Chapter 4 Higher Order Differential Equations

Highest differentiation: $\dfrac{d^n y}{dx^n}, \; n > 1$

Most of the methods in Chapter 4 are applied for the linear DE.

附錄五 DE 的分類

- DE
  - Linear
    4-1～4-9
    - Constant coefficients
      - Homogeneous
      - Nonhomogeneous
    - Cauchy-Euler
      - Homogeneous
      - Nonhomogeneous
    - Others
      - Homogeneous
      - Nonhomogeneous
  - Nonlinear
    4-10

附錄六 Higher Order DE 解法

- linear 4-1
  - homogeneous part
    - reduction of order 4-2
    - auxiliary function 4-3
    - Cauchy-Euler 4-7
  - particular solution
    - "guess" method 4-4 form rule
    - annihilator 4-5
    - variation of parameters 4-6
    - Fourier series (Sec. 11-3)
  - multiple linear DEs 4-8
- nonlinear 4-10
  - reduction of order
  - Taylor series
  - numerical method
- both (but mainly linear)
  - series solution Chap 6
  - transform
    - Laplace transform Chap 7
    - Fourier series Chap 11
    - Fourier cosine /sine series Chap 11
    - Fourier transform Chap 14

4-1 Linear Differential Equations: Basic Theory

The n^{th} order linear DE

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x)=0 \longrightarrow$ homogeneous 名4

$g(x)\neq 0 \longrightarrow$ nonhomogeneous

4.1.1 Nonhomogeneous Equations (可和 page 60 相比較)

Nonhomogeneous linear DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

Part 1

Associated homogeneous DE 名1

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)$$
$$+\cdots+a_1(x)y'(x)+a_0(x)y=0$$

find ***n*** **linearly independent solutions**

$$y_1(x),y_2(x),\cdots\cdots,y_n(x)$$

complementary function 名2

$$c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_n y_n(x)$$

Part 2 名3

particular solution y_p
(**any** solution of the nonhomogeneous linear DE)

☆4

superposition principle (sometimes)

$$g(x)=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$y_p(x)=y_{p_1}(x)+y_{p_2}(x)+\cdots\cdots+y_{p_k}(x)$$

general solution of the nonhomogeneous linear DE

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_ny_n(x)+y_p(x)$$

Theorem 4.1.6 general solution of a nonhomogeneous linear DE

$$y = \underline{y_c} + \underline{y_p}$$

general solution of the associated homogeneous function (complementary function)

名之

particular solution (any solution) of the nonhomogeneous linear DE

$\oplus$

general solution of the nonhomogeneous linear DE

Non-homogeneous Linear DE 解法的步驟 (Also see page 142)

Step 1: Find the general solution (i.e., the complementary function) of the associated homogeneous DE

(Sections 4-2, 4-3, 4-7)

Step 2: Find the particular solution

(Sections 4-4, 4-5, 4-6)

Step 3: Combine the <u>complementary function</u> and the <u>particular solution</u>

Extra Step: Consider the initial (or boundary) conditions

4.1.2 Homogeneous Equations and Complementary Function

4.1.2.1 Definition

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x) = 0 \longrightarrow$ homogeneous

$g(x) \neq 0 \longrightarrow$ nonhomogeneous

- 重要名詞：Associated homogeneous equation

The associated homogeneous equation of a nonhomogeneous DE:

Setting $g(x) = 0$

- Review: Section 2-3, pages 58, 60

[Example] $y''' - 6y'' + 11y' - 6y = 3x$

Associated homogeneous equation: $y''' - 6y'' + 11y' - 6y = 0$

4.1.2.2 Solution of the Homogeneous Equation

★, [Important Theory]: An n^{th} order homogeneous linear DE has n linearly independent solutions. $g(x)=0$

[Theorem 4.1.5]

For an n^{th} order homogeneous linear DE, if

① $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are the solutions of this DE

② $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are linearly independent

then any solution of the homogeneous linear DE can be expressed as:

$$\boxed{y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n}$$

可以和矩陣的概念相比較

From Theorem 4.1.5:

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

$y_1(x), y_2(x), \ldots\ldots, y_n(x)$: fundamental set of solutions 名5

$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$: general solution of the homogenous linear DE

(又稱做 **complementary function**) 名2

也是重要名詞

[Definition 4.1] Linear Dependence / Independence

☆$_2$

If there is no solution other than $c_1 = c_2 = \ldots\ldots = c_n = 0$ for the following equality

$$c_1 y_1(x) + c_2 y_2(x) + \cdots\cdots + c_n y_n(x) = 0 \qquad \text{for all } x$$

$$y_n(x) = -\frac{c_1}{c_n} y_1 - \frac{c_2}{c_n} y_2 - \cdots - \frac{c_{n-1}}{c_n} y_{n-1}$$

then $y_1(x)$, $y_2(x)$, ….., $y_n(x)$ are said to be linearly independent.

$$y_n^{(k)} = -\frac{c_1}{c_n} y_1^{(k)} - \frac{c_2}{c_n} y_2^{(k)} - \cdots - \frac{c_{n-1}}{c_n} y_{n-1}^{(k)}$$

Otherwise, they are linearly dependent.

$$v_n = -\frac{c_1}{c_n} v_1 - \frac{c_2}{c_n} v_2 - \cdots\cdots - \frac{c_{n-1}}{c_n} v_{n-1}$$

判斷是否為 linearly independent 的方法: **Wronskian** wrong scan

[Definition 4.2] Wronskian ☆3

determinant

$$W(y_1, y_2, \cdots\cdots, y_n) = \det \begin{array}{c} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \\ \begin{array}{cccc} \quad v_1 & \quad v_2 & & \quad v_n \end{array} \end{array}$$

$$W(y_1, y_2, \cdots\cdots, y_n) \neq 0 \longrightarrow \text{linearly independent}$$

Note: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

indep. $1, \cos x, \cos 2x, \cos 3x$

indep. $1, x, x^2, x^3$

dependent $1, x, (x^2+x), x^2$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

dependent $1, \cos x, \cos^2 x, \cos 2x$

$\cos 2x = 2\cos^2 x - 1$

4.1.2.3 Examples

[Example 9] (text page 127)

$y''' - 6y'' + 11y' - 6y = 0$

$y_3''' - 6y'' + 11y' - 6y = (27 - 6\times 9 + 11\times 3 - 6)e^{3x}$
$= 0$

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$\det\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1, y_2, and y_3 are linear independent for any x

general solution:

$$\boxed{y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}} \qquad x \in (-\infty, \infty)$$

4.1.3 Particular Solution

Particular solution:
Any solution of the original nonhomogeneous linear DE.

[Example 10] (text page 128)

$$\boxed{y''' - 6y'' + 11y' - 6y = 3x}$$

$$y''' - 6y'' + 11y' - 6y = 0$$

e^{x} e^{2x} e^{3x}

Complementary function

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

(Example 9)

Sec 4-3

Particular solution of

$y''' - 6y'' + 11y' - 6y = 3x$)

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

Sec 4-4

General solution: $\boxed{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x}$

4.1.3.1 Superposition Principle for Particular Solutions

[Theorem 4.1.7] Superposition Principle ☆4

If $y_{p_1}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_1(x)$$

$y_{p_2}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_2(x)$$

$$\vdots$$

$y_{p_k}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_k(x)$$

then $\boxed{y_{p_1}(x)+y_{p_2}(x)+\cdots+y_{p_k}(x)}$ is the particular solution of

$$\boxed{\begin{aligned}&a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)\\&=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)\end{aligned}}$$

[Example 11] (text page 129)

$y_{p_1}(x) = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

(子問題一)

$y_{p_2}(x) = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$

(子問題二)

$y_{p_3}(x) = xe^{x}$ is a particular solution of $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

(子問題三)

$\boxed{y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}}$ is a particular solution of

$$\boxed{y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{\text{part1}} + \underbrace{2e^{2x}}_{\text{part2}} + \underbrace{2xe^{x} - e^{x}}_{\text{part 3}}}$$

