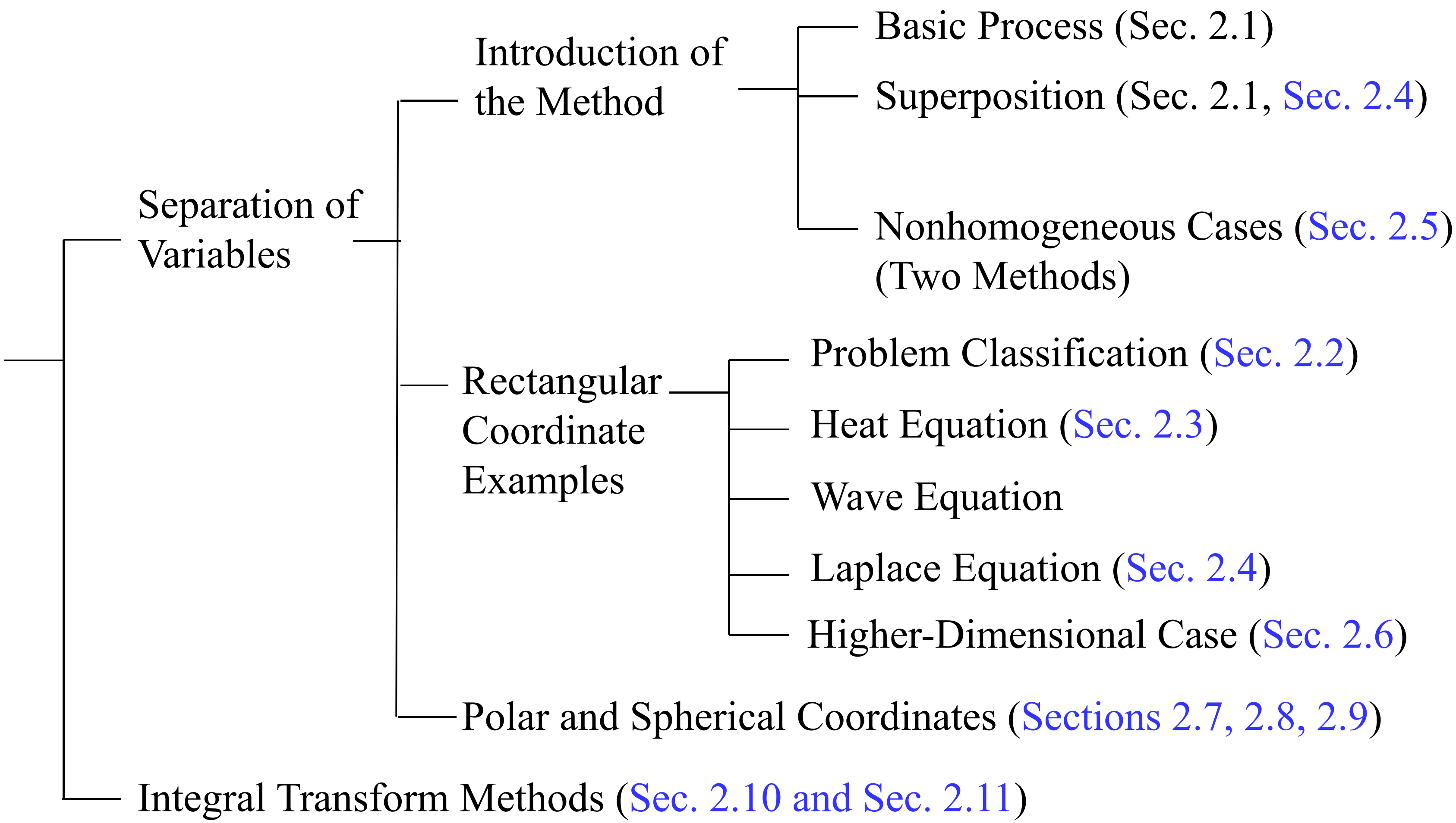
# 2. Partial Differential Equations

[1] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[2] http://djj.ee.ntu.edu.tw/DE.htm

# Solving PDEs



- Separation of Variables
  - Introduction of the Method
    - Basic Process (Sec. 2.1)
    - Superposition (Sec. 2.1, Sec. 2.4)
    - Nonhomogeneous Cases (Sec. 2.5) (Two Methods)
  - Rectangular Coordinate Examples
    - Problem Classification (Sec. 2.2)
    - Heat Equation (Sec. 2.3)
    - Wave Equation
    - Laplace Equation (Sec. 2.4)
    - Higher-Dimensional Case (Sec. 2.6)
  - Polar and Spherical Coordinates (Sections 2.7, 2.8, 2.9)
- Integral Transform Methods (Sec. 2.10 and Sec. 2.11)

# 2.1 Boundary-Value Problem in Rectangular Coordinates

Use the methods of

(1) separation of variables ← Sections 2.1 ~ 2.9

(2) the Laplace / Fourier transforms ← Sections 2.11 and 2.12

to solve the PDE problem.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.1.

linear second order partial differential equation for two independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

7 terms

$B^2-4AC>0$ : hyperbolic, $B^2-4AC=0$ : parabolic

$B^2-4AC<0$ : elliptic

homogeneous : $G(x, y)=0$, nonhomogeneous : $G(x, y)\neq 0$

## 2.1.1 Superposition Principle

【Theorem 2.1.1】 Superposition Principle

If $u_1$, $u_2$, …., $u_k$ are solutions of a homogeneous linear partial differential equation, then

$$u = c_1u_1 + c_2u_2 + \cdots\cdots + c_ku_k$$

is also a solution of the homogeneous linear partial differential equation.

## 2.1.2 Method of Separation of Variables

解 PDE with BVP (or IVP) 的方法

**(1) method of separation of variables**

若 PDE 當中有對 $x$ 及對 $y$ 的偏微分，

假設解為 $u(x, y) = X(x)Y(y)$

**(2) using the Fourier transform (or Fourier cosine transform, Fourier sine transform)** (see Sections 2.10 and 2.11)

**共通的精神： PDE $\longrightarrow$ ODE**

# Method of Separation of Variables 的流程

**(Step 1)** 假設解為 $u(x, y) = X(x)Y(y)$

解法關鍵

**(Step 2)** 將 $u(x, y) = X(x)Y(y)$ 代入 PDE，把 PDE 變成

$$\text{“ function of } X\text{ ”} = \text{“ function of } Y\text{ ”} = -\lambda$$

的型態

$\lambda$ 被稱為 real separation constant

## Steps 3, 4, 5 要分成不同的 Cases 來解

除了trivial 的情形外，所有可能的 cases 都要考慮

**(Step 3)** 將 function of $X = -\lambda$ 的解算出，即為 $X(x)$

註：(a) 如果有等於零的 boundary (initial) conditions，
也要在這一步考慮

(See the Examples in Sections 2.3, 2.4, and 2.5)

(b) 有時，先解 $Y(y)$ 會比較容易

(視 boundary (initial) conditions 而定)

(c) 在這一步中，有的時候，會得出 $\lambda$ 的限制

**(Step 4)** 將 function of $Y = -\lambda$ 的解算出，即為 $Y(y)$

需注意的地方和 Step 3 相同

**(Step 5)** $u(x, y) = X(x)Y(y)$

**(Step 6)** 將所有可能的解全部加起來

**(Step 7)** 用非零的 boundary (initial) conditions 將 coefficients 求出

註： 這一步經常會用到 Fourier series, Fourier cosine series 或 Fourier sine series

※ 若沒有 boundary (initial) conditions，Steps 6, 7 可以省略

Rules:

$x$ 的 BVP (IVP) 簡單 $\longrightarrow$ 先算 $X(x)$

$y$ 的 BVP (IVP) 簡單 $\longrightarrow$ 先算 $Y(y)$

沒有 BVP (IVP) $\longrightarrow$ 先算 $X(x)$ 或 $Y(y)$ 皆可

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0$$

$$u(0,y)=0 \qquad u(L,y)=0$$

$$u(x,0)=f(x) \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0}=g(x)$$

先算 $X(x)$

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}$$

$$u(0,y)=f(y) \qquad u(L,y)=0$$

$$\left.\frac{\partial}{\partial y}u(x,y)\right|_{y=0}=0 \qquad \left.\frac{\partial}{\partial y}u(x,y)\right|_{y=H}=0$$

