# 2.7 Polar Coordinates

Sections 2.7, 2.8, 2.9 are extended from Sec. 2.1, but the polar, cylindrical, and spherical coordinates are adopted.

- 2D — Polar Coordinate (Sec. 2.7)
- 3D
  - Cylindrical Coordinate (Sec. 2.8)
  - Spherical Coordinate (Sec. 2.9)

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.1

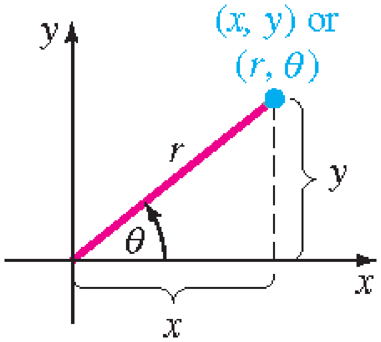



Fig. 2.7.1 Polar coordinates of a point $(x, y)$ are $(r, \theta)$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.1.

$(x, y)$ ⟶ $(r, \theta)$

original coordinate → polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad r^2 = x^2 + y^2$$

the Laplacian of $u$ in $x$-$y$ coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

$$\Downarrow$$

the Laplacian of $u$ in polar coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

In this section we focus only on boundary-value problems involving Laplace's equation $\nabla^2 u = 0$ in polar coordinates:

The key points of this section.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad \Longrightarrow \quad \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(Proof): Since

$$x = r\cos\theta, \quad y = r\sin\theta$$

we have

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta} \qquad \frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+(y/x)^2}\frac{-y}{x^2}\frac{\partial u}{\partial \theta}$$

from $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{-y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial \theta} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial \theta} = \frac{2y}{2\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+(y/x)^2}\frac{1}{x}\frac{\partial u}{\partial \theta}$$

$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\boxed{\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}} \qquad \boxed{\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x}$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial \theta \partial r}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\frac{\partial u}{\partial y} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}\frac{\partial u}{\partial y}$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial r^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

The proof is completed.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

In this section, we focus on the Laplace's equation with steady temperature, i.e.,

$$\nabla^2 u = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

## [Example 1] Steady Temperatures in a Circular Plate

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

subject to $u(c,\theta)=f(\theta),\ 0<\theta<2\pi$

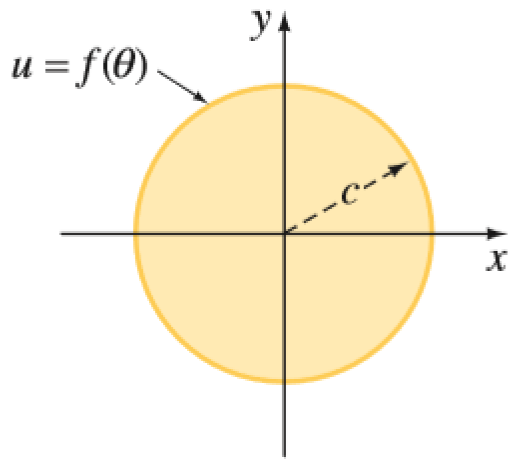



Fig. 2.8.2

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

**SOLUTION**

(Step 1) Suppose that $u(r,\theta)=R(r)\Theta(\theta)$

$$R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$$

$$r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\frac{\Theta''(\theta)}{\Theta(\theta)}=0$$

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda.$$

(Step 2) $$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

$$r^2 R'' + rR' - \lambda R = 0 \qquad \Theta'' + \lambda\Theta = 0.$$

There is no zero boundary condition.

But note that $\Theta(\theta)$ should be periodic: $\Theta(\theta) = \Theta(\theta + 2\pi)$

(Step 3) Then, we try to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \qquad \Theta(\theta) = \Theta(\theta + 2\pi)$$

Case 1: $\lambda = 0, \Longrightarrow \Theta''(\theta) = 0, \Longrightarrow \Theta(\theta) = c_1 + c_2\theta$

from $\Theta(\theta) = \Theta(\theta + 2\pi) \Longrightarrow \Theta(\theta) = c_1$

Case 2: $\lambda < 0$, set $\lambda = -\alpha^2 \Longrightarrow \Theta''(\theta) - \alpha^2\Theta(\theta) = 0,$

