工程數學--微分方程

Differential Equations (DE)

授課者：丁建均

教學網頁：http://djj.ee.ntu.edu.tw/DE.htm
(請上課前來這個網站將講義印好)

歡迎大家來修課！

課程資訊

上課時間： 星期三 第 3, 4 節 (AM 10:20~12:10)

上課地點： 明達205

(課程將錄影，放在 NTUCool)

課本： "Differential Equations-with Boundary-Value Problem," Dennis G. Zill and Michael R. Cullen, 9th edition, 2017. (metric version, international version)

評分方式：四次作業 15%, 期中考 42.5%, 期末考 42.5%

授課者：丁建均

Office： 明達館723室， TEL： 33669652

Office hour： 週一，二，四，五的下午皆可來找我

老師 E-mail: jjding@ntu.edu.tw

教學網頁：http://djj.ee.ntu.edu.tw/DE.htm

個人網頁：http://disp.ee.ntu.edu.tw/

共同教學網頁： http://cc.ee.ntu.edu.tw/~tomme/DE/DE.html

大助教： 洪聲仰

大助教聯絡方式： shengyang@ntu.edu.tw

注意事項：

(1) 本課程採行雙軌制，同學們可以來現場上課，或是可觀看 NTUCool 的影片

(2)請上課前，來這個網頁，將上課資料印好。

http://djj.ee.ntu.edu.tw/DE.htm

(3) 請各位同學踴躍出席 。

(4) 作業不可以抄襲。作業若寫錯但有用心寫仍可以有 40%~90% 的分數，但抄襲或借人抄襲不給分。

11

(5) 每次作業有~~11~~題

8 0

$658 \times 923 \mod 13 = 0$

$1217 \times 672 \mod 11$ 7

7 1

上課日期

Week Number	Date (Wednesday)	Remark
1.	9/4	
2.	9/11	
3.	9/18	
4.	9/25: HW1	
5.	10/2	
6.	10/9	
7.	10/16: HW2	
8.	10/23: Midterms	範圍： (Sections 2-2 ~ 4-5)
9.	10/30	
10.	11/6	
11.	11/13	
12.	11/20: HW3	
13.	11/27	
14.	12/4	
15.	12/11: HW4	
16.	12/18: Finals	範圍： (Sections 4-6 ~ 12-4)

課程大綱

- Introduction (Chap. 1)
- First Order DE
  order = 1
  - 解法 (Chap. 2)
  - 應用 (Chap. 3)
  - 矩陣解 (Chap. 8，範圍外)
- Higher Order DE
  order ≥ 2
  - 解法 (Chap. 4)
  - 應用 (Chap. 5，範圍外)
  - 非線性 (Sections 4-10, 5-3, 工數特論)
  - 多項式解法 (Chap. 6)
- Partial DE
  - 解法 (Sections 12-1, 12-4)
  - 直角座標 (Chapter 12，工數特論)
  - 圓座標 (Chapter 13，工數特論)
- Transforms
  - Laplace Transform (Chap. 7 ，範圍外)
  - Fourier Series (Chap. 11)
  - Fourier Transform (Chap. 14，工數特論)

授課範圍

期中考範圍

Sections 1-1, 1-2, 1-3

Sections 2-1, 2-2, 2-3, 2-4, 2-5, 2-6

Sections 3-1, 3-2

Sections 4-1, 4-2, 4-3, 4-4, 4-5

期末考範圍

Sections 4-6, 4-7

Sections 6-1, 6-2, 6-3, 6-4

Sections 11-1, 11-2, 11-3

Sections 12-1, 12-4

blue colors: 要考的章節

Chapter 1 Introduction to Differential Equations

/tɝmə'nalədʒɪ/

1.1 Definitions and Terminology (術語)

(1) Differential Equation (DE): any equation containing derivation
(text page 3, definition 1.1)

$$\frac{dy(x)}{dx} = 1$$

(2)
x: independent variable 自變數
$y(x)$: dependent variable 應變數

$$\int_0^x \sin(t) f(x-t)\,dt + \frac{d^3 f(x)}{dx^3} = \cos(x)$$

- Note: In the text book, $f(x)$ is often simplified as f

- notations of differentiation

$$\frac{df}{dx}, \quad \frac{d^2 f}{dx^2}, \quad \frac{d^3 f}{dx^3}, \quad \frac{d^4 f}{dx^4}, \quad \ldots\ldots\ldots \quad \text{Leibniz notation}$$

$$f', \quad f'', \quad f''', \quad f^{(4)}, \quad \ldots\ldots\ldots \quad \text{prime notation}$$

$$\dot{f}, \quad \ddot{f}, \quad \dddot{f}, \quad \ddddot{f}, \quad \ldots\ldots\ldots \quad \text{dot notation}$$

$$f_x, \quad f_{xx}, \quad f_{xxx}, \quad f_{xxxx}, \quad \ldots\ldots\ldots \quad \text{subscript notation}$$

$d(\xi)$

$$d_{\xi\xi} = \frac{d^2 d(\xi)}{d\xi^2}$$

常微分

(3) Ordinary Differential Equation (ODE):

平常

differentiation with respect to one independent variable

$$\frac{d^3u}{dx^3}+\frac{d^2u}{dx^2}+\frac{du}{dx}+\cos(6x)u=0 \qquad \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}=2xy+z$$

3rd order DE   1st order DE

偏微分

(4) Partial Differential Equation (PDE):

differentiation with respect to two or more independent variables

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \frac{\partial x}{\partial t}=\frac{\partial y}{\partial \tau}$$

different   2nd order   1st order

(5) Order of a Differentiation Equation: the order of the highest derivative in the equation

$$\frac{d^7u}{dx^7} + 2\frac{d^6u}{dx^6} + 2\frac{d^5u}{dx^5} + 4\frac{d^4u}{dx^4} = 0$$

7th order

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 5y = e^x$$

2nd order

(6) Linear Differentiation Equation:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

(i) For y, only the terms $y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^n y}{dx^n}$ appear.

(ii) All of the coefficient terms $a_m(x)$ $m =$ 1, 2, …, n are independent of y.

Property of linear differentiation equations:

If $a_n(x)\frac{d^n y_1}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y_1}{dx^{n-1}}+\cdots+a_1(x)\frac{dy_1}{dx}+a_0(x)y_1=g_1(x)$

$$a_n(x)\frac{d^n y_2}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y_2}{dx^{n-1}}+\cdots+a_1(x)\frac{dy_2}{dx}+a_0(x)y_2=g_2(x)$$

$k=0,1,\cdots n$

$a_k(x)\frac{d^k}{dx^k}y_3 = a_k(x)\frac{d^k}{dx^k}(by_1+cy_2)$

$= b\,a_k(x)\frac{d^k}{dx^k}y_1 + c\,a_k(x)\frac{d^k}{dx^k}y_2$

and $y_3 = by_1 + cy_2$, then

$$a_n(x)\frac{d^n y_3}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y_3}{dx^{n-1}}+\cdots+a_1(x)\frac{dy_3}{dx}+a_0(x)y_3=bg_1(x)+cg_2(x)$$

(if $y(x)$ is treated as the input and $g(x)$ is the output)

(7) Non-Linear Differentiation Equation

$$(y+3)\frac{d^2y}{dx^2}+\frac{dy}{dx}+2y=x$$ not linear (ii) is not satisfied

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y^2=e^x$$ not linear (i) is not satisfied

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+e^y=e^x$$ not linear (i) is not satisfied

[Example 1.1.2] Linear and Nonlinear ODEs

(a) The equations

$4x\frac{dy}{dx} + y - x = 0$

$$(y - x)dx + 4xdy = 0, \quad y'' - 2y + y = 0, \quad x^3 \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

linear linear linear

are, in turn, *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

nonlinear term: coefficient depends on y

nonlinear term: nonlinear function of y

nonlinear term: power not 1

$$(1 - y)y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \sin y = 0, \quad \text{and} \quad \frac{d^4y}{dx^4} + y^2 = 0$$

nonlinear nonlinear nonlinear

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

明確

(8) Explicit Solution (text page 8)

The solution is expressed as $y = \phi(x)$

(9) Implicit Solution (text page 8)

不明確

Example: $\dfrac{dy^2}{dx} = -x$, $\rightarrow y^2 = -\dfrac{x^2}{2} + c$

Solution: $\dfrac{1}{2}x^2 + y^2 = c$ (implicit solution)

or $y = \sqrt{c - x^2/2}$

$y = -\sqrt{c - x^2/2}$ (explicit solution)

$y = \cdots\cdots$

1.2 Initial Value Problem (IVP)

A differentiation equation always has more than one solution.

for $\frac{dy}{dx} = 1$, $y = x + c$ c is any constant

$y = x$, $y = x+1$, $y = x+2$ … are all the solutions of the above differentiation equation.

General form of the solution: $\boldsymbol{y = x + c}$, where c is any constant.

The initial value (未必在 $x = 0$) is helpful for obtain the unique solution.

$y = x + c$ $x = 0, y = 2$ 代入 $2 = 0 + c$

$c = 2$

$$\frac{dy}{dx} = 1 \text{ and } y(0) = 2 \longrightarrow y = x+2$$

$$\frac{dy}{dx} = 1 \text{ and } y(2) = 3.5 \longrightarrow y = x+1.5$$

$y = x + c$

$3.5 = 2 + c$ $c = 1.5$

The k^{th} order linear differential equation usually requires k independent initial conditions (or k independent boundary conditions) to obtain the unique solution.

2 unknowns need at least 2 constraints

k unknowns need at least k constraints

$$\frac{d^2y}{dx^2} = 1$$

solution: $y = x^2/2 + bx + c$,

$b+c=1.5$

$2b+c=1$

$b=-0.5,\ c=2$

$2=\frac{1}{2}+b+c$ $\quad$ $3=2+2b+c$

b and c can be any constant

$y=\frac{x^2}{2}-\frac{x}{2}+2$

$y(1) = 2$ and $y(2) = 3$ (boundary conditions，在不同點) (10)

$y(0) = 1$ and $y'(0) = 5$ (initial conditions ，在相同點) (11)

$y(0) = 1$ and $y'(3) = 2$ (boundary conditions，在不同點)

from $y(0)=1$, $1=c$

$y'(0)=5$, $y'=x+b$ $\quad$ $5=b$

$y=\frac{x^2}{2}+5x+1$

For the k^{th} order differential equation, the initial conditions can be 0^{th} ~ $(k-1)^{th}$ derivatives at some points.