先算 $Y(y)$

Note: Separation of variables 的方法其實未必可以得出 PDE 所有的解

有些解無法用 $X(x)Y(y)$ 來表示

Separation of variables 的主要好處是比其他方法簡單

## [Example 1]

$$\boxed{\frac{\partial u^2}{\partial x^2} = 4\frac{\partial u}{\partial y}}$$

Step 1 假設解為 $u(x, y) = X(x)Y(y)$ (解法關鍵)

Step 2 將 $u(x, y) = X(x)Y(y)$ 代入 $\frac{\partial u^2}{\partial x^2} = 4\frac{\partial u}{\partial y}$

$$X''(x)Y(y) = 4X(x)Y'(y)$$

$$\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)}$$

real separation constant

令 $\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)} = -\lambda$ (解法關鍵)

$$X''(x) + 4\lambda X(x) = 0 \qquad Y'(y) + \lambda Y(y) = 0$$

(The detail can be reviewed from the PowerPoint in DE1)

$$X''(x)+4\lambda X(x)=0 \qquad Y'(y)+\lambda Y(y)=0$$

**Case 1 for Steps 3, 4, 5** $\lambda=0$

Step 3-1 $X''(x)=0$

auxiliary function $m^2=0$ roots : 0, 0

$$X(x)=c_1+c_2x$$

Step 4-1 $Y'(y)=0$ $Y(y)=c_3$

Step 5-1 $u(x,y)=X(x)Y(y)=(c_1+c_2x)c_3=A_1+B_1x$

$$A_1=c_1c_3 \qquad B_1=c_2c_3$$

**Case 2 for Steps 3, 4, 5** $\lambda < 0$

為了方便起見，令 $\lambda = -\alpha^2$

Step 3-2 $X''(x) - 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $2\alpha, -2\alpha$

$$X(x) = d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$$

通常將解改寫成 $X(x) = c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

Step 4-2 $\dfrac{Y'(y)}{Y(y)} = \alpha^2$ $Y'(y) - \alpha^2 Y(y) = 0$

$Y'(y) - \alpha^2 Y(y) = 0$ $Y(y) = c_6 e^{\alpha^2 y}$

Step 5-2 $u(x,y) = X(x)Y(y) = A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x)$

$A_2 = c_4 c_6$ $B_2 = c_5 c_6$

## Case 3 for Step 3 $\lambda > 0$

為了方便起見，令 $\lambda = \alpha^2$

Step 3-3 $X''(x) + 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $j2\alpha, -j2\alpha$

$$X(x) = c_7 \cos(2\alpha x) + c_8 \sin(2\alpha x)$$

Step 4-3 $\dfrac{Y'(y)}{Y(y)} = -\alpha^2$ $\quad Y'(y) + \alpha^2 Y(y) = 0$ $\qquad Y(y) = c_9 e^{-\alpha^2 y}$

Step 5-3 $u(x,y) = A_3 e^{-\alpha^2 y} \cos(2\alpha x) + B_3 e^{-\alpha^2 y} \sin(2\alpha x)$

若要處理 boundary conditions，或著想得到 general solution，要將所有可能的解都加起來

Step 6

$$u(x,y) = \boxed{A_1 + B_1 x} + \boxed{\sum_{\alpha>0} [A_{2,\alpha} e^{\alpha^2 y} \cosh(2\alpha x) + B_{2,\alpha} e^{\alpha^2 y} \sinh(2\alpha x)]}$$

$$\boxed{+\sum_{\alpha>0} [A_{3,\alpha} e^{-\alpha^2 y} \cos(2\alpha x) + B_{3,\alpha} e^{-\alpha^2 y} \sin(2\alpha x)]}$$

$\alpha$ 是任意實數

(註：nonseparable 的解在這一步得到)

# Section 2.2 Classical PDEs and Boundary-Value Problems

## 2.2.1 本節綱要

(1) one-dimensional heat equation (或簡稱為 heat equation)

$$\boxed{k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}} \qquad k>0$$

(2) one-dimensional wave equation (或簡稱為 wave equation)

$$\boxed{a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}}$$

(3) two-dimensional form of Laplace's equation (或簡稱為 Laplace's equation )

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

名詞：

heat equation, (page 105)

wave equation, (page 107)

Laplace’s equation, (page 110)

Laplacian, (page 111)

Dirichlet condition, (page 113)

Neumann condition, (page 113)

Robin condition (page 113)

本節的重點：七大名詞，和它們所對應的公式

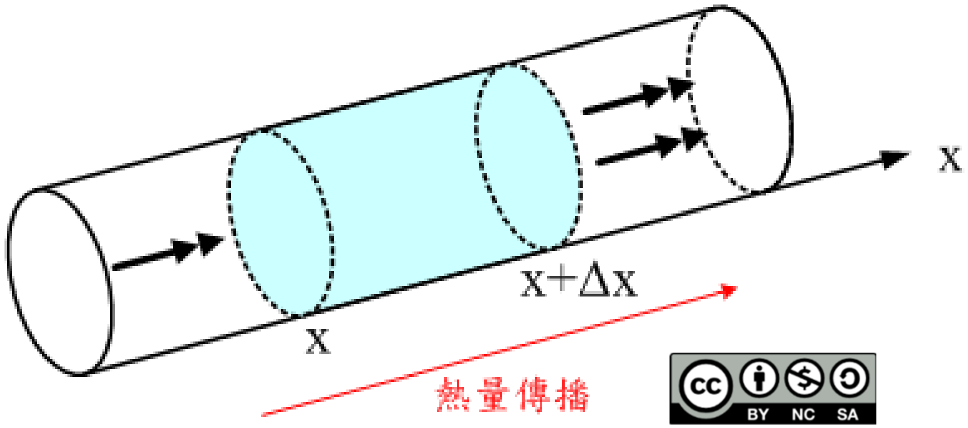
## 2.2.2 One-Dimensional Heat Equation

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

由來：熱傳導的理論

$u(x, t)$: temperature, $t$: time, $x$: location

Fig. 2.2.1



heat equation 別名：diffusion equation

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

Example:

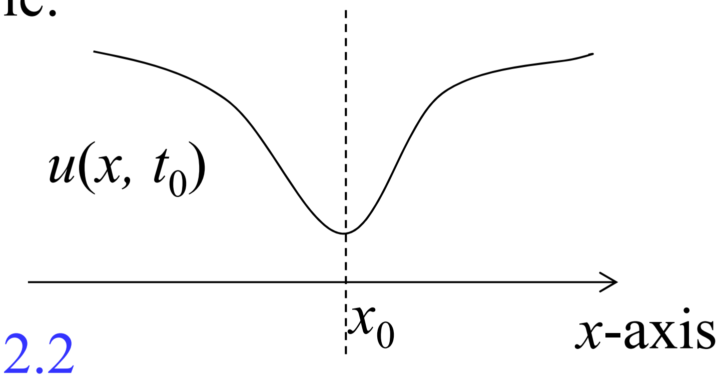


Fig. 2.2.2

$u(x,\ t)$: temperature,
$t$: time,   $x$: location

$x_0$ 的溫度將上升 $\left.\frac{\partial u(x_0,t)}{\partial t}\right|_{t=t_0} > 0$

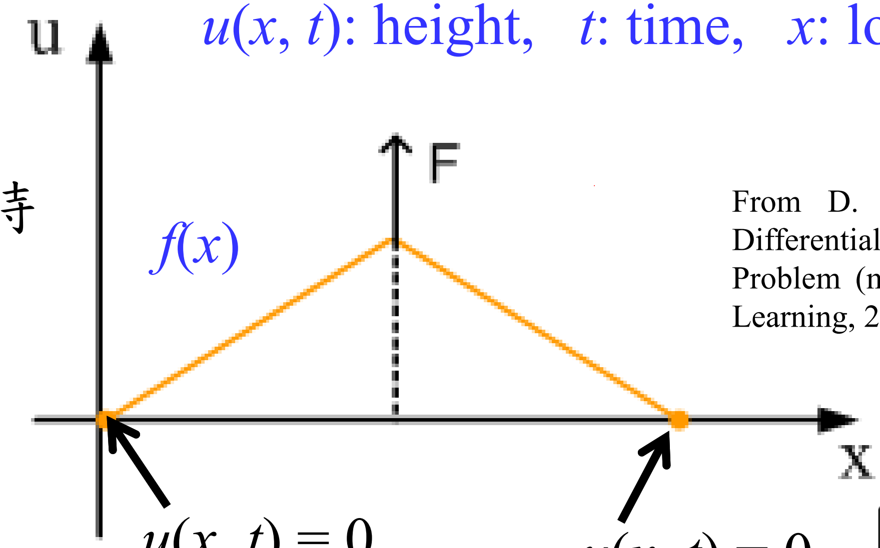
## 2.2.3 One-Dimensional Wave Equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

