Chapter 7 The Laplace Transform

作用：把微分變成乘法

Chapter 4 曾經提過 $\frac{d^k}{dt^k}y(t)$ 可寫成 $D^k y(t)$

Laplace transform可以將 $\frac{d^k}{dt^k}y(t)$ 變成

$$s^k Y(s) - s^{k-1}y(0) - s^{k-2}y'(0) - \cdots\cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)$$

Section 7-1 Definition of the Laplace Transform

7-1-1 Definitions

- **Laplace Transform** of $f(t)$

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt}$$

經常以大寫來代表 transform 的結果

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Laplace Transform is one of the integral transform

- transform:　把一個 function 變成另外一個 function

- integral transform:　可以表示成積分式的 transform

$$F(s)=\int_a^b \underbrace{K(s,t)}_{\text{kernel}} f(t)dt$$

對 Laplace transform 而言

$$K(s,t)=e^{-st}, \qquad a=0, \quad b\to\infty$$

註：Chap. 14 將教到的 Fourier transform, 也是一種 integral transform

7-1-2 Linear Property

$$\int_0^{\infty} e^{-st}\left[\alpha f(t)+\beta g(t)\right]dt=\alpha\int_0^{\infty} e^{-st} f(t)dt+\beta\int_0^{\infty} e^{-st} g(t)dt$$

$$\mathcal{L}\left\{\alpha f(t)+\beta g(t)\right\}=\alpha\mathcal{L}\left\{f(t)\right\}+\beta\mathcal{L}\left\{g(t)\right\}$$

事實上，<u>所有的 integral transform 都有linear property</u>

7-1-3 The Laplace Transforms of Some Basic Functions

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(at)$	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

(彼此密切相關)

Example 1 $\mathcal{L}\{1\}$ (text page 280)

$$\mathcal{L}\{1\}=\int_0^\infty e^{-st}dt=-\frac{e^{-st}}{s}\bigg|_0^\infty=-\frac{e^{-s\cdot\infty}}{s}-(-\frac{e^{-s\cdot 0}}{s})=\frac{1}{s}$$

(1) $-\frac{e^{-s\cdot\infty}}{s}$ 比較正式的寫法是 $\lim_{b\to\infty}-\frac{e^{-s\cdot b}}{s}$

(2) 這裡假設 $\mathcal{Re}(s)>0$, 所以 $-\frac{e^{-s\cdot\infty}}{s}=0$

Example 2 $\mathcal{L}\{t\}$ (text page 280)

$$\int_a^b u(t)v'(t)\,dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)\,dt$$

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,dt$$

$$= -\frac{te^{-st}}{s}\Bigg|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= -\frac{\infty \cdot e^{-s\cdot\infty}}{s} + \frac{0 \cdot e^{-s\cdot 0}}{s} - \frac{e^{-st}}{s^2}\Bigg|_0^\infty$$

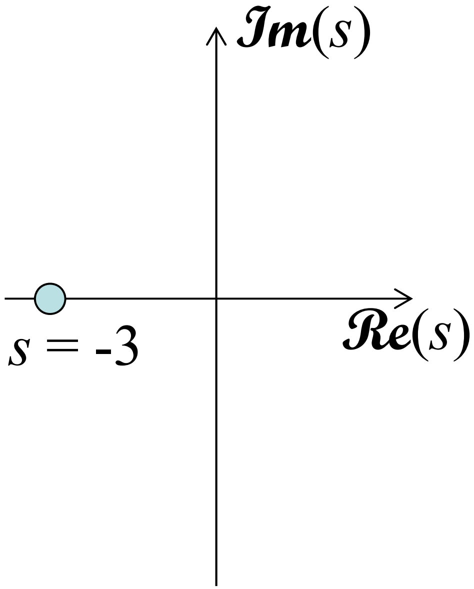
$$= -\frac{e^{-s\cdot\infty}}{s^2} + \frac{e^{-s\cdot 0}}{s^2}$$

$$= \frac{1}{s^2}$$

Example 3 $\mathcal{L}\left\{e^{-3t}\right\} = \dfrac{1}{s+3}$ (text page 280)

Pole (分母為0 的地方) 在複數平面左半邊 $\longrightarrow$ stable

Pole 在複數平面右半邊 $\longrightarrow$ unstable



Example 4 $\mathcal{L}\{\sin(2t)\}$ (text page 281)

除了課本的解法之外，

另一個解法 $\sin(2t) = \frac{1}{2i}\left(e^{i2t} - e^{-i2t}\right)$

$$\mathcal{L}\{\sin(2t)\} = \frac{1}{2i}\mathcal{L}\left\{e^{i2t}\right\} - \frac{1}{2i}\mathcal{L}\left\{e^{-i2t}\right\} = \frac{1}{2i}\frac{1}{s-i2} - \frac{1}{2i}\frac{1}{s+i2}$$

$$= \frac{1}{2i}\frac{s+i2-(s-i2)}{(s-i2)(s+i2)} = \frac{1}{2i}\frac{i4}{s^2+4} = \frac{2}{s^2+4}$$

Example 5 (text page 281)

$$\mathcal{L}\{1+5t\} = \mathcal{L}\{1\} + 5\mathcal{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}$$

$$\mathcal{L}\left\{4e^{-3t} - 10\sin 2t\right\} = 4\mathcal{L}\left\{e^{-3t}\right\} - 10\mathcal{L}\{\sin 2t\} = \frac{4}{s+3} - \frac{20}{s^2+4}$$

7-1-4 When Does the Laplace Transforms Exist?

Constraint 1 for the existence of the Laplace transform :

For a function $f(t)$, there should exist constants c, $M >$ 0, and $T > 0$ such that

