Selected Topics in Engineering Mathematics HW3 solution

(1).

(a). Let $f = \frac{\omega}{2\pi}$ , $G_1(\omega) = G\left(f = \frac{\omega}{2\pi}\right)$ $= \int_{-\infty}^{\infty} g(x)e^{-j2\pi\cdot\left(\frac{\omega}{2\pi}\right)x}dx = \int_{-\infty}^{\infty} g(x)e^{-j\omega x}dx$

(b). $G_2(\omega) = \sqrt{\frac{1}{2\pi}}G_1(\omega)$ $= \sqrt{\frac{1}{2\pi}}G\left(f = \frac{\omega}{2\pi}\right)$

(c). $G_3(\alpha) = G\left(f = \frac{-\alpha}{2\pi}\right)$ $= \int_{-\infty}^{\infty} g(x)e^{-j2\pi\cdot\left(\frac{-\alpha}{2\pi}\right)x}dx = \int_{-\infty}^{\infty} g(x)e^{j\alpha x}dx$

(2).

(a). $g(x) = \begin{cases} -2e^{-2(x+1)}, & x > -1 \\ 0, & \text{otherwise} \end{cases}$

$$G(f) = \mathcal{FT}\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx = \int_{-1}^{\infty} -2e^{-2(x+1)} \cdot e^{-j2\pi fx}dx$$

$$= e^{-2} \cdot \int_{-1}^{\infty} e^{-(2+j2\pi f)x}dx = e^{-2} \cdot \frac{e^{(2+j2\pi f)}}{2+j2\pi f} = \frac{e^{j2\pi f}}{2+j2\pi f}$$

<note> : $e^{-(2+j2\pi f)\cdot\infty} = 0$

(b). Rectangular function : $r_a(x) = \begin{cases} 1, & -a \le x \le a \\ 0, & \text{otherwise} \end{cases}$

(Property 1). The Fourier transform of the rectangular function is sinc function as follows:

$$R_a(f) = \mathcal{FT}\{r_a(x)\} = \int_{-\infty}^{\infty} r_a(x)e^{-j2\pi fx}dx = \int_{-a}^{a} e^{-j2\pi fx}dx = 2a \cdot sinc(2af)$$

<note> : $sinc(2af) = \frac{\sin(2a\pi f)}{2a\pi f}$

(Property 2). Modulation

Cosine modulation : $\mathcal{FT}\{y(x)\cos(2\pi f_0 x)\} = \frac{1}{2}[Y(f-f_0) + Y(f+f_0)]$

Sine modulation : $\mathcal{FT}\{y(x)\sin(2\pi f_0 x)\} = \frac{1}{2j}[Y(f-f_0) - Y(f+f_0)]$

(Property 3). Shift:

$$\mathcal{FT}\{y(x-b)\} = Y(f) \cdot e^{-j2\pi fb}$$

Above the properties, we can infer that

*Rectangular with* $a = 3$ : $r_{a=3}(x) = \begin{cases} 1, & -3 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$ $\xrightarrow[\text{Property1}]{\mathcal{FT}}$ $R_{a=3}(f) = 6sinc(6f)$

↓

*Shift by* 3 : $r_{a=3}(x-3) = \begin{cases} 1, & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$ $\xrightarrow[\text{Property3}]{\mathcal{FT}}$ $R_{a=3}(f) \cdot e^{-j2\pi f\cdot 3}$

↓

*Sine modulation* : $r_{a=3}(x-3) \cdot \sin(2\pi \cdot 3 \cdot x)$

↓ *Property2* $\mathcal{FT}$

Final answer : $\frac{1}{2j}\left\{R_{a=3}(f-3) \cdot e^{-j2\pi(f-3)\cdot 3} - R_{a=3}(f+3) \cdot e^{-j2\pi(f+3)\cdot 3}\right\}$

$$= j3sinc\big(6(f+3)\big)e^{-j6\pi(f+3)} - j3sinc\big(6(f-3)\big)e^{-j6\pi(f-3)}$$

(c).