「拉像皮筋」的模型

$u(x, t)$: height, $t$: time, $x$: location

(1) 剛開始時



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

$u(x, t) = 0$
when $x = 0$

$u(x, t) = 0$
when $x = L$

Fig. 2.2.3

wave equation 別名： telegraph equation

(2) 手放開之後產生振動

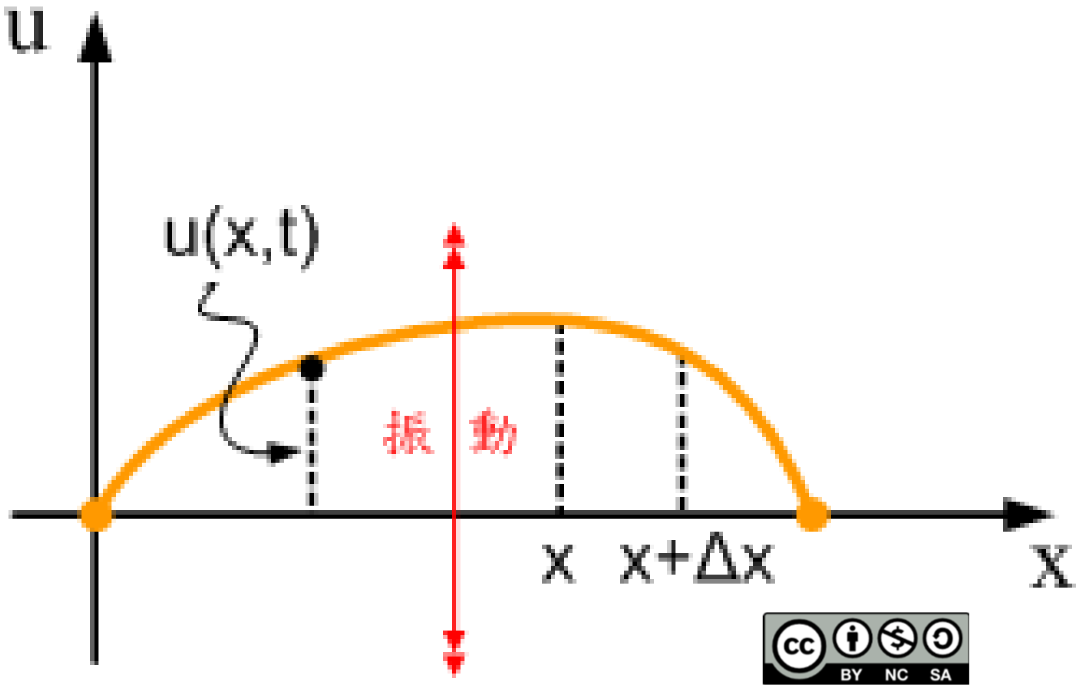


Fig. 2.2.4

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

- Wave equation 其他的應用：

  Theory of high-frequency transmission line

  Fluid mechanics (流體力學)

  Acoustics (聲學)

  Elasticity (彈力學)

  Microwave engineering (電波工程)

## 2.2.4 Two-Dimensional Form of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

溫度隨著位置而變化的模型

$u(x, y)$: temperature,

$x, y$: location

Laplace's Equation 亦可用 Laplacian 表示, $\nabla^2 u(x, y) = 0$

Laplacian: $\nabla^2$

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

# Modification

加上外力，或與外界的交互作用

例：heat equation 的 modification

$$k\frac{\partial^2 u}{\partial x^2} - h\left(u - u_m\right) = \frac{\partial u}{\partial t}$$

例：wave equation 的 modification

$$a^2\frac{\partial^2 u}{\partial x^2} + F\left(x, t, u, u_t\right) = \frac{\partial^2 u}{\partial t^2}$$

- Laplace’s Equation 的其他應用

  Static displacement of membranes

  Edge detection (邊緣偵測)

  Microwave engineering (電波工程)

## 2.2.6 Boundary Conditions 或 Initial Conditions

| | | |
|---|---|---|
| Dirichlet condition | $u = \ldots\ldots..$ | (沒微分) |
| Neumann condition | $\frac{\partial u}{\partial n} = \cdots\cdots$ | (有微分) |
| Robin condition | $\frac{\partial u}{\partial n} + hu = \cdots\cdots$ | (混合) |

$h$ is a constant

# 2.3 Heat Equation

This section can also be viewed as an example of Section 2-1

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0 \qquad (1)$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \qquad (2)$$

$$u(x,0) = f(x), \quad 0 < x < L. \qquad (3)$$

Solution:

(Step 1) $u(x,t) = X(x)T(t)$

$$kX''(x)T(t) = X(x)T'(t)$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.3.

$$kX''(x)T(t) = X(x)T'(t)$$

(Step 2)
$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \qquad (4)$$

$$X'' + \lambda X = 0 \qquad (5)$$

$$T' + k\lambda T = 0. \qquad (6)$$

(From Zero Boundary Conditions)

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0.$$

Since for a nontrivial solution, $T(t)$ cannot be zero,

$X(0) = 0$ and $X(L) = 0$.

We have

$$\begin{cases} X'' + \lambda X = 0, \ \ X(0) = 0, \ \ X(L) = 0. \\ T' + k\lambda T = 0. \end{cases} \qquad (7)$$

(i) $X''+\lambda X=0,\ X(0)=0,\ X(L)=0.$ (ii) $T'+k\lambda T=0.$

**Case 1 for Steps 3, 4, 5 $\lambda=0$**

$$X''=0 \Longrightarrow X(x)=c_1+c_2x$$

From $X(0)=0,\ X(L)=0 \Longrightarrow c_1=c_2=0 \Longrightarrow X(x)=0$

$$u(x,t)=X(x)T(t)=0$$

(trivial solution)

(i) $X''+\lambda X=0,\ \ X(0)=0,\ \ X(L)=0.$ (ii) $T'+k\lambda T=0.$

**Case 2 for Steps 3, 4, 5 $\lambda < 0$**

Set $\lambda = -\alpha^2$

$$X''-\alpha^2 X=0 \Longrightarrow X(x)=c_1e^{\alpha x}+c_2e^{-\alpha x}$$

$$\Longrightarrow X(x)=c_3\cosh(\alpha x)+c_4\sinh(\alpha x)$$

Note:

$$c_3\cosh(\alpha x)+c_4\sinh(\alpha x)=c_3\frac{e^{\alpha x}+e^{-\alpha x}}{2}+c_4\frac{e^{\alpha x}-e^{-\alpha x}}{2}$$

$$=\frac{c_3+c_4}{2}e^{\alpha x}+\frac{c_3-c_4}{2}e^{-\alpha x}$$

From $X(0)=0$, $c_3=0$

From $X(L)=0$, $c_4\sinh(\alpha L)=0$, $c_4=0$

$$\Longrightarrow X(x)=0 \Longrightarrow u(x,t)=X(x)T(t)=0$$

(trivial solution)

(i) $X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0.$ (ii) $T' + k\lambda T = 0.$

**Case 3 for Steps 3, 4, 5 $\lambda > 0$**

Set $\lambda = \alpha^2$

$$X'' + \alpha^2 X = 0 \Longrightarrow X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

$$X(0) = 0, \Longrightarrow c_1 = 0,$$

$$X(L) = 0. \Longrightarrow c_2 \sin \alpha L = 0 \Longrightarrow \alpha = n\pi / L, \quad \lambda = n^2\pi^2 / L^2.$$

$$X(x) = c_2 \sin(\pi n x / L), \quad \lambda = n^2\pi^2 / L^2. \qquad n = 1, 2, 3, \ldots.$$

$$T' + k\frac{n^2\pi^2}{L}T = 0. \Longrightarrow T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$$

$$u_n(x,t) = X(x)T(t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x, \qquad n = 1, 2, 3, \ldots.$$

