Chapter 8 Systems of Linear First-Order Differential Equations

註：本章這學期只教不考

另一種解「聯立微分方程式」的方法

(1) Section 4.9: $\frac{d^k}{dt^k}y(t) \longrightarrow D^k y(t)$

(2) Chapter 7: $\frac{d^k}{dt^k}y(t) \longrightarrow s^k Y(s) - s^{k-1}y(0) - s^{k-2}y'(0) - \cdots\cdots - y^{(k-1)}(0)$

(3) Chapter 8: Using matrix operations

比較

(1) 這 3 種方法都只適用於 linear & constant coefficients 的情形

註：其實 Laplace transform 可用來解 nonlinear & non-constant coefficient DEs, 但過程頗為複雜

(2) Laplace transform 的方法優於 Section 4-8 的方法的地方，在於可以輕易的解決 initial condition 的問題

注意：但是，若 boundary conditions 不是在 $t = 0$ 的地方，用 Laplace transform 需要花一番功夫。

(3) 無論是 Section 4-8 的方法，還是 Laplace transform,

運算量皆不少

Chapter 8 的方法可以減少 1st order 聯立微分方程式的運算量

但 2nd order 以上反而比 Laplace transform 麻煩

Section 8.1 Preliminary Theory

方法的限制：

(a) linear,

(b) 1st order DEs

(c) full rank (n 個 dependent variable 需要 n 個式子)

名詞：

linear system (pp. 525)　homogeneous, nonhomogeneous (pp. 526)

solution vector (pp. 526)　fundamental set of solutions (pp. 530)

general solution (pp. 527)　complementary function (pp. 534)

particular solution (pp. 534)

本節學習秘訣：和 Section 4-1 相比較

8-1-1 表示法和名詞

假設有 n 個 dependent variables $x_1(t)$, $x_2(t)$, ……., $x_n(t)$,

n 個只有針對其中一個dependent variable 做微分的 linear DEs

$$\frac{d}{dt}x_1(t)=a_{11}(t)x_1(t)+a_{12}(t)x_2(t)+\cdots\cdots+a_{1n}(t)x_n(t)+f_1(t)$$

$$\frac{d}{dt}x_2(t)=a_{21}(t)x_1(t)+a_{22}(t)x_2(t)+\cdots\cdots+a_{2n}(t)x_n(t)+f_2(t)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{d}{dt}x_n(t)=a_{n1}(t)x_1(t)+a_{n2}(t)x_2(t)+\cdots\cdots+a_{nn}(t)x_n(t)+f_n(t)$$

稱作 **linear system**

Matrix form of a linear system

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

$$\mathbf{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & \cdots & a_{nn}(t) \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ \vdots \\ f_n(t) \end{bmatrix}$$

$\uparrow$ solution vector

$f_n(t) = 0$ for all n $\longrightarrow$ homogeneous linear system

otherwise $\longrightarrow$ nonhomogeneous linear system

$$\frac{dx}{dt} = x + 3y$$
$$\frac{dy}{dt} = 5x + 3y$$

可改寫成

$$\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \mathbf{X} \qquad \text{其中} \qquad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 2 (text page 335)

$$\mathbf{X}_1 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \qquad \mathbf{X}_2 = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t}$$

皆為 $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \mathbf{X}$ 的解

$$\mathbf{X}_1' = \begin{bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{bmatrix} \qquad \mathbf{A}\mathbf{X}_1 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \mathbf{X}_1'$$

$$\mathbf{X}_2' = \begin{bmatrix} 18e^{6t} \\ 30e^{6t} \end{bmatrix} \qquad \mathbf{A}\mathbf{X}_2 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix} = \begin{bmatrix} 18e^{6t} \\ 30e^{6t} \end{bmatrix} = \mathbf{X}_2'$$

If $x_1(t_0) = r_1$, $x_2(t_0) = r_2$, ………., $x_n(t_0) = r_n$,

linear system 可寫成

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F} \qquad \text{subject to} \qquad \mathbf{X}(t_0) = \mathbf{X_0}$$

$$\mathbf{X}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ \vdots \\ x_n(t_0) \end{bmatrix} \qquad \mathbf{X_0} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ r_n \end{bmatrix}$$

8-1-2 基本定理

將 Section 4-1 的幾個定理改成 vector 和 matrix 的型態

[Theorem 8.1.1] If the entries of **A** and **F** are **continuous** on a common interval that contains the point t_0, then the initial value problem on the previous page has a **unique solution** on this interval.

(比較 Theorem 4.1.1, page 140)

[Theorem 8.1.2] For the homogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ $(\mathbf{F} = \mathbf{0})$

if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ are the solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$

then $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots\cdots + c_n\mathbf{X}_n$ is also a solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$

[Definition 8.1.3 and Theorem 8.1.5] If the size of **A** is $n \times n$ **and** $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ **are** the linearly independent solutions of $\mathbf{X}' = \mathbf{A}\mathbf{X}$, then $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are said to be a fundamental set of solutions.

Then, the general solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is

$$\boxed{\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots\cdots + c_n\mathbf{X}_n}$$

$c_1, c_2, \ldots, c_n$ are arbitrary constants

(比較 Theorem 4.1.5, page 147)

[Theorem 8.1.3] Linearly dependent / independent 判斷方式

$$W\left(\mathbf{X_1},\mathbf{X_2},\cdots\cdots\mathbf{X_n}\right)=\det\begin{bmatrix} x_{11}(t) & x_{12}(t) & \cdots & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & \cdots & x_{nn}(t) \end{bmatrix}$$

(課本用 | | 來表示 det)

$$\mathbf{X_1}=\begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ \vdots \\ x_{n1}(t) \end{bmatrix} \quad \mathbf{X_2}=\begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ \vdots \\ x_{n2}(t) \end{bmatrix} \quad \cdots\cdots \quad \mathbf{X_n}=\begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

Either $W(\mathbf{X}_1, \mathbf{X}_2, \ldots., \mathbf{X}_n) \neq 0$ for every t $\longrightarrow$ linearly independent

or $W(\mathbf{X}_1, \mathbf{X}_2, \ldots., \mathbf{X}_n) = 0$ $\longrightarrow$ dependent