4.1.4 New Notations

Notation: $D^n y = \dfrac{d^n y}{dx^n}$

☆5

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y \xrightarrow{\text{可改寫成}} D^2y + 5Dy + 6y \xrightarrow{\text{可改寫成}} (D^2 + 5D + 6)y$$

可再改寫成

$$\boxed{\begin{gathered} L(y) \\ L = D^2 + 5D + 6 \end{gathered}}$$

限制

(Useful for the linear DE with constant coefficients)

4.1.5 Initial-Value and Boundary Value Problems

4.1.5.1 The n^{th} Order Initial Value Problem

i.e., the n^{th} order linear DE with the constraints at the same point

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0,\quad y'(x_0)=y_1,\quad y''(x_0)=y_2,\quad \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad y^{(n-1)}(x_0)=y_{n-1}$$

n $\boxed{\text{initial conditions}}$ 名6

(given at the same point)

[Theorem 4.1.1] ☆6 only applied for "initial condition"
not applied for "boundary condition"

For an interval I that contains the point x_0

① If $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous at $x = x_0$

② $a_n(x_0) \neq 0$

(很像Section 2-3 當中x_0不是singular point的條件)

then for the problem on page 155, the solution $y(x)$ exists and is unique on the interval I that contains the point x_0

(Interval I 的範圍，取決於何時 $a_n(x) = 0$ 以及 何時 $a_k(x)$ $(k = 0 \sim n)$ 不為continuous)

Otherwise, the solution is either non-unique or does not exist.

(infinite number of solutions) (no solution)

[Example 1] (text page 119)

Sec 4-3

$$3y''' + 5y'' - y' + 7y = 0 \qquad y(1) = 0 \qquad y'(1) = 0 \qquad y''(1) = 0$$

unique solution

[Example 2] (text page 120)

Sec 4-3

$$y'' - 4y' = 12x \qquad y(0) = 4 \qquad y'(0) = 1$$

unique solution

$a_n(x) = x^2$ $\quad x_0 = 0 \quad a_n(x_0) = 0$

- $x^2 y'' - 2xy' + 2y = 6 \qquad y(0) = 3 \qquad y'(0) = 1$

有無限多組解

$y = cx^2 + x + 3$ $\qquad c$ 為任意之常數

- 比較：

$$x^2 y'' - 2xy' + 2y = 6 \qquad y(1) = 3 \qquad y'(1) = 1$$

There is only one solution

$$y = x^2 - x + 3$$

$$x \in (0, \infty)$$

- Note:

The initial value can also be the form as:

$$\alpha y(x_0) + \beta y'(x_0) = y_0$$

$$\sum_{n=0}^{N-1} \alpha_n y^{(n)}(x_0) = y_0 \quad \text{(general initial condition)}$$

4.1.5.2 n^{th} Order Boundary Value Problem

67

Boundary conditions are specified at different points

比較：Initial conditions are specified at the same points

例子：$a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $\quad y(a) = y_0, \quad y(b) = y_1 \qquad a \neq b$

或 $\quad y'(a) = y_0, \quad y(b) = y_1$

或 $\quad \begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$

★6-1

An n^{th} order linear DE with n boundary conditions may have a unique solution, no solution, or infinite number of solutions.

[Example 3] (text page 120)

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

(1) $y(0) = 0 \rightarrow c_1 = 0$ $\quad y(\pi/2) = 0 \rightarrow c_1 = 0$

$y = c_2 \sin(4x)$ $\quad c_2$ is any constant (infinite number of solutions)

(2) $y(0) = 0 \rightarrow c_1 = 0$ $\quad y(\pi/8) = 0 \rightarrow c_2 = 0$

$y = 0$ $\quad$ (unique solution)

4.1.6 本節要注意的地方

(1) Most of the theories in Section 4.1 are applied to the linear DE

(2) 注意 initial conditions 和 boundary conditions 之間的不同

(3) 快速判斷 linear independent

4.1.6.1 名詞

- general solution of the nonhomogenous linear DE (page 142)

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots\cdots + c_n y_n(x) + y_p(x)$$

- associated homogeneous equation, (page 145)
(重要名詞)

- fundamental set of solutions (page 147)

- complementary function (general solution of the homogenous linear DE)
(重要名詞) (page 147)

- Wronskian (page 149)

- particular solution (page 151)

- initial conditions, boundary conditions (pages 155, 159)
(重要名詞)

(補充 1) Theorem 4.1.1 的解釋

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0 \qquad y'(x_0)=y_1 \quad \cdots\cdots\cdots\cdots \quad y^{(n-1)}(x_0)=y_{n-1}$$

When $a_n(x_0) \neq 0$

$$y^{(n)}(x_0)+\frac{a_{n-1}(x_0)}{a_n(x_0)}y^{(n-1)}(x_0)+\cdots+\frac{a_1(x_0)}{a_n(x_0)}y'(x_0)+\frac{a_0(x)}{a_n(x_0)}y(x_0)=\frac{g(x_0)}{a_n(x_0)}$$

find $y^{(n)}(x_0)$

$$y^{(n-1)}(x_0+\Delta)=y^{(n-1)}(x_0)+y^{(n)}(x_0)\Delta$$

find $y^{(n-1)}(x_0+\Delta)$

(根據 $f'(t)=\dfrac{f(t+\Delta)-f(t)}{\Delta}$ ， $f(t+\Delta)=f(t)+f'(t)\Delta$)

以此類推

$$y^{(n-2)}(x_0+\Delta)=y^{(n-2)}(x_0)+y^{(n-1)}(x_0)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+\Delta)$$

$$y^{(n-3)}(x_0+\Delta)=y^{(n-3)}(x_0)+y^{(n-2)}(x_0)\Delta \longrightarrow \text{find } y^{(n-3)}(x_0+\Delta)$$

$$\vdots$$

$$y(x_0+\Delta)=y(x_0)+y'(x_0)\Delta \longrightarrow \boxed{\text{find } y(x_0+\Delta)}$$

$$\downarrow$$

$$y^{(n)}(x_0+\Delta)+\frac{a_{n-1}(x_0+\Delta)}{a_n(x_0+\Delta)}y^{(n-1)}(x_0+\Delta)+\cdots+\frac{a_1(x_0+\Delta)}{a_n(x_0+\Delta)}y'(x_0+\Delta)$$

$$+\frac{a_0(x+\Delta)}{a_n(x_0+\Delta)}y(x_0+\Delta)=\frac{g(x_0+\Delta)}{a_n(x_0+\Delta)} \longrightarrow \text{find } y^{(n)}(x_0+\Delta)$$

$$\downarrow$$

$$y^{(n-1)}(x_0+2\Delta)=y^{(n-1)}(x_0+\Delta)+y^{(n)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-1)}(x_0+2\Delta)$$

$$y^{(n-2)}(x_0+2\Delta)=y^{(n-2)}(x_0+\Delta)+y^{(n-1)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+2\Delta)$$

$$\vdots$$

$$y(x_0+2\Delta) = y(x_0+\Delta) + y'(x_0+\Delta)\Delta \longrightarrow \boxed{\text{find } y(x_0+2\Delta)}$$

$$y^{(n)}(x_0+2\Delta) + \frac{a_{n-1}(x_0+2\Delta)}{a_n(x_0+2\Delta)} y^{(n-1)}(x_0+2\Delta) + \cdots + \frac{a_1(x_0+2\Delta)}{a_n(x_0+2\Delta)} y'(x_0+2\Delta)$$
$$+ \frac{a_0(x+2\Delta)}{a_n(x_0+2\Delta)} y(x_0+2\Delta) = \frac{g(x_0+2\Delta)}{a_n(x_0+2\Delta)}$$

以此類推，可將 $y(x_0+3\Delta)$, $y(x_0+4\Delta)$, $y(x_0+5\Delta)$, ……………

以至於將 $y(x)$ 所有的值都找出來。

(求 $y(x)$ for $x > x_0$ 時, 用正的 Δ 值，

求 $y(x)$ for $x < x_0$ 時, 用負的 Δ 值)

Requirement 1: $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous
是為了讓 $a_k(x_0+m\Delta)$ 皆可以定義

