1.(a)

$$\frac{y''(x)}{(y'(x))^2}=\left(1+x^{-2}\right)\exp\left(-x^2/2\right)$$

Let $u=y'$

$$\frac{u'}{u^2}=\left(1+x^{-2}\right)\exp\left(-x^2/2\right),\quad \frac{1}{u}=\frac{e^{-x^2/2}}{x}+C \text{ , since } y'(1)=e^{1/2},\quad C=0.$$

$$y'=xe^{x^2/2}$$

$$y=e^{x^2/2}+C$$

1.(b)

$$\left(y'(x)\right)^2 y''(x)=-\sin^2 x\cos x$$

Let $u=y'$

$$u^2u'=-\sin^2 x\cos x$$

$$u^2du=-\sin^2 x\cos xdx$$

$$u^3=-\sin^3 x+C,\ C=0$$

$$y'=-\sin x$$

$$y=\cos x+C$$

1.(c)

$$y''(x)=2y(x)y'(x)$$

Let $u=y'$, $\dfrac{d^2}{dx^2}y=\dfrac{d}{dx}u=\dfrac{dy}{dx}\dfrac{d}{dy}u=u\dfrac{d}{dy}u$

$$u\frac{d}{dy}u=2yu$$

$$\frac{d}{dy}u=2y$$

$$u=y^2+C,\ C=0$$

$$y'=y^2$$

$$-\frac{1}{y}=x+C,\ C=-1$$

$$y=\frac{1}{1-x}$$

2.(a)

$$\sin y\frac{\partial u(x,y)}{\partial x}=\cos x\frac{\partial u(x,y)}{\partial y}$$

Let $u(x,y)=X(x)Y(y)$,

$$X'(x)Y(y)\sin y = X(x)Y'(y)\cos x$$

$$\frac{X'(x)}{X(x)\cos x} = \frac{Y'(y)}{Y(y)\sin y} = -\lambda$$

$$\begin{cases} X'(x) + \lambda X(x)\cos x = 0 \\ Y'(y) + \lambda Y(y)\sin y = 0 \end{cases}$$

$$\begin{cases} X = c_1 \exp(-\lambda \sin x) \\ Y = c_2 \exp(\lambda \cos y) \end{cases}$$

$$u = XY = c_1 c_2 \exp(-\lambda \sin x + \lambda \cos y)$$

$$u(x,y) = \sum_{\lambda} c_\lambda \exp\big(\lambda(\cos y - \sin x)\big)$$

2.(b)

$$4\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},$$

Let $u(x,t) = X(x)T(t)$,

$$4X''T = XT', \quad \frac{X''}{X} = \frac{T'}{4T} = -\lambda,$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ T'(t) + 4\lambda T(t) = 0 \end{cases}$$

Case 1 $\lambda = 0$

$$\begin{cases} X''(x) = 0 \\ T'(t) = 0 \end{cases}$$

$$\begin{cases} X = c_1 + c_2 x \\ Y = c_3 \end{cases}$$

since $X'(0) = X'(3) = 0$, $u = XY = C_1$.

Case 2 $\lambda < 0$

$$\lambda = -\alpha^2$$

$$\begin{cases} X''(x) - \alpha^2 X(x) = 0 \\ T'(t) - 4\alpha^2 T(t) = 0 \end{cases}$$

$$\begin{cases} X = c_1 \exp(\alpha x) + c_2 \exp(-\alpha x) \\ Y = c_3 \exp(4\alpha^2 t) \end{cases}$$

since $X'(0) = X'(3) = 0$, $c_1 = c_2 = 0, u = 0$.

Case 3 $\lambda > 0$

$\lambda = \alpha^2$

$$\begin{cases} X''(x) + \alpha^2 X(x) = 0 \\ T'(t) + 4\alpha^2 T(t) = 0 \end{cases}$$

$$\begin{cases} X = c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\ \quad T = c_3 \exp(-4\alpha^2 t) \end{cases}$$

since $X'(0) = X'(3) = 0$, for $X = c_1 \cos\left(\frac{n\pi}{3} x\right)$, $\lambda = \frac{n^2\pi^2}{9}$.