(比較 Wronskian, page 150)

Example 4 (text page 337)

$$\mathbf{X_1} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} \quad \mathbf{X_2} = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$$

$$W(\mathbf{X_1}, \mathbf{X_2}) = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} e^{-2t+6t} = 8e^{4t} \neq 0$$

determinant

[Theorem 8.1.6] General solution for nonhomogeneous system

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F} \quad \text{subject to} \quad \mathbf{X}(t_0) = \mathbf{X_0}$$

$$\begin{aligned}\mathbf{X} &= \mathbf{X_c} + \mathbf{X_p} \\ &= c_1\mathbf{X_1} + c_2\mathbf{X_2} + \cdots\cdots + c_n\mathbf{X_n} + \mathbf{X_p}\end{aligned}$$

$\mathbf{X}_c = c_1\mathbf{X_1} + c_2\mathbf{X_2} + \cdots\cdots + c_n\mathbf{X_n}$ 稱作為 complementary function

$\mathbf{X}_p$ particular solution

(比較講義 page 152)

8-1-3 本節要注意的地方

(1) 大部分的定理和 Section 4-1 相似

(2) 當一個式子出現 2 個 dependent variable 的微分時

先化成講義 page 525 linear system 的型態

Section 8.2 Homogeneous Linear Systems

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

8-2-1 本節摘要

(A) 解法的限制：

同講義 page 524 ，但多了二個限制

(d) homogeneous

(e) 最好是 constant coefficients

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

(B) 解法

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$ size of **A**: $n \times n$ (constant coefficients)

假設解為 $$\mathbf{X}_a = \begin{bmatrix} k_{1,a} \\ k_{2,a} \\ \vdots \\ \vdots \\ k_{n,a} \end{bmatrix} e^{\lambda_a t} = \mathbf{K_a} e^{\lambda_a t}$$ $a = 1, 2, \ldots., n$

其中

λ_a: **A** 的 eigenvalue
$\mathbf{K_a}$: **A** 的 eigenvector ($\mathbf{A}\mathbf{K_a} = \lambda \mathbf{K_a}$)

證明見講義 page 541

General solution:

$$\mathbf{X} = c_1 \mathbf{K_1} e^{\lambda_1 t} + c_2 \mathbf{K_2} e^{\lambda_2 t} + c_3 \mathbf{K_3} e^{\lambda_3 t} + \cdots\cdots + c_n \mathbf{K_n} e^{\lambda_n t}$$

(C) 三種情形

Case 1: **A** has distinct eigenvalues: 解法如前一頁

Case 2: **A** has repeated eigenvalues

當 λ_a 的 multiplicities 為 m

Case 2.1 可以找到 λ_a 的 m 個 linearly independent eigenvectors

解法同前一頁

Case 2.2 無法找到 λ_a 的 m 個 linearly independent eigenvectors

若只有 1 個 linearly independent eigenvector，將解表示成

$$\mathbf{X}_{\mathbf{a,1}} = \mathbf{K}_{\mathbf{a,1}} e^{\lambda_a t}$$

$$\mathbf{X}_{\mathbf{a,2}} = \mathbf{K}_{\mathbf{a,1}} t e^{\lambda_a t} + \mathbf{K}_{\mathbf{a,2}} e^{\lambda_a t}$$

$$\vdots$$

$$\mathbf{X}_{\mathbf{a,m}} = \mathbf{K}_{\mathbf{a,1}} \frac{t^{m-1}}{(m-1)!} e^{\lambda_a t} + \mathbf{K}_{\mathbf{a,2}} \frac{t^{m-2}}{(m-2)!} e^{\lambda_a t} + \cdots\cdots + \mathbf{K}_{\mathbf{a,m}} e^{\lambda_a t}$$

注意：

$$\left(\mathbf{A}-\lambda_a\mathbf{I}\right)\mathbf{K_{a,1}}=0$$

$$\left(\mathbf{A}-\lambda_a\mathbf{I}\right)\mathbf{K_{a,2}}=\mathbf{K_{a,1}}$$

$$\left(\mathbf{A}-\lambda_a\mathbf{I}\right)\mathbf{K_{a,3}}=\mathbf{K_{a,2}}$$

$$\vdots$$

$$\left(\mathbf{A}-\lambda_a\mathbf{I}\right)\mathbf{K_{a,m}}=\mathbf{K_{a,m\text{-}1}}$$

Case 2.3 無法找到 λ_a 的 m 個 linearly independent eigenvectors

有超過 1 個 linearly independent eigenvector

其實，也可以用類似方法求解，但較為複雜

Case 3　若 $\lambda_a = \alpha + j\beta$ 為 **A** 的eigenvalues, **A** 為 real matrix

$\lambda_b = \alpha - j\beta$ 必為 **A** 的eigenvalues

若 $\mathbf{K_a} = \mathbf{B_1} + j\mathbf{B_2}$ 為 λ_a 所對應的 eigenvector

$\mathbf{K_b} = \mathbf{B_1} - j\mathbf{B_2}$ 必為 λ_b 所對應的 eigenvector

此時，可將解改寫成

$$\mathbf{X_a} = [\mathbf{B_1} \cos \beta t - \mathbf{B_2} \sin \beta t] e^{\alpha t}$$

$$\mathbf{X_b} = [\mathbf{B_2} \cos \beta t + \mathbf{B_1} \sin \beta t] e^{\alpha t}$$

(D) 名詞與其他

multiplicity (page 549)

8-2-2 方法

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

假設解為 $$\mathbf{X} = \begin{bmatrix} k_1 e^{\lambda t} \\ k_2 e^{\lambda t} \\ \vdots \\ \vdots \\ k_n e^{\lambda t} \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ k_n \end{bmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t} \qquad \mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ k_n \end{bmatrix}$$

(和 Section 4-3 相似)

$$\mathbf{X}' = \begin{bmatrix} k_1 \lambda e^{\lambda t} \\ k_2 \lambda e^{\lambda t} \\ \vdots \\ \vdots \\ k_n \lambda e^{\lambda t} \end{bmatrix} = \mathbf{K} \lambda e^{\lambda t}$$