$\Longrightarrow \Theta(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta \Longrightarrow \Theta(\theta) = 0$

(trivial)

$$\Theta''(\theta)+\lambda\Theta(\theta)=0, \qquad \Theta(\theta)=\Theta(\theta+2\pi)$$

Case 3: $\lambda>0$, set $\lambda=\alpha^2$ $\Longrightarrow$ $\Theta''(\theta)+\alpha^2\Theta(\theta)=0,$

$\Longrightarrow$ $\Theta(\theta)=c_1\cos\alpha\theta+c_2\sin\alpha\theta$

From $\Theta(\theta)=\Theta(\theta+2\pi)$

$$c_1\cos\alpha\theta+c_2\sin\alpha\theta=c_1\cos(\alpha\theta+\alpha 2\pi)+c_2\sin(\alpha\theta+\alpha 2\pi)$$

$\alpha 2\pi=n2\pi$ where $n$ is a positive integer,

$\alpha=n,\qquad (\lambda=n^2)$

$\Theta(\theta)=c_1\cos n\theta+c_2\sin n\theta$ where $n$ is a positive integer,

Combine the results of Cases 1 and 3

$\Theta(\theta)=c_1\cos n\theta+c_2\sin n\theta \qquad (\lambda=n^2)$

where $n$ is a nonnegative integer

(Step 4)

$$r^2 R'' + rR' - \lambda R = 0$$

This is an important application of the Cauchy-Euler equation on page 35

Since $\lambda_n = n^2,\ n = 0,\ 1,\ 2,\ \ldots\ldots$

$$r^2 R'' + rR' - n^2 R = 0$$ ← Cauchy-Euler

Auxiliary: $m(m-1) + m - n^2 = 0, \quad m = \pm n$

the solutions are

$$R(r) = c_3 + c_4 \ln r, \quad n = 0$$

$$R(r) = c_5 r^n + c_6 r^{-n}, \quad n = 1,\ 2,\ \ldots$$

Since $\ln 0 \to -\infty \qquad 0^{-n} \to \infty$ but $R(0)$ should not be infinite

$c_4 = c_6 = 0$ should be satisfied

⟹ $R(r) = c_3, \quad n = 0, \qquad R(r) = c_5 r^n, \quad n = 1,\ 2,\ \ldots$

$$\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$$ where $n$ is a nonnegative integer

$R(r) = c_3, \quad n = 0,$ $\qquad R(r) = c_5 r^n, \quad n = 1, 2, \ldots.$

(Step 5) $u_n(r,\theta) = R(r)\Theta(\theta)$

$u_0(r,\theta) = A_0$ when $n = 0$,

$u_n(r,\theta) = r^n (A_n \cos n\theta + B_n \sin n\theta)$ when $n = 1, 2, \ldots.$

(Step 6)

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

(Step 7)

By applying the boundary condition $u(c,\theta) = f(\theta), \ 0 < \theta < 2\pi.$

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

Next, solve the unknowns from the formula of the Fourier series

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

From the formula of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$-p < x < p$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$\Rightarrow \begin{cases} A_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)d\theta \\ A_n = \dfrac{1}{c^n\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta \\ B_n = \dfrac{1}{c^n\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta. \end{cases}$$

Since $f(\theta) = f(\theta + 2\pi)$

$$\Rightarrow \begin{cases} A_0 = \dfrac{1}{2\pi}\displaystyle\int_{0}^{2\pi} f(\theta)d\theta \\ A_n = \dfrac{1}{c^n\pi}\displaystyle\int_{0}^{2\pi} f(\theta)\cos n\theta d\theta \\ B_n = \dfrac{1}{c^n\pi}\displaystyle\int_{0}^{2\pi} f(\theta)\sin n\theta d\theta. \end{cases}$$