1.3 Differential Equations as Mathematical Model

differentiation
= amount of variation within a unit of independent variable

Physical meaning of differentiation:

the variation at certain time or certain place

[Example 1]: $v(t) = \dfrac{dx(t)}{dt}$, $a(t) = \dfrac{dv(t)}{dt} = \dfrac{d^2x(t)}{dt^2}$

$F - \beta v = ma \Rightarrow F - \beta \dfrac{dx(t)}{dt} = m\dfrac{d^2x(t)}{dt^2}$

2nd order DE
linear 2nd order DE

$x(t)$: location, $v(t)$: velocity, $a(t)$: acceleration
F: force, β: coefficient of friction, m: mass

[Example 2]: 人口隨著時間而增加的模型 1st order linear DE

$$\frac{dA(t)}{dt} = kA(t)$$

A: population

人口增加量和人口呈正比

$g(t)=k \quad h(A)=A$

Separable

Sec 2-2

$$\frac{dA}{A} = k\,dt$$

$$\ln|A| = kt + c_1$$

$$|A| = e^{kt} e^{c_1}$$

$$A = c e^{kt}$$

$(c = \pm e^{c_1})$

Sec 2-3

$$A' - kA = 0$$

$$e^{\int -k\,dt} = e^{-kt}$$

$$(e^{-kt} A)' = 0$$

$$e^{-kt} A = c$$

$$A = c e^{kt}$$

1st order linear DE

[Example 3]: 開水溫度隨著時間會變冷的模型

$$\frac{dT}{dt} = k(T - T_m)$$

$k<0$

T: 熱開水溫度,

T_m: 環境溫度

t: 時間

Ex: $T(0)=100$ $T_m=20$

$T(1)=60$, then

$T(2)=40$, $T(3)=30$, $T(4)=25 \cdots$

Sec 2-2

$$\frac{dT}{T-T_m} = k\,dt$$

$$\ln|T-T_m| = kt + c_1$$

$$T - T_m = \pm e^{kt} e^{c_1}$$

$$T = c e^{kt} + T_m$$

$(c = \pm e^{c_1})$

Sec 2-3

$$T' - kT = -kT_m$$

$$e^{\int -k\,dt} = e^{-kt}$$

$$(T e^{-kt})' = -kT_m e^{-kt}$$

$$T e^{-kt} = T_m e^{-kt} + c$$

$$T = T_m + c e^{kt}$$

大一微積分所學的：

$\int f(t)dt$ 的解 例如：$\int \frac{1}{t}dt = \ln|t| + c$

$$\boxed{\frac{dA(t)}{dt} = f(t) \Rightarrow A(t) = \int f(t)dt + c}$$

Example: $\frac{dA(t)}{dt} = \frac{1}{t}$ $\longrightarrow$ $A(t) = \ln|t| + c$

$$\frac{dA(t)}{dt} = \frac{1}{t^2+4} \Rightarrow A(t) = \int \frac{1}{t^2+4}dt + c = ?\ \ \frac{1}{2}\tan^{-1}\frac{t}{2} + c$$

Problems

(1) 若等號兩邊都出現 dependent variable (如 pages 19, 20 的例子)

(2) 若 order of DE 大於 1 (如 page 18 的例子)

該如何解？

Review

- dependent variable and independent variable
- DE
- PDE and ODE
- Order of DE
- linear DE and nonlinear DE
- explicit solution and implicit solution
- initial value; boundary value
- IVP

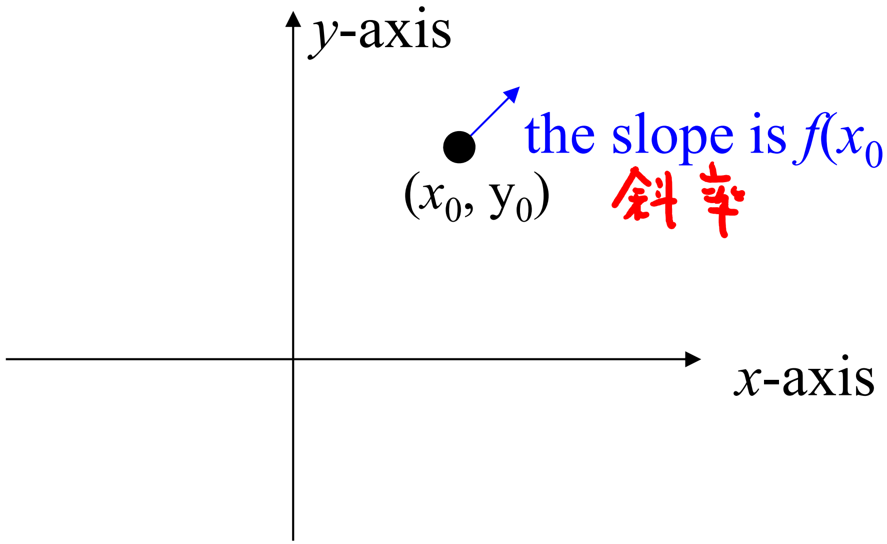
Chapter 2 First Order Differential Equation

2-1 Solution Curves without a Solution

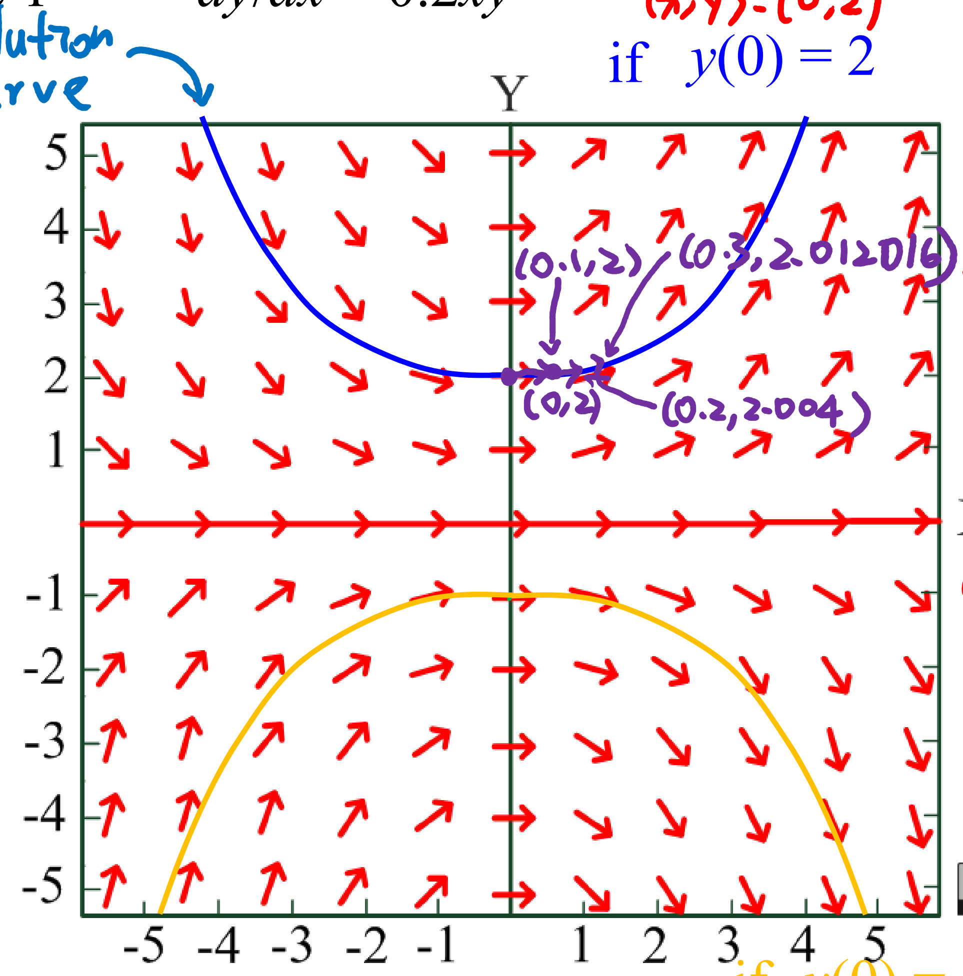
圖解法

Instead of using analytic methods, the DE can be solved by graphs (圖解)

slopes and the field directions: $\dfrac{dy}{dx} = f\left(x, y\right)$



Example 1 $dy/dx = 0.2xy$



① $(x,y) : (0,2)$ $0.2xy = 0$

slope : 0

when $x = 0.1$

y should be

$y(0.1) = \underbrace{2}_{y(0)} + 0.1 \times \underbrace{0}_{\text{slope}} = 2$

② $(x,y) = (0.1, 2)$

slope $= 0.2 \times 0.1 \times 2 = 0.04$

when $x = 0.2$

$y(0.2) = \underbrace{2}_{y(0.1)} + 0.1 \times \underbrace{0.04}_{\text{slope}} = 2.004$

③ $(x,y) : (0.2, 2.004)$

slope $= 0.2 \times 0.2 \times 2.004$

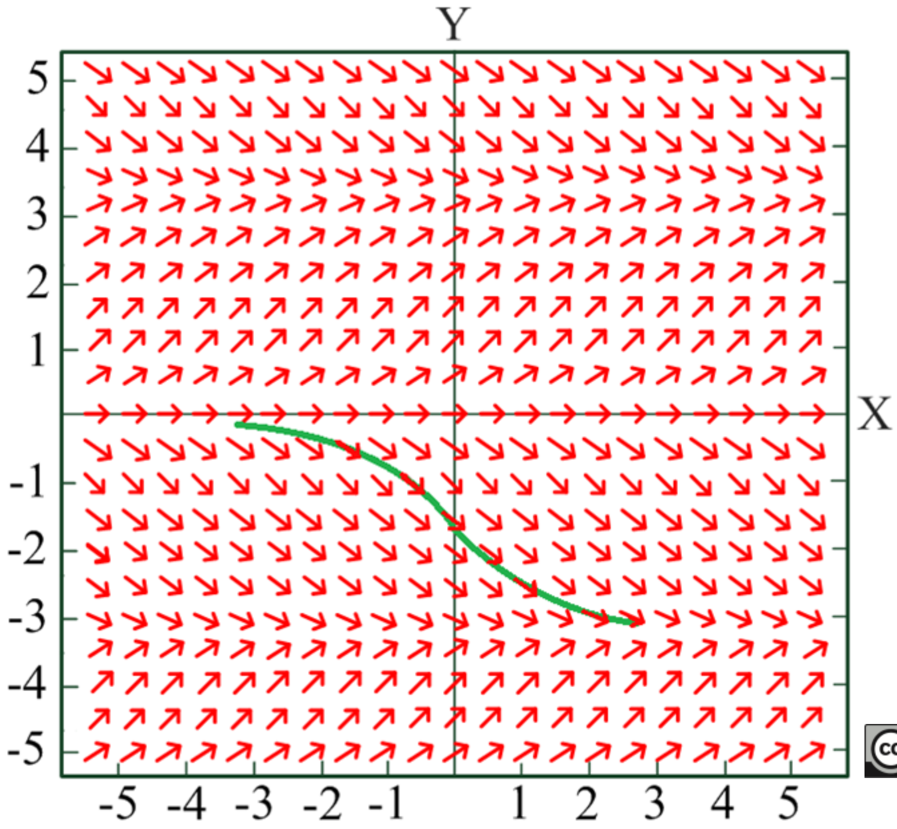
$= 0.08016$

$y(0.3) = 2.004 + 0.1 \times 0.08016$

$= 2.012016$

From： Fig. 2-1-3(a) in “Differential Equations-with Boundary-Value Problem”, 9th ed., Dennis G. Zill and Michael R. Cullen.

Example 2 $dy/dx = \sin(y)$, $y(0) = -3/2$



From： Fig. 2-1-4 in "Differential Equations-with Boundary-Value Problem", 9^{th} ed., Dennis G. Zill and Michael R. Cullen.

With initial conditions, one curve can be obtained

Advantage:

It can solve some 1^{st} order DEs that cannot be solved by mathematics.

Disadvantage:

It can only be used for the case of the 1st order DE.