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

$$\downarrow \qquad \downarrow$$

$$\mathbf{K} \lambda e^{\lambda t} = \mathbf{A}\mathbf{K} e^{\lambda t}$$

$$\lambda \mathbf{K} = \mathbf{A}\mathbf{K}$$

$\mathbf{X}' = \mathbf{AX}$ 的問題變成 $\lambda\mathbf{K} = \mathbf{AK}$

(和 linear algebra 當中解 eigenvector, eigenvalue 的問題相同)

$$\lambda\mathbf{K} = \mathbf{AK}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

λ 是 **A** 的 eigenvalue

由 $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ 算出

(稱作 characteristic equation)

K 是 **A** 的 eigenvector

當 λ 算出後，**K** 為使得

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0$ 成立

的任一個滿足 $\mathbf{K} \neq 0$ 的解

Example 1 (text page 341)

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y$$
$$\Longrightarrow \quad \mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0$$

$$\lambda = -1, 4$$

(i) When $\lambda = -1$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \qquad \begin{cases} 3k_1 + 3k_2 = 0 \\ 2k_1 + 2k_2 = 0 \end{cases} \qquad k_2 = -k_1$$

設 $k_1 = 1, \longrightarrow k_2 = -1 \quad \mathbf{K}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(ii) When $\lambda = 4$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0 \qquad \begin{cases} -2k_1 + 3k_2 = 0 \\ 2k_1 - 3k_2 = 0 \end{cases}$$

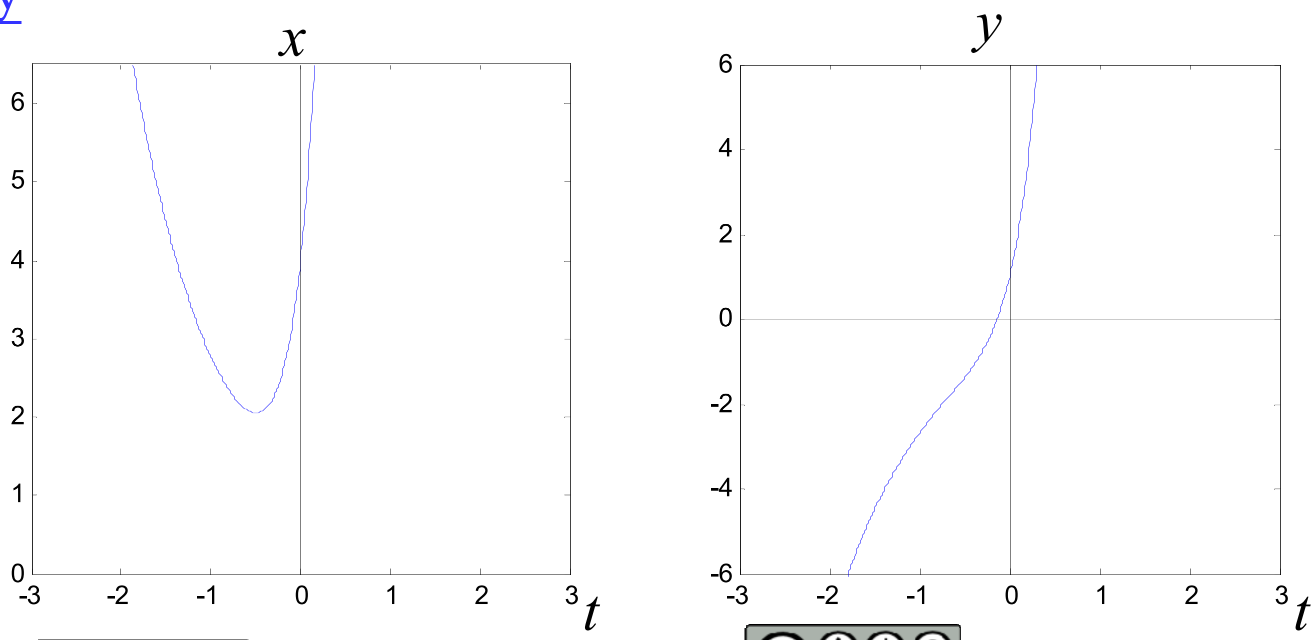
$$k_2 = 2\,k_1/3 \qquad 設\ k_1 = 3, \qquad k_2 = 2 \qquad \mathbf{K_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\mathbf{X_1} = \mathbf{K_1}e^{-t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} \qquad \mathbf{X_2} = \mathbf{K_2}e^{4t} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{4t}$$

$$\mathbf{X} = c_1\mathbf{X_1} + c_2\mathbf{X_2} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{4t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}$$

trajectory



(a) Graph of $x = e^{-t} + 3e^{4t}$ (b) Graph of $y = -e^{-t} + 2e^{4t}$

Fig. 8.2.1

8-2-3 Case 1: Distinct Eigenvalues

根據 eigenvalues ，分成 3 cases

Case 1: Distinct eigenvalues

Case 2: Repeated eigenvalues

Case 3: Complex eigenvalues

Example 2 (text page 343)

$$\frac{dx}{dt} = -4x + y + z$$

$$\frac{dy}{dt} = x + 5y - z$$

$$\frac{dz}{dt} = y - 3z$$

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda+3)(\lambda+4)(\lambda-5) = 0$$

$\lambda = -3, -4, 5$ (distinct)

When $\lambda = -3$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K_1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 8 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

3rd row: $k_2 = 0$

1st row: $-k_1 + k_2 + k_3 = -k_1 + k_3 = 0,\ \ k_1 = k_3$

$$\mathbf{K_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = -4,\ \ \lambda = 5$ (自己練習解解看)

$$X = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix} e^{-4t} + c_3 \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix} e^{5t}$$

8-2-4 Case 2: Repeated Eigenvalues

有時, $\det(\mathbf{A} - \lambda\mathbf{I})$ 會出現 $(\lambda - \lambda_a)^m$

λ_a 被稱作 eigenvalue of multiplicity m

$$\mathbf{A} = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \qquad \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \left(\lambda + 3\right)^2$$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \qquad \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = -\left(\lambda + 1\right)^2\left(\lambda - 5\right)$$

(這種情形較複雜，但也是本節的重點)