**EXAMPLE 2** Steady Temperatures in a Semicircular Plate

Find the steady-state temperature $u(r, \theta)$ in

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

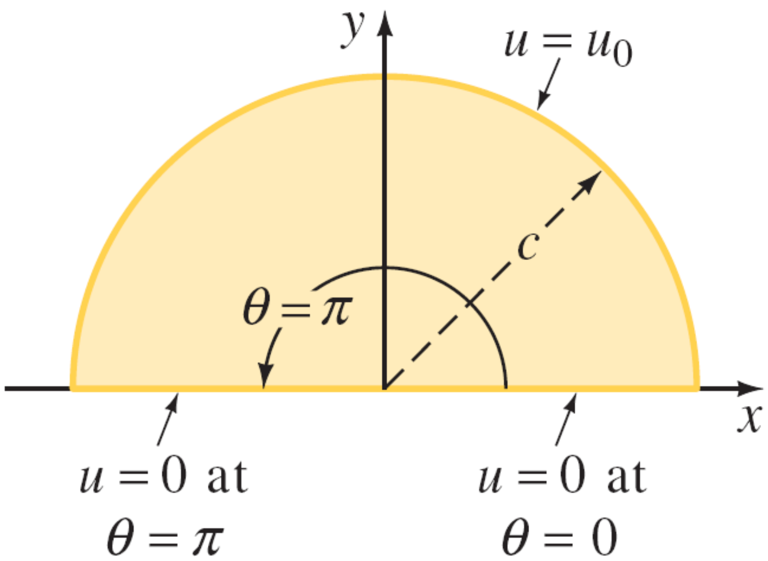



Fig. 2.8.3 Semicircular plate in Example 2

**SOLUTION** The problem can be formulated as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

$$u(c,\theta)=u_0, \quad 0<\theta<\pi$$

$$u(r,0)=0, \quad u(r,\pi)=0, \quad 0<r<c.$$

(Step 1) Suppose that $u(r,\theta)=R(r)\Theta(\theta)$

$$R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$$

$$r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\frac{\Theta''(\theta)}{\Theta(\theta)}=0$$

(Step 2)
$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

$$r^2R''+rR'-\lambda R=0$$

$$\Theta''+\lambda\Theta=0.$$

(Step 3)

From $u(r,0)=0, \quad u(r,\pi)=0,$

$$R(r)\Theta(0)=0, \quad R(r)\Theta(\pi)=0,$$

$$\Theta(0)=0 \text{ and } \Theta(\pi)=0.$$

We then try to solve

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0)=0, \quad \Theta(\pi)=0.$$

Case 1: $\lambda = 0 \Rightarrow \Theta''(\theta) = 0 \Rightarrow \Theta(\theta) = c_1\theta + c_2$

From $\Theta(0)=0, \quad \Theta(\pi)=0 \Rightarrow \Theta(\theta)=0$

Case 2: $\lambda < 0, \quad \text{set } \lambda = -\alpha^2 \Rightarrow \Theta''(\theta) - \alpha^2\Theta(\theta) = 0$

$$\Rightarrow \Theta(\theta) = c_3\cosh\alpha\theta + c_4\sinh\alpha\theta$$

From $\Theta(0)=0, \quad \Theta(\pi)=0 \Rightarrow \Theta(\theta)=0$

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\pi) = 0.$$

Case 3: $\lambda > 0, \quad \text{set } \lambda = \alpha^2 \Rightarrow \Theta''(\theta) + \alpha^2\Theta(\theta) = 0$

$$\Rightarrow \Theta(\theta) = c_5 \cos\alpha\theta + c_6 \sin\alpha\theta$$

From $\Theta(0) = 0, \quad \Theta(\pi) = 0 \Rightarrow c_5 = 0, \quad \alpha = n$

