EM_HW4 solutions

Problem 1

Let $g[n] = IDFT\{DFT(x[n])DFT(y[n])\}$

$g[n] = y[n] *_c x[n]$, where $*_c$ denotes the circular convolution operation.

$$= \sum_{k=0}^{19} y[k]x[((n-k))_{20}]$$

$$= x[n] + x[((n-10))_{20}]$$

Hence,

$g[0] = x[0] + x[10] = 1$, $g[1] = x[1] + x[11] = 0$, $g[2] = x[2] + x[12] = 1$

$g[3] = x[3] + x[13] = 0$, $g[4] = x[4] + x[14] = 1$, $g[5] = x[5] + x[15] = 0$

$g[6] = x[6] + x[16] = 1$, $g[7] = x[7] + x[17] = 0$, $g[8] = x[8] + x[18] = 1$

$g[9] = x[9] + x[19] = 0$, $g[10] = x[0] + x[10] = 1$, $g[11] = x[11] + x[1] = 0$

$g[12] = x[12] + x[2] = 1$, $g[13] = x[13] + x[3] = 0$

$g[14] = x[14] + x[4] = 1$, $g[15] = x[15] + x[5] = 0$

$g[16] = x[16] + x[6] = 1$, $g[17] = x[17] + x[7] = 0$

$g[18] = x[18] + x[8] = 1$, $g[19] = x[19] + x[9] = 0$

Ans : $g[n] = [1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0]$

Problem 2

(a)

1. Calculate eigenvalues and the corresponding eigenvectors of A :

   We can find that A has two eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$ and

   the corresponding eigenvectors are $v_1 = [1, 1]^T$ and $v_2 = [-1, 1]^T$

2. Find A_2 :

$$A_2 = A \otimes A = \begin{bmatrix} A & 3A \\ 3A & A \end{bmatrix}$$

3. Find eigenvalues and the corresponding eigenvectors of A_2 :

   (a) $A_2 * [v_1 \, v_1]^T = 16[v_1 \, v_1]^T$

   (b) $A_2 * [v_1 \; -v_1]^T = -8[v_1 \; -v_1]^T$

   (c) $A_2 * [v_2 \, v_2]^T = -8[v_2 \, v_2]^T$

   (d) $A_2 * [v_2 \; -v_2]^T = 4[v_2 \; -v_2]^T$

Ans : Eigenvalues : 16, -8, -8, 4

Eigenvectors : for $\lambda = 16$: $[1, 1, 1, 1]^T$; for $\lambda = 4$: $[-1, 1, 1, -1]^T$

for $\lambda = -8$: $[-1, 1, -1, 1]^T$, $[1, 1, -1, -1]^T$

(b)

$\lim_{\alpha\to 0}(\|A\|_\alpha)^\alpha = 4$
$\|A\|_1$=1+3+3+1=8
$\|A\|_2=\sqrt{1+9+9+1}=\sqrt{20}$
$\|A\|_\infty$=3

(c)

$\lim_{\alpha\to 0}(\|A_n\|_\alpha)^\alpha = 4^n$
$\|A_n\|_1 = 8^n$
$\|A_n\|_2 = 20^{n/2}$
$\|A_n\|_\infty = 3^n$

Problem 3

(a)

1. Find eigenvalues and the corresponding eigenvectors of A :
   (a) $\lambda_1 = 1$ and $v_1 = [1, 0, 0, 0]^T$
   (b) $\lambda_2 = 0$ with multiplicity 3 and $v_{21} = [0, 0, 1, 0]^T$, $v_{22} = [0, 0, 0, 1]^T$
2. Solve $Av_{23} = v_{21}$ in order to form the Jordan-Canonical form of A
   $\Rightarrow v_{23} = [0, 1/2, 0, 0]^T$
3. Jordan-Canonical form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$sin(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sin(1) & 0 & 0 & 0 \\ 0 & sin(0) & \frac{(-1)^0}{1!}cos(0) & 0 \\ 0 & 0 & sin(0) & 0 \\ 0 & 0 & 0 & sin(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} sin(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4

The Markov model matrix is $\begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}$, where the (i,j) entry denotes the probability that state j transfers to state i (state 1 : sunny day, state 2 : cloudy day, state 3 : rainy day)

(a) $\begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.436 \\ 0.327 \\ 0.237 \end{bmatrix}$

$\therefore$ the probability that the weather is sunny 3 days later is 0.436

(b) 1 is an eigenvalue of the Markov model matrix and its multiplicity is 1. Furthermore, an eigenvector corresponding to eigenvalue 1 is $[1, 1, 1]^T$. Hence, the probability that the weather is sunny infinite number of days later is $\frac{1}{1+1+1} = \frac{1}{3}$