Requirement 2: $a_n(x) \neq 0$ 是為了讓 $a_k(x_0+m\Delta)/a_n(x_0+m\Delta)$ 不為無限大

4-2 Reduction of Order

4.2.1 適用情形

☆₁ (1) (2) (3)

Suitable for the 2^{nd} order linear homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(4) One of the nontrivial solution $y_1(x)$ has been known.

trivial solution: $y=0$

$y_1 \neq 0$

4.2.2 解法

假設 $\boxed{y_2(x) = u(x)\,y_1(x)}$

將 DE 變成 Standard form

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

$$\Downarrow$$

$$y'' + \frac{a_1(x)}{a_2(x)}\,y' + \frac{a_0(x)}{a_2(x)}\,y = 0$$

$$P(x) = \frac{a_1(x)}{a_2(x)}$$

先將DE 變成 Standard form

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

If $\boxed{y(x) = u(x)\,y_1(x)}$, (比較 Section 2-3)

$$y' = uy_1' + u'y_1 \qquad y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$uy_1'' + 2u'y_1' + u''y_1 + P(x)uy_1' + P(x)u'y_1 + Q(x)uy_1 = 0$$

$$u\underbrace{(y_1'' + P(x)\,y_1' + Q(x)\,y_1)}_{\text{zero}} + 2u'y_1' + u''y_1 + P(x)u'y_1 = 0$$

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\downarrow$$

$$\frac{dw}{dx}y_1 + w(2\frac{dy_1}{dx} + P(x)y_1) = 0$$

multiplied by $dx/(y_1 w)$

$$\downarrow$$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable (with 3 variables) Sec 2-2

$$\downarrow$$

$$\int\frac{dw}{w} + 2\int\frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\downarrow$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\downarrow$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$$\ln\left|wy_1^2\right| = -\int P(x)\,dx + c$$

$$wy_1^2 = \pm e^{-\int P(x)dx+c}$$

$c_1 = \pm e^c$

$$w = c_1 e^{-\int P(x)dx} / y_1^2$$

$$\boxed{u = \int w\,dx = \cancel{c_1}\int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx + \cancel{c_2}}$$

We can set $c_1 = 1$ and $c_2 = 0$

(因為我們算 $u(x)$ 的目的，只是為了要算出與 $y_1(x)$ 互相independent的另一個解)

Remember it!

☆☆

$$\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx}$$

where $\underline{P(x) = \frac{a_1(x)}{a_2(x)}}$ (the coefficient of $y'(x)$ in the standard form)

☆2

4.2.3 例子

[Example 1] (text page 132)

Sec 4-3 $\boxed{y'' - y = 0}$

We have known that $y_1 = e^x$ is one of the solution

$e^{-\int P(x)dx} = e^{-c_1} = c$

$P(x) = 0$ $\quad y_2(x) = e^x \int c e^{-2x} dx = -\frac{1}{2} c e^{-x}$

$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$

$\frac{a_1}{a_0} = 0$ $\quad a_1 = 0$, $a_2 = 1$

Specially, set $c = -2$, ($y_2(x)$ 只要 independent of $y_1(x)$ 即可所以 c 的值可以任意設) ☆3

$$\boxed{y_2(x) = e^{-x}}$$

General solution: $\boxed{y(x) = c_1 e^x + c_2 e^{-x}}$

勿忘相加

[Example 2] (text page 133) (將課本 x 的範圍做更改)

Sec 4-7 $(0, \infty)$

$$\boxed{x^2 y'' - 3xy' + 4y = 0} \quad \text{when } x \in (-\infty, 0)$$

$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$

We have known that $y_1 = x^2$ is one of the solution

$x = 0$ is a singular point

Note: the interval of x

If $x \in (0, \infty)$ $(x > 0)$, $\int dx / x = \ln x$ 如課本

$P(x) = -\frac{3}{x}$

If $x < 0$, $\int dx / x = \ln(-x)$

$e^{-\int P(x)\,dx}$

$= e^{3\ln|x| + \cancel{c_1}}$

$$y_2(x) = x^2 \int \frac{e^{3\ln(-x)}}{x^4} dx = x^2 \int \frac{(-x)^3}{(-x)^4} dx$$

$$= -x^2 \int \frac{1}{x} dx = -x^2 \ln|x|$$

$$\boxed{y(x) = c_1 x^2 + c_2 x^2 \ln|x|}$$

4.2.4 本節需注意的地方

(1) 記住公式 $\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx}$

(2) 若不背公式 (不建議)，在計算過程中別忘了對 $w(x)$ 做積分

(3) 別忘了 $P(x)$ 是 "standard form" 一次微分項的 coefficient term

(4) 同樣有 singular point 的問題

(5) 因為 $y_2(x)$ 是 homogeneous linear DE 的 "任意解" ，所以計算時，常數的選擇以方便為原則

(6) 由於 $\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ 的計算較複雜且花時間，所以要多加練習

多算習題

附錄七： Hyperbolic Function

$e^{jx} = \cos x + j\sin x$

$e^{-jx} = \cos x - j\sin x$

☆1 $$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

比較： $$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

☆2 $$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

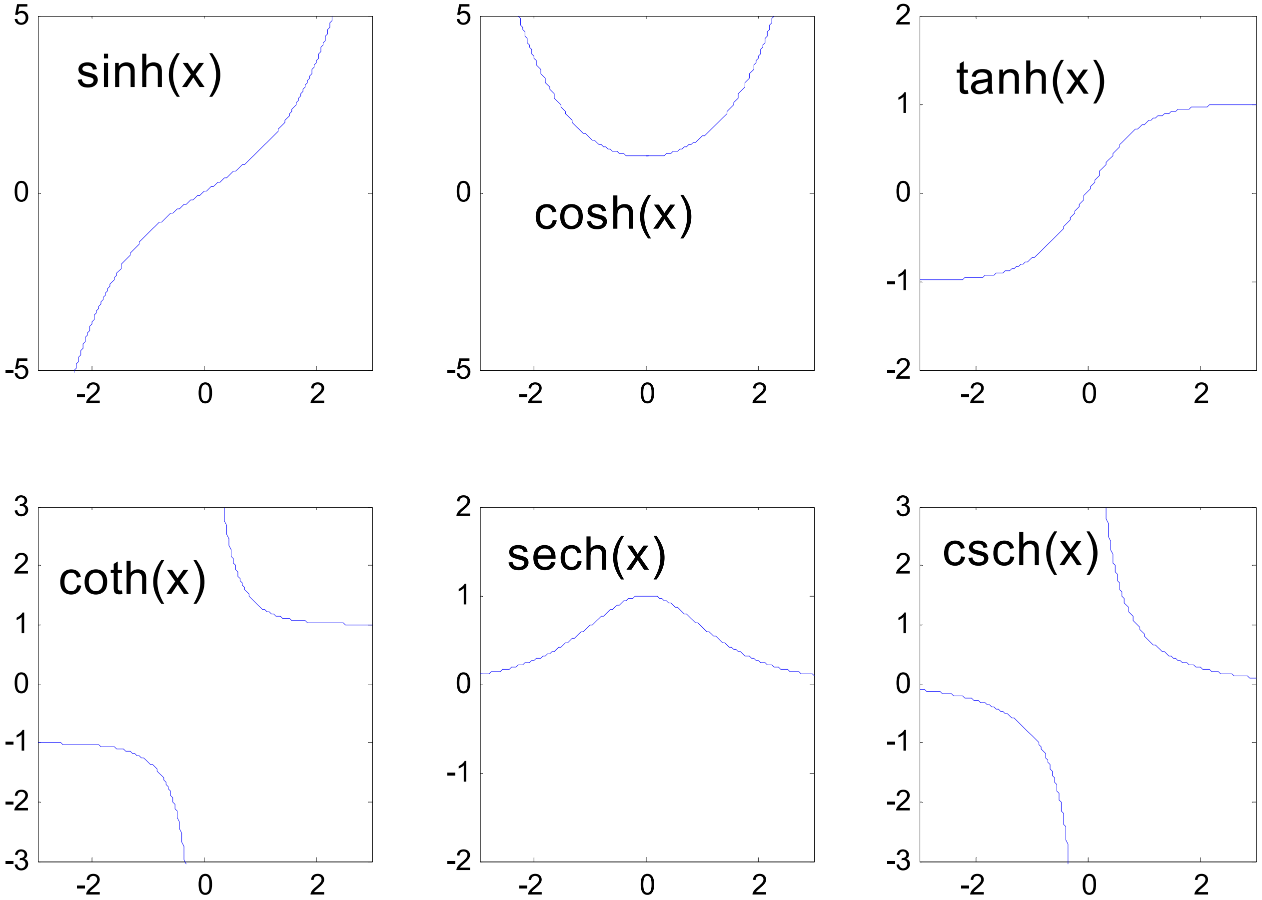
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}}$$