$$\Rightarrow \Theta(\theta) = c_6 \sin n\theta \qquad n = 1, 2, 3, \ldots.$$

The only nontrivial solution for $\Theta(\theta)$ is

$$\boxed{\Theta(\theta) = c_6 \sin n\theta} \qquad n = 1, 2, 3, \ldots.$$

In this case, $\lambda = n^2$

(Step 4) To solve $R(r)$

$$r^2R'' + rR' - \lambda R = 0 \Rightarrow r^2R'' + rR' - n^2R = 0$$

$$R(r) = c_7 r^n + c_8 r^{-n}$$

To be bounded at $r = 0$, $c_8$ must be 0

$$\boxed{R(r) = c_7 r^n} \qquad n = 1, 2, 3, \ldots.$$

(Step 5) $u_n(r,\theta) = R(r)\Theta(\theta) = A_n r^n \sin n\theta,$

(Step 6) $u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$

(Step 7) From $u(c,\theta) = u_0$

$$u_0 = \sum_{n=1}^{\infty} A_n c^n \sin n\theta$$

Using Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$0 < x < p$$

$$b_n = \frac{2}{p}\int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

⇒

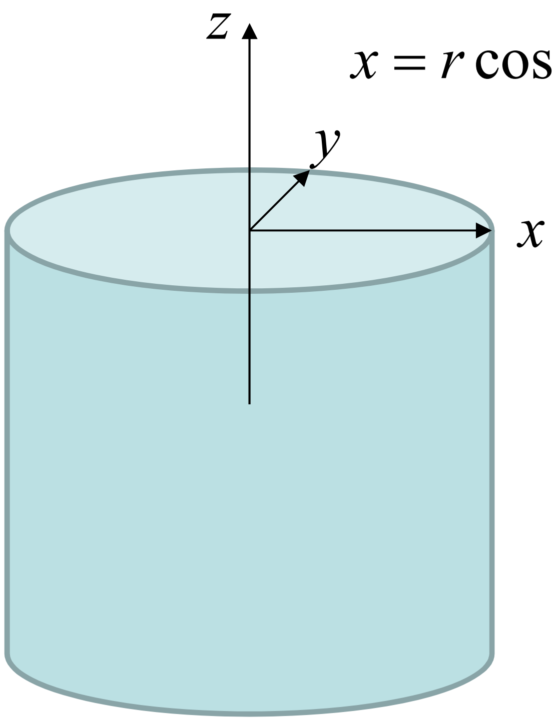
$$A_n c^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta d\theta$$

$$A_n = \frac{2u_0}{\pi c^n} \frac{1-(-1)^n}{n}$$

Solution:

$$u(r,\theta) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \left(\frac{r}{c}\right)^n \sin n\theta$$

# 2.8 CYLINDRICAL COORDINATES




$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z.$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2

## 2.8.1 Review for Special Functions

- **Bessel's equation of order *v***

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$ Solution: $c_1 J_v(x) + c_2 Y_v(x)$

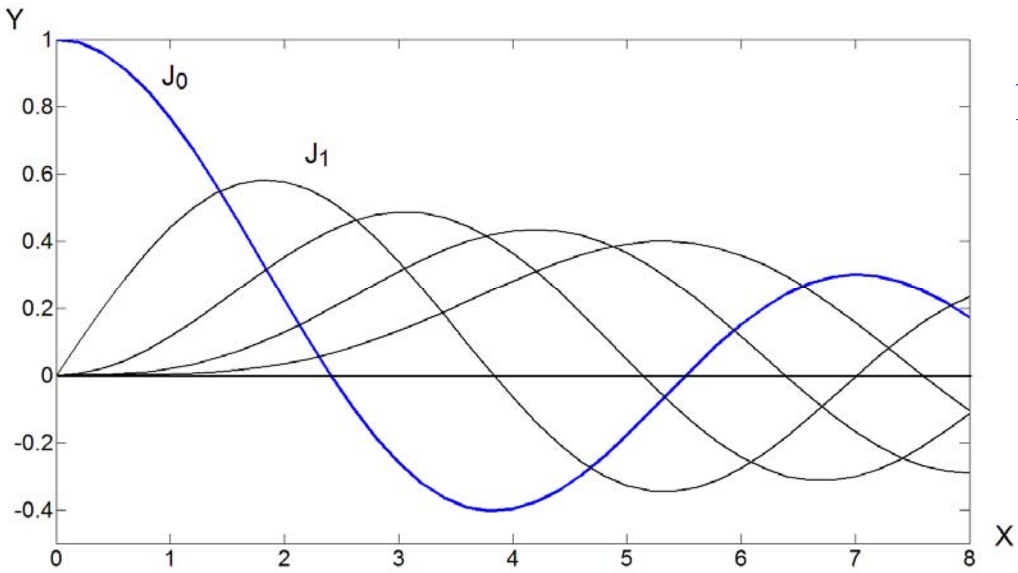
$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$ : 1st kind Bessel function

