

Part 1 公式與定義總整理

(1) Series, Integral, and Transform (非常重要)

把握不同 transform 之間的「關聯性」，多比較彼此之間相同或相異的地方

(1) Laplace Transform	$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$
(2) Fourier series (standard form)	<p>interval: $x \in [-p, p]$</p> $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right),$ $a_0 = \frac{1}{p} \int_{-p}^p f(x) dx, \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx,$ $b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx, \quad a_0, a_n, b_n: \text{ Fourier coefficients}$
(2-1) Fourier series (half range extension form)	<p>interval: $x \in [0, L]$</p> <p>將 Fourier series 的 p 變成 $L/2$</p> $\frac{1}{p} \int_{-p}^p \quad \text{變成} \quad \frac{2}{L} \int_0^L$
(3) Fourier cosine series (cosine series)	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$ $a_0 = \frac{2}{p} \int_0^p f(x) dx, \quad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$ <p>適用情形：</p> <p>(1) interval: $x \in [-p, p], f(x) = f(-x)$</p> <p>(2) interval: $x \in [0, p]$ (half range extension 時)</p>
(4) Fourier sine series (sine series)	$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x \quad b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$ <p>適用情形：</p> <p>(1) interval: $x \in [-p, p], f(x) = -f(-x)$</p> <p>(2) interval: $x \in [0, p]$ (half range extension 時)</p>
(5) Fourier integral (Sec. 14-3)	$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)] d\alpha$ $A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos(\alpha x) dx \quad B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin(\alpha x) dx$

(6) Fourier cosine integral (或 cosine integral)	$f(x) = \frac{2}{\pi} \int_0^{\infty} A(\alpha) \cos(\alpha x) d\alpha, \quad A(\alpha) = \int_0^{\infty} f(x) \cos(\alpha x) dx$
(7) Fourier sine integral (或 sine integral)	$f(x) = \frac{2}{\pi} \int_0^{\infty} B(\alpha) \sin(\alpha x) d\alpha, \quad B(\alpha) = \int_0^{\infty} f(x) \sin(\alpha x) dx$
(8) Fourier transform (Sec. 14-4)	$\mathfrak{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{j\alpha x} dx = F(\alpha)$
(8-1) inverse Fourier transform	$\mathfrak{F}^{-1}[F(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-j\alpha x} d\alpha = f(x)$
(9) Fourier cosine transform	$\mathfrak{F}_c[f(x)] = \int_0^{\infty} f(x) \cos(\alpha x) dx = F(\alpha)$
(9-1) inverse Fourier cosine transform	$\mathfrak{F}_c^{-1}[F(\alpha)] = \frac{2}{\pi} \int_0^{\infty} F(\alpha) \cos(\alpha x) d\alpha = f(x)$
(10) Fourier sine transform	$\mathfrak{F}_s[f(x)] = \int_0^{\infty} f(x) \sin(\alpha x) dx = F(\alpha)$
(10-1) inverse Fourier sine transform	$\mathfrak{F}_s^{-1}[F(\alpha)] = \frac{2}{\pi} \int_0^{\infty} F(\alpha) \sin(\alpha x) d\alpha = f(x)$

(2) 和 Laplace Transform 相關的公式 (很重要)

Laplace transform	$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$
Differentiation $L\{f^{(n)}(t)\} =$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
Multiplication by t $L\{t^n f(t)\} =$	$(-1)^n \frac{d^n}{ds^n} F(s)$
Integration	$L\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$
Multiplication by exp	$L\{e^{at} f(t)\} = F(s-a)$
Translation (I)	$L\{f(t-a)u(t-a)\} = e^{-as} F(s)$
Translation (II)	$L\{g(t)u(t-a)\} = e^{-as} L\{g(t+a)\}$