☆3 Note: $\sinh(x) = -\sinh(-x)$

☆4 $\cosh(x) = \cosh(-x)$


sinh(x)
cosh(x)
tanh(x)
coth(x)
sech(x)
csch(x)

☆5 $$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$

☆6 $$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$

$$\frac{d}{dx}\tanh(ax) = a\,\mathrm{sech}^2(ax)$$

$$\frac{d}{dx}\coth(ax) = -a\,\mathrm{csch}^2(ax)$$

$$\frac{d}{dx}\mathrm{sech}(ax) = -a\,\mathrm{sech}(ax)\tanh(ax)$$

$$\frac{d}{dx}\mathrm{csch}(ax) = -a\,\mathrm{csch}(ax)\coth(ax)$$

☆7 $$\sinh(0) = 0$$

☆8 $$\cosh(0) = 1$$

$$\sinh'(0) = 1$$

$$\cosh'(0) = 0$$

$$\sin(ix) = i\sinh(x)$$

$$\cos(ix) = \cosh(x)$$

比較 $$\frac{d}{dx}\sin(ax) = a\cos(ax)$$

$$\frac{d}{dx}\cos(ax) = -a\sin(ax)$$

$$\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a} + c$$

$$\int \cosh(ax)\,dx = \frac{\sinh(ax)}{a} + c$$

$$\int \tanh(ax)\,dx = \frac{\ln\left|\cosh(ax)\right|}{a} + c$$

$$\int \coth(ax)\,dx = \frac{\ln\left|\sinh(ax)\right|}{a} + c$$

$$\int \operatorname{sech}(ax)\,dx = \frac{2\tan^{-1}\left(\tanh(\frac{a}{2}x)\right)}{a} + c$$

$$\int \operatorname{csch}(ax)\,dx = \frac{\ln\left|\tanh(\frac{a}{2}x)\right|}{a} + c$$

4-3 Homogeneous Linear Equations with Constant Coefficients

輔助

本節使用 auxiliary equation 的方法來解 homogeneous DE

KK: [ɔgˋzɪljərɪ]

4-3-1 限制條件

限制條件: (1) homogeneous

(2) linear

(3) constant coefficients

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

a_0, a_1, a_2, …. , a_n are constants

(the simplest case of the higher order DEs)

4-3-2 解法

解法核心：

Suppose that the solutions has the form of e^{mx} #2

Example: $y''(x) - 3\,y'(x) + 2\,y(x) = 0$

Set $y(x) = e^{mx}$, $m^2 \cancel{e^{mx}} - 3m\,\cancel{e^{mx}} + 2\,\cancel{e^{mx}} = 0$

$m^2 - 3m + 2 = 0 \longrightarrow$ solve m

$(m-2)(m-1)=0$ $m=2,1$

$y_1 = e^{2x}$

$y_2 = e^{x}$

$y = c_1 e^{2x} + c_2 e^{x}$

可以直接把 n 次微分用 m^n 取代，變成一個多項式

這個多項式被稱為 auxiliary equation 名,

- 解法流程

$$\boxed{a_n y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_1 y'(x)+a_0 y=0}$$

$y^{(k)}(x)\rightarrow m^k$

e^{mx}

Step 1-1
auxiliary function

$$\boxed{a_n m^n+a_{n-1}m^{n-1}+\cdots+a_1 m+a_0=0}$$

Step 1-1 Find n roots , m_1, m_2, m_3, …., m_n

(If m_1, m_2, m_3, …., m_n are distinct)

Step 1-2 n linearly independent solutions e^{m_1x} , e^{m_2x} , e^{m_3x} , ……, e^{m_nx}

(有三個 Cases)

Step 1-3 Complementary function

$$\boxed{y=c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\ldots\ldots+c_ne^{m_nx}}$$

4-3-3 Three Cases for Roots (2nd Order DE)

$$\boxed{a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0}$$

$$\downarrow$$

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$

roots $m_1 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$ $\quad m_2 = \dfrac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$

solutions

$a_1^2 - 4a_2a_0 > 0$

Case 1 $\underline{m_1 \neq m_2,}$ m_1, m_2 are real (其實 m_1, m_2 不必限制為 real)

☆3-1

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$a_1^2 - 4a_2a_0 = 0$

Case 2 $m_1 = m_2$ (m_1 and m_2 are of course real)

First solution: $\boxed{y_1 = e^{m_1 x}}$ ☆ 3-2

Second solution: using the method of "Reduction of Order"

Sec 4-2, page 170

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

$P(x) = \frac{a_1}{a_2}$

$y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = 0$

$$= e^{m_1 x}\int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx$$

$$= e^{m_1 x}\int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \qquad m_1 = \frac{-a_1}{2a_2}$$

$$= e^{m_1 x}\int dx = e^{m_1 x}(x + \cancel{c})$$

$$\boxed{y_2(x) = xe^{m_1 x}}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

有重根 → 乘x

$a_1^2 - 4a_2a_0 < 0$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

#3-3

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

$$\alpha = -a_1 / 2a_2, \qquad \beta = \sqrt{4a_2a_0 - a_1^2} / 2a_2$$

Solution: $\boxed{y = C_1 e^{\alpha x + j\beta x} + C_2 e^{\alpha x - j\beta x}}$

Another form:

$$y = e^{\alpha x}\left(C_1 e^{j\beta x} + C_2 e^{-j\beta x}\right)$$

$$= e^{\alpha x}\left(C_1 \cos\beta x + jC_1 \sin\beta x + C_2 \cos\beta x - jC_2 \sin\beta x\right)$$

set $c_1 = C_1 + C_2$ and $c_2 = jC_1 - jC_2$

$\boxed{y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)}$ c_1 and c_2 are some constant

#3-3

[Example 1] (text page 137)

(a) $\boxed{2y'' - 5y' - 3y = 0}$

$$\frac{5 \pm \sqrt{25 + 4 \cdot 6}}{4} = \frac{5 \pm 7}{4} = 3, -\frac{1}{2}$$

$2m^2 - 5m - 3 = 0,\quad m_1 = -1/2,\ m_2 = 3$

$$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$$

(b) $\boxed{y'' - 10y' + 25y = 0}$ $\quad (m-5)^2$

$m^2 - 10m + 25 = 0,\quad m_1 = 5,\ m_2 = 5$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

$$\frac{-4 \pm \sqrt{4^2 - 28}}{2} = -2 \pm \frac{i}{2}\sqrt{12} = -2 \pm i\sqrt{3}$$

(c) $\boxed{y'' + 4y' + 7y = 0}$

$m^2 + 4m + 7 = 0,\quad m_1 = -2 + i\sqrt{3},\quad m_2 = -2 - i\sqrt{3}$

$\alpha = -2$
$\beta = \sqrt{3}$

$$\boxed{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}$$

4-3-4 Three + 1 Cases for Roots (Higher Order DE)

For higher order case $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$

auxiliary function: $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$

roots: $m_1, m_2, m_3, \ldots, m_n$

(1) If $m_p \neq m_q$ for $p = 1, 2, \ldots, n$ and $p \neq q$

(也就是這個多項式在 m_q 的地方只有一個根)

then $\boxed{e^{m_q x}}$ is a solution of the DE. ☆4-1

重覆次數

(2) If the multiplicities of m_q is k (當這個多項式在 m_q 的地方有 k 個根),

$$\boxed{e^{m_q x}, xe^{m_q x}, x^2 e^{m_q x}, \cdots\cdots, x^{k-1} e^{m_q x}}$$ ☆4-2

are the solutions of the DE.