(Step 6) $u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}A_n e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x,$

(Step 7) From the boundary condition, $u(x,0)=f(x)$

$$u(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}x=f(x)$$

From Fourier sine series (附錄四)

$$g(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p g(x)\sin\frac{n\pi}{p}x\,dx$$

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx.$$

Therefore,

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x.$$

# 附錄三： Hyperbolic Function

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

比較： $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

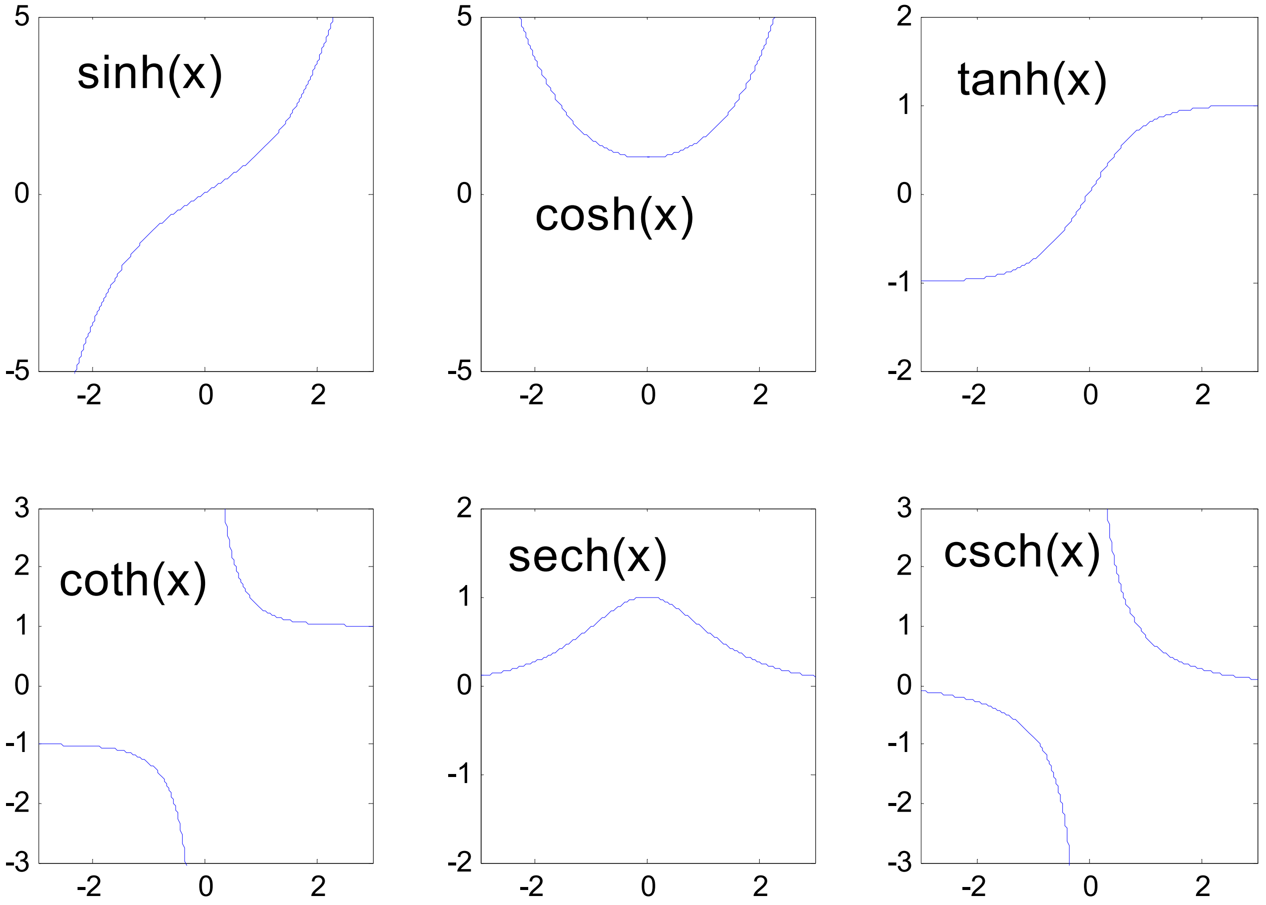
$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}}$$


sinh(x)
cosh(x)
tanh(x)
coth(x)
sech(x)
csch(x)
5
0
-5
2
1
-1
-2
3
-3
-2
0
2

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$

$$\sinh(0) = 0$$

$$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$

$$\cosh(0) = 1$$

$$\frac{d}{dx}\tanh(ax) = a\,\mathrm{sech}^2(ax)$$

$$\sinh'(0) = 1$$

$$\frac{d}{dx}\coth(ax) = -a\,\mathrm{csch}^2(ax)$$

$$\cosh'(0) = 0$$

$$\frac{d}{dx}\mathrm{sech}(ax) = -a\,\mathrm{sech}(ax)\tanh(ax)$$

$$\sin(ix) = i\sinh(x)$$

$$\frac{d}{dx}\mathrm{csch}(ax) = -a\,\mathrm{csch}(ax)\coth(ax)$$

$$\cos(ix) = \cosh(x)$$

# 附錄四 Review for Fourier Series and Fourier Cosine / Sine Series

## Fourier Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx \qquad a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\frac{n\pi}{p}xdx$$

$$b_n=\frac{1}{p}\int_{-p}^{p}f(x)\sin\frac{n\pi}{p}xdx$$

Fourier Series

$f(x)$ is even

Fourier cosine series (或 cosine series)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x$$

$$a_0=\frac{2}{p}\int_0^p f(x)\,dx \qquad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$f(x)$ is odd

Fourier sine series (或 sine series)

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x$$

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

# Section 2.4 Laplace's Equation

## 2.4.1 Section 2.4 綱要

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

(使用method of separation of variables 來解)

「問題 1」 $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$ $\left.\frac{\partial u}{\partial x}\right|_{x=a}=0$ for $0<y<b,$

$u(x,0)=0$ $u(x,b)=f(x)$ for $0<x<a$

「問題 2」 $u(0,y)=0$ $u(a,y)=0$ for $0<y<b,$

$u(x,0)=0$ $u(x,b)=f(x)$ for $0<x<a$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.5.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

「問題 3」 $u(0,y)=F(y)$ $u(a,y)=G(y)$ for $0<x<a$

$u(x,0)=f(x)$ $u(x,b)=g(x)$ for $0<y<b,$

**※ 特別注意 “superposition principle”**

## 2.4.2 Solutions for Laplace's Equations (挑戰解解看)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \qquad \text{for } 0<y<b,$$

$$u(x,0)=0 \qquad u(x,b)=f(x) \qquad \text{for } 0<x<a$$

Step 1 假設解為 $u(x, y) = X(x)Y(y)$

Step 2 代入 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 得出

$$X''(x)Y(y)+X(x)Y''(y)=0 \qquad \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}$$

令 $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$

得出2 個 ODEs $X''(x)+\lambda X(x)=0 \qquad Y''(y)-\lambda Y(y)=0$

## Steps 3, 4, 5 的前處理

(1) 因為 $x$ 的 boundary condition 較簡單，所以先解 $X(x)$

(2) 分成 $\lambda=0,\ \lambda<0,\ \lambda>0$ 三個 cases

(3) 由於 $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$ for all $0<y<b$,

$$\left.\frac{\partial X(x)Y(y)}{\partial x}\right|_{x=0}=X'(0)Y(y)=0$$

$Y(y)$ 不可為 0 (否則 $u(x,y)=X(x)Y(y)=0$)

所以 $\boxed{X'(0)=0}$

同理，由 $\left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \longrightarrow \boxed{X'(a)=0}$

