1.(a)

$$x\frac{\partial u(x,y,z)}{\partial x}+(y+1)\frac{\partial u(x,y,z)}{\partial y}+(z+2)\frac{\partial u(x,y,z)}{\partial z}=0$$

Sol:

Let $u(x,y,z)=X(x)Y(y)Z(z)$, $x\frac{X'(x)}{X(x)}+(y+1)\frac{Y'(y)}{Y(y)}+(z+2)\frac{Z'(z)}{Z(z)}=0$

$x\frac{X'(x)}{X(x)}=-\lambda$, $xX'(x)+\lambda X(x)=0$, $X(x)=c_1x^{-\lambda}$,

$(y+1)\frac{Y'(y)}{Y(y)}=-\mu, Y(y)=c_2(y+1)^{-\mu}$

$(z+2)Z'(z)-(\mu+\lambda)Z(z)=0$, $Z(z)=c_3(z+2)^{\mu+\lambda}$

$u(x,y,z)=\sum_{\lambda}\sum_{\mu}c_{\lambda,\mu}x^{-\lambda}(y+1)^{-\mu}(z+2)^{\mu+\lambda}$

1.(b)

Sol:

Let $u(x,y,z)=X(x)Y(y)T(t)$,

$\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}=\frac{\partial^2 u(x,y,t)}{\partial t^2}$

$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=\frac{T''(t)}{T(t)}$

Since $X(0)=X(1)=Y(0)=Y(1)=0$,

$\frac{X''(x)}{X(x)}=-\lambda=-\alpha^2, \alpha=n\pi, X(x)=c_1\sin(n\pi x)$

$\frac{Y''(y)}{Y(y)}=-\mu=-\beta^2, \beta=m\pi, Y(y)=c_2\sin(m\pi y)$

$$T''(t)-(\mu+\lambda)T(t)=0$$

Since $\partial_t u(x,y,t)\big|_{t=0}=0$, $T(t)=c_3\cos\left(\sqrt{\alpha^2+\beta^2 t}\right)$.

$$u(x,y,t)=\sum_m\sum_n A_{m,n}\cos\left(\sqrt{n^2+m^2}\pi t\right)\sin(n\pi x)\sin(m\pi y)$$

If $t=0$,

$$\sin(\pi x)\sin(\pi y)=\sum_m\sum_n A_{m,n}\sin(n\pi x)\sin(m\pi y)$$

$$A_{m,n}=\delta(n-1)\delta(m-1)$$

$$u(x,y,t)=\cos\left(\sqrt{2}\pi t\right)\sin(\pi x)\sin(\pi y)$$

1.(c)

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

Sol:

Let $u(r,\theta)=R(r)\Theta(\theta)$. Consider that $u(r,0)=u(r,\pi/2)=0$, we have

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda=\alpha^2 \quad , \quad \Theta''+\lambda\Theta=0 \quad , \quad \Theta=c_1\sin\alpha\theta \quad , \quad \alpha=2n \quad ,$$

$\Theta=c_1\sin 2n\theta$ $\quad r^2R''+rR'-\lambda R=0$, $\quad R=c_3 r^\alpha$.

$$u(r,\theta)=\sum_{n=1}^{\infty}A_n r^{2n}\sin 2n\theta,$$

Consider that $u(1,\theta)=1$, we have $1=\sum_{n=1}^{\infty}A_n\sin 2n\theta$,

$$A_n=\frac{2}{\pi/2}\int_0^{\pi/2}\sin 2n\theta d\theta=\frac{2}{\pi}\frac{1-(-1)^n}{n}$$

$$u(r,\theta)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}r^{2n}\sin 2n\theta$$

1.(d)

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=1$$

Sol:

Let $u(r,z)=v(r,z)+\psi(z)$, $\psi''(z)=1$, $\psi_c=c_1+c_2 z$, $\psi_p=Az^2, A=\frac{1}{2}$.

$\psi=\psi_c+\psi_p=c_1+c_2 z+\frac{z^2}{2}$, $v(r,z)=R(r)Z(z)$.

Consider that $u(r,0)=u(r,5)=0,\quad 0<z<5,\quad 0<r<1,$ we have

$$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda=\alpha^2.$$

$rR''+R'+\lambda rR=0$, $Z''-\lambda Z=0$, $\alpha=\frac{n\pi}{5}$, $Z(z)=c_3\sin\left(\frac{n\pi z}{5}\right)$.