(Property 1). The Fourier transform of Gaussian function is also Gaussian function as follows:

$$\mathcal{FT}\{e^{-\alpha x^2}\} = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\pi f)^2}{\alpha}}$$

(Property 2).

$$\mathcal{FT}\{xg(x)\} = \left(\frac{j}{2\pi}\right)\frac{\mathrm{dY}(f)}{\mathrm{df}}$$

Therefore, the problem can be reformed as

$$\mathrm{G}(f) = \mathcal{FT}\{g(x)\} = \int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx = \int_{-\infty}^{\infty}(xe^{-x^2})e^{-j2\pi fx}dx + \int_{-\infty}^{\infty}(e^{-x^2})e^{-j2\pi fx}dx$$

1st term (with property 1,2): $\int_{-\infty}^{\infty}(xe^{-x^2})e^{-j2\pi fx}dx = \left(\frac{j}{2\pi}\right)\cdot\frac{d}{df}\left(\sqrt{\pi}e^{-(\pi f)^2}\right) = \sqrt{\pi}(-j\pi f)e^{-\pi^2 f^2}$

2nd term (with property 1): $\int_{-\infty}^{\infty}(e^{-x^2})e^{-j2\pi fx}dx = \sqrt{\pi}e^{-\pi^2 f^2}$

Above the inference : $\mathrm{G}(f) = \mathcal{FT}\{g(x)\} = \sqrt{\pi}(1 - j\pi f)e^{-\pi^2 f^2}$

(3).

(Property 1): Parseval's Thm: $\int_{-\infty}^{\infty}|g(x)|^2dx = \int_{-\infty}^{\infty}|G(f)|^2df$ , where $\mathrm{G}(f) = \mathcal{FT}\{g(x)\}$

(Property 2). The Fourier transform of the sinc function is rectangular function as follows:

$$R_a(f) = \mathcal{FT}\{sinc(2ax)\} = \frac{1}{2a}R_a(f)$$

<note> : $R_a(f) = \begin{cases} 1, & -a \le \mathrm{f} \le a \\ 0, & otherwise \end{cases}$

(Property 3). Modulation

Cosine modulation : $\mathcal{FT}\{y(x)\cos(2\pi f_0 x)\} = \frac{1}{2}[Y(f - f_0) + Y(f + f_0)]$

Sine modulation : $\mathcal{FT}\{y(x)\sin(2\pi f_0 x)\} = \frac{1}{2j}[Y(f - f_0) - Y(f + f_0)]$

Let $\mathrm{g}(x) = sinc(2x)\cos(\pi x)$

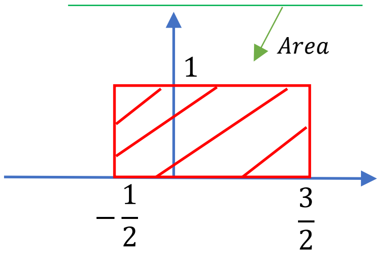
$$g_1(x) = sinc(2x) \rightarrow \quad G_1(f) = \mathcal{FT}\{g_1(x)\} = \frac{1}{2}R_{a=1}(f) \quad \ldots property\ \ 2$$

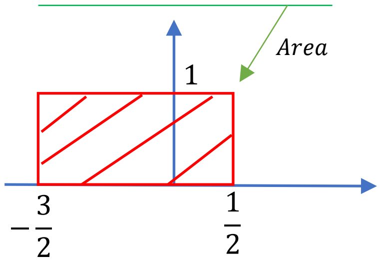
$$\mathrm{g}(x) = g_1(x)\cos(\pi x) \rightarrow \mathrm{G}(f) = \mathcal{FT}\{g(x)\} = \frac{1}{2}\left\{\frac{1}{2}\left[R_{a=1}\left(f - \frac{1}{2}\right) + R_{a=1}\left(f + \frac{1}{2}\right)\right]\right\} \ldots property\ \ 3$$