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$ : 2nd kind Bessel function

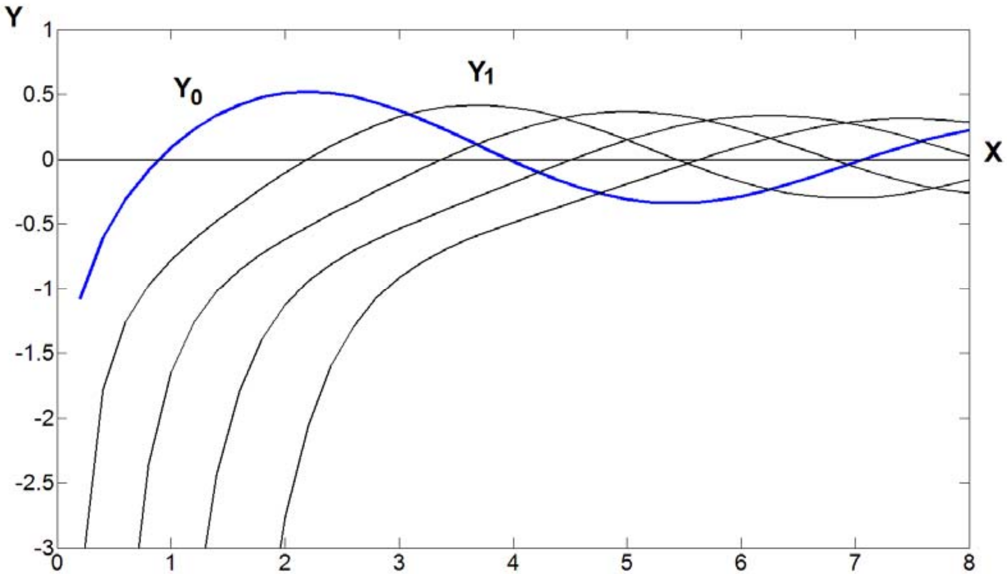
where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt$ (gamma function)

$$\Gamma(n+1) = n!$$

$$\Gamma(x+1) = x\Gamma(x)$$




1[st] kind Bessel function




: 2[nd] kind Bessel function

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

Zeros of $J_0$, $J_1$, $Y_0$, and $Y_1$

| | $J_0(x)$ | $J_1(x)$ | $Y_0(x)$ | $Y_1(x)$ |
|---|---|---|---|---|
| $x_1$ | 2.4048 | 0.0000 | 0.8936 | 2.1971 |
| $x_2$ | 5.5201 | 3.8317 | 3.9577 | 5.4297 |
| $x_3$ | 8.6537 | 7.0156 | 7.0861 | 8.5960 |
| $x_4$ | 11.7915 | 10.1735 | 10.2223 | 11.7492 |
| $x_5$ | 14.9309 | 13.3237 | 13.3611 | 14.8974 |

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

## • Generalization for Bessel's equation of order *v*

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \qquad 解：c_1J_v(x) + c_2Y_v(x)$$

$$x^2y'' + xy' + (\alpha^2x^2 - v^2)y = 0 \qquad 解：c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

Proof: Set $t = \alpha x$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \alpha\frac{dy}{dt}$$

Similarly, $$\frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \alpha\frac{d}{dt}\left(\alpha\frac{dy}{dt}\right) = \alpha^2\frac{d^2y}{dt^2}$$

$$原式 = x^2y'' + xy' + (\alpha^2x^2 - v^2)y = \frac{t^2}{\alpha^2}\alpha^2\frac{d^2y}{dt^2} + \frac{t}{\alpha}\alpha\frac{dy}{dt} + (\alpha^2\frac{t^2}{\alpha^2} - v^2)y$$

$$= t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0$$ → 對 $t$ 而言是 Bessel equation

$$y = c_1J_v(t) + c_2Y_v(t) = c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