$$|f(t)| \le Me^{ct} \quad \text{for all } t > \mathrm{T}$$

In this condition, $f(t)$ is said to be of exponential order c

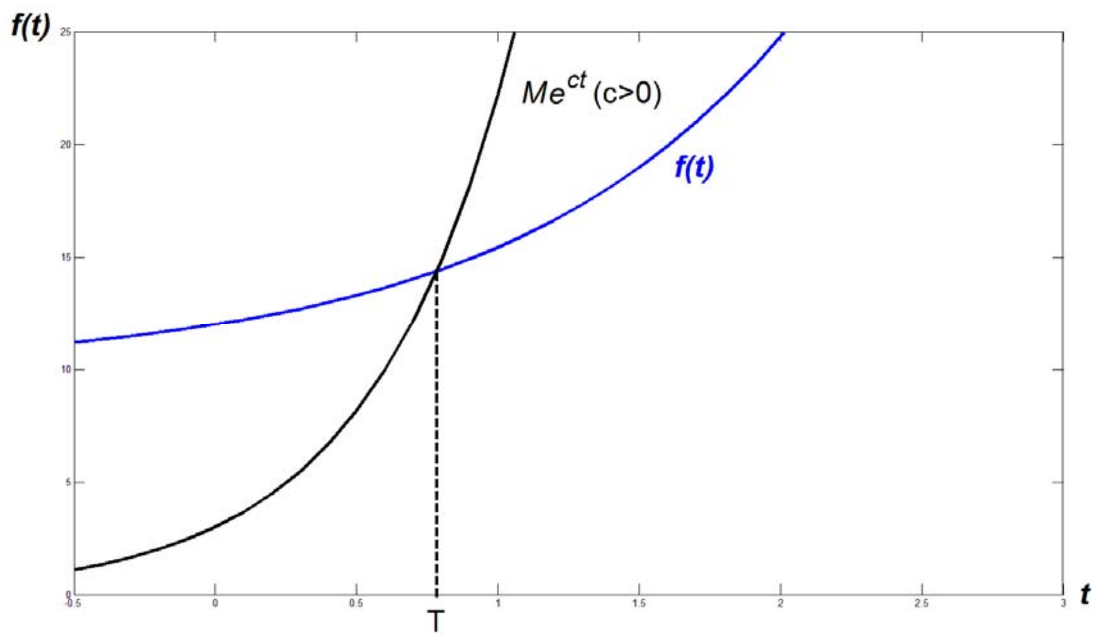


Fig. 7.1.2

Example： $f(t) = t,\ \ e^{-t}, 2\cos t$ 皆為 exponential order 1

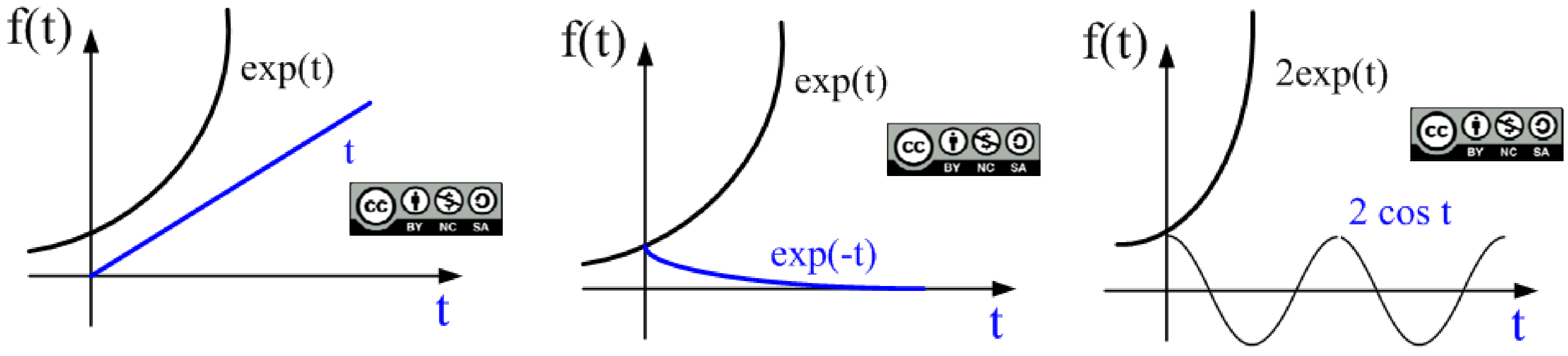


Fig. 7.1.3 **(a)** **(b)** **(c)**

補充：其實，對一個function 而言， exponential order c 不只一個

例子： $f(t) = t^n$ 為 exponential order c, $c > 0$

There exists an M such that $\left|\dfrac{t^n}{e^{ct}}\right| \le M$ if $c > 0$

Example： $f(t) = \exp(t^2)$ 時，並不存在一個 c 使得

$$\left|f(t)\right| \le Me^{ct} \quad \text{for all } t > T$$

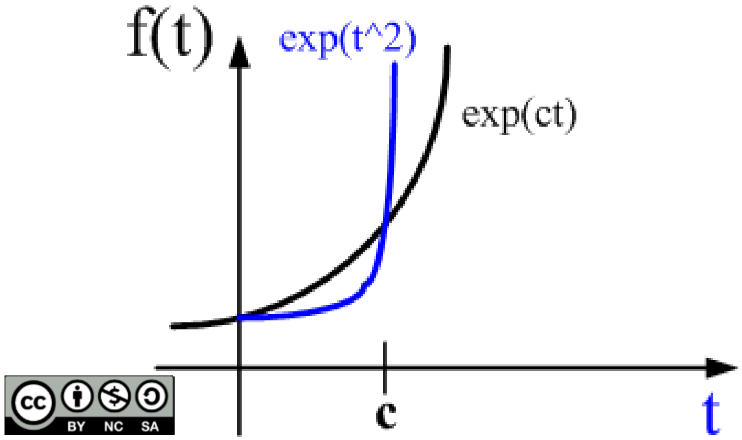


Fig. 7.1.4

只要有一個 c 使得 $\left|f(t)\right| \le Me^{ct}$ for all $t > T$

我們稱 $f(t)$ 為 of exponential order

否則，我們稱 $f(t)$ 為 not of exponential order

Constraint 2 for the existence of the Laplace transform **:**

$f(t)$ should be piecewise continuous on $[0, \infty)$

在任何 $t \in [a, b]$ 的區間內 $(0 \le a \le b < \infty)$

$f(t)$ 為 discontinuous 的點的個數為有限的

稱作是「piecewise continuous」

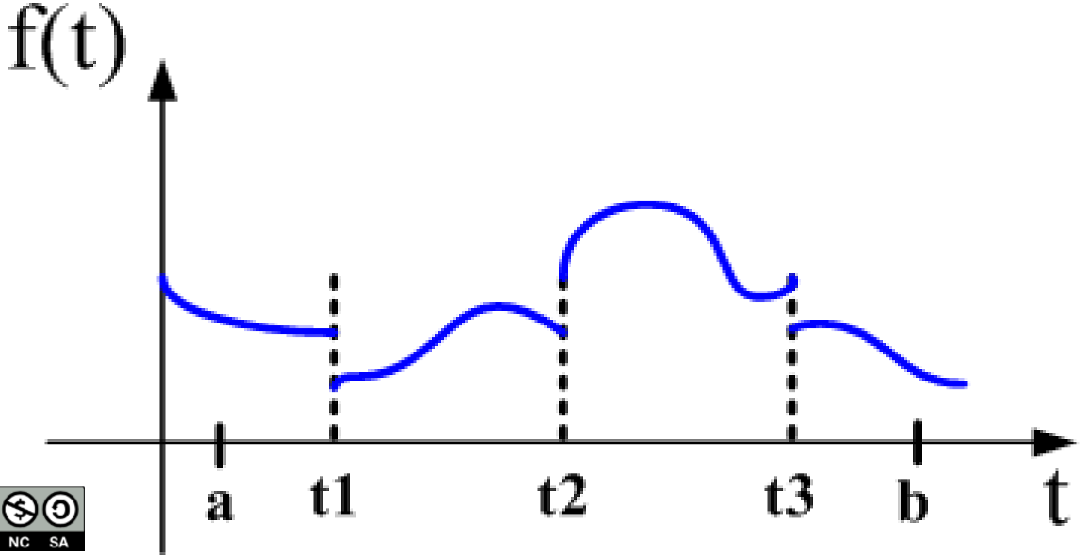


Fig. 7.1.1

注意： $1/t$ 不為 piecewise continuous

Constraints 1 and 2 are "sufficient conditions"

若滿足 $\longrightarrow$ Laplace transform 存在

若不滿足 $\longrightarrow$ Laplace transform 未必不存在

反例：$f(t) = t^{-1/2}$ 不為 piecewise continuous

但是 Laplace transform 存在　$F(s) = s^{-1/2}\sqrt{\pi}$

補充說明：$f(t) = t^{-1/2}$ 不為 piecewise continuous 是因為

$$f(0) \to \infty$$

所以 $f(t)$ 在 $t = 0$ 附近有無限多個不連續點

事實上，只要 $f(t_1) \to \infty$, $|t_1|$ is not infinite，

$f(t)$ 必定不為 piecewise continuous

Theorem 7.1.3

If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\lim_{s \to \infty} F(s) = 0$$

7-1-5 Section 7-1 需要注意的地方

(1) Laplace transform of some basic functions 要背起來

(2) 記公式時，一些地方要小心 sin, sinh, $1/t^n$

$$\sin kt \longrightarrow \frac{k}{s^2+k^2}$$

沒有平方

有平方

(3) 熟悉(a) 包含 exponential function 的積分

以及 (b) $\int_a^b u(t)v'(t)dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)dt$

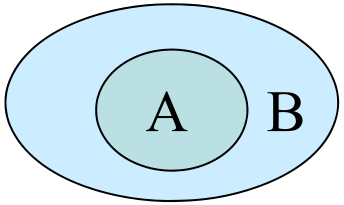
的積分技巧

(4) 要迅速判斷一個式子當 $t \longrightarrow \infty$ 時是否為 0

(5) 小心正負號

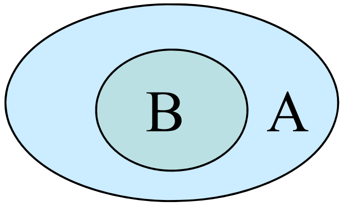
附錄七：充分條件和必要條件的比較

If A is satisfied, then B is also satisfied :



A is the sufficient conditions of B (充分條件)

If B is satisfied, then A is bound to be satisfied :



A is the necessary conditions of B (必要條件)

B is satisfied if and only if A is be satisfied :

A is the necessary and sufficient conditions of B

(充分且必要的條件)

Section 7-2 Inverse Transforms and Transforms of Derivatives

本節有兩大部分：

(1) inverse Laplace transform 的計算 (7-2-1 ~ 7-2-3)

(2) 將微分變成 Laplace transform 當中的乘法 (7-2-4 ~ 7-2-6)

7-2-1 Inverse 方法一： One-to-One Relation

When (1) $f_1(t)$ and $f_2(t)$ are piecewise continuous on $[0, \infty)$, and

(2) $f_1(t)$ and $f_2(t)$ are of exponential order, then

if $f_1(t) \neq f_2(t)$ $\longrightarrow$ then $F_1(s) \neq F_2(s)$

換句話說，在這種情形下，Laplace transform 是 one-to-one 的運算。

If the Laplace transform of $f_1(t)$ is $F_1(s)$,

then the inverse Laplace transform of $F_1(s)$ must be $f_1(t)$.

Table of Inverse Laplace Transforms

$F(s)$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$
$\dfrac{1}{s}$	1
$\dfrac{n!}{s^{n+1}}$	t^n
$\dfrac{1}{s-a}$	$\exp(at)$
$\dfrac{k}{s^2+k^2}$	$\sin(kt)$
$\dfrac{s}{s^2+k^2}$	$\cos(kt)$
$\dfrac{k}{s^2-k^2}$	$\sinh(kt)$
$\dfrac{s}{s^2-k^2}$	$\cosh(kt)$

Example 1 (text page 287) (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}=\frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\}=\frac{1}{4!}t^4$$

Example 2 (text page 287) $\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}=-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$
$$=-2\cos(2t)+3\sin(2t)$$

7-2-2 Inverse 方法 (二） Decomposition of Fractions

Example 3 (text page 288) $\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = Ae^{t} + Be^{2t} + Ce^{-4t}$$

問題：A, B, C 該如何算出？

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$s^2+6s+9 = (A+B+C)s^2 + (2A+3B-3C)s - 8A - 4B + 2C$$

太麻煩

7-2-3 計算分數分解係數的快速法

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

兩邊各乘上 $(s-1)$

$$\frac{s^2+6s+9}{(s-2)(s+4)}=A+(s-1)\frac{B}{s-2}+(s-1)\frac{C}{s+4}$$

把 $s=1$ 代入 $-\frac{16}{5}=A$　　　這二個步驟可以合併

左式乘上 $(s-2)$後，把 $s=2$ 代入

$$B=\left.\frac{s^2+6s+9}{(s-1)\cancel{(s-2)}(s+4)}\right|_{s=2}=\frac{25}{6}$$

左式乘上 $(s+4)$後，把 $s=-4$ 代入

$$C=\left.\frac{s^2+6s+9}{(s-1)(s-2)\cancel{(s+4)}}\right|_{s=-4}=\frac{1}{30}$$

通則：要將一個 fraction 分解　　(Cover up method)

$$\frac{K(s)}{(s-a_1)(s-a_2)\cdots\cdots(s-a_N)}=Q(s)+\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots\cdots+\frac{A_N}{s-a_N}$$

$a_1, a_2, \ldots., a_N$ 互異

(1) 用多項式的除法算出 $Q(s)$

$$\frac{K(s)}{(s-a_1)(s-a_2)\cdots\cdots(s-a_N)}=\underbrace{Q(s)}_{\text{商}}+\frac{\overbrace{K_1(s)}^{\text{餘式}}}{(s-a_1)(s-a_2)\cdots\cdots(s-a_N)}$$

使得 order of $K_1(s) < N$

(2) 算出 A_n

$$A_n=\frac{K_1(s)}{(s-a_1)(s-a_2)\cdots\cdots(s-a_{n-1})\cancel{(s-a_n)}(s-a_{n+1})\cdots\cdots(s-a_N)}\Bigg|_{s=a_n}$$