$T = c_3 \exp\left(-4\frac{n^2\pi^2}{9} t\right)$ $,u = XT = \sum_{n=0}^{\infty} A_n \exp\left(-4\frac{n^2\pi^2}{9} t\right)\cos\left(\frac{n\pi}{3} x\right)$

Since $u(x,0) = \sin^2(\pi x)$ $,\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{3} x\right) = \sin^2(\pi x) = \frac{1-\cos(2\pi x)}{2}$

$A_0 = \frac{1}{2}$, $A_6 = -\frac{1}{2}$, for $n \neq 0, 6$ $, A_n = 0$.

$\Rightarrow u = \frac{1}{2} - \frac{1}{2}\exp(-16\pi^2 t)\cos(2\pi x)$

2.(c)

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0,$$

Let $u(x,y) = X(x)Y(y)$,

$X''Y + XY'' = 0,\quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases}$$

Case 1 $\lambda = 0$

$$\begin{cases} X''(x) = 0 \\ Y''(y) = 0 \end{cases}$$

$$\begin{cases} X = c_1 + c_2 x \\ Y = c_3 + c_4 y \end{cases}$$

since $X(0) = X(2) = 0$, $u = XY = 0$.

Case 2 $\lambda < 0$

$\lambda = -\alpha^2$

$$\begin{cases} X''(x)-\alpha^2 X(x)=0 \\ Y''(y)+\alpha^2 Y(y)=0 \end{cases}$$

$$\begin{cases} X=c_1\cosh(\alpha x)+c_2\sinh(\alpha x) \\ Y=c_3\cos(\alpha y)+c_4\sin(\alpha y) \end{cases}$$

since $X(0)=X(2)=0$, $c_1=c_2=0$, $u=XY=0$.

Case 3 $\lambda>0$

$\lambda=\alpha^2$

$$\begin{cases} X''(x)+\alpha^2 X(x)=0 \\ Y''(y)-\alpha^2 Y(y)=0 \end{cases}$$

$$\begin{cases} X=c_1\cos(\alpha x)+c_2\sin(\alpha x) \\ Y=c_3\cosh(\alpha y)+c_4\sinh(\alpha y) \end{cases}$$

since $X(0)=X(2)=0$, for $X=c_2\sin\left(\frac{n\pi}{2}x\right)$, $\lambda=\frac{n^2\pi^2}{4}$.

Since $Y(0)=0$, $Y=c_4\sinh\left(\frac{n\pi}{2}y\right)$

$u=XT=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{2}x\right)\sinh\left(\frac{n\pi}{2}y\right)$, since $u(x,2)=\begin{cases} x & 0<x\le 1 \\ 2-x & 1<x<2 \end{cases}$,

$$\begin{aligned} A_n &= \frac{1}{\sinh n\pi}\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx \\ &= \frac{1}{\sinh n\pi}\left[\int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx+\int_1^2(2-x)\sin\left(\frac{n\pi}{2}x\right)dx\right] \\ &= \frac{1}{\sinh n\pi}\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

3.

$$x\frac{\partial u(x,y)}{\partial x}+y\frac{\partial u(x,y)}{\partial y}+x^2+y^2=0$$

Let $u(x,y)=v(x,y)+\psi_1(x)+\psi_2(y)$

$$x\left[\frac{\partial v(x,y)}{\partial x}+\psi_1'(x)\right]+y\left[\frac{\partial v(x,y)}{\partial x}+\psi_2'(y)\right]+x^2+y^2=0$$

$$\begin{cases} x\psi_1'(x) + x^2 = 0 \\ y\psi_2'(y) + y^2 = 0 \end{cases}$$

$$\begin{cases} \psi_1(x) = -\dfrac{x^2}{2} + c_1 \\ \psi_2(y) = -\dfrac{y^2}{2} + c_2 \end{cases}$$

$$v(x,y) = X(x)Y(y), \qquad xX'Y + yXY' = 0, \qquad \frac{xX'}{X} = -\frac{yY'}{Y} = -\lambda$$

$$\begin{cases} xX'(x) + \lambda X(x) = 0 \\ yY'(y) - \lambda Y(y) = 0 \end{cases}$$

$$\begin{cases} X = c_3 x^{-\lambda} \\ Y = c_4 y^{\lambda} \end{cases}$$

$$v = XY = C_2 x^{-\lambda} y^{\lambda}.$$

$$u(x,y) = \sum_{\lambda} c_{\lambda} x^{-\lambda} y^{\lambda} - \frac{x^2 + y^2}{2}$$