Convolution property	convolution: $y(t) = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ $L\{y(t)\} = F(s)G(s)$
Periodic input If $f(t) = f(t+T)$	$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$
$L\{1\} =$	$1/s$
$L\{u(t)\} =$	$1/s$
$L\{t^n\} =$	$\frac{n!}{s^{n+1}}$
$L\{\exp(at)\} =$	$\frac{1}{s-a}$
$L\{\sin(kt)\} =$	$\frac{k}{s^2 + k^2}$
$L\{\cos(kt)\} =$	$\frac{s}{s^2 + k^2}$
$L\{\sinh(kt)\} =$	$\frac{k}{s^2 - k^2}$
$L\{\cosh(kt)\} =$	$\frac{s}{s^2 - k^2}$
$L\{u(t-t_0)\} =$	$\frac{e^{-t_0s}}{s}$
$L\{\delta(t)\} =$	1

(3) Chapter 6 的相關公式與定義

(i) ordinary point	先將 DE 變成 <u>standard form</u> : $y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y' + P_0(x)y = 0$
(ii) regular singular point	(i) 若 $P_0(x), P_1(x), \dots, P_{n-1}(x)$, 在 $x = x_0$ 這一點為 analytic, 則 x_0 為 <u>ordinary point</u>
(iii) irregular singular point	(ii) 若 $P_0(x), P_1(x), \dots, P_{n-1}(x)$, 在 $x = x_0$ 不為 analytic, 但 $(x-x_0)^n P_0(x), (x-x_0)^{n-1} P_1(x), \dots, (x-x_0) P_{n-1}(x)$ 在 $x = x_0$ 為 analytic, 則 x_0 為 <u>regular singular point</u>
	(iii) 以上二條件皆不滿足, 則 x_0 為 <u>irregular singular point</u>

regular singular point 的情形下， $r_2 - r_1 =$ integer 時，有時(並非所有情況) 要用這個式子求 $y_2(x)$	$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{y_1^2(x)} dx$
Bessel's equation of order ν	$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$
Legendre's equation of order n	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$
Solutions of Bessel's equation of order ν	$c_1 J_\nu(x) + c_2 Y_\nu(x)$

(4) Chapter 7 的相關公式與定義

Step function	$u(t-a) = 1 \quad \text{for } t > a, \quad u(t-a) = 0 \quad \text{for } t < a,$
convolution (旋積) 很重要，一定要會	$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ 這裡 * 代表旋積
Integration for $\delta(t-t_0)$	$\int_{-\infty}^{\infty} \delta(t-t_0) dt = 1$
Sifting property for $\delta(t-t_0)$	$\int_p^q f(t)\delta(t-t_0) dt = f(t_0)$
Relation between $\delta(t-t_0)$ and $u(t)$	$\int_{-\infty}^t \delta(\tau-t_0) d\tau = u(t-t_0) \quad \frac{d}{dt}u(t-t_0) = \delta(t-t_0)$

(5) Chapter 11 的相關公式與定義

inner product	$(f_1, f_2) = \int_a^b f_1(x)f_2^*(x)dx$ *: conjugate
orthogonal	$(f_1, f_2) = \int_a^b f_1(x)f_2^*(x)dx = 0$
square norm	$\ f(x)\ ^2 = (f(x), f(x)) = \int_a^b f(x)f^*(x)dx = \int_a^b f(x) ^2 dx$
norm	$\ f(x)\ = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x)f^*(x)dx} = \sqrt{\int_a^b f(x) ^2 dx}$
inner product with weight function	$(f_1, f_2) = \int_a^b f_1(x)f_2^*(x)w(x)dx$

orthogonal with respect to a weight function	$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) w(x) dx = 0$
normalize	$\psi(x) \longrightarrow v(x) = \frac{\psi(x)}{\ \psi(x)\ }$ 註: $\ v(x)\ = 1$
orthogonal set	$(\phi_m(x), \phi_n(x)) = 0$ for $m \neq n$, no constraint for $(\phi_n(x), \phi_n(x))$
orthonormal set	$(\phi_m(x), \phi_n(x)) = 0$ for $m \neq n$, $(\phi_n(x), \phi_n(x)) = 1$
orthogonal series expansion	$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$ where $c_n = \frac{(f(x), \phi_n(x))}{(\phi_n(x), \phi_n(x))}$ ← inner products
even and odd	If $f(x)$ is even, $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ If $f(x)$ is odd, $\int_{-a}^a f(x) dx = 0$