同理，由 $u(x,0)=0 \longrightarrow \boxed{Y(0)=0}$

$\boxed{X''(x)+\lambda X(x)=0}$ $\boxed{X'(0)=0}$ $\boxed{X'(a)=0}$

$\boxed{Y''(y)-\lambda Y(y)=0}$ $\boxed{Y(0)=0}$

**Case 1 of Steps 3, 4, 5: $\lambda=0$**

Step 3-1 $X''(x)=0$ solution: $X(x)=c_1+c_2x$

由 boundary conditions $X'(0)=0$ $X'(a)=0$

$c_2=0$

$X(x)=c_1$

Step 4-1 $Y''(y)=0$ $Y(0)=0$

solution: $Y(y)=c_3+c_4y$

根據 boundary condition $Y(0)=0$, $c_3=0$

$Y(y)=c_4y$

Step 5-1

$$u(x,y)=X(x)Y(y)=c_1c_4y=A_0y \qquad A_0=c_1c_4$$

**Case 2 of Steps 3, 4, 5: $\lambda<0$**

令 $\lambda=-\alpha^2$

Step 3-2 $X''(x)-\alpha^2X(x)=0 \qquad X'(0)=0 \qquad X'(a)=0$

solution: $X(x)=d_2e^{\alpha x}+d_3e^{-\alpha x}$

可改寫成 $X(x)=d_4\cosh(\alpha x)+d_5\sinh(\alpha x)$

由 boundary conditions $X'(0)=0 \quad X'(a)=0$

以及 $\frac{d}{dx}\cosh(\alpha x)=\alpha\sinh(\alpha x)$, $\frac{d}{dx}\sinh(\alpha x)=\alpha\cosh(\alpha x)$

$$\begin{cases} d_5\alpha=0 \\ d_4\alpha\sinh(\alpha a)+d_5\alpha\cosh(\alpha a)=0 \end{cases} \Longrightarrow \begin{cases} d_5=0 \\ d_4=0 \end{cases} \Longrightarrow X(x)=0$$

因此， case 2 得出 trivial solution $u(x, y) = X(x)Y(y) = 0$

$u(x,b) = f(x)$ 將無法滿足　　　$\lambda < 0$ 時無解

(不需再算 Steps 4-2, 5-2)

**Case 3 of Steps 3, 4, 5: $\lambda > 0$**

令 $\lambda = \alpha^2$

Step 3-3　$X''(x) + \alpha^2 X(x) = 0$　　$X'(0) = 0$　　$X'(a) = 0$

solution:　$X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

由 boundary conditions　$X'(0) = 0$　$X'(a) = 0$

$$\begin{cases} c_2\alpha = 0 \\ -c_1\alpha\sin(\alpha a) + c_2\alpha\cos(\alpha a) = 0 \end{cases} \Longrightarrow \begin{cases} c_1 = \text{any nonzero constant} \\ \alpha = \dfrac{n\pi}{a} \qquad n \text{ 是任意整數} \\ c_2 = 0 \end{cases}$$

再次注意：不可直接判斷成 $c_1 = 0$ and $c_2 = 0$

應該看看是否有適當的 $\alpha$, 讓第二個式子等於零

$$\boxed{X_n(x) = c_1 \cos\frac{n\pi}{a}x}$$

$n$ 是任意整數 $\lambda = \alpha^2 = \frac{n^2\pi^2}{a^2}$

Step 4-3 $Y''(y) - \frac{n^2\pi^2}{a^2}Y(y) = 0$ since $\lambda = \frac{n^2\pi^2}{a^2}$

$Y(0) = 0$

solution: $Y_n(y) = d_3 e^{\frac{n\pi}{a}y} + d_4 e^{-\frac{n\pi}{a}y}$

經常改寫為 $Y_n(y) = c_3 \cosh\left(\frac{n\pi}{a}y\right) + c_4 \sinh\left(\frac{n\pi}{a}y\right)$

根據 boundary condition $Y(0) = 0$ $c_3 = 0$

$$Y_n(y) = c_4 \sinh\left(\frac{n\pi}{a}y\right)$$

## Step 5-3

$$u(x,y)=X(x)Y(y)=c_1\cos\left(\frac{n\pi}{a}x\right)c_4\sinh\left(\frac{n\pi}{a}y\right)=A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$A_n=c_1c_4$

$n$ 是任意整數

## Step 6 把所有可能的解，全部加起來

$$u(x,y)=A_0y+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

Q: 為什麼 $n$ 是從 1 加到 $\infty$，而非由 $-\infty$ 加到 $\infty$？

討論：既然 $n$ 是任意整數，那為什麼 $n$ 是從1 加到 $\infty$，

而非由 $-\infty$ 加到 $\infty$？

因為 $\cos\left(\frac{n\pi}{a}x\right)=\cos\left(\frac{-n\pi}{a}x\right), \qquad \sinh\left(\frac{n\pi}{a}y\right)=-\sinh\left(\frac{-n\pi}{a}y\right),$

$\sinh(0)=0$

可證明 $\displaystyle\sum_{n=-\infty}^{\infty} B_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{a}x\right)\left[B_n\sinh\left(\frac{n\pi}{a}y\right)-B_{-n}\sinh\left(\frac{n\pi}{a}y\right)\right]$$

$$=\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$A_n=B_n-B_{-n}$$

Step 7 $$\boxed{u(x,y)=A_0 y+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)}$$

nonzero boundary condition: $u(x,b)=f(x)$

$$f(x)=A_0 b+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

**也就是說，$2A_0b$ 和 $A_n\sinh\left(\frac{n\pi}{a}b\right)$ ($n = 1, 2, \ldots., \infty$)**

**是 $f(x)$ 的 Fourier cosine series 的 coefficients**

$$2A_0b=\frac{2}{a}\int_0^a f(x)dx \qquad A_n\sinh\left(\frac{n\pi}{a}b\right)=\frac{2}{a}\int_0^a f(x)\cos\frac{n\pi}{a}xdx$$

$$\boxed{A_0=\frac{1}{ab}\int_0^a f(x)dx} \qquad \boxed{A_n=\frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\cos\frac{n\pi}{a}xdx}$$

## 2.4.3 Laplace's Equations with Dirichlet Problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$$u(0,y)=0 \qquad u(a,y)=0 \qquad 0<y<b,$$

$$u(x,0)=0 \qquad u(x,b)=f(x) \qquad 0<x<a,$$

用 method of separation of variables，經過計算得出

$$u(x,y)=\sum_{n=1}^{\infty} A_n \sinh\frac{n\pi}{a}y\sin\frac{n\pi}{a}x$$

$$A_n=\frac{2}{a\sinh\frac{n\pi}{a}b}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

可自行練習解解看

## 2.4.4 Superposition Principle

Dirichlet Problem 可分解成兩個子問題

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$$u(0,y)=F(y) \qquad u(a,y)=G(y) \qquad \text{for } 0<y<b,$$

$$u(x,0)=f(x) \qquad u(x,b)=g(x) \qquad \text{for } 0<x<a,$$

當四個邊界都不為零時，很難直接用 separation of variable 的方法解出來

子問題 1 $\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0 \quad 0<x<a, \quad 0<y<b,$

$u_1(0,y)=0 \quad u_1(a,y)=0 \quad \text{for } 0<y<b,$

$u_1(x,0)=f(x) \quad u_1(x,b)=g(x) \quad \text{for } 0<x<a,$

子問題 2 $\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0 \quad 0<x<a, \quad 0<y<b,$

$u_2(0,y)=F(y) \quad u_2(a,y)=G(y) \quad \text{for } 0<y<b,$

$u_2(x,0)=0 \quad u_2(x,b)=0 \quad \text{for } 0<x<a,$

假設 $u_1(x, y)$, $u_2(x, y)$ 分別是子問題 1, 子問題 2 的解

則 $\underline{\boldsymbol{u}(x, y)=\boldsymbol{u}_1(x, y)+\boldsymbol{u}_2(x, y)}$ 是原來問題的解