Case 2.1 當 λ_a 的 multiplicity 為 m ($m > 1$) 時，有的時候可以將 m 個 linearly independent eigenvectors 全部找出來。

此時，solutions 解法和 Case 1 相同

注意： 當 $\mathbf{A} = \mathbf{A}^T$ 時， 若 λ_a 的 multiplicity 為 m ，一定可以找到 λ_a 所對應的 m 個 linearly independent eigenvectors

Example 3 (text page 345)

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = -(\lambda + 1)^2 (\lambda - 5)$$

(i) 當 $\lambda = -1$

$$(\mathbf{A} - \lambda \mathbf{I}) \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row operation new 2nd row = old 2nd row + 1st row

new 3rd row = old 3rd row − 1st row

$$\begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2(k_1 - k_2 + k_3) = 0$$

3 個 variables, 1 個式子 ⟶ $3 - 1 = 2$

2 個 linearly independent solutions

$2(k_1 - k_2 + k_3) = 0$ 　　2 個 linearly independent solutions

(第一個 solution) 設 $k_1 = 0, k_2 = 1 \longrightarrow k_3 = 1$

(第二個 solution) 設 $k_1 = 1, k_2 = 0 \longrightarrow k_3 = -1$

Check: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ 的確互為 linearly independent

為 **A** 在 $\lambda = -1$ 時的 eigenvectors

小技巧：任意給定其他 n-1 個 unknowns 的值

再將最後一個 unknown 的值算出來

通常可以得到一個新的 independent solution

(但是也有的時候得到的解不為 independent, 所以要 check)

(ii) 當 $\lambda = 5$ 算出來的 eigenvector 為 $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

General solution for Example 3:

$$\mathbf{X} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-5t}$$

Case 2.2 當 λ_a 的 multiplicity 為 m $(m > 1)$ 時，有的時候只能找出 1 個 linearly independent eigenvector $\mathbf{K_{a,1}}$。

將λ_a 所對應的 m 個解表示成

$$\mathbf{X_{a,1}} = \mathbf{K_{a,1}} e^{\lambda_a t}$$

$$\mathbf{X_{a,2}} = \mathbf{K_{a,1}} t\, e^{\lambda_a t} + \mathbf{K_{a,2}}\, e^{\lambda_a t}$$

$$\mathbf{X_{a,3}} = \mathbf{K_{a,1}} \frac{t^2}{2} e^{\lambda_a t} + \mathbf{K_{a,2}} t\, e^{\lambda_a t} + \mathbf{K_{a,3}} e^{\lambda_a t}$$

$$\vdots$$

$$\mathbf{X_{a,m}} = \mathbf{K_{a,1}} \frac{t^{m-1}}{(m-1)!} e^{\lambda_a t} + \mathbf{K_{a,2}} \frac{t^{m-2}}{(m-2)!} e^{\lambda_a t} + \cdots\cdots + \mathbf{K_{a,m}}\, e^{\lambda_a t}$$

$\mathbf{K_{a,1}}$: 唯一滿足 $\mathbf{AK_{a,1}} = \lambda_a \mathbf{K_{a,1}}$ 的 eigenvector

$\mathbf{K_{a,q}}$ $(q \neq 1)$ 的求法如後頁

當 $$\mathbf{X}_{\mathbf{a,p}} = \mathbf{K}_{\mathbf{a,1}} \frac{t^{p-1}}{(p-1)!} e^{\lambda_a t} + \mathbf{K}_{\mathbf{a,2}} \frac{t^{p-2}}{(p-2)!} e^{\lambda_a t} + \cdots\cdots + \mathbf{K}_{\mathbf{a,p-1}} \frac{t^1}{1!} e^{\lambda_a t} + \mathbf{K}_{\mathbf{a,p}} e^{\lambda_a t}$$

$(p = 1, 2, \ldots, m)$

$$\mathbf{X}'_{\mathbf{a,p}} = \lambda_a \mathbf{K}_{\mathbf{a,1}} \frac{t^{p-1}}{(p-1)!} e^{\lambda_a t} + (\mathbf{K}_{\mathbf{a,1}} + \lambda_a \mathbf{K}_{\mathbf{a,2}}) \frac{t^{p-2}}{(p-2)!} e^{\lambda_a t} + (\mathbf{K}_{\mathbf{a,2}} + \lambda_a \mathbf{K}_{\mathbf{a,3}}) \frac{t^{p-3}}{(p-3)!} e^{\lambda_a t}$$
$$+ \cdots\cdots + (\mathbf{K}_{\mathbf{a,p-2}} + \lambda_a \mathbf{K}_{\mathbf{a,p-1}}) \frac{t^1}{1!} e^{\lambda_a t} + (\mathbf{K}_{\mathbf{a,p-1}} + \lambda_a \mathbf{K}_{\mathbf{a,p}}) e^{\lambda_a t}$$

由 $\mathbf{X}' = \mathbf{AX}$ $\qquad \mathbf{AX} - \mathbf{X}' = 0$

比較 $\mathbf{AX} - \mathbf{X}' = 0$ 當中 $\frac{t^q}{q!} e^{\lambda_a t}$ $(q = 0, 1, \ldots., m-1)$ 的係數得出

$$\begin{cases} (\mathbf{A} - \lambda_a \mathbf{I})\mathbf{K}_{\mathbf{a,1}} = 0 \\ (\mathbf{A} - \lambda_a \mathbf{I})\mathbf{K}_{\mathbf{a,2}} = \mathbf{K}_{\mathbf{a,1}} \\ (\mathbf{A} - \lambda_a \mathbf{I})\mathbf{K}_{\mathbf{a,3}} = \mathbf{K}_{\mathbf{a,2}} \\ \vdots \\ (\mathbf{A} - \lambda_a \mathbf{I})\mathbf{K}_{\mathbf{a,p}} = \mathbf{K}_{\mathbf{a,p-1}} \end{cases} \Longrightarrow$$

