2-4 Exact Equations

2-4-1 方法的條件

任何 first order DE 皆可改寫成

$N(x,y)\frac{dy}{dx}+M(x,y)=0$

$$M(x,y)dx+N(x,y)dy=0$$ 的型態

(1) 當 $\boxed{\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)}$ ☆1-1 成立時， 限制條件

可以用本節的 Exact Equation 的方法來解

(2) 當 $\boxed{\frac{\frac{\partial}{\partial y}M(x,y)-\frac{\partial}{\partial x}N(x,y)}{M(x,y)}}$ ☆1-2 is independent of x

或 $\boxed{\frac{\frac{\partial}{\partial y}M(x,y)-\frac{\partial}{\partial x}N(x,y)}{N(x,y)}}$ ☆1-3 is independent of y

可以用Modified Exact Equation Method 來解 (見講義 2-4-5)

2-4-2 方法的來源

- Review the concept of partial differentiation

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$ ☆2

- Specially, when $f(x, y) = c$ where c is some constant,

等高線

$$\underbrace{\frac{\partial f}{\partial x}}_{M}dx + \underbrace{\frac{\partial f}{\partial y}}_{N}dy = 0$$

補充：

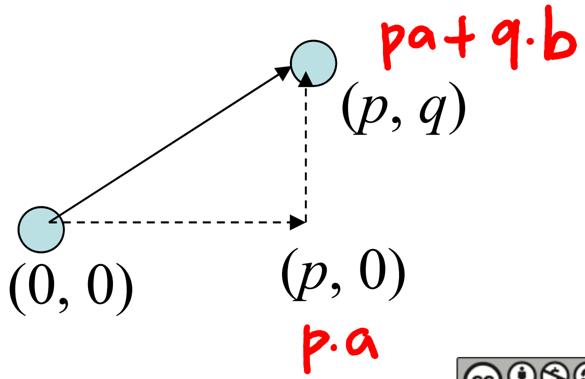
$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$df(x_1,x_2,x_3,\cdots,x_k) = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \cdots\cdots + \frac{\partial f}{\partial x_k}dx_k$$

思考：假設一個人在山坡的某處。若往東走，每走 1 公尺，高度會增加 a 公尺。若往北走，每走 1 公尺，高度會增加 b 公尺。假設這人現在所在的位置是(0, 0)。那麼這人的東北方，座標為 (p, q) 的地方，高度會比 (0, 0) 高多少？

$$a \times p + b \times q$$

$$df(x,y) = \underset{a}{\boxed{\frac{\partial f(x,y)}{\partial x}}}\,\underset{p}{\boxed{dx}} + \underset{b}{\boxed{\frac{\partial f(x,y)}{\partial y}}}\,\underset{q}{\boxed{dy}}$$



[Definition 2.4.1] We can express any 1st order DE as

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

- If there exists some function $f(x, y)$ that satisfies

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)} \quad \text{and} \quad \boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)} ,$$

★3

then we call the 1st order DE the **exact equation**.

- The method for checking whether the DE is an exact equation:

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

★1-1

(Proof): If $\dfrac{\partial f(x,y)}{\partial x} = M(x,y)$ and $\dfrac{\partial f(x,y)}{\partial y} = N(x,y)$,

then $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial}{\partial y}\dfrac{\partial}{\partial x} f(x,y) = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y} f(x,y) = \dfrac{\partial N(x,y)}{\partial x}$

For the exact equation,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\downarrow$$

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0$$

$$\downarrow$$ from page 102

可改寫成 $df(x,y) = 0,$

$$\downarrow$$

$$f(x,y) = c$$ ☆4

2-4-3 解法 ☆☆: $Mdx + Ndy = 0$

Step 0: check whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The Method for Solving the Exact Equation (A):

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)}$$

(1)

$$f(x,y) = \int M(x,y)\,dx + \boxed{g(y)}$$

☆5

g(y) is a constant for x

(2)

$$\boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(2)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

(2)

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

(2)

$$g(y) = \int g'(y)\,dy$$

(3) 代入

(3)

$$\boxed{\int M(x,y)\,dx + g(y) = c}$$

$f(x,y)=c$

(4)

further computation

Solution

double N

(0) Check $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$

整理

Previous Step: Check whether $\boxed{\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}}$ is satisfied.