$y^{(6)}(x) = 0$

$m^6 = 0$

$m = 0, 0, 0, 0, 0, 0$

$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$ $\quad 1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad x^5$

(3) If both $\alpha+j\beta$ and $\alpha-j\beta$ are the roots of the auxiliary function,

then

$$\boxed{e^{\alpha x}\cos(\beta x),\, e^{\alpha x}\sin(\beta x)}$$ ☆4-3

are the solutions of the DE.

(4) If the multiplicities of $\alpha+j\beta$ is k and the multiplicities of $\alpha-j\beta$ is also k, then ☆4-4

$$\boxed{\begin{array}{l} e^{\alpha x}\cos(\beta x),\, xe^{\alpha x}\cos(\beta x),\, x^2e^{\alpha x}\cos(\beta x),\cdots\cdots,\, x^{k-1}e^{\alpha x}\cos(\beta x) \\ e^{\alpha x}\sin(\beta x),\, xe^{\alpha x}\sin(\beta x),\, x^2e^{\alpha x}\sin(\beta x),\cdots\cdots,\, x^{k-1}e^{\alpha x}\sin(\beta x) \end{array}}$$

are the solutions of the DE.

Note: If $\alpha + j\beta$ is a root of a real coefficient polynomial,

then $\alpha - j\beta$ is also a root of the polynomial.

$$a_n(\alpha + j\beta)^n + a_{n-1}(\alpha + j\beta)^{n-1} + \cdots + a_1(\alpha + j\beta) + a_0 = 0$$

a_0, a_1, a_2, …. , a_n are real

[Example 3] (text page 138)

Solve $\boxed{y''' + 3y'' - 4y = 0}$

Step 1-1 $m^3 + 3m^2 - 4 = 0$

$$(m-1)\underbrace{\left(m^2 + 4m + 4\right)}_{(m+2)^2} = 0 \qquad \boxed{m_1 = 1, \quad m_2 = m_3 = -2}$$

Step 1-2 3 independent solutions: e^x , e^{-2x} , xe^{-2x}

Step 1-3 general solution: $\boxed{y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}}$

[Example 4] (text page 138)

Solve $\boxed{y^{(4)}(x)+2y''(x)+y(x)=0}$

Step 1-1 $m^4+2m^2+1=0$

$(m^2+1)^2=0$ four roots: $\boldsymbol{i, i, -i, -i}$

$\alpha=0, \beta=1$ $\alpha \pm i\beta$

Step 1-2 4 independent solutions: $\cos x$, $x\cos x$, $\sin x$, $x\sin x$

Step 1-3 general solution: $\boxed{y=c_1\cos x+c_2 x\cos x+c_3\sin x+c_4 x\sin x}$

4-3-5 How to Find the Roots

(1) Formulas

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0} \Rightarrow m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

$$\boxed{a_3 m^3 + a_2 m^2 + a_1 m + a_0}$$

$$\Rightarrow m_1 = S + T - \frac{a_2}{3a_3}$$

Solutions:

$$m_2 = -\frac{1}{2}(S+T) - \frac{a_2}{3a_3} + i\frac{1}{2}\sqrt{3}(S-T)$$

$$m_3 = -\frac{1}{2}(S+T) - \frac{a_2}{3a_3} - i\frac{1}{2}\sqrt{3}(S-T)$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^3}} \quad , \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^3}}$$

$$Q = \frac{1a_1}{3a_3} - \frac{a_2^2}{9a_3^2} \quad , \quad R = \frac{9a_1 a_2 - 27a_3 a_0 - 2a_2^3}{54a_3^2}$$

(太複雜了)

(2) Observing

例如：1 是否為 root $\longrightarrow$ 看係數和是否為 0

又如：

$$3m^3 + 5m^2 + 10m - 4$$

factor: 1,3 factor: 1,2,4 ★ 5

分母 分子

possible roots: ±1, ±2, ±4, ±1/3, ±2/3, ±4/3

test for each possible root $\longrightarrow$ find that 1/3 is indeed a root

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \frac{1}{3}\right)\left(3m^2 + 6m + 12\right)$$

(3) Solving the roots of a polynomial by software

Maple

Mathematica (by the commands of **Nsolve** and **FindRoot**)

Matlab (by the command of **roots**)

4-3-6 本節需注意的地方

(1) 注意重根和 conjugate complex roots 的情形

(2) 寫解答時，要將 "General solution" 寫出來

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

(3) 因式分解要熟練

(4) 本節的方法，也適用於 1^{st} order 的情形

4-4 Undetermined Coefficients – Superposition Approach

This section introduces some method of "guessing" the particular solution.

4-4-1 方法適用條件

the same as Sec 4-3 but non-homogeneous

Suitable for (1) linear and (2) constant coefficient DE.

$$a_n y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_1 y'(x)+a_0 y(x)=g(x)$$

☆1

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ………contain finite number of terms.

4-4-2 方法

把握一個原則：　　form rule

***g*(*x*) 長什麼樣子，particular solution 就應該是什麼樣子.**

記熟下一頁的規則

(計算時要把 A, B, C, … 這些 unknowns 解出來)

Trial Particular Solutions (from text page 146)

$g(x)$	Form of y_p
1 (any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

It comes from the "form rule". See page 201.

$g(x) = e^{2x} + xe^{3x}$ $\quad y_p = ?$ $\quad Ae^{2x} + (Bx + C)e^{3x}$

$g(x) = \cos(x) + x^2 \sin(2x)$ $\quad y_p = ?$ $\quad A\cos x + B\sin x + (Cx^2 + Dx + E)\cos(2x) + (Fx^2 + Gx + H)\sin(2x)$

$g(x) = \cosh(2x)$ $\quad y_p = ?$ $\quad A\cosh(2x) + B\sinh(2x)$

$\frac{e^{2x} + e^{-2x}}{2}$

or $Ce^{2x} + De^{-2x}$

4-4-3 Examples

[Example 2] $y'' - y' + y = 2\sin 3x$ (text page 144)

Step 1: find the solution of the associated homogeneous equation

$$y_c = e^{x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

auxiliary $m^2 - m + 1 = 0$

$m = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\alpha = \frac{1}{2}$

$\beta = \frac{\sqrt{3}}{2}$

Guess

Step 2: particular solution $\boxed{y_p = A\cos 3x + B\sin 3x}$

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

$$\begin{cases} -8A - 3B = 0 & \cdots (1) \\ 3A - 8B = 2 & \cdots (2) \end{cases} \Longrightarrow A = 6/73,\ B = -16/73$$

$3\times(1) + 8\times(2)$: $-73B = 16$

$$\boxed{y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$$

Step 3: General solution:

$$\boxed{y = e^{x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$$

[Example 3] $y'' - 2y' - 3y = \underline{4x - 5} + \underline{6xe^{2x}}$ (text page 145)

Step 1: Find the solution of

$m^2 - 2m - 3 = 0$ $\quad y'' - 2y' - 3y = 0.$

$(m+1)(m-3) = 0$ $\quad y_c = c_1 e^{3x} + c_2 e^{-x}$

Step 2: Particular solution

from page 152 superposition

*2

$$y'' - 2y' - 3y = 4x - 5 \qquad\qquad y'' - 2y' - 3y = 6xe^{2x}$$

guess

$$\boxed{y_{p_1} = Ax + B}$$

$$y'_{p_1} = A$$

$$y''_{p_1} = 0$$

$$-3Ax - 2A - 3B = 4x - 5$$

$$A = -\frac{4}{3}, \quad B = \frac{23}{9}$$

$$\boxed{y_{p_1} = -\frac{4}{3}x + \frac{23}{9}}$$

guess

$$\boxed{y_{p_2} = Cxe^{2x} + Ee^{2x}}$$

$$y'_{p_2} = 2Cxe^{2x} + Ce^{2x} + 2Ee^{2x}$$

$$y''_{p_2} = 4Cxe^{2x} + 4Ce^{2x} + 4Ee^{2x}$$

$$-3Cxe^{2x} + (2C - 3E)e^{2x} = 6xe^{2x}$$

$$C = -2, \quad E = -\frac{4}{3}$$

$$\boxed{y_{p_2} = -(2x + \frac{4}{3})e^{2x}}$$

Particular solution

$$y_p = y_{p_1} + y_{p_2} = -\frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^x$$

Step 3: General solution

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x} - \frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^{2x}}$$

4-4-4 方法的解釋：Form Rule

Form Rule: y_p should be a linear combination of $g(x)$, $g'(x)$,

☆☆ $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ……………..