It requires a lot of time

Section 2-6 A Numerical Method

數值方法

- Another way to solve the DE without analytic methods
- independent variable $x \xrightarrow{\text{sampling(取樣)}} x_0,\ x_1, x_2, \ldots\ldots\ldots\ldots$
- Find the solution of $\dfrac{dy(x)}{dx} = f(x, y)$

$$\frac{dy(x)}{dx} = \lim_{x_{n+1}-x_n \to 0} \frac{y(x_{n+1}) - y(x_n)}{x_{n+1} - x_n}$$

Since $\dfrac{dy(x)}{dx} = f(x, y) \xrightarrow{\text{approximation}} \dfrac{y(x_{n+1}) - y(x_n)}{x_{n+1} - x_n} = f(x_n, y(x_n))$

$$y(x_{n+1}) = \underbrace{y(x_n)}_{\text{前一點的值}} + \underbrace{f(x_n, y(x_n))}_{\text{slope}} \underbrace{(x_{n+1} - x_n)}_{\text{取樣間格}}$$

$$\frac{dy(x)}{dx} = f(x,y)$$

recursive

$$y(x_{n+1}) = y(x_n) + f(x_n, y(x_n))(x_{n+1} - x_n)$$

sampling points : $x_0, x_1, x_2, x_3 \cdots\cdots$

If $y(x_0)$ is known

n=0 $\quad y(x_1) = y(x_0) + f(x_0, y(x_0))(x_1 - x_0)$

n=1 $\quad y(x_2) = y(x_1) + f(x_1, y(x_1))(x_2 - x_1)$

n=2 $\quad y(x_3) = y(x_2) + f(x_2, y(x_2))(x_3 - x_2)$

$\vdots$

$\vdots$

$\vdots$

$\vdots$

$$\frac{dy(x)}{dx} = f(x, y) \qquad\qquad y(x_{n+1}) = y(x_n) + f(x_n, y(x_n))(x_{n+1} - x_n)$$

Example:

- $dy(x)/dx = 0.2xy \longrightarrow y(x_{n+1}) = y(x_n) + 0.2x_n y(x_n)*(x_{n+1} - x_n)$.

- $dy/dx = \sin(x) \longrightarrow y(x_{n+1}) = y(x_n) + \sin(x_n)*(x_{n+1} - x_n)$.

$y(x) = -\cos x + C$ $\qquad$ $y(x) = -\cos x$

if $y(0) = -1$, $-1 = -1 + C$, $C = 0$

後頁為 $dy/dx = \sin(x)$, $y(0) = -1$,

(a) $x_{n+1} - x_n = 0.01$, $\qquad$ (b) $x_{n+1} - x_n = 0.1$,

(c) $x_{n+1} - x_n = 1$, $\qquad$ (d) $x_{n+1} - x_n = 0.1$, $dy/dx = 10\sin(10x)$ 的例子

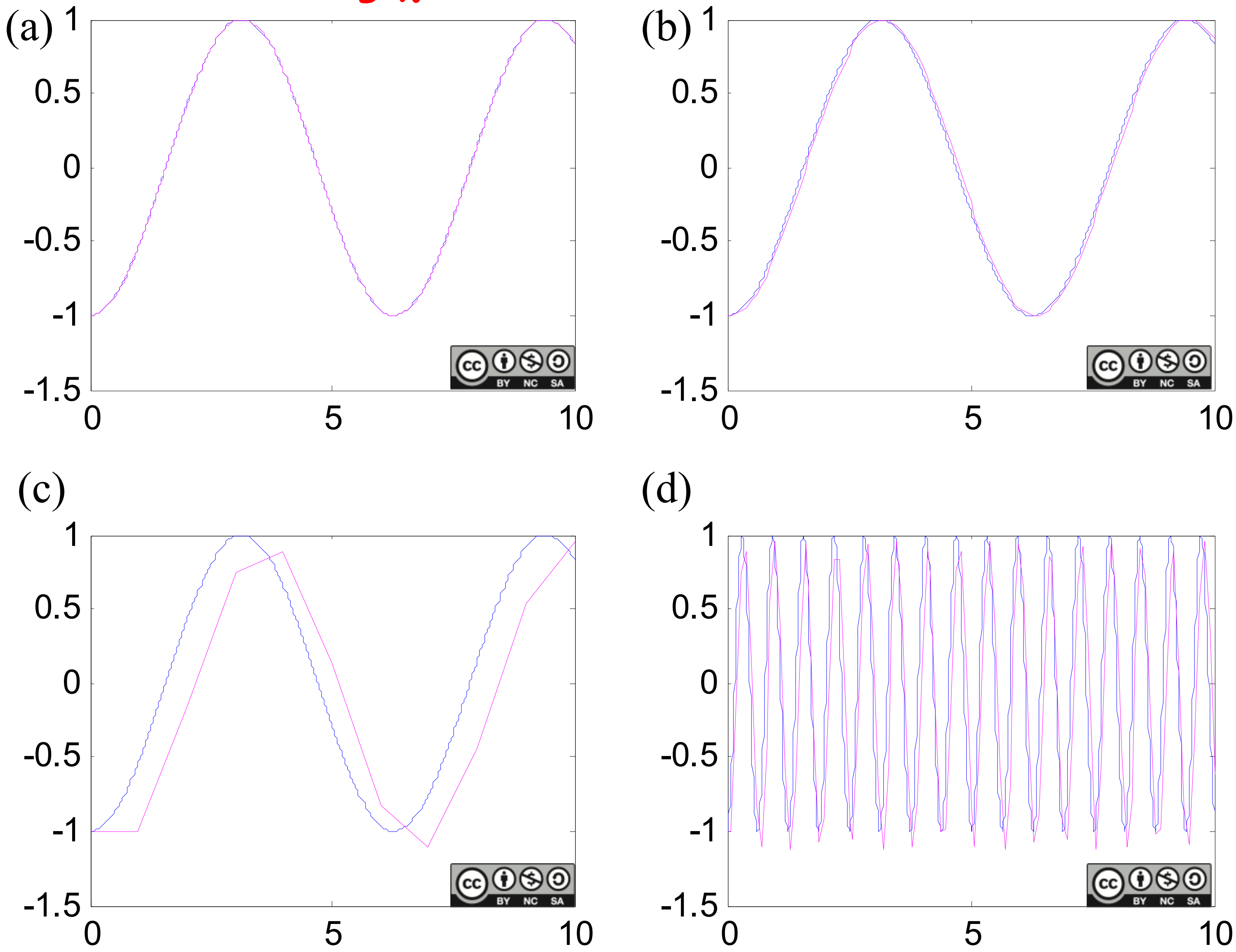
if $y(0) = -1$, $y(x) = -\cos(10x)$

Constraint for obtaining accurate results:

(1) small sampling interval $\quad$ (2) small variation of $f(x, y)$

Blue line: analytic solution; pink line: numerical solution

$-\cos x$



Advantages

-- It can solve some 1st order DEs that cannot be solved by mathematics.

-- can be used for solving a complicated DE (not constrained for the 1^{st} order case)

-- suitable for computer simulation

Disadvantages

-- numerical error (數值方法的課程對此有詳細探討)

附錄一 Table of Integration 複習

$1/x$	$\ln\lvert x\rvert + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$
$\tan(x)$	$-\ln\lvert\cos(x)\rvert + c$
$\cot(x)$	$\ln\lvert\sin(x)\rvert + c$
a^x	$a^x/\ln(a) + c$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c$
$1/\sqrt{a^2-x^2}$	$\sin^{-1}(x/a)+c$
$-1/\sqrt{a^2-x^2}$	$\cos^{-1}(x/a)+c$
$x\,e^{ax}$	$\dfrac{e^{ax}}{a}\left(x-\dfrac{1}{a}\right)+c$
$x^2\,e^{ax}$	$\dfrac{e^{ax}}{a}\left(x^2-\dfrac{2x}{a}+\dfrac{2}{a^2}\right)+c$

Exercises for Practicing

(not homework, but are encouraged to practice)

1-1: 1, 13, 19, 23, 37

1-2: 3, 13, 21, 33

1-3: 2, 7, 28

2-1: 1, 13, 25, 33

2-6: 1, 3

附錄二 Methods of Solving the First Order Differential Equation

- graphic method Sec 2-1
- numerical method Sec 2-6
- analytic methods
  - direct integral (calculus)
  - separable variable Sec 2-2
  - method for linear equation Sec 2-3
  - method for exact equation Sec 2-4
  - homogeneous equation method
  - Bernoulli's equation method
  - method for $Ax + By + c$

  (homogeneous equation method, Bernoulli's equation method, method for $Ax + By + c$: Sec 2-5)
- series solution Chap 6
- matrix solution Chap 8
- transform methods
  - Laplace transform Chap 7
  - Fourier series Chap 11
  - Fourier transform Chap 14

Direct Integral

It is the simplest method for solving the 1st order DE:

constraint : y(x) cannot appear in both sides.

$dy(x)/dx = f(x)$

$$y(x) = \int f(x)dx$$
$$= F(x) + c$$

where $\boxed{\dfrac{dF(x)}{dx} = f(x)}$

Something about Calculating the Integral

(1) Integration 的定義：$\int_{x_0}^{x} f(t)dt$

例：$\int_{x_0}^{x} \cos(t)dt = \sin x + c$

(2) 算完 integration 之後不要忘了加 constant c

(3) If $\int_{x_0}^{x} f(t)dt = g(x) + c$

then $\frac{d}{dx} g(x) = f(x)$

$$\int_{x_0}^{x} f(at)dt = \frac{1}{a} g(ax) + c_1$$

c_1 is also some constant

$$\frac{d}{dx} g(ax) = a\, f(ax)$$

2-2 Separable Variables

2-2-1 方法的限制條件

1st order DE 的一般型態: $dy(x)/dx = f(x, y)$

[Definition 2.2.1] (text page 47)

If $dy(x)/dx = f(x, y)$ and $f(x, y)$ can be separate as

$$\boxed{f(x, y) = g(x)h(y)}$$

constraint ☆1

i.e., $dy(x)/dx = g(x)h(y)$

then the 1st order DE is separable (or have separable variable).

條件： $dy(x)/dx = g(x)h(y)$

$g(x) = \cos x\, e^{x}$, $h(y) = e^{2y}$

$$\frac{dy}{dx} = \cos(x)e^{x+2y}$$ separable Sec 2-2

$$\frac{dy}{dx} = x + y$$ not separable Sec. 2-3

but linear

2-2-2 解法 ☆☆

If $\boxed{\dfrac{dy}{dx} = g(x)h(y)}$, then

Step 1 $\dfrac{dy}{h(y)} = g(x)dx$ 分離變數

$\Sigma p(y)dy = \Sigma g(x)dx$

$\boxed{p(y)dy = g(x)dx}$ where $\boxed{p(y) = 1/h(y)}$

對y 對x作積分

Step 2 $\int p(y)dy = \int g(x)dx$ 個別積分

$P(y) + c_1 = G(x) + c_2$ where $\boxed{\dfrac{dP(y)}{dy} = p(y)}$ $\boxed{\dfrac{dG(x)}{dx} = g(x)}$

$\boxed{P(y) = G(x) + c}$

Extra Step: (a) Initial conditions ☆4

(b) Check the singular solution ☆5 (i.e., the constant solution)

奇解

Extra Step (b) Check the singular solution (常數解):

☆5

Suppose that y is a constant r

$$\frac{dy}{dx} = g(x)h(y)$$

$$\downarrow$$

$$0 = g(x)h(r) \quad \text{for any } x$$

$$\downarrow$$

$$h(r) = 0$$

$$\downarrow$$

solution for r

$$\downarrow$$

See whether the solution is a special case of the general solution.

2-2-3 Examples

[Example 1] (text page 48)

$$\boxed{(1+x)\,dy - y\,dx = 0}$$

$\rightarrow (1+x)\frac{dy}{dx} = y$

$$\frac{dy}{dx} = \frac{y}{1+x}$$

$g(x) = \frac{1}{1+x}$, $h(y) = y$

Step 1 $\frac{dy}{y} = \frac{dx}{1+x}$

$\ln|y| + c_2 = \ln|1+x| + c_3$

$c_1 = c_3 - c_2$

Step 2 $\ln|y| = \ln|1+x| + c_1$

$e^{\ln|y|} = e^{\ln|1+x|} e^{c_1}$

$$|y| = e^{\ln|1+x|} e^{c_1} \longrightarrow |y| = e^{c_1}|1+x|$$

$$y = \pm e^{c_1}|1+x| = \pm e^{c_1}(1+x)$$

$$\boxed{y = c(1+x)} \qquad c = \pm e^{c_1}$$

★2 注意運算細節

Extra Step (b)

check the singular solution

set $y = r$,

$0 = r/(1+x)$

$r = 0,$

$y = 0$

(a special case of the general solution)

$c = 0$

Example 練習小技巧

遮住解答和筆記，自行重新算一次

(任何和解題有關的提示皆遮住)

Practice more and Learn better.