Step 1: Solve $\dfrac{\partial f(x,y)}{\partial x} = M(x,y) \longrightarrow f(x,y) = \int M(x,y)\,dx + g(y)$

Step 2: 將 $f(x, y)$ 代入 $\dfrac{\partial f(x,y)}{\partial y} = N(x,y)$ ，以解出 $g(y)$

Step 3: Substitute $g(y)$ into

$$f(x,y) = \int M(x,y)\,dx + g(y) = c$$

Step 4: Further computation and obtain the solution

Extra Steps: (a) Consider the initial value problem

The Method for Solving the Exact Equation (B):

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)}$$

$$\boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(1)

$$f(x,y) = \int N(x,y)\,dy + h(x) = c$$

$h(x)$ is a constant for y

(2)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int N(x,y)\,dy + h'(x) = M(x,y)$$

(2)

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy$$

(2)

$$h(x) = \int h'(x)\,dx$$

(3) 代入

(2)

(3)

$$\boxed{\int N(x,y)\,dy + h(x) = c}$$

f = c

(4) further computation

Solution

(0) Check $\dfrac{\partial}{\partial y}M(x,y) = \dfrac{\partial}{\partial x}N(x,y)$

2-4-4 例子

[Example 1] (text page 67)

It can be solved by Sections 2-2, 2-3

$$\boxed{2xydx+(x^2-1)dy=0}$$

$$\frac{dy}{dx}+\frac{2xy}{x^2-1}=0$$

$M(x,y)=2xy$

$N(x,y)=x^2-1$ ⇢ Step 0: check whether it is exact

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

Step 1: $\dfrac{\partial f}{\partial x}=2xy$ → $f(x,y)=x^2y+g(y)$ ★5

Step 2: $\dfrac{\partial f}{\partial y}=x^2-1$

Step 2: $\dfrac{\partial f}{\partial y}=x^2+g'(y)=x^2-1$

Step 2: $g'(y)=-1$

Step 2: $g(y)=-y$ $-y+c_1$

Step 3: $x^2y-y=c$

Step 4: $\boxed{y=c/(x^2-1)}$

singular points: ± 1

$(-1,1)$, $(1,\infty)$, or $(-\infty,-1)$

思考: 是否有其他的方法可以解 Example 1?

[Example 2] (text page 67)

$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy$

$\frac{\partial N}{\partial x} = 2e^{2y} - \cos xy + xy\sin xy$

Exact!

$$\boxed{(\overset{M}{e^{2y} - y\cos xy})dx + (\overset{N}{2xe^{2y} - x\cos xy + 2y})dy = 0}$$

$$M(x,y) = e^{2y} - y\cos xy \qquad N(x,y) = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy \qquad \frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Step 1

Step 2

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

\#5

$$e^{2y} - y\cos xy + h'(x)$$

$$= e^{2y} - y\cos xy$$

Step 3

Step 3

$$h'(x) = 0$$

Step 2

$$h(x) = \cancel{c_1} \ 0$$

$$f(x,y) = xe^{2y} - \sin xy + y^2$$

Step 4

\#4

$$c = xe^{2y} - \sin xy + y^2$$

$$xe^{2y} - \sin xy + y^2 + c = 0$$

implicit solution

Step 0:

check for exact

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial N}{\partial x}$$

要注意

(a) 自行由另一個方向 $f(x,y)=\int M(x,y)dx+g(y)$ 來練習，

看是否得出同樣的結果。

(b) 得出的解 $xe^{2y}-\sin xy+y^2+c=0$ 為 implicit solution

(c) 思考：何時用 $f(x,y)=\int M(x,y)dx+g(y)$

何時用 $f(x,y)=\int N(x,y)dy+h(x)$

[Example 3] (text page 68)

$$\boxed{\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y\left(1-x^2\right)}}\qquad y(0)=2$$

$\underbrace{(xy^2-\cos x\sin x)}_{M}dx+\underbrace{y(x^2-1)}_{N}dy=0$

$\frac{\partial M}{\partial y}=2xy \quad \frac{\partial N}{\partial x}=2xy$

$M_y=N_x$ Exact!