(9) Chapter 12 的相關公式與定義

hyperbolic	for $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0$ $B^2 - 4AC > 0$
elliptic	同上, 但 $B^2 - 4AC < 0$
parabolic	同上, 但 $B^2 - 4AC = 0$
heat equation	$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ $k > 0$
wave equation	$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$
Laplace's equation	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
Dirichlet condition	$u = \dots\dots$
Neumann condition	$\frac{\partial u}{\partial n} = \dots\dots$
$X''(x) + \lambda X(x) = 0$ $X(0) = 0 \quad X(L) = 0$	解: $X(x) = c_2 \sin \frac{n\pi}{L} x$, $\lambda = \frac{n^2 \pi^2}{L^2}$ $n = 1, 2, 3, \dots\dots$

(10) Chapter 14 的相關公式與定義

Differentiation property for Fourier transform	$\mathfrak{F}[f'(x)] = -j\alpha F(\alpha) \quad \mathfrak{F}[f^{(n)}(x)] = (-j\alpha)^n F(\alpha)$
Differentiation property for Fourier cosine transform	$\mathfrak{F}_c[f'(x)] = \alpha \mathfrak{F}_s[f(x)] - f(0)$ $\mathfrak{F}_c[f''(x)] = -\alpha^2 \mathfrak{F}_c[f(x)] - f'(0)$
Differentiation property for Fourier sine transform	$\mathfrak{F}_s[f'(x)] = -\alpha \mathfrak{F}_c[f(x)]$ $\mathfrak{F}_s[f''(x)] = -\alpha^2 \mathfrak{F}_s[f(x)] + \alpha f(0)$

(11) 其他重要公式

Taylor series for $f(x)$	$\sum_{m=0}^{\infty} \frac{(x-x_0)^m}{m!} f^{(m)}(x_0)$
Taylor series for $\exp(x)$	$\sum_{m=0}^{\infty} \frac{x^m}{m!}$
Taylor series for $\cos(x)$	$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}$
$\cos(a+b) =$	$\cos(a)\cos(b) - \sin(a)\sin(b)$
$\sin(a+b) =$	$\sin(a)\cos(b) + \cos(a)\sin(b)$
$\cos(a)\cos(b) =$	$[\cos(a+b) + \cos(a-b)]/2$
$\sin(a)\sin(b) =$	$[-\cos(a+b) + \cos(a-b)]/2$
$\sin(a)\cos(b) =$	$[\sin(a+b) + \sin(a-b)]/2$
$\cos(2a) =$	$\cos^2(a) - \sin^2(a) \quad \text{or} \quad 1 - 2\sin^2(a) \quad \text{or} \quad 2\cos^2(a) - 1$
$\sin(2a) =$	$2\sin a \cos a$
$\cosh x =$	$\frac{e^x + e^{-x}}{2}$
$\sinh x =$	$\frac{e^x - e^{-x}}{2}$
$\sinh(0) =$	0
$\cosh(0) =$	1
$\left. \frac{d}{dx} \cosh x \right _{x=0} =$	0

公式雖然多，但是把握彼此之間的關係，注意相同或相異之處，就可以較容易的記起來

Part 2 「解法」總整理

(一) n^{th} order linear DE 的 series solutions 解法

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

$a_k(x)$ ($k = 0, 1, \dots, n$) 必需為多項式，若不是，要先用 Taylor series 展開

(Case 1) 條件：當 x_0 為 ordinary point 時

方法：假設 $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$ ，代入原式

流程：5 個 steps, 參考講義 352 頁，

(Case 2) 條件：當 x_0 為 regular singular point 時

方法：假設 $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ ，代入原式 (即 Frobenius 法)

流程：7 個 steps, 參考講義 372-373 頁，

• Case 2 的特殊情形：

當 $r_1 \neq r_2$ and r_1, r_2 are real, $r_2 - r_1 = \text{integer}$

「有的時候」用講義 372 頁的方法只能得出一個 independent solution $y_1(x)$

必需根據 $y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$ 和長除法解出第二個 independent solution