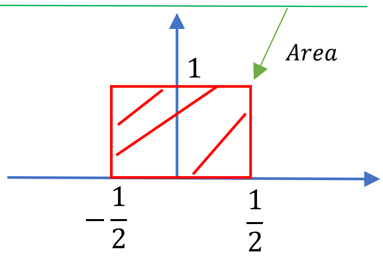
Therefore, by property 1:

$$\int_{-\infty}^{\infty} |sinc(2x)\cos(\pi x)|^2 dx = \int_{-\infty}^{\infty} \left|\frac{1}{2}\left\{\frac{1}{2}\left[R_{a=1}\left(f-\frac{1}{2}\right)+R_{a=1}\left(f+\frac{1}{2}\right)\right]\right\}\right|^2 df$$

$$= \frac{1}{16}\left[\int_{-\infty}^{\infty}\left(R_{a=1}\left(f-\frac{1}{2}\right)\right)^2 df + \int_{-\infty}^{\infty}\left(R_{a=1}\left(f+\frac{1}{2}\right)\right)^2 df + \int_{-\infty}^{\infty} 2\cdot R_{a=1}\left(f-\frac{1}{2}\right)\cdot R_{a=1}\left(f+\frac{1}{2}\right) df\right]$$




$$= \frac{1}{16}\left[\int_{-\frac{1}{2}}^{\frac{3}{2}}(1)^2 df + \int_{-\frac{3}{2}}^{\frac{1}{2}}(1)^2 df + \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\cdot 1\cdot 1 df\right] = \frac{1}{16}[2+2+2] = \frac{3}{8}$$

(4).

(a).

(Property 1):

$$\mathcal{FT}\{g_1(x) * g_2(x)\} = G_1(f)\cdot G_2(f) \leftrightarrow g_1(x) * g_2(x) = \mathcal{FT}^{-1}\{G_1(f)\cdot G_2(f)\}$$

(Property 2). The Fourier transform of the sinc function is rectangular function as follows:

$$R_a(f) = \mathcal{FT}\{sinc(2ax)\} = \frac{1}{2a}R_a(f)$$

<note> : $R_a(f) = \begin{cases} 1, & -a \le f \le a \\ 0, & otherwise \end{cases}$

(Property 3). Modulation

Cosine modulation : $\mathcal{FT}\{\cos(2\pi f_0 x)\} = \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$

Sine modulation : $\mathcal{FT}\{\sin(2\pi f_0 x)\} = \frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$

<note> : Dirac-delta function

$\delta(f) = \begin{cases} \infty, & f = 0 \\ 0 & , f \neq 0 \end{cases}$

Let $g_1(x) = \cos(4\pi x)\sin(6\pi x), g_2(x) = sinc(8x), g_3(x) = sinc(6x), g_4(x) = sinc(4x)$

<note> : $\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b)+\sin(a-b)]$

$$g_1(x) = \cos(4\pi x)\sin(6\pi x) = \frac{1}{2}[\sin(10\pi x)+\sin(2\pi x)] = \frac{1}{2}[\sin(2\pi\cdot 5\cdot x)+\sin(2\pi\cdot 1\cdot x)]$$

$$G_1(f) = \mathcal{FT}\{g_1(x)\} = \frac{1}{2}\left\{\frac{1}{2j}[\delta(f-5)-\delta(f+5)+\delta(f-1)-\delta(f+1)]\right\} \ldots property\ 3$$

$$G_2(f) = \mathcal{FT}\{g_2(x)\} = \frac{1}{8}R_{a=4}(f) = \begin{cases} \frac{1}{8}, & -4 \le \mathrm{x} \le 4 \\ 0, & \text{otherwise} \end{cases} \quad \ldots property\ \ 2$$

$$G_3(f) = \mathcal{FT}\{g_3(x)\} = \frac{1}{6}R_{a=3}(f) = \begin{cases} \frac{1}{6}, & -3 \le \mathrm{x} \le 3 \\ 0, & \text{otherwise} \end{cases} \quad \ldots property\ \ 2$$

$$G_4(f) = \mathcal{FT}\{g_4(x)\} = \frac{1}{4}R_{a=2}(f) = \begin{cases} \frac{1}{4}, & -2 \le \mathrm{x} \le 2 \\ 0, & \text{otherwise} \end{cases} \quad \ldots property\ \ 2$$