當 $u(x, y) = u_1(x, y) + u_2(x, y)$

$$u(0, y) = u_1(0, y) + u_2(0, y) = 0 + F(y) = F(y)$$

$$u(a, y) = u_1(a, y) + u_2(a, y) = 0 + G(y) = G(y)$$

$$u(x, 0) = u_1(x, 0) + u_2(x, 0) = f(x) + 0 = f(x)$$

$$u(x, b) = u_1(x, b) + u_2(x, b) = g(x) + 0 = g(x)$$

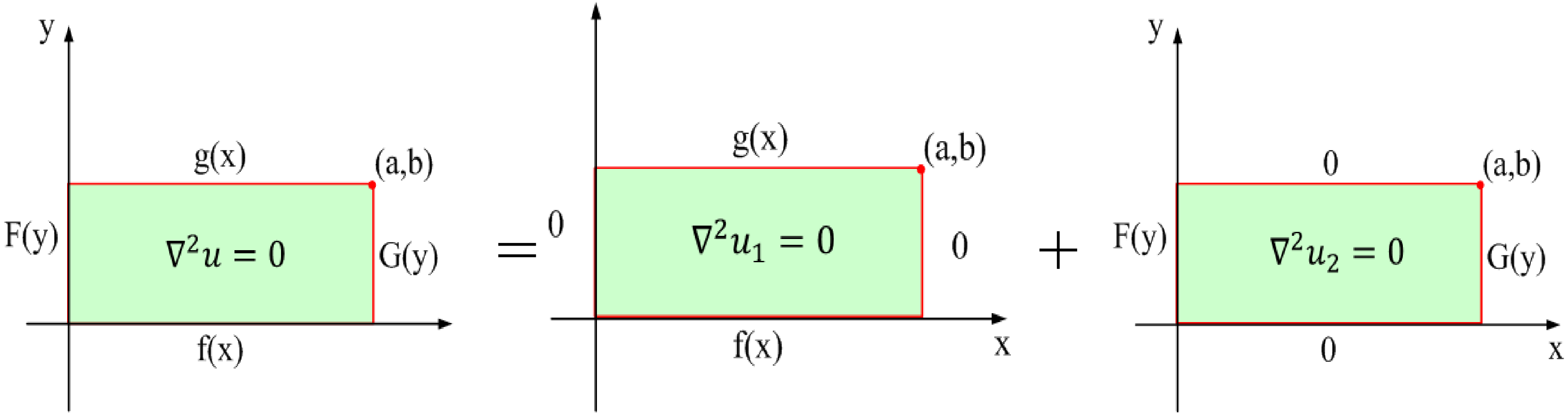


Fig. 2.5.1

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.5.

子問題 1 的解 $u_1(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{a}y+B_n\sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x$

$$A_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$B_n=\frac{1}{\sinh\left(\frac{n\pi}{a}b\right)}\left[\frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx-A_n\cosh\left(\frac{n\pi}{a}b\right)\right]$$

子問題 2 的解 $u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{b}x+B_n\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$

$$A_n=\frac{2}{b}\int_0^b F(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

$$B_n=\frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\left[\frac{2}{b}\int_0^b G(y)\sin\left(\frac{n\pi}{b}y\right)dy-A_n\cosh\left(\frac{n\pi}{b}a\right)\right]$$

原來問題的解 $u_1(x,y)+u_2(x,y)$

## 2.4.5 Sections 2.1~2.4需要注意的地方

(1) Method of separation of variables 解 PDE 的過程雖然長，但是把握住講義 pages 94-96 的 7 個 steps，就大致上沒問題。

(2) 注意，

若 boundary conditions 出現 $u(0, y) = 0,\ u(L, y) = 0,$

最後的解總是和 sine 有關 $X(x) = c_2 \sin\frac{n\pi}{L}x$ 週期為 $2L/n$

若 boundary conditions 出現 $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

最後的解總是和 cosine 或 constant 有關

$X(x) = c_1$ or $X_n(x) = c_1 \cos\frac{n\pi}{L}x$ 週期也為 $2L/n$

(3) 經驗足夠後，看到 $u(x, y)$ 的 boundary conditions

出現 $u(a, y) = 0 \longrightarrow$ 就知道 $X(a) = 0$ ，

看到 $u(x, b) = 0 \longrightarrow$ 就知道 $Y(b) = 0$ 。

看到 $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \longrightarrow$ 就知道 $X'(a) = 0$，

看到 $\left.\frac{\partial u}{\partial y}\right|_{y=b} = 0 \longrightarrow$ 就知道 $Y'(b) = 0$

(4) 要熟悉 $\cosh(x)$, $\sinh(x)$ 的性質

(5) Method of separation of variables 在計算上容易出錯的地方

(以 講義 pages 127-135 Laplace equations 為例)

(a) $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$

(b) Steps 3, 4, 5 要考慮所有 cases

(c) 不可直接由 $c_1 = 0$ 及 $c_1 \cos\alpha x + c_2 \sin\alpha x = 0$ 判斷 $c_1 = c_2 = 0$

因為 $\alpha$ 可以是 $\pi n/L$, 如講義 page 132 所述

(d) 在 Step 6，要將所有可能的解加起來，才是 $u(x, t)$ 的一般解

如講義 page 134 所述

# 2.5 Nonhomogeneous Boundary-Value Problems

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Nonhomogeneous: $G\neq 0$

Key ideas: Separate the original problem into two or more problems

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.6.

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Method 1 $u(x,y)=v(x,y)+\psi(x)$

Method 2 $u(x,y)=v(x,y)+\psi(y)$

Method 3 $u(x,y)=v(x,y)+\psi_1(x)+\psi_2(y)$

Method 4 $u(x,y)=v(x,y)+\psi(x,y)$

(Method 4 只教不考)

Extra Methods: Expansion by Fourier series, Fourier cosine series,
or Fourier sine series

## 2.6.1 Method 1

**Method 1** $u(x,y)=v(x,y)+\psi(x)$

Constraint: $G$ is independent of $y$

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)$$

$$+A\psi''(x)+D\psi(x)+F\psi(x)=G(x)$$

Problem A:

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)=0$$

Problem B:

$$A\psi''(x)+D\psi'(x)+F\psi(x)=G(x)$$

## [Example 1]

Solve $$k\frac{\partial^2 u}{\partial x^2}+r=\frac{\partial u}{\partial t},\quad 0<x<1,\quad t>0$$

$$u(0,t)=0,\quad u(1,t)=u_1,\quad t>0$$

$$u(x,0)=f(x),\quad 0<x<1$$

$r$ and $u_1$ are nonzero constants

(Solution): Since $G=r$, $u(0,t)=0$, $u(1,t)=u_1$

independent of $t$ constants

Method 1 can be applied.

$$u(x,t)=v(x,t)+\psi(x),$$

$$k\frac{\partial^2 v(x,t)}{\partial x^2}+k\frac{d^2\psi(x)}{dx^2}+r=\frac{\partial v(x,t)}{\partial t}+\frac{d\psi(x)}{dt}$$

$$k\frac{\partial^2 v(x,t)}{\partial x^2}+k\frac{d^2\psi(x)}{dx^2}+r=\frac{\partial v(x,t)}{\partial t}$$

Problem A $k\psi''(x)+r=0, \qquad \psi(0)=0, \quad \psi(1)=u_1$

Problem B
$$k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}, \quad 0<x<1, \quad t>0$$
$$v(0,t)=0, \quad v(1,t)=0, \quad t>0$$
$$v(x,0)=f(x)-\psi(x), \quad 0<x<1$$

For Problem A

$$k\psi''(x)+r=0, \qquad \psi(0)=0, \quad \psi(1)=u_1$$

$$\psi''(x)=-r/k \qquad \psi(x)=-\frac{r}{2k}x^2+c_1x+c_0$$

$$\psi(0)=0, \quad \psi(1)=u_1 \Longrightarrow c_0=0, \quad -\frac{r}{2k}+c_1=u_1$$

$$\psi(x)=-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_1\right)x$$

For Problem B, from Section 2-3

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin\left(n\pi x\right)$$

where $A_n = 2\int_0^1 \left[ f\left(x\right) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_1\right)x \right] \sin\left(n\pi x\right) dx$

Therefore,

$$u(x,t) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_1\right)x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin\left(n\pi x\right)$$

## 2.6.2 Methods 2 and 3

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

**Method 2** $u(x,y)=v(x,y)+\psi(y)$

Constraint: $G$ is independent of $x$

**Method 3** $u(x,y)=v(x,y)+\psi_1(x)+\psi_2(y)$

Constraint: $G = G_1(x) + G_2(y)$
$G_1(x)$ is independent of $y$
$G_2(y)$ is independent of $x$

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

$$u\left(x,y\right)=v\left(x,y\right)+\psi_1\left(x\right)+\psi_2\left(y\right)$$

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)$$

$$+A\psi_1''\left(x\right)+D\psi_1'\left(x\right)+F\psi_1\left(x\right)+C\psi_2''\left(x\right)+E\psi_2'\left(x\right)+F\psi_2\left(x\right)=G_1\left(x\right)+G_2\left(y\right)$$

Problem A:

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)=0$$

Problem B:

$$A\psi_1''\left(x\right)+D\psi_1\left(x\right)+F\psi_1\left(x\right)=G_1\left(x\right)$$

Problem C:

$$A\psi_2''\left(x\right)+D\psi_2\left(x\right)+F\psi_2\left(x\right)=G_2\left(x\right)$$

## 2.6.3 Method 4 (只教不考)

**Method 4** $u(x,t) = v(x,t) + \psi(x,t)$

Constraint of Method 4: Not applicable for Laplace's equation.