Why? 如此一來，在比較係數時才不會出現多餘的項

form rule 不適用例子

$g(x) = \ln x \qquad \ln x \to \frac{1}{x} \to, \frac{1}{x^2} \to, \frac{1}{x^3} \to \frac{1}{x^4} \to \cdots$

$x^{0.5} \qquad x^{0.5} \to x^{-0.5} \to x^{-1.5} \to x^{-2.5} \to \cdots$

When g(x) = x^n

$$x^n \rightarrow x^{n-1} \rightarrow x^{n-2} \rightarrow x^{n-3} \rightarrow \cdots\cdots \rightarrow 1 \rightarrow 0$$

$$y_p = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots\ldots + A_0$$

When g(x) = cos kx

$$\begin{tikzcd} \cos kx \arrow[r] & \sin kx \arrow[l, bend left] \end{tikzcd}$$

$$y_p = A_1 \cos kx + A_2 \sin kx$$

When g(x) = exp(kx)

$$\begin{tikzcd} e^{kx} \arrow[loop right] \end{tikzcd}$$

$$y_p = A \exp(kx)$$

When $g(x) = x^n \exp(kx)$

$$g'(x) = nx^{n-1}e^{kx} + kx^n e^{kx}$$

$$g''(x) = n(n-1)x^{n-2}e^{kx} + 2nkx^{n-1}e^{kx} + k^2 x^n e^{kx}$$

$$g'''(x) = n(n-1)(n-2)x^{n-3}e^{kx} + 3kn(n-1)x^{n-2}e^{kx} + 3k^2 n x^{n-1}e^{kx} + k^3 x^n e^{kx}$$

$$\vdots$$

會發現 $g(x)$ 不管多少次微分，永遠只出現

$$x^n e^{kx},\ x^{n-1}e^{kx},\ x^{n-2}e^{kx},\ x^{n-3}e^{kx},\ \cdots\cdots,\ e^{kx}$$

$$\boxed{y_p = c_n x^n e^{kx} + c_{n-1}x^{n-1}e^{kx} + c_{n-2}x^{n-2}e^{kx} + \cdots\cdots + c_0 e^{kx}}$$

4-4-5 Glitch of the method:

失靈；小缺陷

[Example 4] $y'' - 5y' + 4y = 8e^x$ (text page 146)

Particular solution guessed by Form Rule:

$$y_p = Ae^x$$

$$y_p'' - 5y_p' + 4y_p = Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

$$0 = 8e^x$$ (no solution)

Why?

Glitch condition 1: The **particular solution** we guess belongs to the complementary function.

$m^2-5m+4=0 \quad m=4,1$

For Example 4 $\quad y''-5y'+4y=8e^x$

Complementary function $\quad y_c = c_1e^x + c_2e^{4x} \qquad \underline{Ae^x} \in \underline{y_c}$

解決方法：再乘一個 x ☆3-1

$$\boxed{y_p = Axe^x} \qquad y_p' = Axe^x + Ae^x$$

$$y_p'' = Axe^x + 2Ae^x$$

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x \Longrightarrow A = -8/3$$

$$\boxed{y_p = -\frac{8}{3}xe^x}$$

$$\boxed{y = c_1e^x + c_2e^{4x} - \frac{8}{3}xe^x}$$

$m^2-2m+1=0$, $m=0,0$

[Example 7] $y''-2y'+y=e^x$ (text page 148)

$$y_c = c_1e^x + c_2xe^x$$

From Form Rule, the particular solution is Ae^x

$$Ae^x \in y_c$$

$$Axe^x \in y_c$$

★3-2

如果乘一個 x 不夠，則再乘一個 x

$$\boxed{y_p = Ax^2e^x}$$

$$y'_p = (Ax^2+2Ax)e^x$$

$$y''_p = (Ax^2+4Ax+2A)e^x$$

$$y''_p - 2y'_p + y_p = 2Ae^x = e^x \implies A = 1/2$$

$$\boxed{y_p = x^2e^x/2}$$

$$\boxed{y = c_1e^x + c_2xe^x + x^2e^x/2}$$

[Example 8] (text page 148)

$m^2+1=0$

$m=\pm i$

$$y''+y=\underbrace{4x}_{\text{子問題1}}+\underbrace{10\sin x}_{\text{子問題2}} \qquad y(\pi)=0 \qquad y'(\pi)=2$$

Step 1 $\quad \alpha=0 \quad \beta=1$ $\quad y_c=c_1\cos x+c_2\sin x$

Step 2 $\quad y_p=Ax+B+C\sin x+E\cos x$

修正 $\quad y_p=Ax+B+Cx\sin x+Ex\cos x$

注意： sin$x$, cos$x$ 都要乘上 x

同一個子問題的 particular solution 乘上相同的東西 ☆3-3

$$y_p=4x-5x\cos x$$

Step 3 $\quad \boxed{y=c_1\cos x+c_2\sin x+4x-5x\cos x}$

Step 4 $\quad$ Solving c_1 and c_2 by initial conditions (最後才解 IVP)

$$y(\pi)=-c_1+4\pi+5\pi=0 \Longrightarrow c_1=9\pi$$

$$y'=-c_1\sin x+c_2\cos x+4-5\cos x+5x\sin x$$

$$y'(\pi)=-c_2+9=2 \Longrightarrow c_2=7$$

$$\boxed{y=9\pi\cos x+7\sin x+4x-5x\cos x}$$

[Example 11] (text page 149) 子問題1 $y_{p,1} = A$

$y_{p,2} = Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}$

$m^4+m^3=0$

$(m+1)m^3=0$

$m=-1,0,0,0$

$$y^{(4)} + y''' = 1 - x^2e^{-x}$$

子問題2

$$y_c = \boxed{c_1} + c_2x + c_3x^2 + \boxed{c_4e^{-x}}$$

From Form Rule

$$y_p = \boxed{A} + Bx^2e^{-x} + Cxe^{-x} + \boxed{Ee^{-x}}$$

y_p 只要有一部分和 y_c 相同就作修正

乘上 x^3 修正 (一起乘)乘上 x

☆3-3

$$\boxed{y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}}$$

來自同一個子問題的項，乘的要相同

If we choose $y_p = A + Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}$

X

$$y_p^{(4)} + y_{(p)}''' = \underline{-2Bxe^{-x} + (6B-C)e^{-x}} = 1 - x^2e^{-x}$$

沒有 $1, x^2e^{-x}$ 兩項，不能比較係數，無解

If we choose $\boxed{y_p = Ax^3 + Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}}$

X

$$y_p^{(4)} + y_{(p)}''' = \underline{6A - 2Bxe^{-x} + (6B - C)e^{-x}} = 1 - x^2e^{-x}$$

沒有 x^2e^{-x} 這一項，不能比較係數，無解

If we choose $\boxed{y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}}$

O

$$y_p^{(4)} + y_{(p)}'''$$

$$= 6A - 3Bx^2e^{-x} + \left(18B - 2C\right)xe^{-x} + (-18B + 6C - E)e^{-x}$$

$$= 1 - x^2e^{-x}$$

$A = 1/6,\ B = 1/3,\ C = 3,\ E = 12$

$$\boxed{y_p = \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}}$$

$$\boxed{y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}}$$

Glitch condition 2: $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, …………… contain **infinite number of terms**.