Therefore, by property 1:

$$g(x) = g_1(x) * g_2(x) * g_3(x) * g_4(x) \rightarrow \mathrm{G}(f) = \mathcal{FT}\{g(x)\} = G_1(f)G_2(f)G_3(f)G_4(f)$$

➔ $\mathrm{G}(f) = \frac{1}{8}\cdot\frac{1}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\left\{\frac{1}{2j}[\delta(f-1) - \delta(f+1)]\right\} = \frac{1}{384}\left\{\frac{1}{2j}[\delta(f-1) - \delta(f+1)]\right\}$

➔ $g(x) = \mathcal{FT}^{-1}\{\mathrm{G}(f)\} = \frac{1}{384}\cdot\sin(2\pi x)$

(b).

(Property 1):

$$\mathcal{FT}\{g_1(x) * g_2(x)\} = G_1(f)\cdot G_2(f) \leftrightarrow g_1(x) * g_2(x) = \mathcal{FT}^{-1}\{G_1(f)\cdot G_2(f)\}$$

(Property 2). The Fourier transform of Gaussian function is also Gaussian function as follows:

$$\mathcal{FT}\{e^{-\alpha x^2}\} = \sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\pi f)^2}{\alpha}}$$

(Property 3). Modulation

Cosine modulation : $\mathcal{FT}\{\cos(2\pi f_0 x)\} = \frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)]$

Sine modulation : $\mathcal{FT}\{\sin(2\pi f_0 x)\} = \frac{1}{2j}[\delta(f-f_0) - \delta(f+f_0)]$

<note> : Dirac-delta function

$$\delta(\mathrm{f}) = \begin{cases} \infty, & \mathrm{f} = 0 \\ 0 & ,\mathrm{f} \neq 0 \end{cases}$$

(Property 4).

$$\mathcal{FT}\{\delta'(x)\} = j2\pi f$$

$$Filtering: \int_{-\infty}^{\infty} \delta(f-f_0)Y(f)e^{j2\pi f x}dx = Y(f_0)e^{j2\pi f_0 x}$$

Let $g_1(x) = e^{-\pi x^2}, g_2(x) = \cos(4\pi x), g_3(x) = \delta'(x)$

$$G_1(f) = \mathcal{FT}\{g_1(x)\} = e^{-\pi f^2} \ldots property\ \ 2$$

$$G_2(f) = \mathcal{FT}\{g_2(x)\} = \frac{1}{2}[\delta(f-2) + \delta(f+2)] \ldots property\ \ 3$$

$$G_3(f) = \mathcal{FT}\{g_3(x)\} = j2\pi f$$

$$g(x) = g_1(x) * g_2(x) * g_3(x) \rightarrow \mathrm{G}(f) = \mathcal{FT}\{g(x)\} = G_1(f)G_2(f)G_3(f)$$

$$\mathrm{G}(f) = \frac{j2\pi f}{2}\cdot e^{-\pi f^2}\cdot[\delta(f-2) + \delta(f+2)]$$

$$g(x) = \mathcal{FT}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f)e^{j2\pi fx}dx$$

$$= \int_{-\infty}^{\infty} \left\{\frac{j2\pi f}{2} \cdot e^{-\pi f^2} \cdot [\delta(f-2) + \delta(f+2)]\right\} e^{j2\pi fx}dx$$