$\frac{\partial f}{\partial x}=xy^2-\cos x\sin x$, $\frac{\partial f}{\partial y}=y(x^2-1)$

$f=\frac{y^2}{2}(x^2-1)+h(x)$

$xy^2+h'(x)=xy^2-\cos x\sin x$

$h'(x)=-\frac{1}{2}\sin(2x)$

$\sin(2x)=2\sin x\cos x$

$h(x)=\frac{1}{4}\cos(2x)$

$f=\frac{y^2}{2}(x^2-1)+\frac{1}{4}\cos(2x)=C$

自修，但注意

(a) initial value problem,

(b) 何時用 $f(x,y)=\int M(x,y)dx+g(y)$

何時用 $f(x,y)=\int N(x,y)dy+h(x)$

(c) 得出的 implicit solution 為 $y^2\left(1-x^2\right)-\cos^2 x=3$, 範圍：$x\in(-1, 1)$

from $x=0$, $y=2$

而 explicit solution 為 $\boxed{y=\sqrt{\frac{3+\cos^2 x}{1-x^2}}}$, 範圍：$x\in(-1, 1)$

$2\cdot(-1)+\frac{1}{4}=C$, $C=-\frac{7}{4}$

$2y^2(x^2-1)+\cos(2x)=-7$

$2y^2(x^2-1)+2\cos^2 x-1=-7$

$y^2(x^2-1)+\cos^2 x=-3$

singular points: $x=\pm 1$

注意 ★6

$\cancel{(-\infty,-1)}$ $(-1, 1)$ $\cancel{(1,\infty)}$

$(\because y(0)=2)$

為何 $y=-\sqrt{\frac{3+\cos^2 x}{1-x^2}}$ 不為解？

$\cos(2x)=\cos^2 x-\sin^2 x$

$=2\cos^2 x-1$

2-4-5 Modified Exact Equation Method

Technique: Use the integrating factor $\mu(x, y)$ to convert the 1st order DE into the exact equation.

$$M(x,y)dx + N(x,y)dy = 0$$

$$\downarrow$$

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

such that $$\frac{\partial \mu(x,y)M(x,y)}{\partial y} = \frac{\partial \mu(x,y)N(x,y)}{\partial x}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

It is hard to find μ.

Simplification: To make ① $\mu_x = 0$ or ② $\mu_y = 0$.

$$\boxed{\mu_x N - \mu_y M = (M_y - N_x)\mu}$$

1

(1) If we want $\mu_x = 0$, i.e, μ is dependent on y alone, then

$$-\mu_y M = (M_y - N_x)\mu$$

independent of x ←

$$\frac{\mu_y}{\mu} = -\frac{M_y - N_x}{M}$$

(should be independent of x)

$\Longrightarrow$ $\boxed{(M_y - N_x)/M}$ should be a function of y alone.

☆1-2

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{M_y - N_x}{M}$$

用 separable variable 的方法

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{M}dy$$

$$\ln|\mu| = \int \frac{N_x - M_y}{M}dy + c_1$$

$$\mu = \cancel{c}\, e^{\int \frac{N_x - M_y}{M}dy}$$

$$c = \pm e^{c_1}$$

$$\boxed{\mu(y) = e^{\int \frac{(N_x - M_y)}{M}dy}}$$

☆☆2

(注意正負號)

$$\boxed{\mu_x N - \mu_y M = (M_y - N_x)\mu}$$

(1) If we want $\mu_y = 0$, i.e, μ is dependent on x alone, then

$$\mu_x N = (M_y - N_x)\mu$$

independent of y

$$\leftarrow \frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$$

independent of y

$\Rightarrow$ $\boxed{(M_y - N_x)/N}$ should be a function of x alone.