Method 1 can be applied to the wave equation and Laplace's equation, but Method 4 cannot.

Example:

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$$

$$u(0,t) = u_0(t), \quad u(L,t) = u_1(t), \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < L,$$

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\quad 0<x<L,\quad t>0$$

$$u(0,t)=u_0(t),\quad u(L,t)=u_1(t),\quad t>0$$

$$u(x,0)=f(x),\quad 0<x<L,$$

Set $u(x,t)=v(x,t)+\psi(x,t)$

Since $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 \psi}{\partial x^2}$ $\qquad$ $\dfrac{\partial u}{\partial t}=\dfrac{\partial v}{\partial t}+\dfrac{\partial \psi}{\partial t}$

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t}\Longrightarrow k\frac{\partial^2 v}{\partial x^2}+\underline{k\frac{\partial^2 \psi}{\partial x^2}}+F(x,t)=\frac{\partial v}{\partial t}+\frac{\partial \psi}{\partial t}$$

$$u(0,t)=u_0(t)\Longrightarrow v(0,t)+\underline{\psi(0,t)}=\underline{u_0(t)}$$

$$u(L,t)=u_1(t)\Longrightarrow v(L,t)+\underline{\psi(L,t)}=\underline{u_1(t)}$$

$$u(x,0)=f(x)\Longrightarrow v(x,0)+\psi(x,0)=f(x)$$

Therefore, after setting $u(x,t)=v(x,t)+\psi(x,t)$ , we separate

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\quad 0<x<L,\quad t>0$$

$$u(0,t)=u_0(t),\quad u(L,t)=u_1(t),\quad t>0$$

$$u(x,0)=f(x),\quad 0<x<L,$$

into two sub-problems:

Problem A: $k\dfrac{\partial^2\psi}{\partial x^2}=0,\quad \psi(0,t)=u_0(t),\qquad \psi(L,t)=u_1(t)$

Problem B: $k\dfrac{\partial^2 v}{\partial x^2}+G(x,t)=\dfrac{\partial v}{\partial t},\quad 0<x<L,\quad t>0$

$$v(0,t)=0,\quad v(L,t)=0,\quad t>0$$

$$v(x,0)=f(x)-\psi(x,0),\quad 0<x<L$$

where $G(x,t)=F(x,t)-\dfrac{\partial\psi}{\partial t}$

Problem A: $k\frac{\partial^2\psi}{\partial x^2}=0, \quad \psi(0,t)=u_0(t), \quad \psi(L,t)=u_1(t)$

$$\psi(x,t)=c_1(t)x+c_0(t)$$

$$\psi(0,t)=u_0(t) \Longrightarrow c_0(t)=u_0(t)$$

$$\psi(L,t)=u_1(t) \Longrightarrow c_1(t)L+u_0(t)=u_1(t)$$

$$\psi(x,t)=u_0(t)+\frac{x}{L}\left(u_1(t)-u_0(t)\right)$$

To Solve Problem B:

$$k\frac{\partial^2 v}{\partial x^2}+G(x,t)=\frac{\partial v}{\partial t},\quad 0<x<L,\quad t>0$$

$$v(0,t)=0,\quad v(L,t)=0,\quad t>0$$

$$v(x,0)=f(x)-\psi(x,0),\quad 0<x<L$$

where $G(x,t)=F(x,t)-\dfrac{\partial \psi}{\partial t}$

An assumption can be applied
(from the associated homogeneous PDE).

$$v\left(x,t\right)=\sum_{n=1}^{\infty}v_n\left(t\right)\sin\frac{n\pi}{L}x \qquad G\left(x,t\right)=\sum_{n=1}^{\infty}G_n\left(t\right)\sin\frac{n\pi}{L}x$$

Try to solve $v_n(t)$ and $G_n(t)$.

# Summary for the Process of Method 4

(Step 1) Use $u(x,t)=v(x,t)+\psi(x,t)$ to separate the original problem into two sub-problems.

(Step 2) Solve Problem A

(Step 3) Use the associated homogeneous PDE to express the solution of Problem B by Fourier sine series

(Step 4) Expand $G_n(t)$ to solve $v_n(t)$

(Step 5) Use $u(x, 0)$ to solve the unknowns of $v_n(t)$

(Step 6) Add the solutions of Problems A and B and obtain $u(x, t)$.

**[Example 2]**

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$u(0,t) = \cos t, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad 0 < x < 1.$$

(Solution):

(Step 1) $u(x,t) = v(x,t) + \psi(x,t)$

Problem A: $\dfrac{\partial^2 \psi}{\partial x^2} = 0, \qquad \psi(0,t) = \cos t, \quad \psi(1,t) = 0$

Problem B:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial \psi}{\partial t} = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = -\psi(x,0), \quad 0 < x < 1$$

(Step 2)

Problem A: $\frac{\partial^2\psi}{\partial x^2}=0, \qquad \psi(0,t)=\cos t, \quad \psi(1,t)=0$

$$\psi(x,t)=c_1(t)x+c_0(t)$$

Solution: $\psi(x,t)=[0-\cos t]x+\cos t=(1-x)\cos t$

Problem B: $\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t}, \quad 0<x<1, \quad t>0$

$$v(0,t)=0, \quad v(1,t)=0, \quad t>0$$

$$v(x,0)=x-1, \quad 0<x<1$$

Problem B: $$\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$$
$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$
$$v(x,0)=x-1,\quad 0<x<1$$

(Step 3) From the associated homogeneous PDE

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$$
$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$
$$v(x,0)=x-1,\quad 0<x<1$$

$$v(x,t)=X(x)T(t)$$

$$X''(x)T(t)=X(x)T'(t) \qquad \frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=-\lambda$$

$$X''(x)+\lambda X(x)=0 \qquad X(0)=0 \qquad X(1)=0$$

$$T'(t)+\lambda T(t)=0$$

$$X''(x) + \lambda X(x) = 0 \qquad X(0) = 0 \qquad X(1) = 0$$

After checking the three cases, the non-trivial solution exists only when

$$\lambda = n^2\pi^2 > 0$$

In this case,

$$X''(x) + n^2\pi^2 X(x) = 0 \qquad X(x) = c\sin n\pi x$$

Therefore, the solution of Problem B should has the following form:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x$$

to be solved

$$\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$$

$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$

$$v(x,0)=x-1,\quad 0<x<1$$

$$v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin n\pi x$$

to be solved

(Step 4) First, express the non-homogeneous term $(1-x)\sin t$ as

$$(1-x)\sin t=\sum_{n=1}^{\infty} G_n(t)\sin n\pi x$$

From the Fourier sine series (附錄四)

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

$$G_n(t)=\frac{2}{1}\int_0^1(1-x)\sin t\sin\frac{n\pi}{1}x\,dx=\frac{2}{n\pi}\sin t$$

$$(1-x)\sin t=\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin t\sin n\pi x$$

$$\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t} \qquad v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin n\pi x$$