例子：$\boxed{\dfrac{s^4-8s^3+31s^2-36s+20}{(s-1)(s-2)(s-3)^2}=Q(s)+\dfrac{A_1}{s-1}+\dfrac{A_2}{s-2}+\dfrac{A_3+A_4(s-3)}{(s-3)^2}}$

$$\frac{s^4-8s^3+31s^2-36s+20}{(s-1)(s-2)(s-3)^2}=1+\frac{s^3+2s^2+3s+2}{(s-1)(s-2)(s-3)^2} \qquad Q(s)=1$$

$$A_1=\left.\frac{s^3+2s^2+3s+2}{\cancel{(s-1)}(s-2)(s-3)^2}\right|_{s=1}=-\frac{8}{4}=-2$$

$$A_2=\left.\frac{s^3+2s^2+3s+2}{(s-1)\cancel{(s-2)}(s-3)^2}\right|_{s=2}=24$$

$$\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}=(s-3)^2\frac{A_1}{s-1}+(s-3)^2\frac{A_2}{s-2}+A_3+A_4(s-3)$$

$$A_3=\left.\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}\right|_{s=3}=\frac{56}{2}=28$$

$$\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}=(s-3)^2\frac{A_1}{s-1}+(s-3)^2\frac{A_2}{s-2}+A_3+A_4(s-3)$$

$$A_4=\frac{d}{ds}\left.\frac{s^3+2s^2+3s+2}{(s-1)(s-2)}\right|_{s=3}=\left.\frac{(3s^2+4s+3)(s-1)(s-2)-(s^3+2s^2+3s+2)(2s-3)}{(s-1)^2(s-2)^2}\right|_{s=3}$$

$$=-21$$

$$\boxed{\frac{s^4-8s^3+31s^2-36s+20}{(s-1)(s-2)(s-3)^2}=1-\frac{2}{s-1}+\frac{24}{s-2}+\frac{28-21(s-3)}{(s-3)^2}}$$

- 小技巧：其實，如果只剩下一個未知數，我們可以將 s 用某個數代入， 快速的將未知數解出

例如，前面的例子，將 $s = 0$ 代入原式

$$\frac{2}{18} = -A_1 - \frac{A_2}{2} + \frac{A_3 - 3A_4}{9} \qquad A_4 = (-1 - 9A_1 - \frac{9A_2}{2} + A_3)/3 = -21$$

例子：$$\boxed{\frac{s^2+2s+3}{(s-1)(s^2+2s+2)}=\frac{A_1}{s-1}+\frac{A_2s+A_3}{s^2+2s+2}}$$

$$A_1=\left.\frac{s^2+2s+3}{\cancel{(s-1)}(s^2+2s+2)}\right|_{s=1}=\frac{6}{5}$$

$$\frac{s^2+2s+3}{(s-1)(s^2+2s+2)}-\frac{6/5}{s-1}=\frac{1}{5}\frac{-s^2-2s+3}{(s-1)(s^2+2s+2)}=\frac{1}{5}\frac{-s-3}{s^2+2s+2}$$

$$\boxed{\frac{s^2+2s+3}{(s-1)(s^2+2s+2)}=\frac{6/5}{s-1}-\frac{1}{5}\frac{s+3}{s^2+2s+2}}$$

7-2-4 Transforms of Derivatives

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = \underline{e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt}$$

$$= 0 - f(0) + s\mathcal{L}\{f(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\int_a^b u(t)v'(t)dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)dt$$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s\mathcal{L}\{f''(t)\} - f''(0) = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

Theorem 7.2.2 Derivative Property of the Laplace Transform

$$\mathcal{L}\left\{f^{(n)}(t)\right\}=s^{n}F(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

7-2-5 Solving the Constant Coefficient Linear DE by Laplace Transforms

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

$$\Big\downarrow \text{Laplace transform}$$

$$\boxed{\begin{aligned} & a_n\left[s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots\cdots - s y^{(n-2)}(0) - y^{(n-1)}(0)\right] \\ & + a_{n-1}\left[s^{n-1} Y(s) - s^{n-2} y(0) - s^{n-3} y'(0) - \cdots\cdots - s y^{(n-3)}(0) - y^{(n-2)}(0)\right] \\ & + \cdots\cdots \\ & + a_1\left[s Y(s) - y(0)\right] + a_0 Y(s) = G(s) \end{aligned}}$$

$$\boxed{\begin{aligned}&a_n\left[s^nY(s)-s^{n-1}y(0)-s^{n-2}y'(0)-\cdots\cdots-sy^{(n-2)}(0)-y^{(n-1)}(0)\right]\\&+a_{n-1}\left[s^{n-1}Y(s)-s^{n-2}y(0)-s^{n-1}y'(0)-\cdots\cdots-sy^{(n-3)}(0)-y^{(n-2)}(0)\right]\\&+\cdots\cdots\\&+a_1\left[sY(s)-y(0)\right]+a_0Y(s)=G(s)\end{aligned}}$$

$$\boxed{P(s)Y(s)-Q(s)=G(s)}$$

$$P(s)=a_ns^n+a_{n-1}s^{n-1}+\cdots\cdots a_1s+a_0$$ (auxiliary)

$$\begin{aligned}Q(s)=&a_n\left[s^{n-1}y(0)+s^{n-2}y'(0)+\cdots\cdots+sy^{(n-2)}(0)+y^{(n-1)}(0)\right]\\&+a_{n-1}\left[s^{n-2}y(0)+\cdots\cdots+sy^{(n-3)}(0)+y^{(n-2)}(0)\right]\\&+\cdots\cdots\\&+a_2\left[sy(0)+y'(0)\right]\\&+a_1\left[y(0)\right]\end{aligned}$$

$$Y(s)=\frac{Q(s)}{P(s)}+\frac{G(s)}{P(s)} \qquad Y(s)=W(s)Q(s)+W(s)G(s)$$

$$W(s)=\frac{1}{P(s)}$$

$G(s)$: Laplace transform of the input

$Q(s)$: caused by initial conditions

$Y(s)$: Laplace transform of the response

$W(s)$: transform function

$\mathcal{L}^{-1}[W(s)Q(s)]$: zero-input response or state response

$\mathcal{L}^{-1}[W(s)G(s)]$: zero-state response or input response

Example 4 (text page 290)

$$y'(t)+3y(t)=13\sin 2t \qquad y(0)=6$$

(Step 1) Laplace Transform

$$sY(s)-y(0)+3Y(s)=13\frac{2}{s^2+4}$$

$$(s+3)Y(s)=6+\frac{26}{s^2+4}$$

(Step 2) Decompose

$$Y(s)=\frac{6}{s+3}+\frac{26}{(s+3)(s^2+4)}$$

$$\left.\frac{26}{\cancel{(s+3)}(s^2+4)}\right|_{s=-3}=\frac{26}{13}=2$$

$$Y(s)=\frac{8}{s+3}+\frac{-2s+6}{s^2+4}$$

$$\frac{26}{(s+3)(s^2+4)}-\frac{2}{s+3}$$

$$=\frac{-2s^2+18}{(s+3)(s^2+4)}=\frac{-2s+6}{s^2+4}$$

$$Y(s)=\frac{8}{s+3}+-2\frac{s}{s^2+4}+3\frac{2}{s^2+4}$$

(Step 3) Inverse Laplace Transform

$$y(t)=8e^{-3t}-2\cos 2t+3\sin 2t$$

Example 5 (text page 291)

$$y''(t)-3y'(t)+2y(t)=e^{-4t} \qquad y(0)=1 \qquad y'(0)=5$$

(Step 1) Laplace

$$\Big\downarrow \text{快速法}$$

$$(s^2-3s+2)Y(s)=s+2+\frac{1}{s+4}$$

(Step 2) Decompose

$$Y(s)=\frac{s+2}{s^2-3s+2}+\frac{1}{(s^2-3s+2)(s+4)}$$

$$=\frac{s+2}{(s-1)(s-2)}+\frac{1}{(s-1)(s-2)(s+4)}$$

$$=-\frac{3}{s-1}+\frac{4}{s-2}-\frac{1}{5}\frac{1}{s-1}+\frac{1}{6}\frac{1}{s-2}+\frac{1}{30}\frac{1}{s+4}$$

$$=-\frac{16}{5}\frac{1}{s-1}+\frac{25}{6}\frac{1}{s-2}+\frac{1}{30}\frac{1}{s+4}$$

(Step 3) Inverse

$$y(t)=-\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

7-2-6 快速法

(A) 求 $P(s)$ $\qquad \dfrac{d^k}{dt^k} \rightarrow s^k$

$$y''(t) - 3y'(t) + 2y(t) \rightarrow P(s) = s^2 - 3s + 2$$

很像 Sec. 4-3 的…

(B) 求 $Q(s)$

$$\begin{aligned}
Q(s) = a_n &\left[s^{n-1}y(0) + s^{n-2}y'(0) + \cdots\cdots + sy^{(n-2)}(0) + y^{(n-1)}(0) \right] \\
+a_{n-1} &\left[s^{n-2}y(0) + \cdots\cdots + sy^{(n-3)}(0) + y^{(n-2)}(0) \right] \\
+\cdots\cdots & \\
+a_2 &\left[sy(0) + y'(0) \right] \\
+a_1 &\left[y(0) \right]
\end{aligned}$$

$$
\begin{array}{lccccc}
 & s^{n-1} & s^{n-2} & \cdots & s & 1 \\
a_n \times & y(0) & y'(0) & \cdots & y^{(n-2)}(0) & y^{(n-1)}(0) \\
a_{n-1} \times & & y(0) & \cdots & y^{(n-3)}(0) & y^{(n-2)}(0) \\
\vdots & & & \ddots & \vdots & \vdots \\
a_2 \times & & & & y(0) & y'(0) \\
a_1 \times & & & & & y(0) \\
\hline
\end{array}
$$

相加

例如，page 454的例子

$$
\begin{array}{ccc}
 & s & 1 \\
1\times & 1 & 5 \\
-3\times & & 1 \\
\end{array}
=
\begin{array}{cc}
s & 1 \\
1 & 5 \\
 & -3 \\
\hline
s & 2
\end{array}
\Longrightarrow Q(s)
$$