☆1-3

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N}$$

用 separable variable 的方法

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{N}dx$$

$$\boxed{\mu(x) = e^{\int \frac{(M_y - N_x)}{N}dx}}$$

☆☆3

☆☆4

前面 2-4-3 (pages 106~108) 的解法流程再加一個步驟：

Step 0:

Whether $M_y = N_x$

Yes → **Case 1**: Use page 106 or 108

No → Whether $\dfrac{M_y - N_x}{M}$ is independent of x

- Yes → **Case 2**: $\mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}$

正負號和條件相反
(小心易背錯)
☆☆2

- No → Whether $\dfrac{M_y - N_x}{N}$ is independent of y
  - Yes → **Case 3**: $\mu(x) = e^{\int \frac{(M_y - N_x)}{N} dx}$

正負號同條件
☆☆3

  - No → **Case 4**: Use other methods

In Cases 2 and 3,

$$M(x, y)dx + N(x, y)dy = 0$$

$$\downarrow$$

$$\mu M(x, y)dx + \mu N(x, y)dy = 0$$

$$\downarrow$$

Use the process of page 106 or 108 but $M(x, y)$ is modified as $\mu M(x, y)$

$N(x, y)$ is modified as $\mu N(x, y)$

[Example 4] (text page 69)

$$\boxed{xydx+(2x^2+3y^2-20)dy=0}$$

$\frac{\partial M}{\partial y}$ $\frac{\partial N}{\partial x}$

$M_y \neq N_x$ ∴ not exact!

Step 0: $M=xy$ $N=2x^2+3y^2-20$

$$\boxed{M_y-N_x=x-4x=-3x}$$

$$\frac{M_y-N_x}{N}=\frac{-3x}{2x^2+3y^2-20}$$

not satisfy Case 3

$$\boxed{\frac{M_y-N_x}{M}=-\frac{3}{y}}$$

(independent of x)
(Case 2)

$-\frac{3}{y} \to \frac{3}{y}$

$$\mu(y)=e^{\int\frac{3}{y}dy}=e^{3\ln|y|}=y^3$$

Q: 為何 c 以及 ± 可省略?

If $M=xy^4$
$N=2x^2y^3+3y^5-20y^3$

$M_y=4xy^3$
$N_x=4xy^3$ ∴ exact!

$$\boxed{xy^4dx+(2x^2y^3+3y^5-20y^3)dy=0}$$

double N

Steps 1~4: $$\boxed{\frac{1}{2}x^2y^4+\frac{1}{2}y^6-5y^4=c}$$

2-4-6 本節需要注意的地方

(1) 使本節方法時，要先將 DE 改成如下的型態

$$M(x,y)dx + N(x,y)dy = 0$$

並且假設

$$\frac{\partial}{\partial x} f(x,y) = M(x,y), \qquad \frac{\partial}{\partial y} f(x,y) = N(x,y)$$

(2) 對 x 而言，$g(y)$ 是個常數；對 y 而言，$h(x)$ 是個常數

(3) 本節很少有 singular solution 的問題，

但是可能有 singular point 的問題

(4) 背熟三個判別式，二種情況的 integrating factor (小心勿背錯)

並熟悉解法的流程

2-5 Solutions by Substitutions

介紹 3 個特殊解法

Question: 尚有不少的 1st order DE 無法用 Sections 2-2~2-4 的方法來解

本節所提到的特殊解法的共通點：

用新的變數 u 來取代 y

對 症 下 藥

2-5-1 特殊解法 1: Homogeneous Equations

If $g(tx, ty) = t^{\alpha} g(x, y)$,　　　　　　　　 **★, 定義 (不同 Sec 2-3)**

then $g(x, y)$ is a **homogeneous function of degree α.**

Which one is homogeneous?　　　　　　 homogeneous　degree $=3$

$g(x, y) = x^3 + y^3$　　 $g(tx,ty) = t^3x^3 + t^3y^3 = t^3 g(x,y)$

$g(x, y) = x^3 + y^3 + 1$　 $g(tx,ty) = t^3x^3 + t^3y^3 + 1$

not homogeneous

(key: The sum of powers should be the same).

Note：課本中，homogeneous 有兩種定義

一種是 Section 2-3 的定義 (較常用)

一種是這裡的定義

兩者並不相同

■ For a 1st order DE:

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

If $M(x, y)$ and $N(x, y)$ are (i) homogeneous functions of (ii) the same degree,

then the 1st order DE is homogeneous.

解法的限制條件 ☆ 2-1

It can by solved by setting $u = \frac{y}{x}$

$$\boxed{y = xu, \qquad dy = u\,dx + x\,du,}$$ ☆☆, (from page 103)

and use the separable value method.