(多訓練手感)

[Example 2] (with **initial condition** and **implicit solution**, text page 49)

$f(x,y) = -\frac{x}{y}$ $x_0 = 4,\ y_0 = -3$

$\frac{\partial f}{\partial y} = \frac{x}{y^2}$ } both are continuous at $x_0 = 4, y_0 = -3$ (satisfies page 48)

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}},\qquad \boxed{y(4) = -3}$$

$g(x) = -x$

$h(y) = \frac{1}{y}$

Extra Step (b)

check the singular solution

$y = r$

$0 = -\frac{x}{r}$

(no solution)

Step 1 $\quad ydy = -xdx$

Step 2 $\quad y^2/2 = -x^2/2 + c$

$y^2 = -x^2 + C$

$y(4) = -3 \quad (-3)^2 = -4^2 + C \quad C = 25$

Extra Step (a)

$4.5 = -8 + c, \quad c = 12.5$

$x^2 + y^2 = 25$ (implicit solution)

$y = \sqrt{25 - x^2}$ invalid → 不满足 $y(4) = -3$

☆3 $\boxed{y = -\sqrt{25 - x^2}}$ valid

(explicit solution)

[Example 3] (with singular solution, text page 49)

$$\boxed{\frac{dy}{dx} = y^2 - 4}$$

$g(x)=1$

$h(y)=y^2-4$

separable

Step 1

$$\frac{dy}{y^2-4} = dx$$

$\frac{1}{y^2-4} = \frac{1}{(y-2)(y+2)}$

$= \frac{A}{y-2} + \frac{B}{y+2}$

$A(y+2)+B(y-2)=1$

$A+B=0,\ 2A-2B=1$

$B=-A \quad 4A=1$

$A=\frac{1}{4},\ B=-\frac{1}{4}$

Step 2

$$\frac{1}{4}\frac{dy}{y-2} - \frac{1}{4}\frac{dy}{y+2} = dx$$

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + 4c_1$$

$\left|\frac{y-2}{y+2}\right| = e^{4x}e^{4c_1}$

$$\frac{y-2}{y+2} = \pm e^{4x+4c_1} = ce^{4x} \longrightarrow \boxed{y = 2\frac{1+ce^{4x}}{1-ce^{4x}}} \quad \text{or} \quad \boxed{y = \pm 2}$$

$$c = \pm e^{4c_1}$$

$y-2 = ce^{4x}(y+2)$

When $c=0$, $y=2$

$c \to \infty$, $y=-2$

Extra Step (b)

check the singular solution

☆5

$$\frac{dy}{dx} = y^2 - 4$$

set $y = r$,

$0 = r^2 - 4$

$r = \pm 2$,

$y = \pm 2$

$\frac{dy}{dx} = g(x)h(y)$ separable

$g(x) = \frac{\sin 2x}{\cos x}$

$h(y) = \frac{e^y}{e^{2y}-y}$ $\frac{1}{h(y)} = \frac{e^{2y}-y}{e^y}$

[Example 4] (text page 50)

$$\boxed{\left(e^{2y}-y\right)\cos x\frac{dy}{dx} = e^y \sin 2x} \qquad y(0)=0$$

Step 1 $\left(e^y - ye^{-y}\right)dy = \frac{\sin 2x}{\cos x}dx = 2\sin x dx$

Note: $\sin 2x = 2\sin x \cos x$

Step 2 $e^y + (y+1)e^{-y} = -2\cos x + c$

Note: $\frac{d}{dy}(ay+b)e^{-y} = -ye^{-y}$

$(-ay+a-b)e^{-y} = -ye^{-y}$

$a=b=1$

Extra Step (a)

from $y(0)=0$

$2=-2+c$

$$\boxed{e^y + ye^{-y} + e^{-y} = 4-2\cos x}$$ (implicit solution)

Extra Step (b)

set $y=r$

$0=e^r \sin 2x$

no solution for r

$\int -ye^{-y}dy = (ay+b)e^{-y}$

Example in the top of text page 51

$$\frac{dy}{dx} = xy^{1/2}, \qquad y(0) = 0$$

Step 1 $y^{-\frac{1}{2}}dy = xdx$

Step 2 $2y^{\frac{1}{2}} = \frac{x^2}{2} + c_1$

$y^{\frac{1}{2}} = \frac{x^2}{4} + c \qquad c = \frac{c_1}{2}$

$y = \left(\frac{x^2}{4} + c\right)^2 = \frac{1}{16}(x^2 + 4c)^2$

Extra Step (a)

from $y(0)=0$

$0 = c^2, \; c = 0$

$y = \frac{x^4}{16}$

Solution: $\boxed{y = \frac{1}{16}x^4}$ or $\boxed{y = 0}$

$f(x,y) = xy^{\frac{1}{2}} \qquad x_0 = 0, \; y_0 = 0$

$\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-\frac{1}{2}}$ is not continuous at $x_0 = 0, \; y_0 = 0$

(not satisfies page 48)

Extra Step (b)

Check the singular solution ✱5

$y = r$

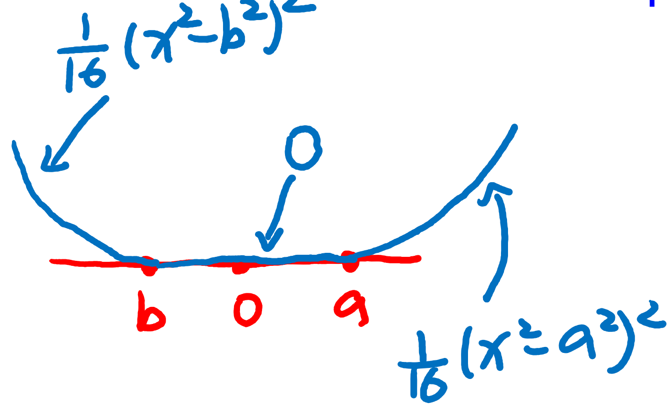
$0 = xr^{\frac{1}{2}}$

$r = 0$

$y = 0$ satisfies $y(0) = 0$

infinite number of solutions

補充：其實，這一題還有更多的解



$$\frac{dy}{dx} = xy^{1/2}, \qquad y(0) = 0$$

solutions: (1) $y = \frac{1}{16}x^4$ (2) $y = 0$

$$(3) \quad y = \begin{cases} \frac{1}{16}\left(x^2 - b^2\right)^2 & \text{for } x \le b \\ 0 & \text{for } b < x < a \\ \frac{1}{16}\left(x^2 - a^2\right)^2 & \text{for } x \ge a \end{cases} \qquad b \le 0 \le a$$

2-2-4 IVP 是否有唯一解？

★6

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

這個問題有唯一解的條件：(Theorem 1.2.1, text page 17)

如果 (i) $f(x, y)$, (ii) $\dfrac{\partial}{\partial y} f(x, y)$ 在 $x = x_0, y = y_0$ 的地方為 continuous

則必定存在一個 h，使得 IVP 在 $x_0 - h < x < x_0 + h$ 的區間當中有唯一解

證明可參考

J. Ratzkin, *Existence and Uniqueness of Solutions to First Order Ordinary Differential Equations*, 2007.

The Existence and Uniqueness Theorem for First-Order Differential Equations, www.math.uiuc.edu/~tyson/existence.pdf

2-2-5 Solutions Defined by Integral

(1) $$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x)$$

(2) If

$dy/dx = g(x)$ and $y(x_0) = y_0$

then

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt}$$ ☆

$$y(x_0) = y_0 + \underbrace{\int_{x_0}^{x_0} g(t)\,dt}_{0}$$

$= y_0$

難以計算積分 (integral, antiderivative) 的 function，

被稱作是 nonelementary function

如 e^{-x^2} ， $\sin x^2$

此時，solution 就可以寫成 $y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt$ 的型態

[Example 5] (text page 51)

$$\frac{dy}{dx} = e^{-x^2} \qquad y(3) = 5$$

Solution $y(x) = 5 + \int_3^x e^{-t^2} dt$ $= 5 + \int_3^\infty e^{-t^2} dt - \int_x^\infty e^{-t^2} dt$

$= 5 + \frac{\sqrt{\pi}}{2} erfc(3) - \frac{\sqrt{\pi}}{2} erfc(x)$

或者可以表示成 complementary error function

$$y(x) = 5 + \frac{\sqrt{\pi}}{2}\left(erfc(3) - erfc(x)\right)$$

- error function (useful in probability)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

- complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \operatorname{erf}(x)$$

