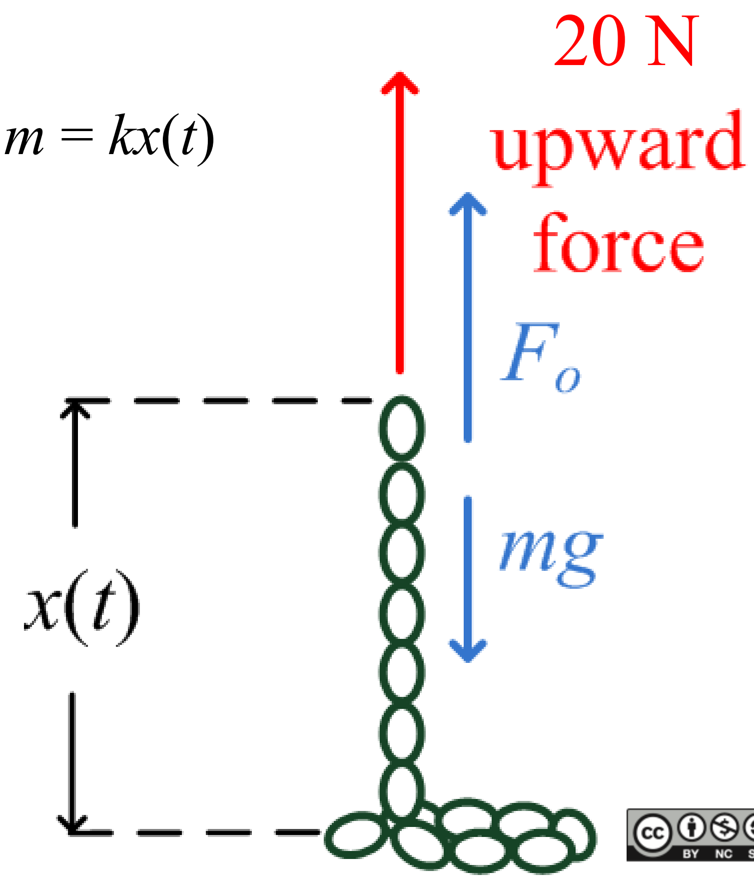
m: 質量, v: 速度, mv: 動量

m 會隨著 x 而改變 (拿鏈子的例子), $m = kx(t)$

In Example 4, chain weight = 1 N/m

重量 (weight) = $x(t)$

質量 (mass) = $x(t)/9.8$



$$F_0 - mg = v\frac{d}{dt}m + m\frac{d}{dt}v, \qquad (F = F_0 - mg)$$

$$F_0 - kgx(t) = \left(\frac{d}{dt}x(t)\right)\frac{d}{dt}kx(t) + kx(t)\frac{d^2}{dt^2}x(t),$$

$$kx(t)\frac{d^2}{dt^2}x(t) + k\left(\frac{d}{dt}x(t)\right)^2 + kgx(t) = F_0$$

F_0: 施力, k: 每單位長的質量, $x(t)$ 高度 (如前頁)

小常識：使用公制時， $g = 9.8$ metres per s^2

使用英制時， $g = 32$ feet per s^2

Example 4 (text page 227)

$$x\frac{d^2x}{dt^2}+(\frac{dx}{dt})^2+9.8x=196$$

$k = 1/9.8$

$g = 9.8$

$F_0 = 20$

(1) Cauchy-Euler equation 缺乏應用

(2) 許多情形還是只能用 Numerical Method 來解

5-3-3 本節需注意的地方

(1) 大部分的情形，用到的 DE 都還算很簡單

但是練習將將物理問題變成 DE 的問題。

(2) 正負號 (和方向有關）易出錯

(3) $\tan\theta =$ 斜率

(4) 儘可能用比較簡單的方法來計算一個問題

Reviews for Higher Order DE:

(A) Linear DE Complementary Function 3 大解法

(1) Reduction of Order (Section 4-2)

$$\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx}$$

適用情形：

(2) Auxiliary Function (Section 4-3)

$$\boxed{a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

$$\downarrow$$

$$\boxed{a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0}$$

適用情形：

4 Cases (See pages 182, 183)

(3) Cauchy-Euler Equation (Section 4-7)

$$\boxed{a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0}$$

$$\downarrow$$

$$\boxed{a_n \frac{m!}{(m-n)!} + a_{n-1}\frac{m!}{(m-n+1)!} + \cdots + a_1 \frac{m!}{(m-1)!} + a_0 = 0}$$

適用情形：

(B) Linear DE Particular solution 3 大解法

(1) Guess (Section 4-4) (熟悉講義 page 194 的表)

要訣： y_p should be a linear combination of $g(x)$, $g'(x)$,

$g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ...