7-2-7 Section 7.2 需要注意的地方

(1) 熟悉分數分解

(2) 可以簡化運算的方法，能學則學

鼓勵各位同學多發揮創意，多多研究能簡化計算的快速法

數學上.......並沒有標準解法的存在

(3) Derivative 公式 initial conditions 的順序別弄反

$$\mathcal{L}\left\{f^{(n)}(t)\right\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Section 7-3 Operational Properties I

介紹兩個可以簡化 Laplace transform 計算的重要性質

First Translation Theorem (translation for *s*)

$$\boxed{\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)}$$

Second Translation Theorem (translation for *t*)

$$\boxed{\mathcal{L}\left\{f(t-a)u(t-a)\right\} = e^{-as}F(s)}$$

$u(t)$: unit step function

(注意兩者之間的異同)

7-3-1 First Translation Theorem (Translation for *s*)

$$\boxed{\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)}$$

Proof:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

7-3-1-1 Inverse of "Translation for *s*"

When $f(t)$ is piecewise continuous and of exponential order

$$\boxed{\mathscr{L}^{-1}\left\{F\left(s-a\right)\right\}=e^{at}f\left(t\right)}$$

(一對一)

註： Sections 7-3 和 7-4 其他的定理亦如此

7-3-1-2 Examples

Example 1 (text page 295)

(a) $\mathcal{L}\left\{e^{5t}t^3\right\} = \mathcal{L}\left\{t^3\right\}\Big|_{s\to s-5} = \left.\frac{3!}{s^4}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$

(b) $\mathcal{L}\left\{e^{-2t}\cos 4t\right\} = \mathcal{L}\left\{\cos 4t\right\}\Big|_{s\to s-(-2)} = \left.\frac{s}{s^2+16}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$

Example 2 (text page 296)

(a) $$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-3)+11}{(s-3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11t\,e^{3t}$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{(s+2)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+2)/2+2/3}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} + \frac{\sqrt{2}}{3}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Example 3 (text page 297)

$$y'' - 6y' + 9y = t^2 e^{3t} \qquad y(0) = 2, \quad y'(0) = 17$$

$$\begin{array}{rccccc}
 & s & 1 & & & \\
1\times & 2 & 17 & \multirow{2}{*}{=} & 2 & 17 \\
-6\times & & 2 & & & -12 \\
\hline
Q(s) = & & & & 2s & +\,5
\end{array}$$

$$\mathcal{L}\left\{t^2 e^{3t}\right\} = \mathcal{L}\left\{t^2\right\}\Big|_{s \to s-3} = \left.\frac{2}{s^3}\right|_{s \to s-3} = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$(s^2-6s+9)Y(s)=2s+5+\frac{2}{(s-3)^3}$$

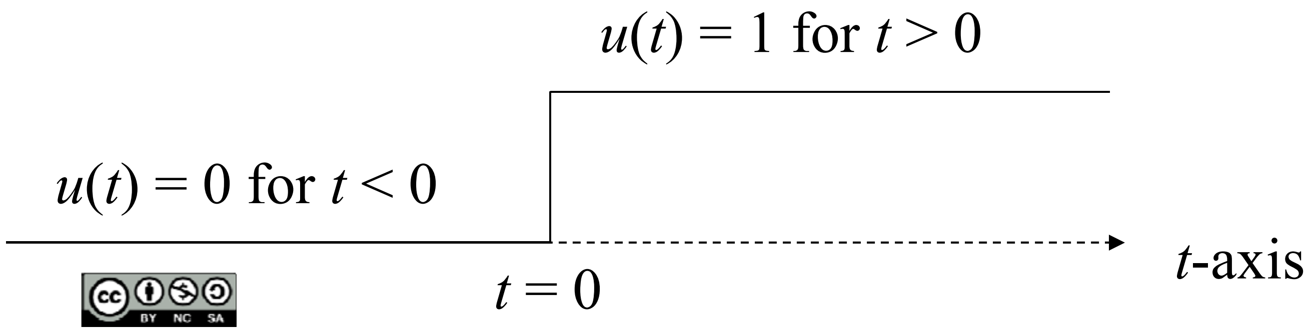
$$Y(s)=\frac{2s+5}{(s-3)^2}+\frac{2}{(s-3)^5}=\frac{2(s-3)+11}{(s-3)^2}+\frac{2}{(s-3)^5}$$

$$=\frac{2}{s-3}+\frac{11}{(s-3)^2}+\frac{1}{12}\frac{4!}{(s-3)^5}$$

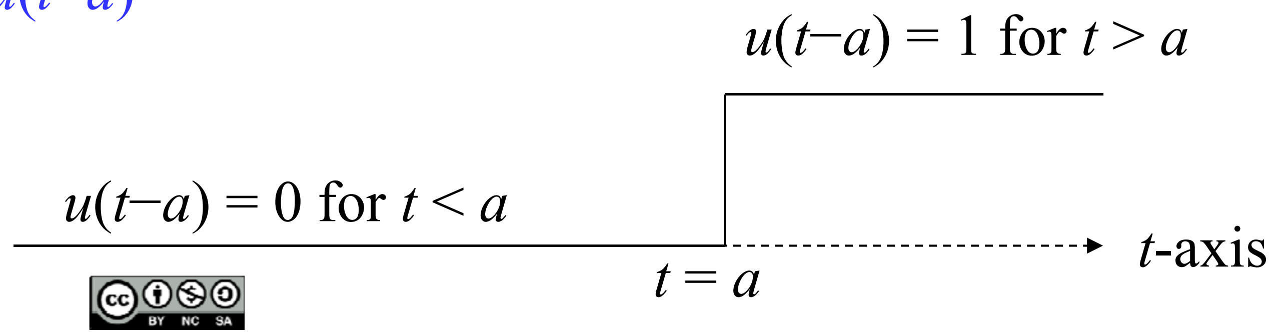
$$y(t)=2e^{3t}+11te^{3t}+\frac{1}{12}t^4e^{3t}$$

7-3-2 Step Function

$u(t)$: unit step function



$u(t-a)$

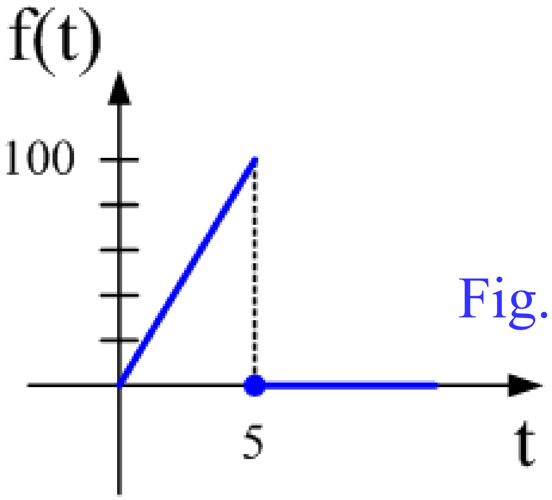


The unit step function acts as a switch (開關).

- Any piecewise continuous function can be expressed as the unit step function for $t \geq 0$

Example 5 (text page 298)

$$f(t) = \begin{cases} 20t & \text{for } 0 \leq t < 5 \\ 0 & \text{for } t > 5 \end{cases}$$



$$f(t) = 20t \cdot u(t) - 20t \cdot u(t-5)$$

Fig. 7.3.5

In general,

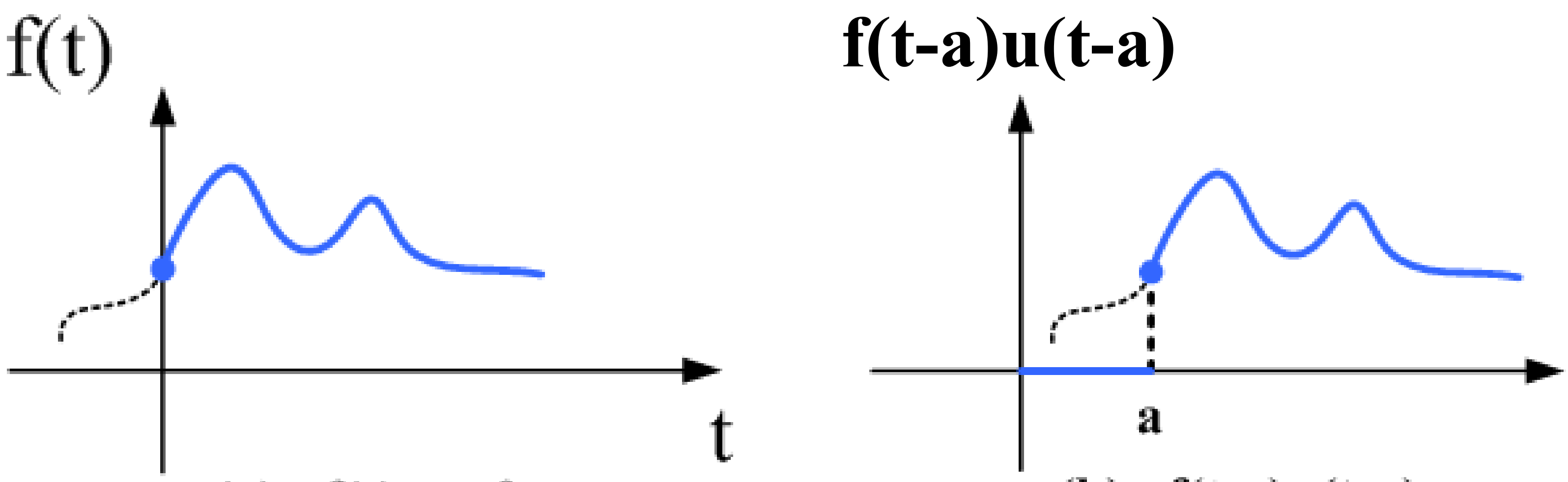
$$f(t) = \begin{cases} h_1(t) & \text{for } 0 \le t < a \\ h_2(t) & \text{for } t > a \end{cases}$$

$$f(t) = h_1(t) \cdot u(t) + (h_2(t) - h_1(t)) \cdot u(t-a)$$

7-3-3 Second Translation Theorem (Translation for t)

$$\boxed{\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)} \qquad a > 0$$

或 $$\boxed{\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}} \qquad a > 0$$