用 t 取代 x 以做區別

See text page 60 in Section 2.3

2-2-6 本節要注意的地方

(1) 複習並背熟幾個重要公式的積分

(2) 別忘了加 c

並且熟悉什麼情況下 c 可以合併和簡化

(3) 若時間允許，可以算一算 singular solution

(4) 多練習，加快運算速度

附錄三 微分方程查詢

http://integrals.wolfram.com/index.jsp

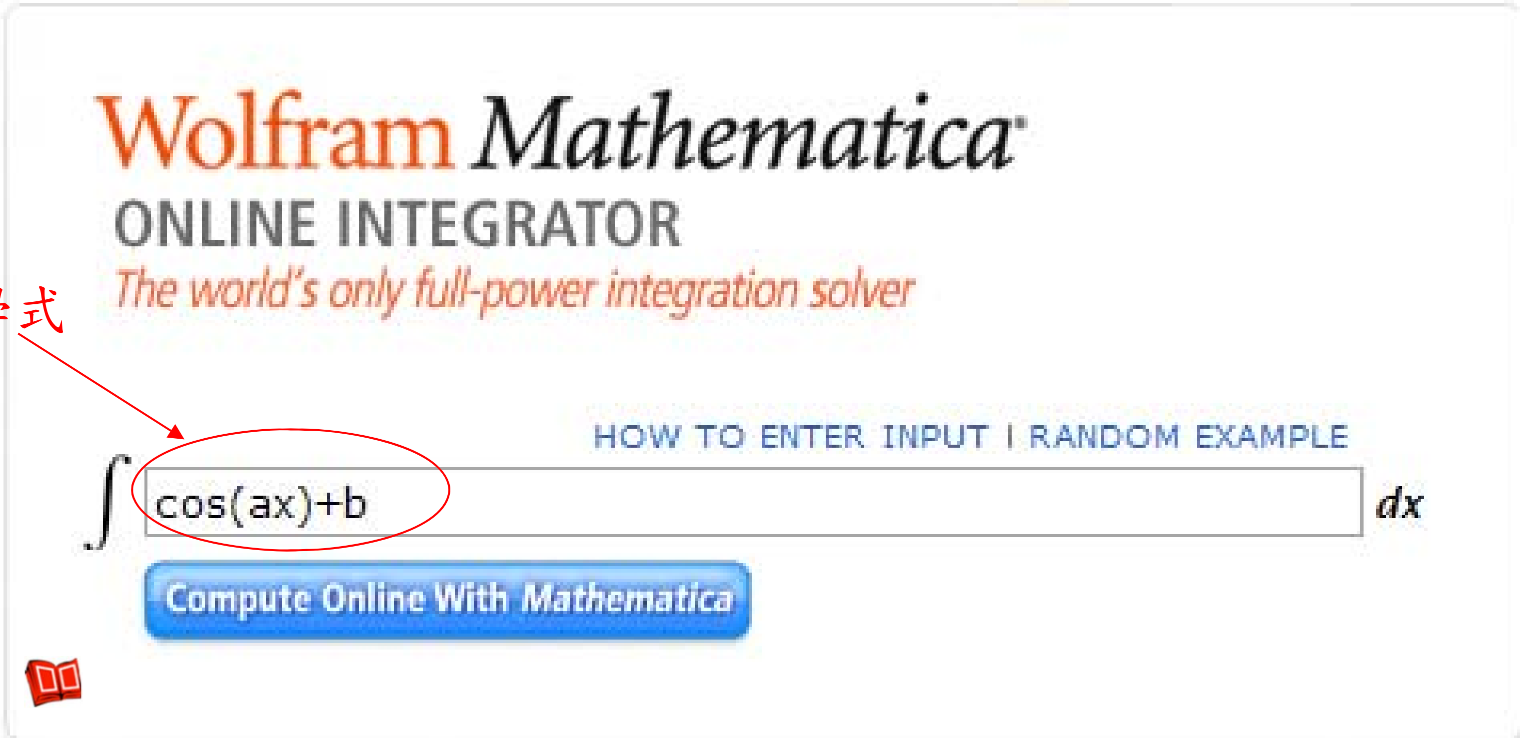
輸入數學式，就可以查到積分的結果

範例：

(a) 先到integrals.wolfram.com/index.jsp 這個網站

(b) 在右方的空格中輸入數學式，例如

數學式

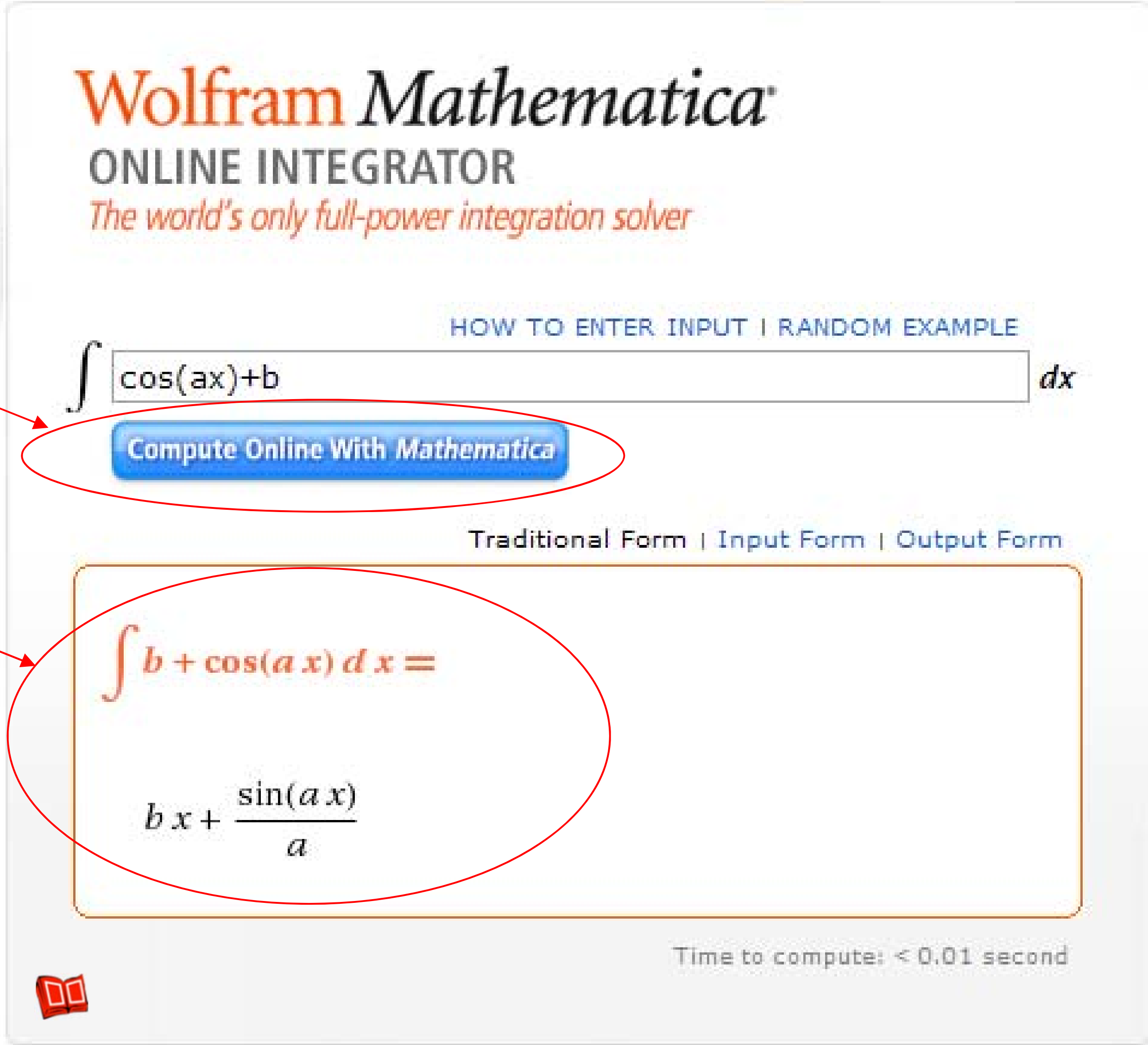


(c) 接著按 “Compute Online with Mathematica”

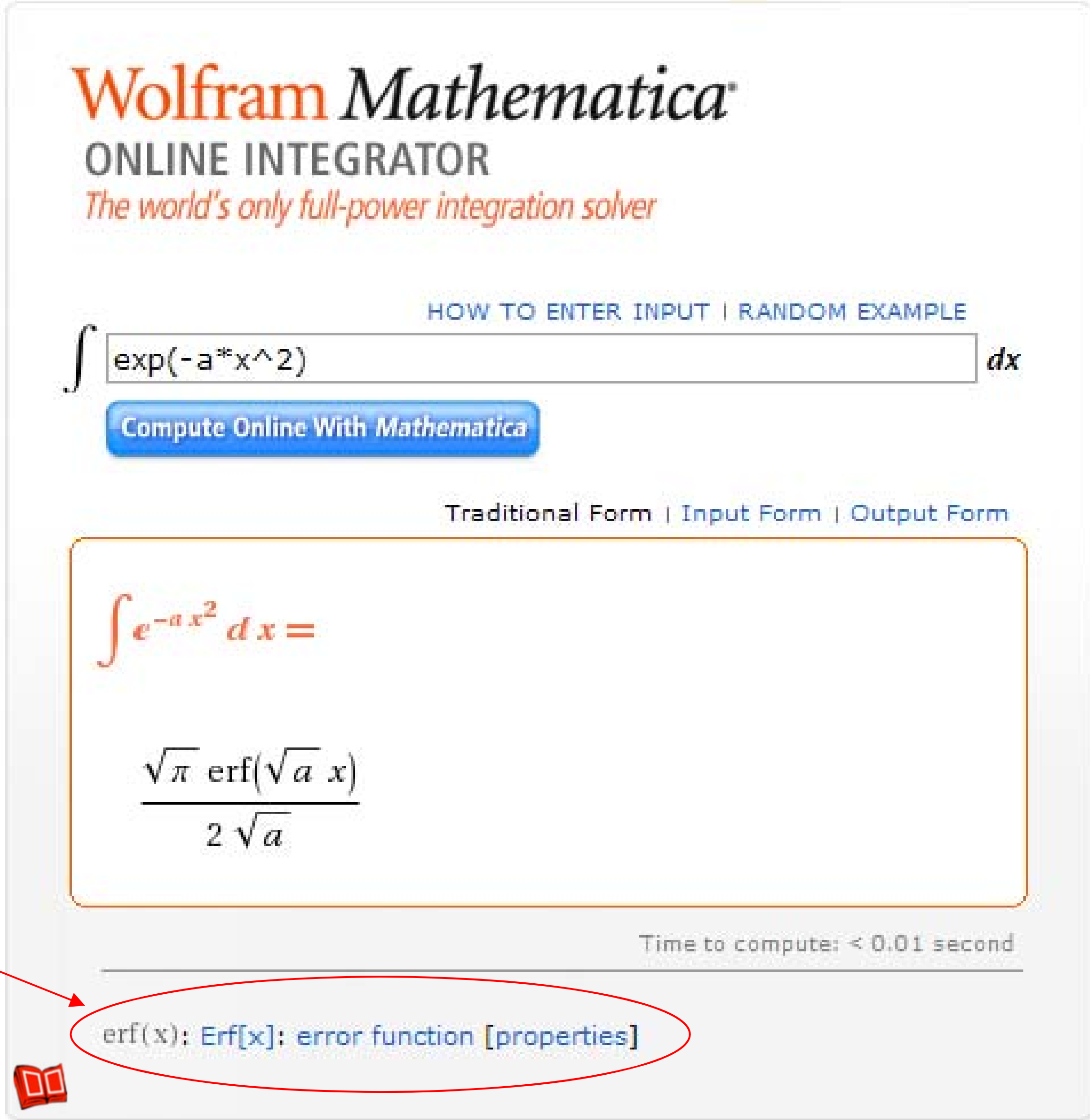
就可以算出積分的結果

按

結果



(d) 有時，對於一些較複雜的數學式，下方還有連結，點進去就可以看到相關的解說



連結

其他有用的網站

http://mathworld.wolfram.com/

對微分方程的定理和名詞作介紹的百科網站

http://www.sosmath.com/tables/tables.html

眾多數學式的 mathematical table (不限於微分方程)

http://www.seminaire-sherbrooke.qc.ca/math/Pierre/Tables.pdf

眾多數學式的 mathematical table，包括 convolution, Fourier transform, Laplace transform, Z transform

軟體當中， Maple, Mathematica, Matlab, Python 皆有微積分結果查詢有功能

Python 微積分查詢

```
from sympy import *
x = symbols('x')

integrate(1/(4+x**2), x)          # Find the integral of
```
$\frac{1}{4+x^2}$

```
diff(cos(x**3), x)                # Find the differentiation of
```
$\cos(x^3)$

```
integrate(1/(x**2), (x, 1,2))     # Find
```
$\int_1^2 \frac{1}{x^2}dx$

2-3 Linear Equations

"friendly" form of DEs

2-3-1 方法的適用條件

[Definition 2.3.1] The first-order DE is a linear equation if it has the following form:

$$\boxed{a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)}$$

限制條件式,

$g(x) = 0$: homogeneous

$g(x) \neq 0$: nonhomogeneous

名詞,,

If $g(x)=0$

$$\frac{dy}{dx}=-\frac{a_0(x)}{a_1(x)}y$$

☆2

Standard form: $\boxed{\dfrac{dy}{dx} + P(x)y = f(x)}$

最高階微分係數為1

$P(x) = \dfrac{a_0(x)}{a_1(x)}$

$f = \dfrac{g(x)}{a_1(x)}$

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \longrightarrow \frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

許多自然界的現象，皆可以表示成 linear first order DE

2-3-2 解法的推導

$$\boxed{\frac{dy}{dx}+P(x)y=f(x)}$$

☆3

子問題 1

$$\frac{dy_c}{dx}+P(x)y_c=0$$

Find the general solution $y_c(x)$

(homogeneous solution)

complementary function 名3

子問題 2

$$\frac{dy_p(x)}{dx}+P(x)y_p(x)=f(x)$$

Find any solution $y_p(x)$

(particular solution)

名4

Solution of the DE

$$\boxed{y(x)=y_c(x)+y_p(x)}$$

- $y_c + y_p$ is a solution of the linear first order DE, since

$$\frac{d(y_c + y_p)}{dx} + P(x)(y_c + y_p)$$
$$= \left(\frac{dy_c}{dx} + P(x)y_c\right) + \left(\frac{dy_p}{dx} + P(x)y_p\right)$$
$$= 0 + f(x) = f(x)$$

- All solutions of the linear first order DE should have the form $y_c + y_p$.