$y = xu$

$dy = \frac{\partial y}{\partial x} dx + \frac{\partial y}{\partial u} du$

$= u\,dx + x\,du$

If $M(x,y)dx + N(x,y)dy = 0$ is homogeneous

$$M(tx,ty) = t^{\alpha}M(x,y) \qquad N(tx,ty) = t^{\alpha}N(x,y)$$

$u = \frac{y}{x}$ 以 $t = 1/x$ 得出

then

$$M(1,u) = x^{-\alpha}M(x,y) \qquad N(1,u) = x^{-\alpha}N(x,y)$$

$$M(x,y) = x^{\alpha}M(1,u) \qquad N(x,y) = x^{\alpha}N(1,u)$$

where $u = y/x$, $\boldsymbol{y = xu}$

$$\boxed{dy = udx + xdu}$$

$$\cancel{x^{\alpha}}M(1,u)dx + \cancel{x^{\alpha}}N(1,u)(udx + xdu) = 0$$

$$[M(1,u) + uN(1,u)]dx + xN(1,u)du = 0$$

$$\boxed{\frac{dx}{x} = -\frac{N(1,u)du}{M(1,u) + uN(1,u)}} \quad \text{(separable)}$$

Procedure for solving the homogeneous 1st order DE

Previous Step: Conclude whether the DE is homogeneous

(快速判斷法：看 powers (指數) 之和是否一致)

Step 1: Set $\boldsymbol{u = y/x}$ $(y = ux)$, $dy = udx + xdu$

並化簡

Step 2: Convert into the separable 1st order DE

Step 3: Solve the separable 1st order DE (用 Sec. 2-2 的方法)

Step 4: Substitute $\boxed{u = y/x}$ (別忘了這個步驟)

☆3

[Example 1] (text page 73)

sum of powers $1+1=2$
$x^1 y^1$

$$\underbrace{(x^2+y^2)}_{M(x,y)}dx+\underbrace{\left(x^2-xy\right)}_{N(x,y)}dy=0$$

$M(tx, ty) = t^2M(x, y)$ $N(tx, ty) = t^2N(x, y)$

Previous Step:

Conclude whether the DE is homogeneous

homogeneous DE degree = 2

Step 1: Set $y = ux$, $dy = udx + xdu$

原式

$$(x^2+u^2x^2)dx+\left(x^2-ux^2\right)(udx+xdu)=0$$

$(x^2 + \cancel{u^2x^2} + ux^2 - \cancel{u^2x^2})dx$
$+ (x^3 - ux^3)dx$

$$(1+u^2)dx+\left(1-u\right)(udx+xdu)=0$$

$$(1+u)dx+\left(1-u\right)xdu=0 \longrightarrow \boxed{\left[\frac{1-u}{1+u}\right]du+\frac{dx}{x}=0}$$

Step 2: Convert into the separable 1st order DE

$$\left[\frac{1-u}{1+u}\right]du+\frac{dx}{x}=0$$

Step 3: Solve the separable 1st order DE

$$\downarrow$$

$$\int\left[-1+\frac{2}{1+u}\right]du+\int\frac{dx}{x}=0$$

$$\downarrow$$

$$-u+2\ln|1+u|+\ln|x|+c_1=0$$

$$\downarrow$$

$$\ln[|(1+u)|^2|x|]=u-c_1$$

$$\downarrow$$

$$(1+u)^2|x|=e^{u-c_1}$$

$$\downarrow$$

$$(1+u)^2x=c_2e^u \qquad (c_2=\pm e^{-c_1})$$

Step 4 代回 $u=y/x$

☆3

$$\downarrow$$

$$(1+y/x)^2x=c_2e^{y/x} \longrightarrow \boxed{(x+y)^2=c_2xe^{y/x}}$$

implicit solution

2-5-2 特殊解法 2: Bernoulli's Equations

伯努利

When n=0 → linear

【定義】 Bernoulli's equation: ☆2-2

$$\boxed{\frac{dy}{dx}+P(x)y=f(x)y^n}$$

so $\underline{y^{-n}\frac{dy}{dx}}+P(x)\underline{y^{1-n}}=f(x)$

$u^{\frac{-n}{1-n}}\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx} \quad u$

$=\frac{1}{1-n}\frac{du}{dx}$

☆☆2

We can set $\boxed{u=y^{1-n}\qquad \frac{dy}{dx}=\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}}$, and the method of solving the 1^st^ order linear DE to solve the Bernoulli's equation.