(a) f(t), t≥0 (b) f(t-a)u(t-a)

Proof:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_0^\infty e^{-s(t_1+a)} f(t_1)\,dt_1 \longleftarrow \text{令 } t_1 = t-a$$

$$= e^{-as}\int_0^\infty e^{-st_1} f(t_1)\,dt_1 = e^{-as}F(s)$$

Example 7(a) (text page 300) $\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}u(t-2)$$

Example 8 (text page 300) $\boxed{\mathcal{L}\{\cos t \cdot u(t-\pi)\}}$

$$\mathcal{L}\{\cos(t+\pi)\} = -\mathcal{L}\{\cos(t)\} = -\frac{s}{s^2+1}$$

$$\mathcal{L}\{\cos t \cdot u(t-\pi)\} = -\frac{s}{s^2+1}e^{-\pi s}$$

Example 9 (text page 301)

$$y' + y = f(t) \qquad y(0) = 5 \qquad f(t) = \begin{cases} 0 & \text{for } 0 \le t < \pi \\ 3\cos t & \text{for } t \ge \pi \end{cases}$$

$$f(t) = 3\cos t \cdot u(t - \pi)$$

$$\mathcal{L}\{\cos(t+\pi)\} = -\mathcal{L}\{\cos(t)\} = -\frac{s}{s^2+1}$$

$$\mathcal{L}\{3\cos t \cdot u(t-\pi)\} = -\frac{3s}{s^2+1}e^{-\pi s}$$

$$(s+1)Y(s) = 5 - \frac{3s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right]e^{-\pi s}$$

$$Y(s)=\frac{5}{s+1}-\frac{3}{2}\underbrace{\left[-\frac{1}{s+1}+\frac{1}{s^2+1}+\frac{s}{s^2+1}\right]}_{-e^{-t}+\sin(t)+\cos(t)}e^{-\pi s}$$

$$y(t)=5e^{-t}+\frac{3}{2}\left[e^{-(t-\pi)}-\sin(t-\pi)-\cos(t-\pi)\right]u(t-\pi)$$

$$=5e^{-t}+\frac{3}{2}\left[e^{-(t-\pi)}+\cos(t)+\sin(t)\right]u(t-\pi)$$

7-3-4 本節需要注意的地方

(1) 套用 "translation for t" 的公式時，

先將 input 變成 $g(t+a)$ 再作 Laplace transform

(例如 Example 7)

(2) Second translation theorem (translation for t) 當 $a > 0$ 時才適用

(3) 套用公式時，注意「順序」

Section 7-4 Operational Properties II

7-4-1 Derivatives of Transforms

$$\boxed{\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)}$$

比較：

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

微分 $\xrightarrow{\text{Laplace}}$ 乘 s^n

乘 t^n $\xrightarrow{\text{Laplace}}$ 微分

Proof of the Theorem of Derivatives of Transforms:

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{d}{ds}[e^{-st}]f(t)dt=-\int_0^\infty e^{-st}tf(t)dt=-\mathcal{L}\left\{tf(t)\right\}$$

$$\frac{d^n}{ds^n}F(s)=\frac{d^n}{ds^n}\int_0^\infty e^{-st}f(t)dt=\int_0^\infty \frac{d^n}{ds^n}[e^{-st}]f(t)dt=\int_0^\infty e^{-st}(-t)^n f(t)dt$$

$$=\mathcal{L}\left\{(-t)^n f(t)\right\}=(-1)^n\mathcal{L}\left\{t^n f(t)\right\}$$

Example 1 (text page 307) $\boxed{\mathcal{L}\{t\sin kt\}}$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \qquad \mathcal{L}\{t\sin kt\} = -\frac{d}{ds}\frac{k}{s^2+k^2} = \frac{2ks}{(s^2+k^2)^2}$$

練習：為何 $\mathcal{L}\{t\cos kt\} = \dfrac{s^2-k^2}{(s^2+k^2)^2}$

7-4-2 Convolution (旋積)

Definition of convolution:

$$\boxed{f(t)*g(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau}$$ (標準定義)

* 代表 convolution

$$\boxed{f(t)*g(t)=\int_{0}^{t} f(\tau)g(t-\tau)d\tau}$$ (課本關於 convolution 的定義)

When $f(t) = 0$ for $t < 0$ and $g(t) = 0$ for $t < 0$,

上方的式子可以簡化為下方的式子

Convolution 的物理意義 (重要)

當 $y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$

Input $f(\tau)$ 對 output $y(t)$ 的影響為 $g(t-\tau)$

$g(t-\tau)$ 只和 t 與 τ 之間的差有關

Input $f(\tau)$ 對 output $y(t)$ 的影響，決定於 t 與 τ 之間的差

$$
\begin{aligned}
y(t) &= \int_0^t f(\tau)g(t-\tau)d\tau \\
&= \{f(0)g(t-0)+f(\Delta)g(t-\Delta)+f(2\Delta)g(t-2\Delta) \\
&\quad +f(3\Delta)g(t-3\Delta)+\cdots\cdots\cdots+f(t)g(t-t)\}\Delta
\end{aligned}
$$

$$y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$
$$=\left\{f(0)g(t-0)+f(\Delta)g(t-\Delta)+f(2\Delta)g(t-2\Delta)\right.$$
$$\left.+f(3\Delta)g(t-3\Delta)+\cdots\cdots\cdots+f(t)g(t-t)\right\}\Delta$$

例如：$f(\tau)$ 是在 τ 這個時間點上太陽照射到某個地方的熱量
$g(t\text{-}\tau)$ 可想像成是經過了$t\text{-}\tau$ 的時間之後，還未幅射回外太熱量比例
$y(t)$ 可想像成是溫度

7-4-3 Convolution Theorem

$$\boxed{\mathcal{L}\{f(t)*g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)}$$

Convolution $\longrightarrow$ Multiplication

Proof: $F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)d\tau\right)\left(\int_0^\infty e^{-s\beta} g(\beta)d\beta\right)$

$$= \int_0^\infty \int_0^\infty e^{-s(\tau+\beta)} f(\tau)g(\beta)d\beta d\tau$$

note (A)
見後頁說明

令 $t = \tau + \beta$

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dt d\tau$$

note (B)
見後頁說明

$$= \int_0^\infty e^{-st}[\int_0^t f(\tau)g(t-\tau)d\tau]dt = \mathcal{L}[f*g]$$

note (A)

定理：$\iint \cdots\cdots\cdots dxdy = \iint \cdots\cdots\cdots C^{-1} dwdv$

$$C = \left| \det \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right|$$

note (B)

積分範圍的改變：

$$\left| \det \begin{bmatrix} \frac{\partial t}{\partial \beta} & \frac{\partial t}{\partial \tau} \\ \frac{\partial \tau}{\partial \beta} & \frac{\partial \tau}{\partial \tau} \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right| = 1$$

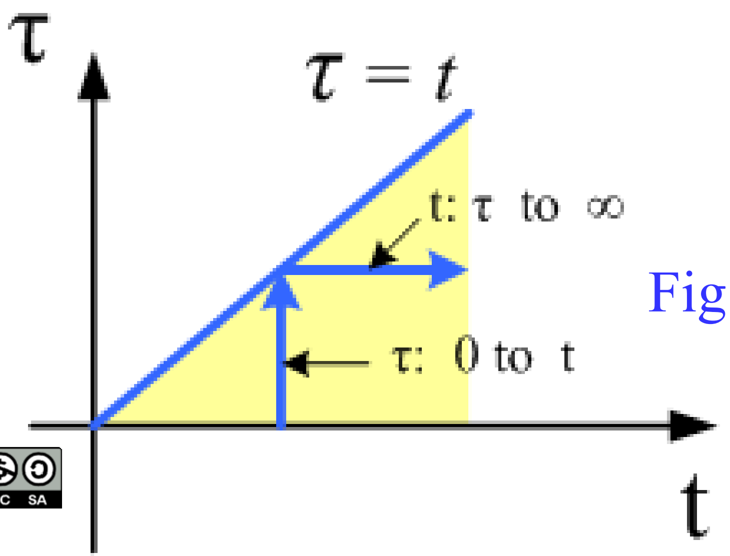


Fig. 7.4.1

Example 3 (text page 308)

Compute $e^t * \sin t$

$$e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)d\tau = \frac{1}{2}\left[e^\tau \sin(t-\tau) + e^\tau \cos(t-\tau)\right]\Big|_0^t$$

$$= \frac{1}{2}\left[-\sin(t) - \cos(t) + e^t\right]$$

$$\mathcal{L}\left(\frac{1}{2}\left[-\sin(t) - \cos(t) + e^t\right]\right) = \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s-1}\right)$$

Example 4 (text page 309)

$$\mathcal{L}\left\{\int_0^t e^\tau \sin(t-\tau)d\tau\right\} = \mathcal{L}\left\{e^t * \sin t\right\} = \frac{1}{s-1}\frac{1}{s^2+1}$$

Example 5 (text page 309)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}(\sin kt * \sin kt) = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)d\tau$$

7-4-4 Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \mathcal{L}\{f(t)*1\} = \frac{F(s)}{s}$$

(想成 “負一次微分”)

Example:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=\int_0^t \sin\tau\, d\tau=-\cos t+1$$

Example:

$$L_1\frac{di(t)}{dt}+Ri(t)+\frac{Q(t)}{C}=E(t)$$

$$L_1\frac{di(t)}{dt}+Ri(t)+\frac{Q(0)}{C}+\frac{1}{C}\int_0^t i(\tau)\,d\tau=E(t)$$

$$L_1 sI(s)-L_1 i(0)+RI(s)+\frac{Q(0)}{C}\frac{1}{s}+\frac{1}{C}\frac{I(s)}{s}=\mathcal{L}\{E(t)\}$$