由 $\mathbf{K}_{\mathbf{a,1}}$ 求出 $\mathbf{K}_{\mathbf{a,2}}$

由 $\mathbf{K}_{\mathbf{a,2}}$ 求出 $\mathbf{K}_{\mathbf{a,3}}$

$\vdots$

由 $\mathbf{K}_{\mathbf{a,p-1}}$ 求出 $\mathbf{K}_{\mathbf{a,p}}$

註：(1) Text page 345 中

$$\mathbf{K}_{11} = \mathbf{K}_{21} = \ldots. = \mathbf{K}_{m1}$$

$$\mathbf{K}_{22} = \mathbf{K}_{32} = \ldots. = \mathbf{K}_{m2}$$

$$\mathbf{K}_{33} = \mathbf{K}_{43} = \ldots. = \mathbf{K}_{m3}$$

$$\vdots$$

$$\vdots$$

(2) $(\mathbf{A} - \lambda_a \mathbf{I})\mathbf{K}_{a,b+1} = \mathbf{K}_{a,b}$ 經常有多個 linearly independent 解

在這種情形下，我們只需找出<u>其中一個解</u>即可

(但是必需以可以繼續解下去為條件，如 page 558)

Example 5 (text page 348)

$$\mathbf{X}' = \mathbf{AX} \qquad \mathbf{A} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)^3 \qquad \text{eigenvalues: } 2,\ 2,\ 2$$

$$(\mathbf{A} - 2\mathbf{I})\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5k_3 = 0, \quad k_2 + 6k_3 = 0 \Longrightarrow k_2 = k_3 = 0$$

only one independent solution: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \mathbf{K}_{\mathbf{a,1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$(\mathbf{A}-2\mathbf{I})\mathbf{K}_{\mathbf{a},\mathbf{2}}=\mathbf{K}_{\mathbf{a},\mathbf{1}}, \qquad \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$5k_3=0, \quad k_2+6k_3=1$ 選擇其中一個 solution: $\mathbf{K}_{\mathbf{a},\mathbf{2}}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

注意： 若選擇 $\mathbf{K}_{\mathbf{a},\mathbf{2}}$ 為其他的值，最後的解還是一樣的

$(\mathbf{A}-2\mathbf{I})\mathbf{K}_{\mathbf{a,3}} = \mathbf{K}_{\mathbf{a,2}}$, $\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$5k_3 = 1, \quad k_2 + 6k_3 = 0$ <u>其中一個</u> solution: $\mathbf{K}_{\mathbf{a,3}} = \begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix}$

General solution of Example 5

$$\mathbf{X} = c_1\mathbf{K}_{\mathbf{a,1}}e^{2t} + c_2(\mathbf{K}_{\mathbf{a,1}}te^{2t} + \mathbf{K}_{\mathbf{a,2}}e^{2t}) + c_3(\mathbf{K}_{\mathbf{a,1}}\frac{t^2}{2}e^{2t} + \mathbf{K}_{\mathbf{a,2}}te^{2t} + \mathbf{K}_{\mathbf{a,3}}e^{2t})$$

$$\mathbf{X} = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2t} + c_2\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{2t}\right\} + c_3\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{t^2}{2}e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix}e^{2t}\right\}$$

Case 2.3 當 λ_a 的 multiplicity 為 m ($m > 1$) 時，有的時候只能找出 $2 \sim m-1$ 個 linearly independent eigenvectors 。

例子：Section 8-2 Exercises 31 and 50

$$\mathbf{X}' = \mathbf{AX} \qquad \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad \det\left(\mathbf{A} - \lambda\mathbf{I}\right) = (2-\lambda)^5$$

three independent solutions: $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$$\text{Set}\quad \mathbf{K_{a,1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (\mathbf{A} - 2\mathbf{I}) \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_2 = 1,\ k_4 = 0 \quad \Rightarrow \quad \text{Choose}\quad \mathbf{K_{a,2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

但 $(\mathbf{A} - 2\mathbf{I})\mathbf{K_{a,3}} = \mathbf{K_{a,2}}$ 無解

Set $\mathbf{K}_{\mathbf{a},1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\Longrightarrow$ $(\mathbf{A} - 2\mathbf{I})\mathbf{K}_{\mathbf{a},2} = \mathbf{K}_{\mathbf{a},1}$ 無解

Set $\mathbf{K}_{\mathbf{a},1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\Longrightarrow$ $(\mathbf{A} - 2\mathbf{I})\mathbf{K}_{\mathbf{a},2} = \mathbf{K}_{\mathbf{a},1}$ 的解為

$$\mathbf{K}_{\mathbf{a},2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(當一個解無法繼續算時，嚐試由其他的解來算)

General solution for Exercises 31 and 50

$$\mathbf{X} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

$$+ c_4 \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t} \right\} + c_5 \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} \right\}$$

8-2-5 Case 3: Complex Conjugated Eigenvalues

其實和 Case 1 (distinct eigenvalues) 相同

只是用不同的方式來表示 solutions

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

當 $\lambda_a = \alpha + j\beta$ 和 $\lambda_b = \alpha - j\beta$ (α, β 為 real) 皆為 **A** 的 eigenvector

且 **A** 為 real matrix

若 $\mathbf{K_a} = \mathbf{B_1} + j\mathbf{B_2}$ ($\mathbf{B_1}$, $\mathbf{B_2}$ 為 real)是 λ_a 所對應的 eigenvector

則 $\mathbf{K_b} = \mathbf{B_1} - j\mathbf{B_2}$ 必為是 λ_b 所對應的 eigenvector

Proof: $\mathbf{A}\mathbf{K_a} = \lambda_a \mathbf{K_a}$ $\quad \overline{\mathbf{A}\mathbf{K_a}} = \overline{\lambda_a \mathbf{K_a}}$ $\quad \mathbf{A}\overline{\mathbf{K_a}} = \overline{\lambda_a}\,\overline{\mathbf{K_a}}$

$\mathbf{A}\mathbf{K_b} = \lambda_a \mathbf{K_b}$

此時，可將解改寫成

$$\mathbf{X_a} = \left[\mathbf{B_1}\cos\beta t - \mathbf{B_2}\sin\beta t\right]e^{\alpha t}$$

$$\mathbf{X_b} = \left[\mathbf{B_2}\cos\beta t + \mathbf{B_1}\sin\beta t\right]e^{\alpha t}$$