$rR''+R'-\left(\frac{n\pi}{5}\right)^2 rR=0$, $R(r)=c_4 I_0\left(\frac{n\pi}{5}r\right)$,

$$v(r,z)=\sum_{n=1}^{\infty}A_n I_0\left(\frac{n\pi}{5}r\right)\sin\left(\frac{n\pi z}{5}\right),$$

$$u(r,z)=\sum_{n=1}^{\infty}A_n I_0\left(\frac{n\pi}{5}r\right)\sin\left(\frac{n\pi z}{5}\right)+c_1+c_2 z+\frac{z^2}{2}$$

Consider that $u(r,0)=u(r,5)=0$, we have $c_1=0, c_2=-\frac{5}{2}$.

$$u(r,z)=\sum_{n=1}^{\infty}A_n I_0\left(\frac{n\pi}{5}r\right)\sin\left(\frac{n\pi z}{5}\right)-\frac{5}{2}z+\frac{z^2}{2}$$

Consider that $u(0,z)=f(z)$, we have

$$f(z)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi z}{5}\right)-\frac{5}{2}z+\frac{z^2}{2}$$

$$A_n=\frac{2}{5}\int_0^5\left(f(z)+\frac{5}{2}z-\frac{z^2}{2}\right)\sin\left(\frac{n\pi z}{5}\right)dz$$

2.

$$\frac{\partial^2 u(x,y)}{\partial x^2}+4\frac{\partial^2 u(x,y)}{\partial y^2}=0$$

Sol:

$$\frac{\partial^2 U(x,s)}{\partial x^2}+4\left[s^2U(x,s)-e^{-x}\right]=0$$

$$U''+4s^2U=4e^{-x}, U_c=c_1\cos 2sx+c_2\sin 2sx, U_p=Ae^{-x}, A+4s^2A=4,$$

$$U_p=\frac{4}{1+4s^2}e^{-x} \quad ,\mathcal{L}\{u(0,s)\}=\frac{1}{s^2+1/4}=U(0,s),\ \ c_1=0$$

$$U=U_c+U_p=c_1\cos 2sx+c_2\sin 2sx+\frac{4}{1+4s^2}e^{-x}$$

$$U=c_2\sin 2sx+\frac{4}{1+4s^2}e^{-x},-\left.\frac{\partial U}{\partial x}\right|_{x=0}=-\left[c_2 2s\cos 2sx-\frac{4}{1+4s^2}e^{-x}\right]_{x=0}=\frac{1}{s^2+1/4}$$

$$c_2=0 \quad , \quad U=\frac{4}{1+4s^2}e^{-x}$$

$$u=2\sin\left(\frac{y}{2}\right)e^{-x}$$

3.

$$9\frac{\partial u(x,t)}{\partial x}=\frac{\partial u(x,t)}{\partial t}$$

Sol:

$$9(i2\pi f)U(f,t)=\frac{\partial U(f,t)}{\partial t}$$

$$U(f,t)=c_1e^{i18\pi ft}$$

$$U(f,0)=c_1=\frac{1}{2}\left[\delta\left(f-\frac{1}{2\pi}\right)+\delta\left(f-\frac{1}{2\pi}\right)\right]$$

$$U(f,t)=\frac{1}{2}\left[\delta\left(f-\frac{1}{2\pi}\right)+\delta\left(f+\frac{1}{2\pi}\right)\right]e^{i18\pi ft}=\frac{1}{2}e^{i9t}\delta\left(f-\frac{1}{2\pi}\right)+\frac{1}{2}e^{-i9t}\delta\left(f-\frac{1}{2\pi}\right)$$

$$u(x,t)=\frac{1}{2}e^{i9t}e^{ix}+\frac{1}{2}e^{-i9t}e^{-ix}=\cos(x+9t)$$

4.(a)

$$v_1(x) = \frac{1}{\sqrt{\int_0^4 1 \cdot dx}} = \frac{1}{2}$$

$$u_2(x) = x - \frac{1}{2}\int_0^4 \frac{1}{2} \cdot x dx = x - 2$$

$$v_2(x) = \frac{x-2}{\sqrt{\int_0^4 (x-2)^2 dx}} = \frac{x-2}{\sqrt{\frac{16}{3}}} = \frac{\sqrt{3}}{4}(x-2)$$

4.(b)

$$c_1 = \int_0^4 g(x) \cdot v_1(x) dx = 1 \quad , c_2 = \int_0^4 g(x) \cdot v_2(x) dx = -\frac{2\sqrt{3}}{3}$$

$$q(x) = c_1 v_1(x) + c_2 v_2(x) = -\frac{1}{2}x + \frac{3}{2}$$