7-4-5 Transform of a Periodic Function

Theorem 7.4.3 When $f(t+T)=f(t)$

then $$\mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

Proof: 令 $f_1(t)=f(t)$ when $0\le t<T$

$f_1(t)=0$ otherwise

$$f(t)=f_1(t)+f_1(t-T)+f_1(t-2T)+f_1(t-3T)+\ldots\ldots\ldots\ldots\ldots$$

$$=f_1(t)+f_1(t-T)u(t-T)+f_1(t-2T)\,u(t-2T)+f_1(t-3T)\,u(t-3T)$$

$$+\ldots\ldots\ldots\ldots\ldots$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_1(t-T)u(t-T)\} + \mathcal{L}\{f_1(t-2T)u(t-2T)\}$$
$$+\mathcal{L}\{f_1(t-3T)u(t-3T)\} + \ldots\ldots\ldots\ldots\ldots$$

$$\mathcal{L}\{f_1(t)\} = \int_0^T e^{-st} f_1(t)\,dt$$

$$\mathcal{L}\{f_1(t-T)u(t-T)\} = e^{-sT}\mathcal{L}\{f_1(t)\}$$

$$\mathcal{L}\{f_1(t-2T)u(t-2T)\} = e^{-2sT}\mathcal{L}\{f_1(t)\}$$

$$\mathcal{L}\{f_1(t-3T)u(t-3T)\} = e^{-3sT}\mathcal{L}\{f_1(t)\}$$

$$\vdots$$

$$\vdots$$

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Example 8 (text page 314)

Square Wave (方波) 的例子

$$E(t)=\begin{cases}1 & \text{for } 0\le t<1\\ 0 & \text{for } 1\le t<2\end{cases} \qquad E(t+2)=E(t)$$

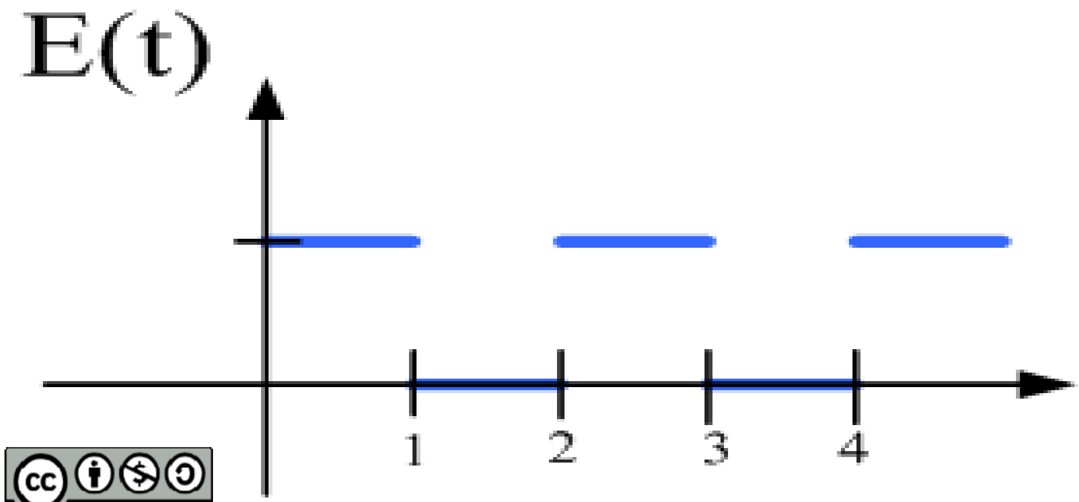


Fig. 7.4.4

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}}\left[\int_0^1 1\cdot e^{-st}\,dt + \int_1^2 0\cdot e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\,\frac{1-e^{-s}}{s} = \frac{1}{(1-e^{-s})(1+e^{-s})}\,\frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$

Example 9 (text page 314)

$$L_1 \frac{d}{dt} i + Ri = E(t)$$

$i(0) = 0$

$E(t)$ 為 page 487 之方波

$$sL_1 I(s) - L_1 i(0) + RI(s) = \frac{1}{s(1+e^{-s})}$$

$$I(s) = \frac{1/L_1}{(s+R/L_1)s(1+e^{-s})}$$

$$I(s)=\frac{1/L}{(s+R/L)s}\cdot\frac{1}{1+e^{-s}}=\left(\frac{1/R}{s}-\frac{1/R}{s+R/L}\right)\cdot\frac{1}{1+e^{-s}}$$

$$=\left(\frac{1/R}{s}-\frac{1/R}{s+R/L}\right)\cdot\left(1-e^{-s}+e^{-2s}-e^{-3s}+\cdots\cdots\right)$$

長除法 $\frac{1}{1+x}=1-x^1+x^2-x^3+\cdots\cdots$

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\cdots\cdots\right)$$

$$-\frac{1}{R}\left(\frac{1}{s+R/L}-\frac{e^{-s}}{s+R/L}+\frac{e^{-2s}}{s+R/L}-\frac{e^{-3s}}{s+R/L}+\cdots\cdots\right)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}=1 \qquad \mathscr{L}^{-1}\left\{\frac{e^{-ks}}{s}\right\}=u(t-k)$$

$$k=0,\ 1,\ 2,\ 3,\ \ldots..$$

$$\frac{e^{-ks}}{s+R/L}=e^{-ks}\times\frac{1}{s+R/L}$$

先使用 $\mathscr{L}^{-1}\left\{\frac{1}{s+R/L}\right\}=e^{-\frac{R}{L}t}u(t)$ 的公式

再算出 $\mathscr{L}^{-1}\left\{\frac{e^{-ks}}{s+R/L}\right\}=e^{-\frac{R}{L}(t-k)}u(t-k)$

註：雖然也可以用 $\frac{e^{-ks}}{s+R/L}=\left.\frac{e^{-k(s-R/L)}}{s}\right|_{s\to s+R/L}$ 來算

但是較麻煩且容易出錯

7-4-6 Section 7.4 要注意的地方

(1) 注意代公式的順序 (例：Page 491 例子)

(2) 熟悉 convolution

(3) 變成積分時，別忘了加上 initial value

如 $$\frac{Q(t)}{C}=\frac{Q(0)}{C}+\frac{1}{C}\int_0^t i(\tau)d\tau$$

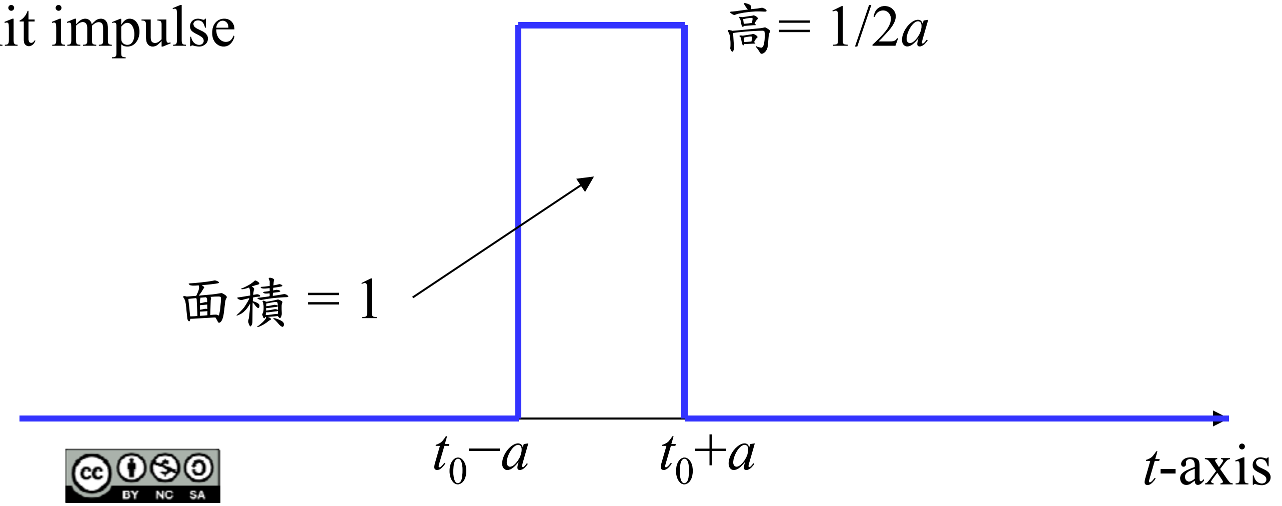
(4) 一定要記熟幾個重要的 properties (7 大性質)

Section 7.5 The Dirac Delta Function

7-5-1 Unit Impulse

$$\delta_a(t-t_0)=\begin{cases}0 & \text{for } t<t_0-a \quad \text{or } t>t_0+a \\ \dfrac{1}{2a} & \text{for } t_0-a\le t\le t_0+a\end{cases}$$

稱作 unit impulse



7-5-2 Dirac Delta Function

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$$

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)=\begin{cases}\infty & \text{for } t=t_0\\ 0 & \text{for } t\neq t_0\end{cases}$$