If $g(x) = \ln x$

$$\ln x \rightarrow \frac{1}{x} \rightarrow \frac{1}{x^2} \rightarrow \frac{1}{x^3} \rightarrow \cdots\cdots$$

If $g(x) = \exp(x^2)$

$$g'(x) \rightarrow 2xe^{x^2}$$

$$g''(x) \rightarrow (4x^2 + 2)e^{x^2}$$

$$g'''(x) \rightarrow (8x^3 + 12x)e^{x^2}$$

$$\vdots$$

$$\vdots$$

4-4-6 本節需要注意的地方

(1) 記住 Table 4.1 的 particular solution 的假設方法
(其實和 “form rule” 有相密切的關聯)

(2) 注意 “glitch condition”

另外，“同一類” 的 term 要乘上相同的東西 (參考 Example 11)

(3) 所以要先算 complementary function，再算 particular solution

(4) 同樣的方法，也可以用在 1^{st} order 的情形

(5) 本方法只適用於 linear, constant coefficient DE

4-5 Undetermined Coefficients – Annihilator Approach

殲滅

/ə'naɪəletər/ 零化子

For a linear DE:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

Annihilator Operator:

能夠「殲滅」$g(x)$ 的 operator

4-5-1 方法適用條件

(1) Linear， (2) Constant coefficients ★1 (和 Sec 4-4 相同)

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ………contain finite number of terms.

4-5-2 Find the Annihilator

D: $\frac{d}{dx}$ (page134)

適用於常數係數

[Example 1] (text page 153)

$g(x) = 1 - 5x^2 + 8x^3 \longrightarrow$ annihilator: D^4

$$D^k g(x) = \frac{d^k}{dx^k} g(x)$$

$g(x) = e^{-3x} \longrightarrow$ annihilator: $D + 3$

$$\frac{d}{dx} g(x) + 3g(x) = 0$$

$$g(x) = 4e^{2x} - 10xe^{2x} \longrightarrow \text{annihilator: } (D-2)^2$$

$$(D-2)^2 = D^2 - 4D + 4$$

$$\frac{d^2}{dx^2}g(x) - 4\frac{d}{dx}g(x) + 4g(x) = 0$$

$$(D-2)g(x) = 8e^{2x} - (20x+10)e^{2x} - 8e^{2x} + 20xe^{2x} = -10e^{2x}$$

$$(D-2)^2 g(x) = 0$$

註：當各個微分項的 coefficients 皆為 constants 時，function of D 的計算方式和 function of x 的計算方式相同

$$(x-2)^2 = x^2 - 4x + 4$$

$$\Rightarrow (D-2)^2 = D^2 - 4D + 4$$

e^{2x} annihilator $D-2$

$(-10x+4)$ annihilator D^2

General rule 1:

If $g(x)=\left(a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_0\right)e^{\alpha x}$

then the annihilator is $[D-\alpha]^{n+1}$ ☆ 2-1

page214
$\alpha=2,\ n=1$

注意： annihilator 和 $\underbrace{a_0, a_1, \ldots\ldots, a_n}$ 無關

只和 α, n 有關

The annihilator is <u>independent of the constant multiplied in the front</u> of each term.

$(D-\alpha-i\beta)(D-\alpha+i\beta)$
$= D^2-2\alpha D+\alpha^2+\beta^2$

General rule 2:

If $g(x)=\left(a_n x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_0\right)e^{\alpha x}\left(b_1 \cos\beta x + b_2 \sin\beta x\right)$

$D-\alpha-i\beta$ annihilates $e^{\alpha x+i\beta x}$

$b_1 \neq 0$ or $b_2 \neq 0$

then the annihilator is $\left[D^2-2\alpha D+\left(\alpha^2+\beta^2\right)\right]^{n+1}$ ☆ 2-2

☆ 2-2

$D-\alpha+i\beta$ annihilates $e^{\alpha x-i\beta x}$

Example 2: (text page 154) $g(x)=5e^{-x}\cos 2x-9e^{-x}\sin 2x$

$n=0,\ \alpha=-1,\ \beta=2$

annihilator D^2+2D+5

Example 5: (text page 156) $g(x)=x\cos x-\cos x$

annihilator $\left[D^2+1\right]^2$

$n=1,\ \alpha=0,\ \beta=1$

Example 6: (text page 157) $g(x)=10e^{-2x}\cos x$

annihilator D^2+4D+5

$n=0,\ \alpha=-2,\ \beta=1$

General rule 3:

L_1 L_2 L_k

If $g(x) = g_1(x) + g_2(x) + \ldots\ldots + g_k(x)$

$L_h[g_h(x)] = 0$ but $L_h[g_m(x)] \neq 0$ if $m \neq h$,

then the annihilator of $g(x)$ is the product of L_h $(h = 1 \sim k)$

$$\boxed{L_k L_{k-1} \cdots L_2 L_1}$$

☆ 2-3

Proof: $L_k L_{k-1} \cdots L_3 L_2 L_1 \left[g_1 + g_2 + g_3 + \cdots\cdots + g_k \right]$

$$= L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 + L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 +$$
$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_3 + \cdots\cdots + L_k L_{k-1} \cdots L_3 L_2 L_1 g_k$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 = L_k L_{k-1} \cdots L_3 L_2 \left[L_1 g_1 \right] = 0$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 = L_k L_{k-1} \cdots L_3 L_1 \left[L_2 g_2 \right] = 0$$

(因為 L_1, L_2 為 linear DE with constant coefficient,

$L_1 L_2 = L_2 L_1$)

Similarly,

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_3 = L_k L_{k-1} \cdots L_4 L_2 L_1 \left[L_3 g_3 \right] = 0$$

$$\vdots$$

$$\vdots$$

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_k = L_{k-1} \cdots L_4 L_3 L_2 L_1 \left[L_k g_k \right] = 0$$

Therefore,

$$\begin{aligned} &L_k L_{k-1} \cdots L_3 L_2 L_1 \left[g_1 + g_2 + g_3 + \cdots\cdots + g_k \right] \\ &= 0 + 0 + 0 + \cdots\cdots + 0 \\ &= 0 \end{aligned}$$

[Example 7] (text page 157)

$$g(x) = \underline{5x^2 - 6x} + \underline{4x^2e^{2x}} + \underline{3e^{5x}}$$

annihilator: D^3

annihilator: $(D-2)^3$ page 215

$n=2,\ \alpha=2$

annihilator: $D-5$

annihilator of $g(x)$: $D^3\,(D-2)^3\,(D-5)$

4-5-3 Using the Annihilator to Find the Particular Solution

☆☆

Step 2-1 Find the annihilator L_1 of $g(x)$

Step 2-2 如果原來的 linear & constant coefficient DE 是

$$L(y) = g(x)$$

那麼將 DE 變成如下的型態：

$$\boxed{L_1\left[L(y)\right] = L_1\left[g(x)\right] = 0}$$

$L_1 Ly=$

(homogeneous linear & constant coefficient DE)

註： If $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$

then $L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

Step 2-3 Use the method in Section 4-3 to find the solution of

$$\boxed{L_1[L(y)]=0}$$

Step 2-4 Find the particular solution.