Its proof is as follows. If y is a solution of the DE, then

$$\frac{dy}{dx} + P(x)y - \left(\frac{dy_p}{dx} + P(x)y_p\right) = f(x) - f(x) = 0$$
$$\frac{d(y - y_p)}{dx} + P(x)(y - y_p) = 0$$

Thus, $y - y_p$ should be the solution of $\frac{dy_c}{dx} + P(x)y_c = 0$

y should have the form of $y = y_c + y_p$

Solving the homogeneous solution $y_c(x)$（子問題一）

$$\boxed{\frac{dy_c}{dx} + P(x)y_c = 0}$$

separable variable

$$\frac{dy_c}{y_c} = -P(x)\,dx$$

$$\ln|y_c| = \int -P(x)\,dx + c_1$$

$$|y_c| = e^{\int -P(x)dx} e^{c_1}$$

$$y_c = \pm e^{c_1} e^{\int -P(x)dx}$$

$$\boxed{y_c = ce^{-\int P(x)dx}}$$

Set $y_1 = e^{-\int P(x)dx}$, then $y_c = cy_1$

Solving the particular solution $y_p(x)$ (子問題二)

$$\boxed{\frac{dy_p(x)}{dx} + P(x)y_p(x) = f(x)}$$

u(x) is to be solved

Set $y_p(x) = u(x)\,y_1(x)$ (猜測 particular solution 和 homogeneous solution 有類似的關係)

$$u(x)\frac{dy_1(x)}{dx} + y_1(x)\frac{du(x)}{dx} + P(x)u(x)y_1(x) = f(x)$$

$$\downarrow$$

$$y_1(x)\frac{du(x)}{dx} + u(x)\underbrace{\left[\frac{dy_1(x)}{dx} + P(x)y_1(x)\right]}_{\text{equal to zero}} = f(x)$$

Since y_1 is one of the homogeneous solution

$$\downarrow$$

$$y_1(x)\frac{du(x)}{dx} = f(x)$$

$$\downarrow$$

$du(x) = \dfrac{f(x)}{y_1(x)}dx$ ignore '+c'

$$\frac{du(x)}{dx} = \frac{f(x)}{y_1(x)} \longrightarrow u(x) = \int \frac{f(x)}{y_1(x)}dx \longrightarrow \boxed{y_p(x) = y_1(x)\int \frac{f(x)}{y_1(x)}dx}$$

$y_1 = e^{-\int P(x)dx}$ $\quad y_p = y_1 \int \frac{f(x)}{y_1} dx$

$y_c = c\,y_1$

$$\boxed{y_c = ce^{-\int P(x)dx}} \qquad \boxed{y_p(x) = e^{-\int P(x)dx} \int [e^{\int P(x)dx} f(x)]dx}$$

solution of the linear 1st order DE:

$$\boxed{y(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx} \int [e^{\int P(x)dx} f(x)]dx}$$

where c is any constant

★ $\boxed{e^{\int P(x)dx} \ : \text{integrating factor}} = \frac{1}{y_1}$

#05

$$e^{\int P(x)dx} y(x) = c + \int [e^{\int P(x)dx} f(x)]dx$$

$$\boxed{\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} f(x)}$$

2-3-3 解法 ☆☆

(Step 1) Obtain the **standard form** and find $P(x)$ (page 59, ☆₂)

(Step 2) Calculate $e^{\int P(x)dx}$

$e^{\int P(x)}y' + P(x)e^{\int P(x)}y = e^{\int P(x)}f(x)$

(Step 3a) The standard form of the linear 1st order DE can be rewritten as:

$$\boxed{\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)}$$

remember it ☆☆₁

(Step 3b) Integrate both sides of the above equation

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)dx + c,$$

$$\boxed{y = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx + ce^{-\int P(x)dx}}$$

☆☆₂

or remember it, skip Step 3a

(Extra Step) (a) Initial value

(c) Check the Singular Point ☆₅ (不同於 singular solution)

奇點

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \qquad \frac{dy}{dx}+P(x)y=f(x)$$

$P=\frac{a_0}{a_1}$ $f=\frac{g}{a_1}$

Singular points: the locations where $a_1(x) = 0$ ☆5

i.e., $P(x) \to \pm\infty$

More generally, even if $a_1(x) \neq 0$ but $P(x) \to \pm\infty$ or $f(x) \to \pm\infty$, then the location is also treated as a singular point.

(a) Sometimes, the solution may not be defined on the interval including the singular points. (such as Example 4)

(b) Sometimes the solution can be defined at the singular points, such as Example 3

More generally, even if $a_1(x) \neq 0$ but $P(x) \to \infty$ or $f(x) \to \infty$, then the location is also treated as a singular point.

Exercise 33

$$(x+1)\frac{dy}{dx} + y = \ln|x|$$

singular points: $(-1, 0)$

since $\ln 0 \to -\infty$

solution intervals

$x \in (0, \infty)$ or $(-1, 0)$ or $(-\infty, -1)$

2-3-4 例子

[Example 2] (text page 57)

$$\boxed{\frac{dy}{dx}-3y=6}$$

Step 1 $P(x)=-3$

Extra Step (c)
check the singular point

Step 2 $e^{\int P(x)dx}=e^{-3x}$

page65 (1)

為何在此時可以將

$\underline{-3x+c}$ 簡化成 $\underline{-3x}$? ☆4

$\frac{d}{dx}\left(e^{-3x\cancel{+c}}y\right)=6e^{-3x\cancel{+c}}$

Step 3 $\frac{d}{dx}\left[e^{-3x}y\right]=6e^{-3x}$

或著，跳過 Step 3，直接代公式

Step 4 $e^{-3x}y=-2e^{-3x}+c$

$$y=e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx+ce^{-\int P(x)dx}$$

$$\boxed{y=-2+ce^{3x}}$$

[Example 3] (text page 58)

$$\boxed{x\frac{dy}{dx}-4y=x^6e^x}$$

÷x

standard

Step 1 $\dfrac{dy}{dx}-4\dfrac{y}{x}=x^5e^x,\quad P(x)=-\dfrac{4}{x}$

Step 2 $e^{\int P(x)dx}=e^{-4\ln|x|}=|x|^{-4}$

若只考慮 $x>0$ 的情形, $e^{\int P(x)dx}=x^{-4}$

page 65 (1)

Step 3 $\dfrac{d}{dx}\left[x^{-4}y\right]=xe^x$ note $f(x)=x^5e^x$

Step 4 $x^{-4}y=(x-1)e^x+c$

$$\boxed{y=(x^5-x^4)e^x+cx^4}$$

$\boxed{x \text{ 的範圍: } (0,\infty)}$ or $x\in(-\infty,0)$

both do not include $x=0$

不可寫成 $(0,\infty)$ and $(-\infty,0)$

Extra Step (c)

check the singular point

$x=0$ since $P(0)\to-\infty$

思考： $x<0$ 的情形

$e^{\int P(x)dx}=(-x)^{-4}$

$=x^{-4}(-1)^{-4}$

$=x^{-4}$

[Example 4] (text page 58)

$$\boxed{\left(x^2-9\right)\frac{dy}{dx}+xy=0}$$

Extra Step (c)

check the singular point

$x=\pm 3$

Step1 ☆2

$$\frac{dy}{dx}+\frac{x}{x^2-9}y=0$$

$$P(x)=\frac{x}{x^2-9}$$

$$\int\frac{g'(x)}{g(x)}=\ln|g(x)|$$

$$\int\frac{2x}{x^2-9}dx=\ln|x^2-9|$$

Step2

$$e^{\int\frac{x}{x^2-9}dx}=e^{\frac{1}{2}\ln\left|x^2-9\right|}=\sqrt{|x^2-9|}$$

Step3

$$\frac{d}{dx}\left(\sqrt{|x^2-9|}\cdot y\right)=0$$

$$\sqrt{|x^2-9|}\cdot y=c$$

$$\boxed{y=\frac{c}{\sqrt{|x^2-9|}}}$$

✓

defined for $x\in(-\infty,-3)$, $(-3,3)$, or $(3,\infty)$

not includes the points of $x=-3$, 3

[Example 6] (text, page 59)

$$\boxed{\frac{dy}{dx}+y=f(x)} \qquad y(0)=0 \qquad f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & x>1\end{cases}$$

$p(x)=1$

check the singular point

0 1

Step2 $e^{\int P(x)dx}=e^{x}$

Step3 $\frac{d}{dx}(e^{x}y)=e^{x}f(x)$ ★6

Case 1 $0\le x\le 1$

$\frac{d}{dx}(e^{x}y)=e^{x}$

$e^{x}y=e^{x}+c_1$

$y=1+c_1e^{-x}$ ★6-1

consider $y(0)=0$ $0=1+c_1$ from initial condition

$\boxed{y=1-e^{-x}}$

case 2 $x>1$

$\frac{d}{dx}(e^{x}y)=0$

$e^{x}y=c_2$

$y=c_2e^{-x}$

要求 $y(x)$ 在 $x=1$ 的地方為 continuous ★6-2

$\boxed{y=(e-1)e^{-x}}$

Case 1: $y(x)=1-e^{-x}$

$y(1)=1-e^{-1}$

Case 2, $y=c_2e^{-x}$, $y(1)=c_2e^{-1}$

$1-e^{-1}=c_2e^{-1}$, $c_2=e-1$

stable: 0 transcient: $(e-1)e^{-x}$

2-3-5 名詞和定義

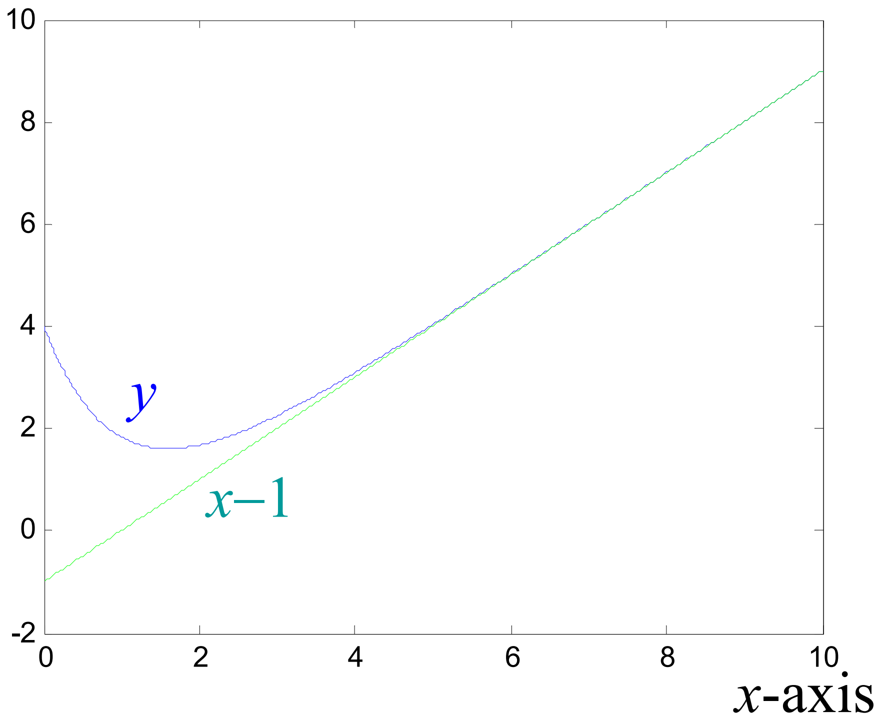
名7 名8

(1) transient term, stable term

Example 5 (text page 59) 的解為 $y = x - 1 + 5e^{-x}$

$5e^{-x}$: transient term 當 x 很大時會消失

$x - 1$: stable term



(2) piecewise continuous

A function $g(x)$ is piecewise continuous in the region of $[x_1, x_2]$ if $g(x)$ exists for any $x \in [x_1, x_2]$.