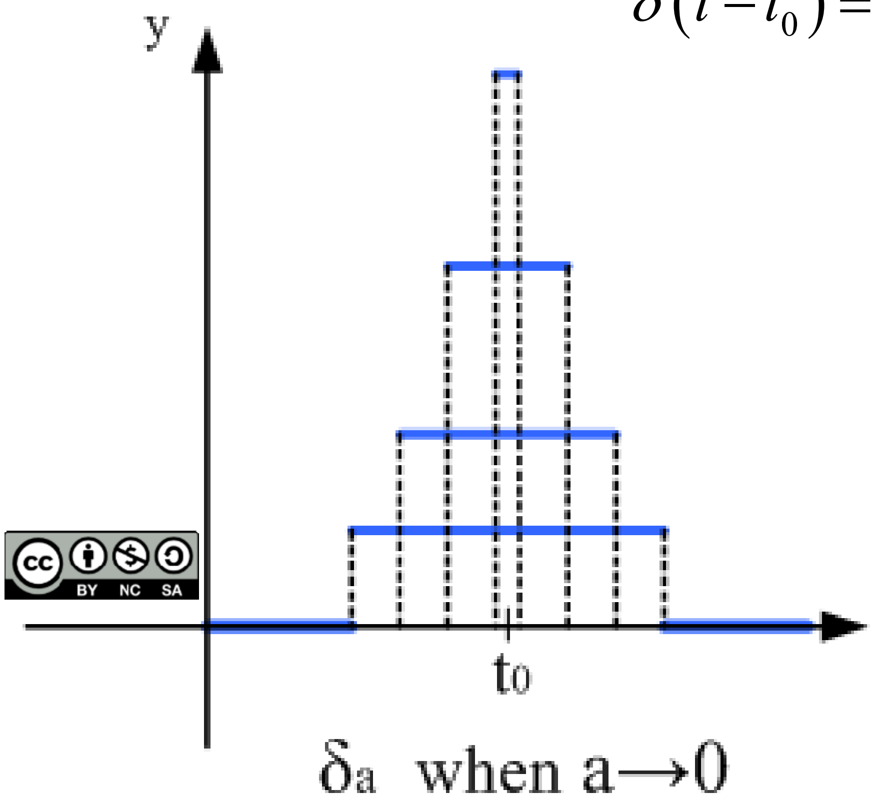


Fig. 7-5-2

7-5-3 Properties of the Dirac Delta Function

(1) Integration $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

(2) Sifting $\int_{p}^{q} f(t)\delta(t-t_0)\,dt = f(t_0)$ when $t_0 \in [p, q]$

Proof: $\int_{p}^{q} f(t)\delta(t-t_0)\,dt = \lim_{a\to 0}\int_{p}^{q} f(t)\delta_a(t-t_0)\,dt$

$\cong f(t_0)\lim_{a\to 0}\int_{p}^{q} \delta_a(t-t_0)\,dt = f(t_0)$

當 a 很小的時候，$f(t) \cong f(t_0)$ for $t_0 - a \le t \le t_0 + a$

(3) Laplace transform of $\delta(t - t_0)$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s} \qquad \text{when } t_0 > 0$$

Proof: $\int_0^\infty e^{-st}\delta(t-t_0)dt =$

(from the sifting property)

(4) Relation with the unit step function

$$\int_{-\infty}^{t} \delta(\tau - t_0)d\tau = u(t-t_0)$$

$$\frac{d}{dt}u(t-t_0) = \delta(t-t_0)$$

7-5-4 Example

Example 1(a) (text page 319)

$$y'' + y = 4\delta(t - 2\pi) \qquad y(0) = 1 \qquad y'(0) = 0$$

$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + 4\frac{e^{-2\pi s}}{s^2+1} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\begin{aligned} y(t) &= \cos t + 4\sin(t - 2\pi)u(t - 2\pi) \\ &= \cos t + 4\sin t \cdot u(t - 2\pi) \end{aligned}$$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$

7-5-5 幾個名詞

$$P(s)Y(s)-Q(s)=G(s) \longrightarrow Y(s)=W(s)Q(s)+W(s)G(s)$$

where $W(s)=\dfrac{1}{P(s)}$

(1) $w(t)=\mathcal{L}^{-1}\{W(s)\}$ 稱作 weight function 或 impulse response

Note: When $Q(s) = 0$ (no initial condition) and $G(s) = 1$ ($g(t) = \delta(t)$),

$Y(s) = W(s)$, $y(t) = w(t)$.

(2) 許多文獻把 Dirac delta function $\delta(t - t_0)$ 亦稱作 delta function ，impulse function，或 unit impulse function

7-5-6 本節要注意的地方

(1) Dirac delta function 不滿足 Theorem 7.1.3

(2) 幾個定理記熟，本節即可應付自如

Section 7-6 Systems of Linear Differential Equations

Chapter 7 的應用題

比較：類似的問題，也曾經在 Section 4-9 出現過

7-6-1 雙彈簧的例子

7-6-2 電路學的例子

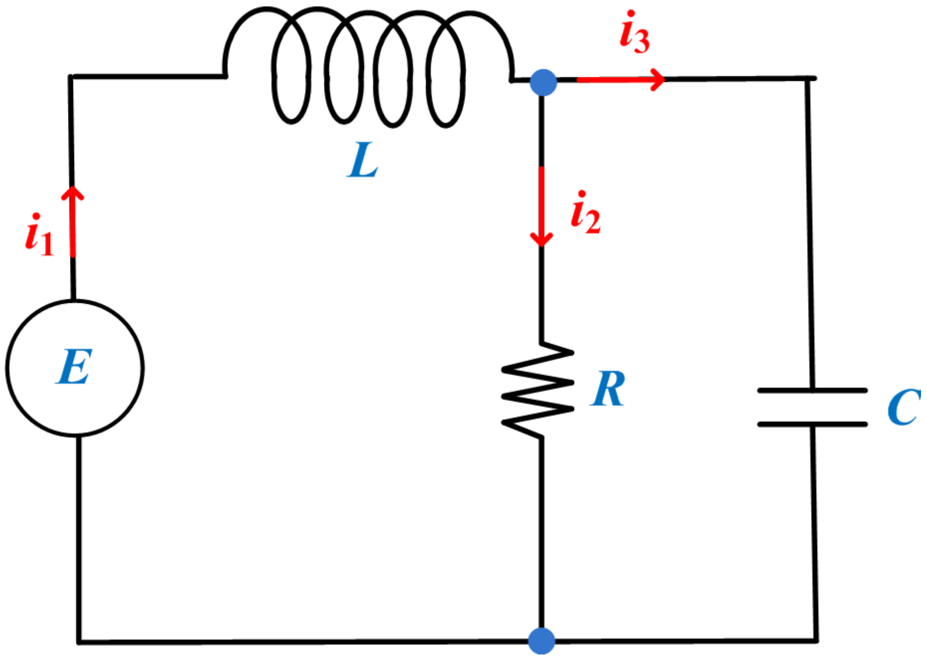


Fig. 3-3-4

Fig. 7-6-2

$$\begin{cases} L\dfrac{di_1(t)}{dt} + i_2R_2 = E(t) \\ \dfrac{q_3}{C} = i_2R \\ i_1 = i_2 + i_3 \end{cases}$$

(由第2, 3 個式子)

$$i_1 = i_2 + \frac{d}{dt}q_3 = i_2 + \frac{d}{dt}RCi_2$$

$$\begin{cases} L\dfrac{di_1(t)}{dt} + i_2R_2 = E(t) \\ RC\dfrac{d}{dt}i_2 + i_2 - i_1 = 0 \end{cases}$$

$$\begin{cases} L\dfrac{di_1(t)}{dt} + i_2R_2 = E(t) \\ RC\dfrac{d}{dt}i_2 + i_2 - i_1 = 0 \end{cases}$$

Example 2 (text page 323)

E(t) = 60 V, L = 1 H, R = 50 Ω, $C = 10^{-4}$ F, $i_1(t) = i_2(t) = 0$

$$\begin{cases} \dfrac{di_1(t)}{dt} + 50i_2 = 60 \\ 0.005\dfrac{d}{dt}i_2 + i_2 - i_1 = 0 \end{cases} \longrightarrow \begin{cases} sI_1(s) + 50I_2(s) = \dfrac{60}{s} \quad \ldots\ldots\ldots\ldots (式1) \\ -I_1(s) + (0.005s+1)I_2(s) = 0 \ldots\ldots (式2) \end{cases}$$

$(式1) \times 1 + (式2) \times s$

$$(0.005s^2 + s + 50)I_2(s) = \frac{60}{s} \qquad (s^2 + 200s + 10000)I_2(s) = \frac{12000}{s}$$

$$I_2(s)=\frac{12000}{s(s+100)^2}=\frac{6/5}{s}-\frac{120}{(s+100)^2}-\frac{6/5}{s+100}$$

複習：分子如何算出？

$$\boxed{i_2(t)=\frac{6}{5}-120te^{-100t}-\frac{6}{5}e^{-100t}}$$

將 $I_2(s)=\frac{6/5}{s}-\frac{120}{(s+100)^2}-\frac{6/5}{s+100}$ 代入式 (1)

$$sI_1(s)=\frac{60}{s}-\frac{60}{s}+\frac{6000}{(s+100)^2}+\frac{60}{s+100}$$

$$I_1(s)=\frac{6000}{s(s+100)^2}+\frac{60}{s(s+100)}$$

$$I_1(s)=\frac{6000}{s(s+100)^2}+\frac{60}{s(s+100)}=\frac{a}{s}+\frac{b+c(s+100)}{(s+100)^2}$$

$$a=\left.\frac{6000}{(s+100)^2}+\frac{60}{(s+100)}\right|_{s=0}=\frac{6}{5}$$

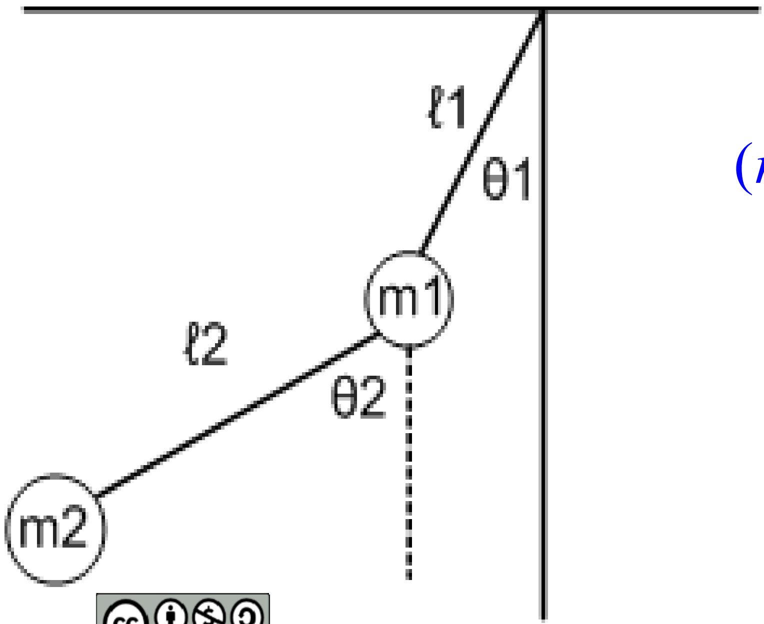
$$b=\left.\frac{6000}{s}+\frac{60(s+100)}{s}\right|_{s=-100}=-60$$