(證明如後)

$$
\begin{aligned}
& c_a(\mathbf{B_1} + j\mathbf{B_2})e^{(\alpha+j\beta)t} + c_b(\mathbf{B_1} - j\mathbf{B_2})e^{(\alpha-j\beta)t} \\
&= c_a e^{\alpha t}(\mathbf{B_1} + j\mathbf{B_2})(\cos\beta t + j\sin\beta t) + c_b e^{\alpha t}(\mathbf{B_1} - j\mathbf{B_2})(\cos\beta t - j\sin\beta t) \\
&= c_a e^{\alpha t}(\mathbf{B_1}\cos\beta t - \mathbf{B_2}\sin\beta t) + c_a e^{\alpha t}(j\mathbf{B_1}\sin\beta t + j\mathbf{B_2}\cos\beta t) \\
&\quad + c_b e^{\alpha t}(\mathbf{B_1}\cos\beta t - \mathbf{B_2}\sin\beta t) - c_b e^{\alpha t}(j\mathbf{B_1}\sin\beta t + j\mathbf{B_2}\cos\beta t) \\
&= (c_a + c_b)e^{\alpha t}(\mathbf{B_1}\cos\beta t - \mathbf{B_2}\sin\beta t) + j(c_a - c_b)e^{\alpha t}(\mathbf{B_1}\sin\beta t + \mathbf{B_2}\cos\beta t)
\end{aligned}
$$

因此，兩個 linearly independent solutions 可改寫為

$$\mathbf{X_a} = \left[\mathbf{B_1}\cos\beta t - \mathbf{B_2}\sin\beta t\right]e^{\alpha t}$$

$$\mathbf{X_b} = \left[\mathbf{B_2}\cos\beta t + \mathbf{B_1}\sin\beta t\right]e^{\alpha t}$$

Example 6 (text page 351)

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \mathbf{A} = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$$

已知 $\lambda = 2i$ 為其中一個 eigenvalue，所對應的 eigenvector

為 $\begin{bmatrix} 2+2i \\ -1 \end{bmatrix}$

可以迅速判斷 2 個 independent solutions 為

$$\mathbf{X}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \sin 2t \qquad \mathbf{X}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \sin 2t$$

8-2-6 高階線性聯立微分方程的解法

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 x_2'' = -k_2 (x_2 - x_1)$$

解法：將問題變成 1st order DE

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

$$x_1' = x_3$$

$$x_2' = x_4$$

$$m_1 x_3' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 x_4' = -k_2 (x_2 - x_1)$$

$\Rightarrow$

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} - \frac{k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & 0 \end{bmatrix}$$

8-2-7 Section 8-2 要注意的地方

(1) 方法適用的情形 (a) linear, (b) 1st order DEs, (c) full rank (n 個 dependent variable 需要 n 個式子), (d) homogeneous, (e) constant coefficients

(2) 複習並熟悉算 eigenvector 的方法
(可以研究快速法)
(我們只要得出任何一個 eigenvector 或任何一組 linearly independent eigenvectors 即可，因此可以選擇當中較簡單的)

(3) Case 2 比較複雜，要多加練習

(4) 注意 page 552 找 independent solution 的小技巧

(5) Case 2.2 $(\mathbf{A}-\lambda_a\mathbf{I})\mathbf{K}_{\mathbf{a,b+1}}=\mathbf{K}_{\mathbf{a,b}}$ 選擇其中一組解即可 (但是要可以繼續解下去)

(6) 計算前，確定 $\frac{d}{dt}x_k$ 的係數皆為 1 (standard form)

Section 8.3 Nonhomogeneous Linear Systems

8.3.1 Section 8.3 摘要

本節討論如何找 $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$ 的 particular solution

(方法 1) undetermined coefficients

猜 particular solutions，類似 Section 4-4

(方法 2) variation of parameters ，類似 Section 4-6

$$\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

$\Phi(t)$: fundamental matrix，定義見 page 579

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

(方法 2) variation of parameters ，with initial conditions

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F} \qquad \mathbf{X}(t_0) = \mathbf{X_0}$$

$$\boxed{\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X_0} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(\tau)\mathbf{F}(\tau)\,d\tau}$$

名詞： fundamental matrix (page 579)

8.3.2 方法一: Undetermined Coefficients

和 Section 4.4 的方法相似

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

根據 $F(t)$ 來「猜」 particular solution

複習講義 page 194

(1) 出現 t^n $\longrightarrow$

(2) 出現 $\cos(at)$ $\longrightarrow$

(3) 出現 $\exp(bt)$ $\longrightarrow$

(4) 出現「綜合」

(5) 只要 $F(t)$ 其中有一個 entry 有某一項

則 particular solution 其他每一個 entry 都要根據這一項來猜

particular solution 的型態 (見 page 575的注意)

(這一點和 Section 4.4 稍有所不同)

(6) 和 homogeneous solution 有重覆時，不只乘 t，原來的 term 也保留 (見 page 577)

(這一點也和 Section 4.4 有所不同)

Example 3 (text page 357)

$$\frac{dx}{dt} = 5x + 3y - 2e^{-t} + 1$$

$$\frac{dy}{dt} = -x + y + e^{-t} - 5t + 7$$

solving the complementary function

$$\mathbf{X}_C' = \mathbf{A}\mathbf{X}_C \quad \mathbf{A} = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$$ eigenvalues of **A**: 2, 4

corresponding eigenvectors for $\lambda = 2$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

corresponding eigenvectors for $\lambda = 4$: $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

complementary function $\mathbf{X}_c = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{4t}$

解 particular solution

因為 $\mathbf{F}(t)=\begin{bmatrix} -2e^{-t}+1 \\ e^{-t}-5t+7 \end{bmatrix}$，所以假設 particular solution 為

$$\mathbf{X_p}(t)=\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}+\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}t+\begin{bmatrix} a_3 \\ b_3 \end{bmatrix}e^{-t}=\begin{bmatrix} a_1+a_2t+a_3e^{-t} \\ b_1+b_2t+b_3e^{-t} \end{bmatrix}$$