適用情形：

遇到重覆，乘 x 或 $\ln x$

(2) Annihilator (Section 4-5)

若原本的 DE 為 $L[y(x)] = g(x)$　　Annihilator: $L_1[\,g(x)] = 0$

Particular solution 為 $L_1\{L[y(x)]\} = 0$ 的解

(扣去和 $L[y(x)] = 0$ 的解重複的部分)

$y = y_c + y_p$

適用情形：

Annihilator 算法三大規則： Pages 213-215

(3) Variation of parameters (Section 4-6)

$$y_p = u_1 y_1 + u_2 y_2 + \cdots\cdots + u_n y_n$$

$$u_k'(x) = \frac{W_k}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

W_k : replace the k^{th} column of W by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$

適用情形：

$$f(x) = \frac{g(x)}{a_n(x)}$$

(4) For Cauchy-Euler Equation (Section 4-7)

可採用二種方法

(1) 先用

$$a_n \frac{m!}{(m-n)!} + a_{n-1} \frac{m!}{(m-n+1)!} + \cdots + a_1 \frac{m!}{(m-1)!} + a_0 = 0$$

解 complementary function

再用 Variation of parameters 解 particular solution

(2) Use the method on pages 261, 262

Set $x = e^t$, $t = \ln x$

then $x^k \frac{d^k y}{dx^k} = (D_t - k + 1) \cdots (D_t - 1) D_t y$

(C) Combination of Linear DEs (Section 4-9)

方法：

Step 1: 將 $\dfrac{d^n}{dt^n}$ 變成 D^n

Steps 2, 3: 用代數消去法，變成只含有一個 dependent variable 的 DE，再將這個 dependent variable 解出來

Step 4: 代回原式，求各 dependent variable 的常數 c_k 之間的關係

適用情形：

(D) Nonlinear DE 的3大解法 (Section 4-10)

(1) Reduction of Order

(1-1) $F\left(x, \frac{d}{dx}y, \frac{d^2}{dx^2}y\right) = 0 \longrightarrow F\left(x, u, \frac{d}{dx}u\right) = 0$

Set $u = \frac{d}{dx}y$

(1-2) $F\left(y, \frac{d}{dx}y, \frac{d^2}{dx^2}y\right) = 0 \longrightarrow F\left(y, u, u\frac{d}{dy}u\right) = 0$

Set $u = \frac{d}{dx}y$

(2) Taylor Series

適用情形：$y^{(n)} = F\left(x, y, y', y'', \cdots\cdots, y^{(n-1)}\right)$

$$F\left(x, y, y', y'', \cdots\cdots, y^{(n-1)}\right) \neq \infty$$

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \\ + \frac{y^{(4)}(x_0)}{4!}(x - x_0)^4 + \cdots\cdots$$

(3) Numerical Method

適用情形：$y^{(n)} = F\left(x, y, y', y'', \cdots\cdots, y^{(n-1)}\right)$

$$F\left(x, y, y', y'', \cdots\cdots, y^{(n-1)}\right) \neq \infty$$

Exercises for practicing

Section 4-9	5, 8, 10, 14, 17, 18, 20, 22, 23
Section 4-10	1, 4, 5, 8, 10, 12, 15, 16, 19, 21, 22, 23
Review 4	43, 44, 48, 50
Section 5-1	1, 11, 29, 43, 44, 49, 52, 56, 60
Section 5-3	14, 15, 16
Review 5	12, 21, 22

註：應用題 2012版本單位用英制，2016, 2017版本用公制