Since $(1-x)\sin t=\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin t\sin n\pi x$

$$\frac{\partial^2}{\partial x^2}v(x,t)=\sum_{n=1}^{\infty} v_n(t)\left(-n^2\pi^2\right)\sin n\pi x \qquad \frac{\partial}{\partial t}v(x,t)=\sum_{n=1}^{\infty} v_n'(t)\sin n\pi x$$

we have

$$\sum_{n=1}^{\infty}\left[v_n(t)\left(-n^2\pi^2\right)+\frac{2}{n\pi}\sin t\right]\sin n\pi x=\sum_{n=1}^{\infty} v_n'(t)\sin n\pi x$$

$$v_n'(t)+n^2\pi^2 v_n(t)=\frac{2\sin t}{n\pi}$$

$$v_n(t)=2\frac{n^2\pi^2\sin t-\cos t}{n\pi\left(n^4\pi^4+1\right)}+C_n e^{-n^2\pi^2 t}$$

$$v(x,t)=\sum_{n=1}^{\infty}\left(2\frac{n^2\pi^2\sin t-\cos t}{n\pi\left(n^4\pi^4+1\right)}+C_n e^{-n^2\pi^2 t}\right)\sin n\pi x$$

$$\frac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = x-1, \quad 0 < x < 1$$

$$v(x,t) = \sum_{n=1}^{\infty} \left( 2\frac{n^2\pi^2 \sin t - \cos t}{n\pi\left(n^4\pi^4+1\right)} + C_n e^{-n^2\pi^2 t} \right) \sin n\pi x$$

(Step 5) To determine $C_n$, we can apply $v(x, 0) = x–1$

$$x-1 = \sum_{n=1}^{\infty} \left( \frac{-2}{n\pi\left(n^4\pi^4+1\right)} + C_n \right) \sin n\pi x$$

From the Fourier sine series

$$\frac{-2}{n\pi\left(n^4\pi^4+1\right)} + C_n = 2\int_0^1 (x-1)\sin n\pi x dx = \frac{-2}{n\pi}$$

$$C_n = \frac{2}{n\pi\left(n^4\pi^4+1\right)} - \frac{2}{n\pi}$$

$$v(x,t)=\sum_{n=1}^{\infty}\left(2\frac{n^2\pi^2\sin t-\cos t}{n\pi\left(n^4\pi^4+1\right)}+C_n e^{-n^2\pi^2 t}\right)\sin n\pi x$$

$$C_n=\frac{2}{n\pi\left(n^4\pi^4+1\right)}-\frac{2}{n\pi}$$

$$v(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2\sin t-\cos t+e^{-n^2\pi^2 t}}{n\left(n^4\pi^4+1\right)}-\frac{e^{-n^2\pi^2 t}}{n}\right)\sin n\pi x$$

(Step 6) $u(x,t)=v(x,t)+\psi(x,t)$

$$\boxed{u(x,t)=(1-x)\cos t+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2\sin t-\cos t+e^{-n^2\pi^2 t}}{n\left(n^4\pi^4+1\right)}-\frac{e^{-n^2\pi^2 t}}{n}\right)\sin n\pi x}$$

# 2.6 Higher-Dimensional Problems

Modifying the method in Section 2-1 just a little.

**Two-dimensional heat equation**

$$h\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}.$$

**Two-dimensional wave equation**

$$a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}.$$

$u(x,y,t)=X(x)Y(y)T(t)$

$$\frac{\partial^2 u}{\partial x^2}=X''YT,\quad \frac{\partial^2 u}{\partial y^2}=XY''T,\quad \text{and}\quad \frac{\partial u}{\partial t}=XYT'.$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.8.

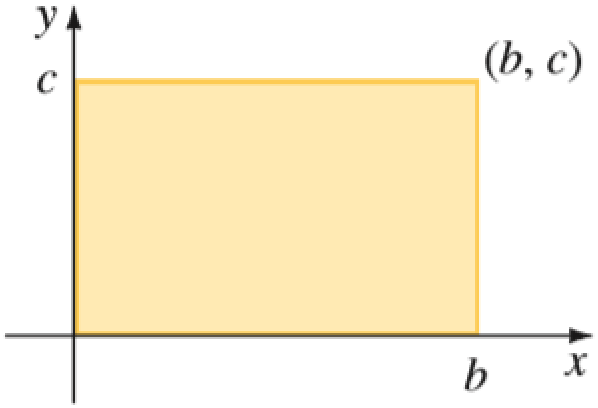
## **[Example 1]** Temperatures in a Plate

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\quad 0<x<b,\quad 0<y<c,\quad t>0$$

$$u(0,y,t)=0,\quad u(b,y,t)=0,\quad 0<y<c,\quad t>0$$

$$u(x,0,t)=0,\quad u(x,c,t)=0,\quad 0<x<b,\quad t>0$$

$$u(x,y,0)=f(x,y),\quad 0<x<b,\quad 0<y<c.$$



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.8.

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t} \qquad u(x,y,t)=X(x)Y(y)T(t),$$

$$k(X''YT+XY''T)=XYT'$$

Divided by *XYT*

$$k\left(\frac{X''}{X}+\frac{Y''}{Y}\right)=\frac{T'}{T} \qquad \frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}$$

Set

$$\begin{tikzcd}
 & \frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}=-\lambda \arrow[dl] \arrow[dr] & & \\
X''+\lambda X=0 & & \frac{Y''}{Y}=\frac{T'}{kT}+\lambda. \arrow[dl] \arrow[dr] & \\
 & \frac{Y''}{Y}=-\mu & & \frac{T'}{kT}+\lambda=-\mu
\end{tikzcd}$$

$$Y''+\mu Y=0 \qquad T'+k(\lambda+\mu)T=0.$$

$$u(0,y,t)=0,\quad u(b,y,t)=0, \Longrightarrow \quad X(0)=0,\quad X(b)=0,$$

$$u(x,0,t)=0,\quad u(x,c,t)=0 \Longrightarrow \quad Y(0)=0,\quad Y(c)=0$$

$$X''+\lambda X=0,\quad X(0)=0,\quad X(b)=0$$

$$Y''+\mu Y=0,\quad Y(0)=0,\quad Y(c)=0.$$

$$T'+k(\lambda+\mu)T=0.$$

There are 3 cases for $X$: $\lambda=0$, $\lambda<0$, and $\lambda>0$.

There is non-trivial solution for $X$ only when $\lambda_m=\dfrac{m^2\pi^2}{b^2}>0$

In this case, $X(x)=c_2\sin\dfrac{m\pi}{b}x$

There are 3 cases for $Y$: $\mu=0$, $\mu<0$, and $\mu>0$.

There is non-trivial solution for $Y$ only when $\mu_n=\dfrac{n^2\pi^2}{c^2}>0$

In this case, $Y(y)=c_4\sin\dfrac{n\pi}{c}y$

$$\lambda_m = \frac{m^2\pi^2}{b^2} \qquad \mu_n = \frac{n^2\pi^2}{c^2}.$$

$$T' + k(\lambda + \mu)T = 0 \quad \Longrightarrow \quad T' + k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{c^2}\right)T = 0$$

$$\Longrightarrow \quad T(t) = c_5 e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t}.$$

$$u(x,y,t) = X(x)Y(y)T(t),$$

$$u_{mn}(x,y,t) = A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y,$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y.$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y.$$

$$u(x,y,0)=f(x,y)\qquad 0<x<b,\quad 0<y<c.$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y=f(x,y)$$

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi}{c}y\right)\sin\frac{m\pi}{b}x=f(x,y)$$

From the Fourier sine series along the *x*-axis

$$g(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x\qquad b_n=\frac{2}{p}\int_0^p g(x)\sin\frac{n\pi}{p}x\,dx$$

$$\left(\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi}{c}y\right)=\frac{2}{b}\int_0^b f(x,y)\sin\frac{m\pi}{b}x\,dx$$

From the Fourier sine series along the *y*-axis

$$A_{mn}=\frac{2}{c}\int_0^c\frac{2}{b}\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}x\right)dx\sin\left(\frac{n\pi}{c}y\right)dy$$

$$\boxed{u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y.}$$

where

$$\boxed{A_{mn}=\frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}x\right)dx\sin\left(\frac{n\pi}{c}y\right)dy}$$

# Double Sine Series (Two Dimensional Sine Series)

$$f\left(x,y\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{m,n}\sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

where

$$B_{m,n}=\frac{4}{bc}\int_{0}^{c}\int_{0}^{b}f\left(x,y\right)\sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{c}y\right)dxdy$$