$$= \int_{-\infty}^{\infty} \left\{\frac{j2\pi f}{2} \cdot e^{-\pi f^2} \cdot \delta(f-2)\right\} e^{j2\pi fx}dx + \int_{-\infty}^{\infty} \left\{\frac{j2\pi f}{2} \cdot e^{-\pi f^2} \cdot \delta(f+2)\right\} e^{j2\pi fx}dx \ldots property\ \ 4$$

$$= \left\{\frac{j2\pi \cdot 2}{2} \cdot e^{-\pi 2^2} \cdot e^{j2\pi \cdot 2x}\right\} + \left\{\frac{j2\pi \cdot (-2)}{2} \cdot e^{-\pi(-2)^2} \cdot e^{j2\pi\cdot(-2)x}\right\}$$

$$= -4\pi e^{-4\pi}\left[\frac{1}{2j}\left(e^{j4\pi x} - e^{-j4\pi x}\right)\right] = -4\pi e^{-4\pi}\sin(4\pi x)$$

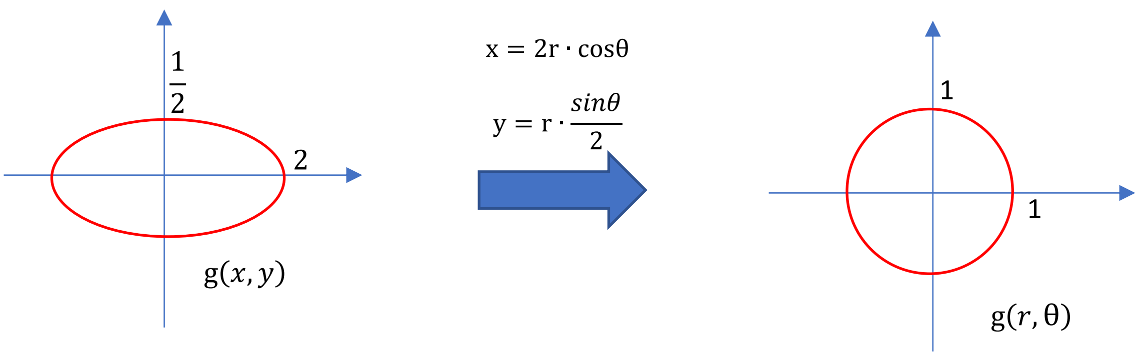
(5).

Reference : https://mathworld.wolfram.com/HankelTransform.html

Let $x = 2r \cdot \cos\theta, \quad y = r \cdot \frac{sin\theta}{2}, \quad f = s \cdot \frac{cos\phi}{2}, \ h = 2s \cdot \sin(\phi)$

By coordinate transformation :

$$g(r,\theta) = g\left(x = 2r\cdot\cos\theta, \quad y = r\cdot\frac{sin\theta}{2}\right) = \begin{cases}1, & |r| < 1\\ 0, & \text{otherwise}\end{cases}$$




(property) : Hankel transform

$g(s) = 2\pi \int_0^\infty f(r)J_0(2\pi sr)rdr$ : if $f(r) = R_a(r) == \begin{cases}1, & -a \le r \le a\\ 0, & \text{otehrwise}\end{cases}$, then $g(s) = \frac{a\cdot J_1(2\pi s)}{s}$

$$G(f,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi fx}e^{-j2\pi hy}g(x,y)dxdy = G\left(f = s\cdot\frac{cos\phi}{2}, h = 2s\cdot\sin(\phi)\right)$$

$$= \int_0^1\int_0^{2\pi} e^{-j2\pi sr\cdot\cos(\phi-\theta)}rd\theta dr = 2\pi\int_0^1 J_0(2\pi sr)rdr = \frac{J_1(2\pi s)}{s} = \frac{J_1\left(2\pi\sqrt{4f^2 + \frac{h^2}{4}}\right)}{\sqrt{4f^2 + \frac{h^2}{4}}}$$

Where $\quad s = \sqrt{4f^2 + \frac{h^2}{4}}$