$y=u^{\frac{1}{1-n}}$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{du^{\frac{1}{1-n}}}{\underline{du}}\frac{du}{dx}=\frac{1}{\underline{1-n}}\underline{u^{\frac{n}{1-n}}}\frac{du}{dx}$$

(Chain rule)

Procedure for solving the Bernoulli's equation

Previous Step : Conclude whether the DE is a Bernoulli's equation

Step 1: Set $\boxed{\boldsymbol{u = y^{1-n}}}$, $\boxed{y = u^{\frac{1}{1-n}}}$, $\boxed{\frac{dy}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}}$

Step 2: Convert the Bernoulli's equation into the 1st order linear DE

Step 3: Solve the 1st order linear DE (use the method in Sec. 2-3)

Step 4: Substitute $\boldsymbol{u = y^{1-n}}$ (別忘了)

[Example 2] (text page 74)

$$\boxed{x\frac{dy}{dx}+y=x^2y^2}$$

$\downarrow$

Previous Step: 判斷 (Bernoulli, $\underline{n=2}$)

Step 1: set $\boldsymbol{u=y^{-1}}$ ($y=u^{-1}$) $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=-u^{-2}\dfrac{du}{dx}$

$u=y^{1-n}=y^{-1}$

(Chain rule)

Step 2: Convert into the 1st order linear DE (standard form)

$$\text{原式}\longrightarrow -xu^{-2}\frac{du}{dx}+u^{-1}=x^2u^{-2}\xrightarrow[\text{乘}-\frac{u^2}{x}]{}\frac{du}{dx}-\frac{1}{x}u=-x$$

Sec 2-3

$P(x)=-\frac{1}{x}$

$e^{\int P(x)}=e^{-\ln|x|}$

$=x^{-1}$

Step 3: Obtain the solution of the 1st order DE

$$u=-x^2+cx$$

$(x^{-1}u)'=-1$

$x^{-1}u=-x+c$

Step 4: 代回 $u=y^{-1}$ $\boxed{y=\dfrac{1}{-x^2+cx}}$

☆3

2-5-3 特殊解法 3

If the 1st order DE has the form,

$$\boxed{\frac{dy}{dx} = f\left(Ax + By + C\right)} \quad (B \neq 0)$$

☆2-3

(解法的限制條件)

☆☆3

we can set $\boxed{u = Ax + By + C}$ to solve it.

$$\boxed{\frac{dy}{dx} = \frac{1}{B}\frac{du}{dx} - \frac{A}{B}}$$

乘 $\frac{1}{Bdx}$

Since $\boxed{du = Adx + Bdy}$ (這式子也許較好記)

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= Adx + Bdy$$

Procedure for solving $\dfrac{dy}{dx} = f\left(Ax + By + C\right)$

Previous Step: Conclude

Step 1: Set $\boxed{\boldsymbol{u = Ax + By + C}} \longrightarrow \boxed{du = Adx + Bdy} \longrightarrow \boxed{\dfrac{dy}{dx} = \dfrac{1}{B}\dfrac{du}{dx} - \dfrac{A}{B}}$

Step 2: Converting (轉化成用其他方法可以解出來的 DE
未必一定是轉化成 separable variable DE)

Step 3: Solving

Step 4: Substitute $u = Ax + By + C$ (別忘了)

[Example 3] (text page 74)

$$\boxed{\frac{dy}{dx} = (-2x+y)^2 - 7}, \qquad y(0) = 0$$

page 103

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Previous Step: 判斷

Step 1: Set $\boxed{u = -2x + y}$ $\boxed{du = -2dx + dy}$ $\boxed{\frac{dy}{dx} = \frac{du}{dx} + 2}$