The particular solution y_p is a solution of

☆3 $$L_1[L(y)]=0$$

$L_1 L y_p = 0$

but $L y_p = 0$

☆3 $\because L y = g$

$L y_p = g$

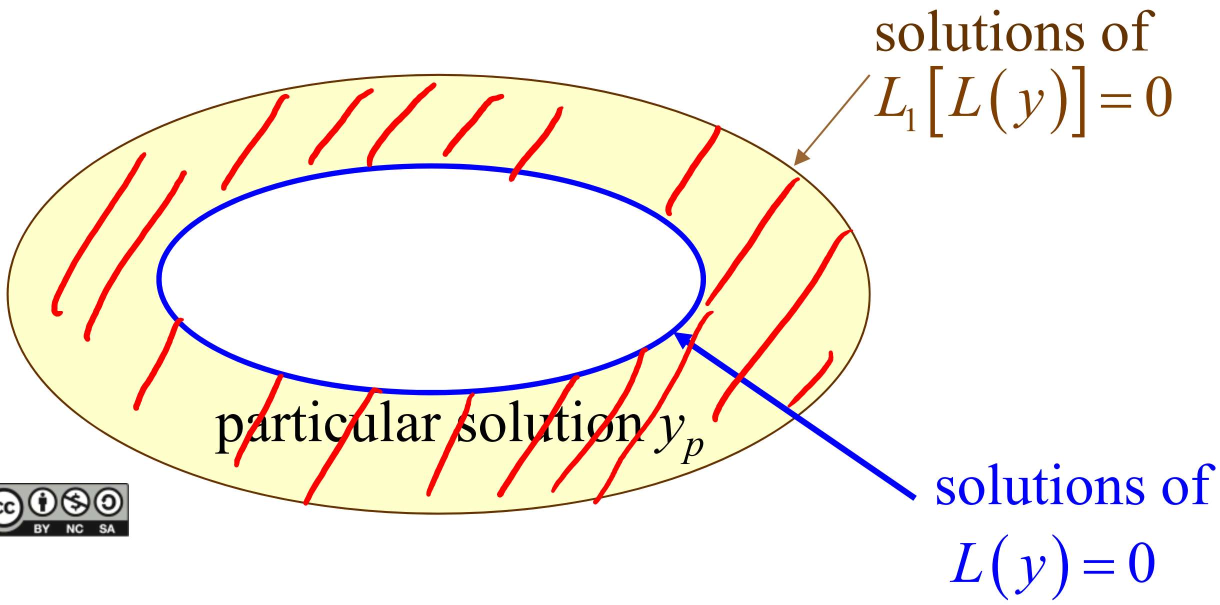
but not a solution of

☆3 $$L(y)=0$$

(Proof): Since $L(y_p)=g(x)$, if $g(x)\neq 0$, $L(y_p)$ should be nonzero.

Moreover, $L_1[L(y_p)]=L_1[g(x)]=0.$

Step 2-5 Solve the unknowns



particular solution y_p $\in$ solutions of $L_1[L(y)]=0$

$\notin$ solutions of $L(y)=0$

★3

本節核心概念

4-5-4 Examples

[Example 3] (text page 155)

$$y'' + 3y' + 2y = 4x^2 \qquad (L = D^2 + 3D + 2)$$

$m^2+3m+2=0$

$m=-2,-1$

$Ly=4x^2$

Step 1: Complementary function

(solution of the associated homogeneous function)

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2-1: Annihilation: D^3 $\qquad (L_1 = D^3)$

$$L_1[L(y)] = L_1[g(x)] = 0$$

Step 2-2: $D^3(D^2+3D+2)y = 0$

Sec 4-3

Step 2-3: auxiliary function $m^3(m^2+3m+2)=0$

roots: $m_1 = m_2 = m_3 = 0,\ m_4 = -1,\ m_5 = -2$

Solution for $L_1[L(\tilde{y})] = 0$:

$$\tilde{y} = \boxed{d_1 + d_2 x + d_3 x^2} + d_4 e^{-x} + d_5 e^{-2x}$$

移除和 complementary function 相同的部分

$y_p = d_1 + d_2 x + d_3 x^2$

Step 2-4: particular solution $y_p = A + Bx + Cx^2$ $\quad y'_p = B + 2Cx$

$$y''_p = 2C$$

Step 2-5: $y''_p + 3y'_p + 2y = 2Cx^2 + (2B + 6C)x + (2A + 3B + 2C) = 4x^2$

$$\begin{cases} 2C = 4 \\ 2B + 6C = 0 \\ 2A + 3B + 2C = 0 \end{cases} \Longrightarrow \begin{aligned} C &= 2 \\ B &= -6 \\ A &= 7 \end{aligned}$$

$$\boxed{y_p = 7 - 6x + 2x^2}$$

Step 3: $\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2}$

Sec4-4 $Ae^{3x}+B\sin x+C\cos x$

⇓ modified

$y_p = Axe^{3x}+B\sin x+C\cos x$

[Example 4] (text page 156)

$$y''-3y'=8e^{3x}+4\sin x \qquad (L=D^2-3D)$$

$n=0,\ \alpha=0,\ \beta=1$

from page 216 annihilator: D^2+1

Step 1: Complementary function

From auxiliary function, $m^2-3m=0$, roots: 0, 3

$$y_c=c_1+c_2e^{3x}$$

Step 2-1: Find the annihilator

$D-3$ annihilate $8e^{3x}$ but cannot annihilate $4\sin x$

(D^2+1) annihilate $4\sin x$ but cannot annihilate $8e^{3x}$

⇓

$L_1=(D-3)(D^2+1)$ is the annihilator of $8e^{3x}+4\sin x$

Step 2-2: $\boxed{(D-3)(D^2+1)\left(D^2-3D\right)y=0}$

Step 2-3: auxiliary function: $\underline{(m-3)}(m^2+1)(\underline{m^2-3m})$ 易犯錯的地方

$= \underline{m}\underline{(m-3)^2}(m^2+1)=0$

★4

solution of $(D-3)(D^2+1)(D^2-3D)\tilde{y}=0$:

$\tilde{y}=\cancel{d_1}+\cancel{d_2e^{3x}}+\boxed{d_3xe^{3x}+d_4\cos x+d_5\sin x}$

$L_1L_y=0$

Step 2-4: particular solution

$y_p = d_3xe^{3x}+d_4\cos x+d_5\sin x$

$\Downarrow$ 代回原式 並比較係數

Step 2-5: $y_p=\frac{8}{3}xe^{3x}+\frac{6}{5}\cos x-\frac{2}{5}\sin x$

Step 3: general solution $\boxed{y=c_1+c_2e^{3x}+\frac{8}{3}xe^{3x}+\frac{6}{5}\cos x-\frac{2}{5}\sin x}$

4-5-5 本節要注意的地方

(1) 所以要先算 complementary function，再算 particular solution

(2) 若有兩個以上的 annihilator，選其中較簡單的即可

(3) 計算 auxiliary function 時有時容易犯錯

(4) $L_1\left[L\left(\tilde{y}\right)\right]=0$ 的解和 $L\left(y\right)=0$ 的解不一樣。

(5) 這方法，只適用於 constant coefficient linear DE

(因為，還需借助 auxiliary function)

The thing that can be done by the annihilator approach can always be done by the "guessing" method in Section 4-4, too.

附錄九：Reviews for Higher Order DE

(A) Linear DE Complementary Function 3 大解法的前 2個

(1) Reduction of Order (Section 4-2)

$$\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx}$$

適用情形：

(2) Auxiliary Function (Section 4-3)

$$\boxed{a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

$$\downarrow$$

$$\boxed{a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0}$$

4 Cases (See pages 181-183, 185-186)

適用情形：

(B) Linear DE Particular solution 3 大解法的前 2個

(1) Guess (Section 4-4) (熟悉講義 page 196 的表)

要訣： y_p should be a linear combination of $g(x)$, $g'(x)$,

$g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ………………

適用情形：

遇到重覆，乘 x (Sec. 4-4, constant coefficients)

或乘 $\ln x$ (Sec 4-7, Cauchy-Euler)

(2) Annihilator (Section 4-5)

若原本的 DE 為 $L[y(x)] = g(x)$ Annihilator: $L_1[\, g(x)] = 0$

Particular solution 為 $L_1\{L[y(x)]\} = 0$ 的解

(扣去和 $L[y(x)] = 0$ 的解重複的部分)

$y = y_c + y_p$

適用情形：

Annihilator 算法三大規則： Pages 215-217

練習題

Sec. 4-1: 3, 7, 8, 10, 13, 20, 24, 29, 33, 36

Sec. 4-2: 2, 4, 9, 13, 14, 16, 18, 19, 22

Sec. 4-3: 7, 16, 20, 22, 24, 28, 33, 39, 41, 52, 54, 56, 59, 61, 63

Section 4-4 5, 6, 14, 17, 18, 24, 26, 32, 33, 39, 42

Section 4-5 2, 7, 8, 13, 18, 31, 45, 60, 62, 69, 70

Review 4 2, 21, 22, 25, 33, 34, 37