In Example 6, $f(x)$ is piecewise continuous in the region of [0, 1) or $(1, \infty)$

片段 連續 的

(3) Integral (積分) 有時又被稱作 antiderivative

(4) error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \operatorname{erf}(x)$$

(5) sine integral function

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

Fresnel integral function

$$S(x) = \int_0^x \sin\left(\pi t^2/2\right)dt$$

(6) $$\frac{dy}{dx} + P(x)y = f(x)$$

$f(x)$ 常被稱作 input 或 driving function

Solution $y(x)$ 常被稱作 output 或 response

2-3-6 小技巧

When $\dfrac{dy}{dx}$ is not easy to calculate:

Try to calculate $\dfrac{dx}{dy}$ 好

Example: $\dfrac{dy}{dx} = \dfrac{1}{x+y^2}$ (not linear, not separable)

$$\downarrow$$

$\dfrac{dx}{dy} = x + y^2$ (linear) for x

$$\downarrow$$

$x = -y^2 - 2y - 2 + ce^y$ (implicit solution)

$\frac{dx}{dy} - x = y^2 \quad P(y) = -1$

$e^{\int -1\,dy} = e^{-y}$

$(e^{-y}x)' = y^2 e^{-y}$

$e^{-y}x = (ay^2 + by + d)e^{-y}$

$= (-y^2 - 2y - 2)e^{-y} + c$

$\frac{d}{dx}(ay^2 + by + d)e^{-y} = y^2 e^{-y}$

$-ay^2 - by - d + 2ay + b = y^2$

$a = -1,\ b = -2,\ d = -2$

2-3-7 本節要注意的地方

(1) 要先將 linear 1st order DE 變成 **standard form**

(2) 別忘了 singular point

注意：singular point 和 Section 2-2 提到的 singular solution 不同

(3) 記熟公式

$$\boxed{\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=e^{\int P(x)dx}f(x)}$$

或

$$\boxed{y=e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx+ce^{-\int P(x)dx}}$$

(4) 計算時，$e^{\int P(x)dx}$ 的常數項可以忽略

太多公式和算法，怎麼辦？

最上策： realize + remember it

上策： realize it

中策： remember it

下策： read it without realization and remembrance

最下策： rest z…..z..…z……

Chapter 3 Modeling with First-Order Differential Equations

應用題

(1) Convert a question into a 1^{st} order DE.

將問題翻譯成數學式

(2) Many of the DEs can be solved by

Separable variable method or

Linear equation method

(with integration table remembrance)

3-1 Linear Models

Growth and Decay (Examples 1~3)

Change the Temperature (Example 4)

Mixtures (Example 5)

Series Circuit (Example 6)

可以用 Section 2-3 的方法來解

[Example 1] (an example of growth and decay, text page 85)

Initial: A culture（培養皿）initially has P_0 number of bacteria.

翻譯 $\rightarrow A(0) = P_0$

The other initial condition: At $t = 1$ h, the number of bacteria is measured to be $3P_0/2$.

翻譯 $\rightarrow A(1) = 3P_0/2$

關鍵句: If the **rate of growth** is **proportional to the number** of bacteria $A(t)$ presented at time t,

翻譯 $\rightarrow \boxed{\dfrac{dA}{dt} = kA}$ k is a constant

Question: determine the time necessary for the number of bacteria to triple

翻譯 $\rightarrow$ find t such that $A(t) = 3P_0$

這裡將課本的 $P(t)$ 改成 $A(t)$

$$\boxed{\frac{dA}{dt} = kA}$$

$A(0) = P_0,\ A(1) = 3P_0/2$ 可以用 什麼方法解？

Extra Step (b)
check singular solution

Step 1 $\frac{dA}{A} = kdt$

Step 2 $\ln|A| = kt + c_1$

$|A| = e^{kt+c_1}$

$A = ce^{kt} \qquad c = \pm e^{c_1}$

Extra Step (a)

(1) $P_0 = c \cdot 1$ ⇨ $c = P_0$

(2) $3P_0/2 = ce^{k}$ ⇨ $k = \ln(3/2) = 0.4055$

$$\boxed{A = P_0 e^{0.4055t}}$$

more one unknown k
∴ two constraints are required.

針對這一題的問題

$3P_0 = P_0 e^{0.4055t}$ $\boxed{t = \ln(3)/0.4055 \approx 2.71h}$

課本用 linear (Section 2.3) 的方法來解 Example 1

思考：為什麼此時需要兩個 initial values 才可以算出唯一解？

[Example 4] (an example of temperature change, text page 88)

Initial: When a cake is removed from an oven, its temperature is measured at 149° C.

翻譯 → $T(0) = 149$

The other initial condition: Three minutes later its temperature is 85 ° C.

翻譯 → $T(3) = 85$

question: Suppose that the room temperature is 21° C. How long will it take for the cake to cool off to 22° C? (註：這裡將課本的問題做一些修改)

翻譯 → find t such that $T(t) = 22$.

另外，根據題意，了解這是一個物體溫度和周圍環境的溫度交互作用的問題，所以 $T(t)$ 所對應的 DE 可以寫成

$$\boxed{\frac{dT}{dt} = k(T - 21)}$$

k is a constant

$$\boxed{\frac{dT}{dt} = k\left(T - 21\right)}$$

Constraints: $T(0) = 149$ $T(3) = 85$

課本用 separable variable 的方法解

如何用 linear 的方法來解？

Sec 2-3

$T' - kT = -k21$

$P(t) = -k$

$e^{\int P(t)dt} = e^{-kt}$

$(e^{-kt}T)' = -ke^{-kt}21$

$e^{-kt}T = e^{-kt} \cdot 21 + C$

$T = 21 + Ce^{kt}$

$T(0) = 149 \rightarrow 149 = 21 + C, \; C = 128$

$T(3) = 85 \rightarrow 85 = 21 + 128e^{3k}$

$e^{3k} = \frac{1}{2} \quad k = \frac{-\ln 2}{3}$

$T = 21 + 128e^{\frac{-\ln 2}{3}t} = 21 + 128 \cdot \left(\frac{1}{2}\right)^{\frac{t}{3}}$

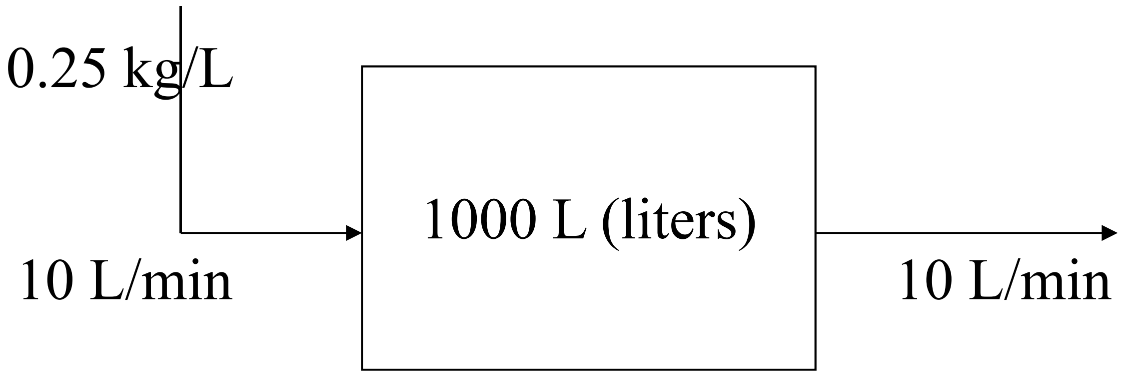
If $21 + 128\left(\frac{1}{2}\right)^{\frac{t}{3}} = 22$

$\left(\frac{1}{2}\right)^{\frac{t}{3}} = \frac{1}{128}, \quad \frac{t}{3} = 7$

$t = 21$

[Example 5] (an example for mixture, text page 88)

Concentration:



水槽

A: the amount of salt in the tank $\qquad A(0)=25$

$A = 250 + Ce^{-\frac{t}{100}}$

$A = 250 - 225e^{-\frac{t}{100}}$

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt})$$

$$= 10 \cdot 0.25 - \frac{10}{1000}A$$

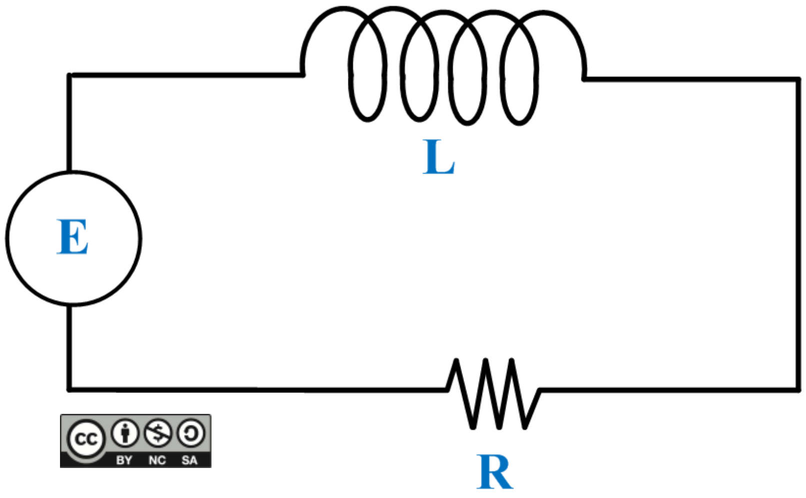
$A' + \frac{A}{100} = 2.5$

$P: \frac{1}{10} \quad e^{\int P dt} = e^{100t}$

$(Ae^{\frac{1}{100}t})' = 2.5e^{\frac{t}{100}}$

$Ae^{\frac{t}{100}} = 250e^{\frac{t}{100}} + C$


80
學年度
NTUEE



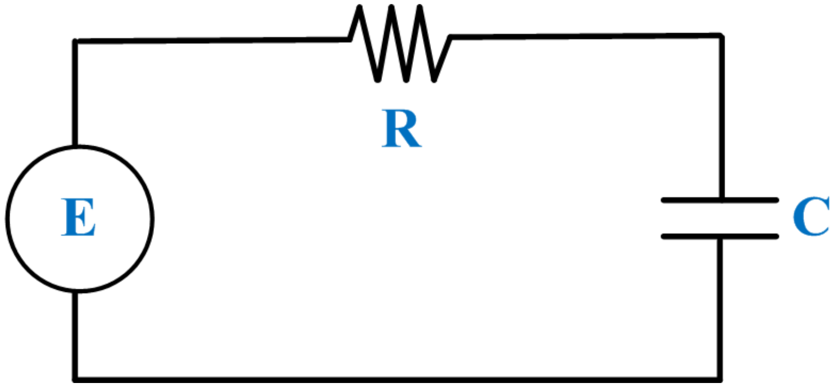
LR series circuit

From Kirchhoff's second law

$$L\frac{di}{dt}+Ri=E\left(t\right)$$

$$\downarrow$$

$$L\frac{d^2}{dt}q$$



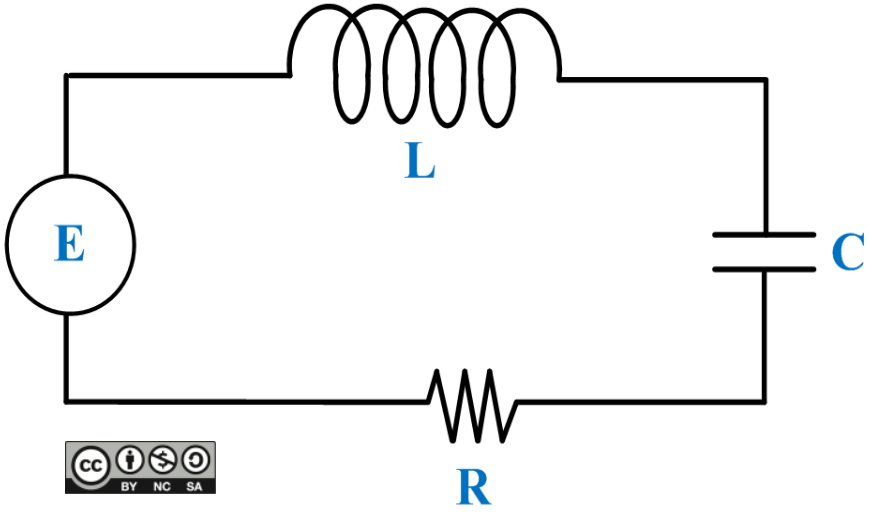
RC series circuit

$$\frac{q}{C} + Ri = E(t)$$

q: 電荷

$i = \frac{dq}{dt}$

$$\frac{q}{C} + R\frac{dq}{dt} = E(t)$$



How about an *LRC* series circuit?