Step 2: Converting

$$\text{原式} \longrightarrow \frac{du}{dx} + 2 = u^2 - 7 \longrightarrow \boxed{\frac{du}{u^2 - 9} = dx}$$

separable

Step 3: Obtain the solution (別忘了在運算過程中，代回 $u = Ax + By$)

$$\frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right) du = dx$$

$$\ln|u-3| - \ln|u+3| = 6x + c_1$$

$$\frac{u-3}{u+3} = c_2 e^{6x} \qquad c_2 = \pm e^{c_1}$$

$$\frac{u-3}{u+3} = c_2 e^{6x}$$

$$\boxed{u = \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}}$$

Step 4: 代回 $u = Ax + By + C$ ☆3

$$\boxed{-2x + y = \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}} \qquad y = 2x + \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}$$

Extra: From $y(0) = 0$ $\qquad 0 = 0 + \frac{3c_2 + 3}{1 - c_2}$ $\qquad c_2 = -1$

$$\boxed{y = 2x + \frac{3(1 - e^{6x})}{1 + e^{6x}}}$$

2-5-4 本節要注意的地方

(1) 對症下藥，先判斷 DE 符合什麼樣的條件，再決定要什麼方法來解 (部分的 DE 可以用兩個以上的方法來解)

(2) 別忘了，寫出解答時，要將 u 用 y/x, y^{1-n}, 或 $Ax + By + C$ 代回來

(3) 本節方法皆有五大步驟

Previous Step: 判斷用什麼方法
Step 1: Set $u = \ldots$, $du/dx = \ldots$
Step 2: Converting，
Step 3: Solving，
Step 4: 將 u 用 x, y 代回來

附錄四　整理：Methods of solving the 1st order DE

(1)Direct integral

條件：$\dfrac{dy}{dx}=f(x)$

破解法：直接積分

$$y=\int f(x)dx+c$$

(2) Separable variable

條件：$\dfrac{dy}{dx}=g(x)h(y)$

破解法：x, y 各歸一邊後積分

$$\int\frac{dy}{h(y)}=\int g(x)dx$$

(3) Linear DE

條件：$\dfrac{dy}{dx}=-P(x)y+f(x)$

破解法：算 $e^{\int P(x)dx}$

$$\boxed{\left(e^{\int P(x)dx}y(x)\right)'=e^{\int P(x)dx}f(x)}$$

(4) Exact equation

條件：$\dfrac{dy}{dx}=-\dfrac{M(x,y)}{N(x,y)}$

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$$

破解法：pages 106, 108 (double N)

先處理 $\dfrac{\partial f}{\partial x}=M(x,y)$

(或反過來)

再處理 $\dfrac{\partial f}{\partial y}=N(x,y)$

(4-1) Exact equation 變型

條件：$\dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$

$(M_y - N_x)/M$ independent of x

破解法：$\mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}$

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

is exact

(4-2) Exact equation 變型

條件：$\dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$

$(M_y - N_x)/N$ independent of y

破解法：$\mu(x) = e^{\int \frac{(M_y - N_x)}{N} dx}$

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

is exact

(5) Homogeneous equation

條件：$\dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$

$$M(tx,ty) = t^{\alpha}M(x,y)$$

$$N(tx,ty) = t^{\alpha}N(x,y)$$

破解法：$u = y/x, \quad (y = xu)$

$$dy = udx + xdu$$

再用 separable variable method

(6) Bernoulli's Equation

條件：$\dfrac{dy}{dx}=-P(x)y+f(x)y^n$

破解法： $u=y^{1-n}$

$$\frac{dy}{dx}=\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}$$

再用 linear DE 的方法

(7) $Ax+By+C$

條件：$\dfrac{dy}{dx}=f(Ax+By+C)$

破解法： $u=Ax+By+c$

$$\frac{dy}{dx}=\frac{1}{B}\frac{du}{dx}-\frac{A}{B}$$

注意 (a) 速度的訓練

(b) Exercises in Review 2 多練習

(c) 行有餘力，觀察 singular solution 和 singular point

練習題

Sec. 2-4: 3, 8, 13, 17, 20, 25, 29, 32, 34, 35, 38, 42

Sec. 2-5: 3, 5, 10, 13, 14, 17, 20, 22, 24, 25, 29, 31

Chap. 2 Review: 2, 13 , 16, 17, 18, 19, 22, 23, 24, 26, 33