♦ 注意，每一個 entry 皆有 1, t, e^{-t}

From $\mathbf{X}'=\mathbf{AX}+\mathbf{F}$

$$\begin{bmatrix} a_2-a_3e^{-t} \\ b_2-b_3e^{-t} \end{bmatrix}=\begin{bmatrix} 5a_1+3b_1+(5a_2+3b_2)t+(5a_3+3b_3)e^{-t} \\ -a_1+b_1+(-a_2+b_2)t+(-a_3+b_3)e^{-t} \end{bmatrix}+\begin{bmatrix} -2e^{-t}+1 \\ e^{-t}-5t+7 \end{bmatrix}$$

$$\begin{bmatrix} a_2 - a_3 e^{-t} \\ b_2 - b_3 e^{-t} \end{bmatrix} = \begin{bmatrix} 5a_1 + 3b_1 + (5a_2 + 3b_2)t + (5a_3 + 3b_3)e^{-t} \\ -a_1 + b_1 + (-a_2 + b_2)t + (-a_3 + b_3)e^{-t} \end{bmatrix} + \begin{bmatrix} -2e^{-t} + 1 \\ e^{-t} - 5t + 7 \end{bmatrix}$$

$$-5a_1 + a_2 - 3b_1 = 1 \qquad a_1 - b_1 + b_2 = 7$$

$$5a_2 + 3b_2 = 0 \qquad -a_2 + b_2 = 5$$

$$6a_3 + 3b_3 = 2 \qquad a_3 - 2b_3 = 1$$

$$a_1 = \frac{35}{32}, \quad b_1 = -\frac{89}{32}, \quad a_2 = -\frac{15}{8}, \quad b_2 = \frac{25}{8}, \quad a_3 = \frac{7}{15}, \quad b_3 = -\frac{4}{15}$$

$$\mathbf{X} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{4t} + \begin{bmatrix} \frac{35}{32} \\ -\frac{89}{32} \end{bmatrix} + \begin{bmatrix} -\frac{15}{8} \\ \frac{25}{8} \end{bmatrix} t + \begin{bmatrix} \frac{7}{15} \\ -\frac{4}{15} \end{bmatrix} e^{-t}$$

補充的範例 (Example 1 in text page 355 的變型)

$$\mathbf{X}' = \mathbf{AX} + \mathbf{F} \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$\lambda = 0, 2 \qquad \text{eigenvector}: \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ , \ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{X_c} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

♦注意，這一項要保留

設 particular solution 為 $\mathbf{X_p}(t) = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} t$

乘 t

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1 + b_1 \end{bmatrix} + t \begin{bmatrix} a_2 + b_2 \\ a_2 + b_2 \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1 + b_1 \end{bmatrix} + t \begin{bmatrix} a_2 + b_2 \\ a_2 + b_2 \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \end{bmatrix}$$

$$\begin{cases} a_2 = a_1 + b_1 - 8 \\ b_2 = a_1 + b_1 + 3 \\ a_2 + b_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 - b_2 = -11 \\ a_2 + b_2 = 0 \end{cases} \Rightarrow a_2 = -\frac{11}{2} \quad , \quad b_2 = \frac{11}{2}$$

$$a_1 + b_1 = \frac{5}{2}$$ ← choose $a_1 = 0 \quad , \quad b_1 = \frac{5}{2}$

$$\mathbf{X} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} + \begin{bmatrix} -\frac{11}{2} \\ \frac{11}{2} \end{bmatrix} t$$

8.3.3.1 方法二: Variation of Parameters

先找 complementary function (solution of the associated homogeneous DE)

$$\mathbf{X}(t) = c_1 \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix} + c_2 \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix} + \cdots\cdots + c_n \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$$

fundamental matrix $\mathbf{\Phi}(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{bmatrix}$

令 $\mathbf{X}_p(t)=\mathbf{\Phi}(t)\mathbf{U}(t)$ $\mathbf{U}(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

$$\mathbf{X}'=\mathbf{A}\mathbf{X}+\mathbf{F}$$

$$\mathbf{X}_p'=\mathbf{\Phi}'(t)\mathbf{U}(t)+\mathbf{\Phi}(t)\mathbf{U}'(t)$$

$$\mathbf{\Phi}'(t)\mathbf{U}(t)+\mathbf{\Phi}(t)\mathbf{U}'(t)=\mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t)+\mathbf{F}(t)$$

由於 $\mathbf{\Phi}(t)$每個 column 都是 associated homogeneous DE 的解

$$\mathbf{\Phi}'(t)=\mathbf{A}\mathbf{\Phi}(t)$$

$$\mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t)+\mathbf{\Phi}(t)\mathbf{U}'(t)=\mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t)+\mathbf{F}(t)$$

$$\mathbf{\Phi}(t)\mathbf{U}'(t)=\mathbf{F}(t) \qquad \mathbf{U}'(t)=\mathbf{\Phi}^{-1}(t)\mathbf{F}(t)$$

$$\mathbf{U}'(t) = \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)$$

$$\mathbf{U}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

$$\mathbf{X}_p(t) = \mathbf{\Phi}(t)\mathbf{U}(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

$$\boxed{\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt}$$

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ some constants

Example 4 (text page 359)

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F} \qquad \mathbf{A} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$$

eigenvalues of **A**: $\lambda = -2, -5$

eigenvectors of **A** : $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\mathbf{X}_{\mathbf{c}}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-5t}$$

fundamental matrix $\quad \mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}$

$$\mathbf{\Phi}(t)=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix} \qquad \det(\mathbf{\Phi}(t))=-3e^{-7t}$$

$$\mathbf{\Phi}^{-1}(t)=\frac{1}{\det(\mathbf{\Phi}(t))}\begin{bmatrix} -2e^{-5t} & -e^{-5t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}=\begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}$$

$$\mathbf{X}_p(t)=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\int\begin{bmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{bmatrix}\begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}dt$$

$$=\begin{bmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{bmatrix}\begin{bmatrix} \int(2te^{2t}+\frac{1}{3}e^{t})dt \\ \int(te^{5t}-\frac{1}{3}e^{4t})dt \end{bmatrix}=\begin{bmatrix} \frac{6}{5}t-\frac{27}{50}+\frac{1}{4}e^{-t} \\ \frac{3}{5}t-\frac{21}{50}+\frac{1}{2}e^{-t} \end{bmatrix}$$

8.3.3.2 和 initial value problems 相結合

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F} \qquad \mathbf{X}(t_0) = \mathbf{X_0}$$