(二) Laplace transform 解 DE 的方法

方法：

DE \rightarrow Laplace transform \rightarrow 計算 \rightarrow 分解因式(若需要的話) \rightarrow inverse Laplace transform

範例：講義 453, 454 頁

主要精神：把微分簡化為乘法

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x) \longrightarrow P(s)Y(s) = Q(s) + G(s)$$

$P(s)$: 即 auxiliary function, $Q(s)$: 來自 initial conditions

計算 $Q(s)$ 的快速法

參考講義 455, 456 頁

• 分解因式的方法 (Cover up method)

$$\frac{K(s)}{(s-a_1)(s-a_2)\cdots(s-a_n)^2\cdots(s-a_N)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n + B_n(s-a_n)}{(s-a_n)^2} + \cdots + \frac{A_N}{s-a_N}$$

其中 a_1, a_2, \dots, a_N 互異, [分子的 order 要小於分母的 order](#)

$$\text{則 } A_n = \left. \frac{K(s)}{(s-a_1)(s-a_2)\cdots(s-a_{n-1})\cancel{(s-a_n)}(s-a_{n+1})\cdots(s-a_N)} \right|_{s=a_n}$$

$$B_n = \left. \frac{d}{ds} \frac{K(s)}{(s-a_1)(s-a_2)\cdots(s-a_{n-1})(s-a_{n+1})\cdots(s-a_N)} \right|_{s=a_n}$$

(三) 用 Fourier Series 來解 Particular Solutions

精神：當 $f(t) = f(t+2p)$ 時，用 Fourier series, Fourier cosine series, 或 Fourier sine series

將 $f(t)$ 表示成 $\cos\left(\frac{n\pi}{p}t\right), \sin\left(\frac{n\pi}{p}t\right)$ 的 linear combination

流程：見講義 655, 656 頁

(四) Partial Differential Equations 的解法 (一)

用 Separation of Variables

精神：例如當 independent variables 為 x and y 時，

假設 $u(x, y) = X(x)Y(y)$ ，代入原式

使得 **PDE** \longrightarrow **ODE**

流程：7 個 Steps, 講義 724-726 頁 (非常重要，請熟悉)

注意：(1) 其中 Steps 3, 4, 5 要分成不同的 cases 來解

(2) 要將 Steps 3, 4, 5 所有的解都**加起來** (Step 6)

(尤其是處理 boundary value problems 時)

(3) 經常把 $d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$ 表示成 $c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

(4) “等於零”的 IVP 或 BVP 先於 Steps 3, 4 處理

例如， $u(L, y) = 0 \rightarrow X(L) = 0$

$$\left. \frac{\partial u}{\partial y} \right|_{y=b} = 0 \rightarrow Y'(b) = 0.$$

“不等於零”的 IVP 或 BVP，要在 Step 7 當中處理

- (5) 當 IVP, BVP 皆不為零時，要用 superposition principle 分成兩個子問題
(參考講義 779, 780 頁)
- (6) 其他需注意的地方：整理於講義 783-786 頁

(五) Partial Differential Equations 的解法 (二)

用 Fourier Transform (或 Fourier Cosine, Sine Transforms)

精神：藉由 Fourier transform (或 Fourier cosine transform, Fourier sine transform)，將針對其中一個變數的偏微分變成乘法

方法：(A) interval 為 $-\infty < v < \infty$ 時 → 用 Fourier transform

(B) interval 為 $0 < v < \infty$

而且有 $u(v, \dots) = 0$ or a constant when $v = 0$ 的 initial condition 時

→ 用 Fourier sine transform

(C) interval 為 $0 < v < \infty$

而且有 $\frac{\partial}{\partial v} u(v, \dots) = 0$ or a constant when $v = 0$ 的 initial condition 時

→ 用 Fourier cosine transform

流程：5 個 Steps, 講義 704 頁 (有一些複雜，請多練習)

需注意的地方：整理於講義 712, 713 頁

(尤其注意其中第五點)

Part 3 補充

同學們若覺得以上的整理，還漏掉哪些公式、定義、或解法，就在這邊補充吧！