$$\frac{q}{C}+R\frac{dq}{dt}+L\frac{d^2q}{dt^2}=E(t)$$

[Example 7] (text page 90) *LR* series circuit

- $E(t)$: 12 volt, • inductance: 1/2 henry,
- resistance: 10 ohms, • initial current: 0

$i(0)=0$

step 1

$$\boxed{\frac{1}{2}\frac{di}{dt}+10i=12} \dashrightarrow \frac{di}{dt}+20i=24 \dashrightarrow P(t)=20 \dashrightarrow e^{\int P(t)dt}=e^{20t+c_1}$$

這裡 $+c_1$ 可省略

$$\frac{d}{dt}e^{20t}i=24e^{20t} \dashrightarrow e^{20t}i=\frac{6}{5}e^{20t}+c \dashrightarrow \boxed{i(t)=\frac{6}{5}+ce^{-20t}}$$

$$i(0)=0$$

$$0=\frac{6}{5}+c$$

$$\boxed{i(t)=\frac{6}{5}-\frac{6}{5}e^{-20t}}$$

stable

transient

Circuit problem for t is small and $t \longrightarrow \infty$

For the LR circuit: L R

transient stable

For the RC circuit: R C

transient stable

3-2 Nonlinear Models

Sec 2-3 cannot be applied

可以用 separable variable 或其他的方法來解

3-2-1 Logistic Equation

logic

used for describing the growth of population

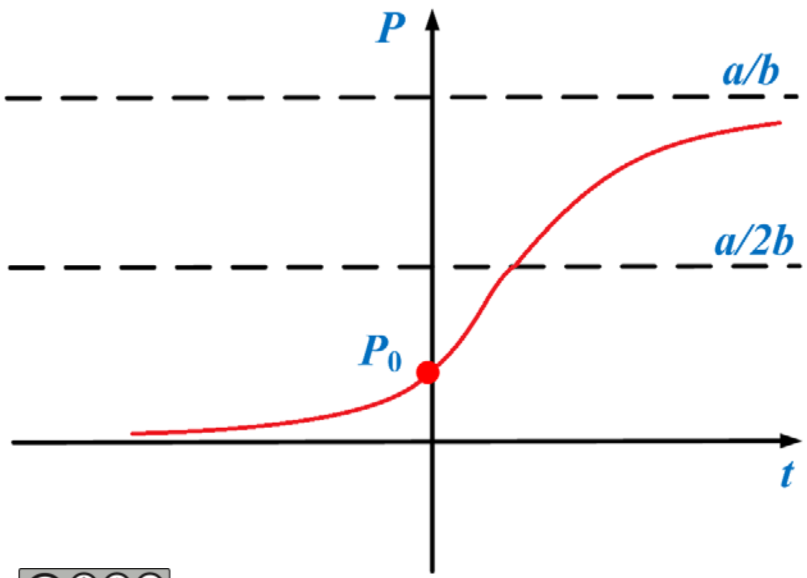
$$\boxed{\frac{dP}{dt} = P(a - bP) = bP(\frac{a}{b} - P)}$$

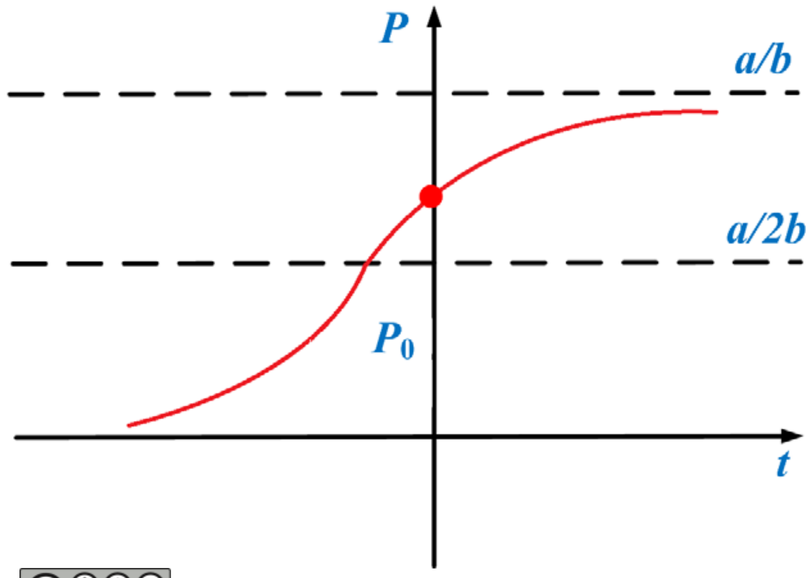
$\frac{a}{b}$: upper bound

The solution of a **logistic equation** is called the **logistic function**.

Two stable conditions: $\boxed{P = 0}$ and $\boxed{P = \frac{a}{b}}$.

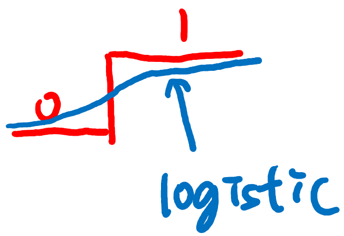
$$\frac{dP}{dt} = bP\left(\frac{b}{a} - P\right)$$





Logistic curves for differential initial conditions

logic



Solving the logistic equation

$$\boxed{\frac{dP}{dt} = P(a-bP)}$$

Sec. 2-2

separable variable

$$\downarrow$$

$$\frac{dP}{P(a-bP)} = dt$$

$$\frac{1}{P(a-bP)} = \frac{A}{P} + \frac{B}{a-bP}$$

$$Aa - AbP + BP = 1$$

$$A = \frac{1}{a}, \quad B = \frac{b}{a}$$

$$\left(\frac{1/a}{P} + \frac{b/a}{a-bP}\right) dP = dt$$

$$\downarrow$$

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + c$$

註： $\int \frac{-b}{a-bP}\,dP = \int \frac{\frac{d}{dP}(a-bP)}{a-bP}\,dP = \ln|a-bP| + c_0$

$$\downarrow$$

$$\ln\left|\frac{P}{a-bP}\right| = at + ac$$

$$\downarrow$$

(with initial condition $P(0) = P_0$)

$$\frac{P}{a-bP} = c_1 e^{at} \longrightarrow \boxed{P(t) = \frac{ac_1}{bc_1 + e^{-at}}} \longrightarrow \boxed{P(t) = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}}$$

$$c_1 = \pm e^{ac}$$

logistic function

[Example 1] (text page 99) There are 1000 students.

- Suppose a student carrying a flu virus returns to an isolate college campus of 1000 students.

  翻譯 $\rightarrow x(0) = 1$

- If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected,

  $1000-x$

  翻譯 $\rightarrow \boxed{\dfrac{dx(t)}{dt} = kx(1000-x)}$ k is a constant

- determine the number of infected students after 6 days

  翻譯 $\rightarrow$ find $x(6)$

- if it is further observed that after 4 days $x(4) = 50$

整個問題翻譯成

$$\boxed{\frac{dx(t)}{dt} = kx(1000 - x)}$$

Constraints: $x(0) = 1,\ \ x(4) = 50$

find $x(6)$

可以用separable variable 的方法

(The solution is on the next page)

$$\boxed{\frac{dx(t)}{dt} = kx(1000 - x)}$$

$$\frac{dx(t)}{x(1000-x)} = kdt$$

$$\frac{1}{1000}\left(\frac{dx}{x} + \frac{dx}{1000-x}\right) = kdt$$

$$\boxed{\frac{dx}{x} - \frac{dx}{x-1000} = 1000kdt}$$

$$\ln|x| - \ln|x-1000| = 1000kt + c_1$$

$$\left|\frac{x}{x-1000}\right| = e^{1000kt+c_1}$$

$$\frac{x}{x-1000} = c_2 e^{1000kt} \quad (c_2 = \pm e^{c_1})$$

$$(c_2 e^{1000kt} - 1)x = c_2 1000 e^{1000kt}$$

$$\boxed{x = \frac{1000}{1 - ce^{-1000kt}}} \quad (c = c_2^{-1})$$

$$x(0) = 1$$

$$1 = \frac{1000}{1-c}$$

$$c = -999$$

$$x = \frac{1000}{1 + 999e^{-1000kt}}$$

$$x(4) = 50$$

$$50 = \frac{1000}{1 + 999e^{-4000k}}$$

$$-1000k = -0.9906$$

$$\boxed{x = \frac{1000}{1 + 999e^{-0.9906t}}} \dashrightarrow x(6) \approx 276$$

Logistic equation 的變形

(1) $\dfrac{dP}{dt}=P(a-bP)\pm h$ 人口有遷移的情形

(2) $\dfrac{dP}{dt}=P(a-bP)-cP$ 遷出的人口和人口量呈正比

(3) $\dfrac{dP}{dt}=P(a-bP)+ce^{-kP}$ 人口越多，遷入的人口越少

(4) $\dfrac{dP}{dt}=P(a-b\ln P)$ Gompertz DE

$=bP(a/b-\ln P)$ 飽合人口為 $e^{a/b}$

人口增加量，和 $\ln\dfrac{\text{飽合人口}}{P}$ 呈正比

3-2-2 化學反應的速度

$A + B \rightarrow C$

$2H_2 + O_2 \rightarrow 2H_2O$

$H_2 + \frac{1}{2}O_2 \rightarrow H_2O$

- Use compounds A and B to for compound C
- $x(t)$: the amount of C
- To form a unit of C requires s_1 units of A and s_2 units of B

  $s_1 = 1$, $s_2 = \frac{1}{2}$
- a: the original amount of A
- b: the original amount of B
- The rate of generating C is proportional to the product of the amount of A and the amount of B

  $a - s_1 x$ $b - s_2 x$

$$\frac{dx(t)}{dt} = k\left(a - s_1 x\right)\left(b - s_2 x\right)$$

Sec. 2-2 separable variables

See Example 2

練習題

(not homework, but are encouraged to practice)

Section 2-2: 4, 6, 8, 10, 12, 14, 16, 21, 25, 28, 30, 36, 46, 48, 50, 54(a)

Section 2-3: 7, 9, 14, 18, 21, 29, 30, 33, 36, 40, 45, 47, 48, 58

Section 3-1: 5, 6, 10, 15, 20, 29, 32

Section 3-2: 2, 5, 14, 15

Review 3: 3, 4, 13, 14