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt$$

在此時，可改寫成定積分的型態

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(\tau)\mathbf{F}(\tau)\,d\tau$$

Since $\mathbf{X}(t_0) = \mathbf{\Phi}(t_0)\mathbf{C} = \mathbf{X_0}$ thus $\mathbf{C} = \mathbf{\Phi}^{-1}(t_0)\mathbf{X_0}$

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X_0} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(\tau)\mathbf{F}(\tau)\,d\tau$$

8.3.4 Section 8.3 需要注意的地方

(1) 2×2 matrix 的 eigenvector 快速算法

(2) 注意 undetermined coefficient 的方法和 Section 4.4 異同處

(3) Variation of parameters 的部分，關鍵在是否能將公式背起來

(4) 通常 undetermined coefficient 的方法會比較容易解

而 variation of parameters 較複雜，但適用於任何情形

(5) 同樣記得先算 complementary function (homogeneous 部分的 solution)，再算 particular solution

Section 8.4 Matrix Exponential

8.4.1 Section 8.4 摘要

把 linear system 當成一般 1st order DE 來解

$$x'(t) = ax(t) \longrightarrow x(t) = ce^{at}$$ (比較 Section 2-3)

$$x'(t) = ax(t) + f(t) \longrightarrow x(t) = ce^{at} + e^{at}\int e^{-at} f(t)\,dt$$

(1) $\mathbf{X}' = \mathbf{AX} \longrightarrow \mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$

(2) $\mathbf{X}' = \mathbf{AX} + \mathbf{F} \longrightarrow \mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt$

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

(3) $\mathbf{X}' = \mathbf{AX} + \mathbf{F} \longrightarrow \mathbf{X}(t) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{X}_0 + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{F}(\tau)\,d\tau$

(With initial condition $\mathbf{X}(t_0) = \mathbf{X}_0$)

其中 $e^{\mathbf{A}t}$ 可以由 Laplace transform $e^{\mathbf{A}t} = L^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$ (see page 589)

或 eigenvector-eigenvalue decomposition (see page 591) 算出

8.4.2 For Homogeneous Systems

$\mathbf{X}' = \mathbf{A}\mathbf{X}$ $\qquad x'(t) = ax(t) \longrightarrow x(t) = ce^{at}$

solution: $\boxed{\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}}$

$e^{\mathbf{A}t}$ 的定義

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots\cdots = \sum_{k=0}^{\infty}\mathbf{A}^k\frac{t^k}{k!}$$

$$\frac{d}{dt}e^{\mathbf{A}t} = \sum_{k=1}^{\infty}\mathbf{A}^k\frac{t^{k-1}}{(k-1)!} = \sum_{h=0}^{\infty}\mathbf{A}^{h+1}\frac{t^h}{h!} = \mathbf{A}e^{\mathbf{A}t}$$

$$h = k - 1$$

8.4.3 For Nonhomogeneous Systems

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}$$

solution: $$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int e^{-\mathbf{A}t}\mathbf{F}(t)\,dt$$

或 $$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{F}(\tau)\,d\tau$$

比較：$x'(t) = ax(t) + f(t) \longrightarrow x(t) = ce^{at} + e^{at}\int e^{-at} f(t)\,dt$

with initial conditions $\mathbf{X}(t_0) = \mathbf{X}_0$

$$\mathbf{X}(t) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{X}_0 + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{F}(\tau)\,d\tau$$

8.4.4 Computation of $e^{\mathbf{A}t}$

$$\mathbf{X}' = \mathbf{AX} \qquad \mathbf{X} = e^{\mathbf{A}t}\mathbf{C}$$

一定有這樣的 initial condition $\mathbf{X}(0) = \mathbf{C}$

constant column vector

令 $\mathbf{X}(s)$ 為 $\mathbf{X}(t)$ 的 Laplace transform

$$s\mathbf{X}(s) - \mathbf{X}(0) = \mathbf{AX}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{C}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}$$

$$e^{\mathbf{A}t}\mathbf{C} = L^{-1}\left[\mathbf{X}(s)\right] = L^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{C}$$

$$\boxed{e^{\mathbf{A}t} = L^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]}$$

Example 2 (text page 364)

$$\mathbf{A}=\begin{bmatrix}1 & -1\\ 2 & -2\end{bmatrix} \qquad \text{Determine } e^{\mathbf{A}t}$$

$$s\mathbf{I}-\mathbf{A}=\begin{bmatrix}s-1 & 1\\ -2 & s+2\end{bmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s(s+1)}\begin{bmatrix}s+2 & -1\\ 2 & s-1\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{s}-\frac{1}{s+1} & -\frac{1}{s}+\frac{1}{s+1}\\ \frac{2}{s}-\frac{2}{s+1} & -\frac{1}{s}+\frac{2}{s+1}\end{bmatrix}$$

$$e^{\mathbf{A}t}=\begin{bmatrix}2-e^{-t} & -1+e^{-t}\\ 2-2e^{-t} & -1+2e^{-t}\end{bmatrix}$$

殺雞焉用牛刀……

複習 linear algebra 當中，$e^{\mathbf{A}t}$ 的求法

(1) eigenvector-eigenvalue decomposition for **A**

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} & \cdots & \mathbf{e_n} \end{bmatrix}$$

$\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \ldots\ldots, \mathbf{e_n}$ 皆為 **A** 的 eigenvectors, 皆為 $n \times 1$ 的 column

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ 為 $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \ldots\ldots, \mathbf{e_n}$ 所對應的eigenvalues

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

$$e^{\mathbf{A}t} = \mathbf{E}e^{\mathbf{D}t}\mathbf{E}^{-1}$$

$$e^{\mathbf{D}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ 0 & 0 & e^{\lambda_3 t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

例如， Example 1 當中

$$e^{\mathbf{D}t} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{E}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

8.4.5 注意

(1) 本節可以解的問題，用 Sections 8-2, 8-3 的方法也可以解

(2) 熟悉公式和 $e^{\mathbf{A}t}$ 的計算

(3) 使用 eigenvalue-eigenvector decomposition 的方法時，別忘了將 λ 變成 $e^{\lambda t}$