$$c=\left.\frac{d}{ds}\left(\frac{6000}{s}+\frac{60(s+100)}{s}\right)\right|_{s=-100}=\left.-\frac{6000}{s^2}-\frac{6000}{s^2}\right|_{s=-100}=-\frac{6}{5}$$

$$I_1(s)=\frac{6/5}{s}-\frac{60}{(s+100)^2}-\frac{6/5}{s+100}$$

$$\boxed{i_1(t)=\frac{6}{5}-60te^{-100t}-\frac{6}{5}e^{-100t}}$$

7-6-3 Double Pendulum (雙單擺) 的例子



$$(m_1 + m_2)l_1^2\theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2)l_1 g\theta_1 = 0$$

$$m_2 l_2^2\theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 l_2 g\theta_2 = 0$$

$$(m_1+m_2)l_1^2\theta_1''+m_2l_1l_2\theta_2''+(m_1+m_2)l_1g\theta_1=0$$

$$m_2l_2^2\theta_2''+m_2l_1l_2\theta_1''+m_2l_2g\theta_2=0$$

Example 3 (text page 324)

$$m_1=3,\ m_2=1,\ l_1=l_2=16,\ \theta_1(0)=1,\ \theta_2(0)=-1,$$

$$\theta'_1(0)=0,\ \theta'_2(0)=0$$

$$1024\theta_1''+256\theta_2''+64g\theta_1=0 \qquad\qquad 16\theta_1''+4\theta_2''+g\theta_1=0$$

$$256\theta_1''+256\theta_2''+16g\theta_2=0 \longrightarrow 16\theta_1''+16\theta_2''+g\theta_2=0$$

Laplace

$$16s^2\Phi_1(s)+4s^2\Phi_2(s)+g\Phi_1(s)=16s-4s=12s$$

$$16s^2\Phi_1(s)+16s^2\Phi_2(s)+g\Phi_2(s)=16s-16s=0$$

$$(16s^2+g)\Phi_1(s)+4s^2\Phi_2(s)=12s \qquad \cdots\cdots\cdots\cdots\cdots (式1)$$

$$16s^2\Phi_1(s)+(16s^2+g)\Phi_2(s)=0 \qquad \cdots\cdots\cdots\cdots\cdots (式2)$$

$$(式1)\times(16s^2+g)-(式2)\times 4s^2$$

$$[(16s^2+g)^2-64s^4]\Phi_1(s)=12s(16s^2+g)$$

$$[192s^4+32s^2g+g^2]\Phi_1(s)=12s(16s^2+g)$$

$$\Phi_1(s)=\frac{12s(16s^2+g)}{192s^4+32s^2g+g^2}=\frac{192s^3+12gs}{(24s^2+g)(8s^2+g)}=\frac{as+b}{24s^2+g}+\frac{cs+d}{8s^2+g}$$

$$\underline{(8a+24c)s^3}+(8b+24d)s^2+\underline{(a+c)gs}+bg+dg=192s^3+12gs$$

$$\begin{cases}8a+24c=192\\ a+c=12\end{cases}\Rightarrow\begin{cases}a=6\\ c=6\end{cases} \qquad \begin{cases}8b+24b=0\\ b+d=0\end{cases}\Rightarrow\begin{cases}b=0\\ d=0\end{cases}$$

$$\Phi_1(s)=\frac{6s}{24s^2+g}+\frac{6s}{8s^2+g}=\frac{6}{24}\frac{s}{s^2+(g/24)}+\frac{6}{8}\frac{s}{s^2+(g/8)}$$

$$\boxed{\theta_1(t)=\frac{1}{4}\cos\left(\frac{\sqrt{g}}{2\sqrt{6}}t\right)+\frac{3}{4}\cos\left(\frac{\sqrt{g}}{2\sqrt{2}}t\right)}$$

將 $\Phi_1(s)=\dfrac{12s(16s^2+g)}{192s^4+32s^2g+g^2}$ 代入 (式2)

$$\Phi_2(s)=-\frac{16s^2}{16s^2+g}\Phi_1(s)=-\frac{192s^3}{(24s^2+g)(8s^2+g)}=\frac{as+b}{24s^2+g}+\frac{cs+d}{8s^2+g}$$

直接用之前的式子

$$\begin{cases}8a+24c=-192\\ a+c=0\end{cases}\Rightarrow\begin{cases}a=12\\ c=-12\end{cases}\qquad\begin{cases}8b+24b=0\\ b+d=0\end{cases}\Rightarrow\begin{cases}b=0\\ d=0\end{cases}$$

$$\Phi_2(s) = \frac{12s}{24s^2+g} - \frac{12s}{8s^2+g} = \frac{12}{24}\frac{s}{s^2+(g/24)} - \frac{12}{8}\frac{s}{s^2+(g/8)}$$

$$\boxed{\theta_2(t) = \frac{1}{2}\cos\left(\frac{\sqrt{g}}{2\sqrt{6}}t\right) - \frac{3}{2}\cos\left(\frac{\sqrt{g}}{2\sqrt{2}}t\right)}$$

分式分解快速驗算技巧

將 $s = 0$, $s = 1$, 或其他的值代入，看等號是否成立

7-6-4 本節需要注意的地方

(1) 正負號勿寫錯

(2) 要熟悉聯立方程式的變數消去法

(3) 多學習，甚至多「研發」簡化計算的技巧

Review of Chapter 7

(1) Laplace transform 定義

$$F(s)=\int_0^{\infty} e^{-st} f(t)dt$$

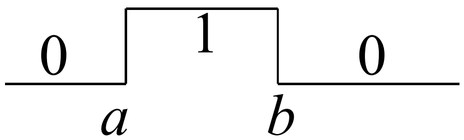
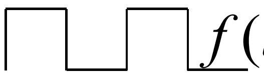
Inverse Laplace transform

If $f(t) \xrightarrow{Laplace} F(s)$ and $f(t)$ is piecewise continuous of exponential order

then $F(s) \xrightarrow{inverse\ \ Laplace} f(t)$

(2) 7 大transform pairs

(看講義 page 423)

$f(t)$	$F(s)$
$t\sin(kt)$	$\dfrac{2ks}{(s^2+k^2)^2}$
$t\cos(kt)$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh(kt)$	$\dfrac{2ks}{(s^2-k^2)^2}$
$t\cosh(kt)$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
$u(t-a)$	e^{-as}/s
$\delta(t)$	1
	$\dfrac{e^{-as}-e^{-bs}}{s}$
 $f(t)=f(t+2a)$	$\dfrac{1}{s(1+e^{-as})}$

(3) 7 大 properties

input	Laplace transform
(1) Differentiation (Sec 7-2) $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots$ $- s f^{(n-2)}(0) - f^{(n-1)}(0)$
(2) Multiplication by t (Sec7-4) $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
(3) Integration (Sec 7-4) $\int_0^t f(\tau) d\tau$	$\dfrac{F(s)}{s}$

(續)

input	Laplace transform
(4) Multiplication by exp (Sec7-3) $e^{at} f(t)$	$F(s-a)$
(5.1) Translation (Sec 7-3) $f(t-a)u(t-a)$	$e^{-as}F(s)$
(5.2) Translation (Sec 7-3) $g(t)u(t-a)$	$e^{-as}\mathcal{L}\{g(t+a)\}$
(6) Convolution (Sec 7-4) $y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$	$Y(s)=F(s)G(s)$
(7) Periodic Input (Sec 7-4) $f(t)=f(t+T)$	$\dfrac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

Properties 補充

input	Laplace transform
Scaling $f(t / a)$	$aF(as)$
Multiple Integrations $\int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_3} \int_0^{\tau_2} f(\tau_1) d\tau_1 d\tau_2 \cdots d\tau_{n-1} d\tau_n$	$\dfrac{F(s)}{s^n}$
Integration for s $f(t) / t$	$\int_s^{\infty} F(s_1) ds_1$

(4) 簡化運算的方法

分式分解 (see pages 442-447)

Initial conditions (see pages 455, 456)

(5) Delta function 的四大性質

Pages 495, 496

(6) General solutions

Laplace transform 的 general solution，可以用 initial conditions 來表示。

例子： $f''(t)-4f(t)=0$　　用Sec. 4-3 的方法解出　　$f(t)=c_1e^{2t}+c_2e^{-2t}$

用 Laplace transform ：

$$s^2F(s)-sf(0)-f'(0)-4F(s)=0$$

$$F(s)=\frac{sf(0)+f'(0)}{s^2-4}=f(0)\frac{s}{s^2-4}+\frac{f'(0)}{2}\frac{2}{s^2-4}$$

$$f(t)=f(0)\cosh 2t+\frac{f'(0)}{2}\sinh 2t$$

和 Section 4-3 的解互相比較

$$f(t) = \frac{2f(0) + f'(0)}{4}e^{2t} + \frac{2f(0) - f'(0)}{4}e^{-2t}$$

$$f(0) = c_1 + c_2$$

$$f'(0) = 2c_1 - 2c_2$$

將 $f(0) = c_1 + c_2$ $f'(0) = 2c_1 - 2c_2$ 代入

$$f(t) = c_1 e^{2t} + c_2 e^{-2t}$$

Exercise for Practice

Sec. 7-1:	5, 8, 9, 18, 32, 33, 36, 38, 41, 49, 54, 55, 56, 57, 58
Sec. 7-2:	11, 20, 23, 26, 27, 30, 40, 41, 45, 46, 49
Sec. 7-3:	10, 16, 19, 20, 24, 34, 35, 42, 44, 56, 58, 62, 64, 68, 70, 74, 83
Sec. 7-4:	8, 13, 29, 32, 34, 42, 47, 50, 56, 57, 58, 63, 65, 67, 70
Sec. 7-5:	5, 6, 8, 11, 12, 17
Sec. 7-6:	8, 11, 12, 14, 15
Review 7:	12, 24, 25, 29, 38, 40, 41, 44, 45, 46