**• Modified Bessel's equation of order *v***

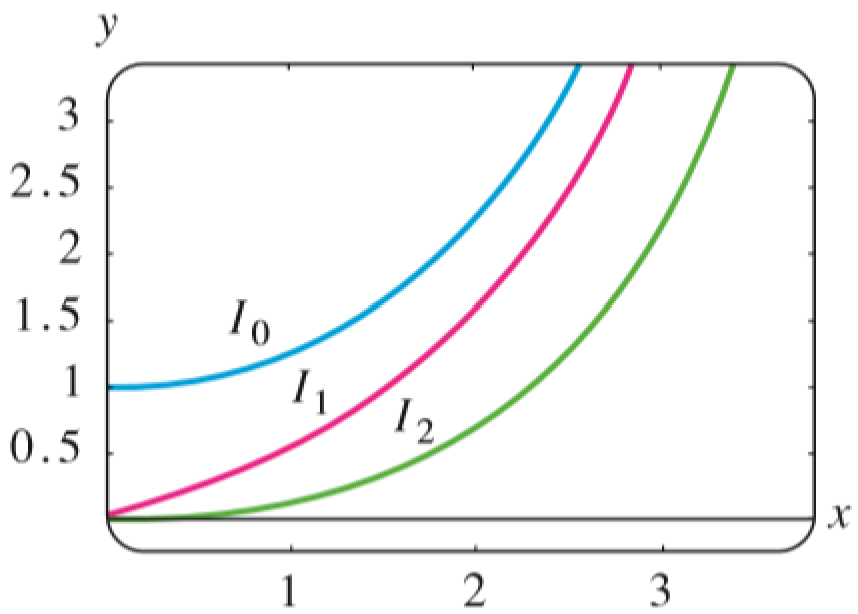
$$x^2y'' + xy' - (x^2 + v^2)y = 0$$ 解：$c_1 I_v(x) + c_2 K_v(x)$

$$x^2y'' + xy' - (\alpha^2 x^2 + v^2)y = 0$$ 解：$c_1 I_v(\alpha x) + c_2 K_v(\alpha x)$

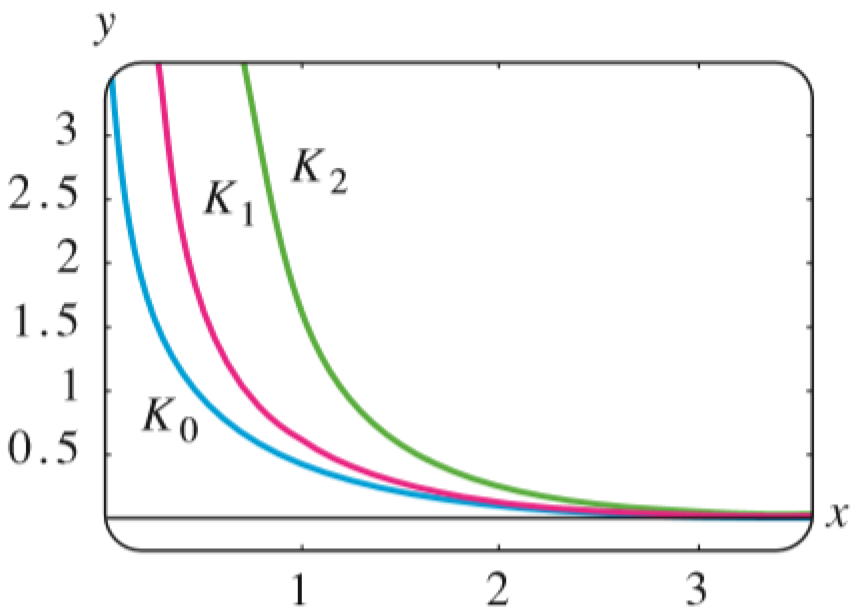
其中 $I_v(x) = i^{-v} J_v(ix)$ 稱作是 modified Bessel function of the first kind of order *v*

$$K_v(x) = \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin v\pi}$$ 稱作是 modified Bessel function of the second kind of order *v*

當 *v* 為整數時，也是取 limit




Modified Bessel functions of the first kind for $n = 0, 1, 2$




Modified Bessel functions of the second kind for $n = 0, 1, 2$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

## • Legendre's equation of order *n*

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

One of the solution: Legendre polynomials $P_n(x)$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

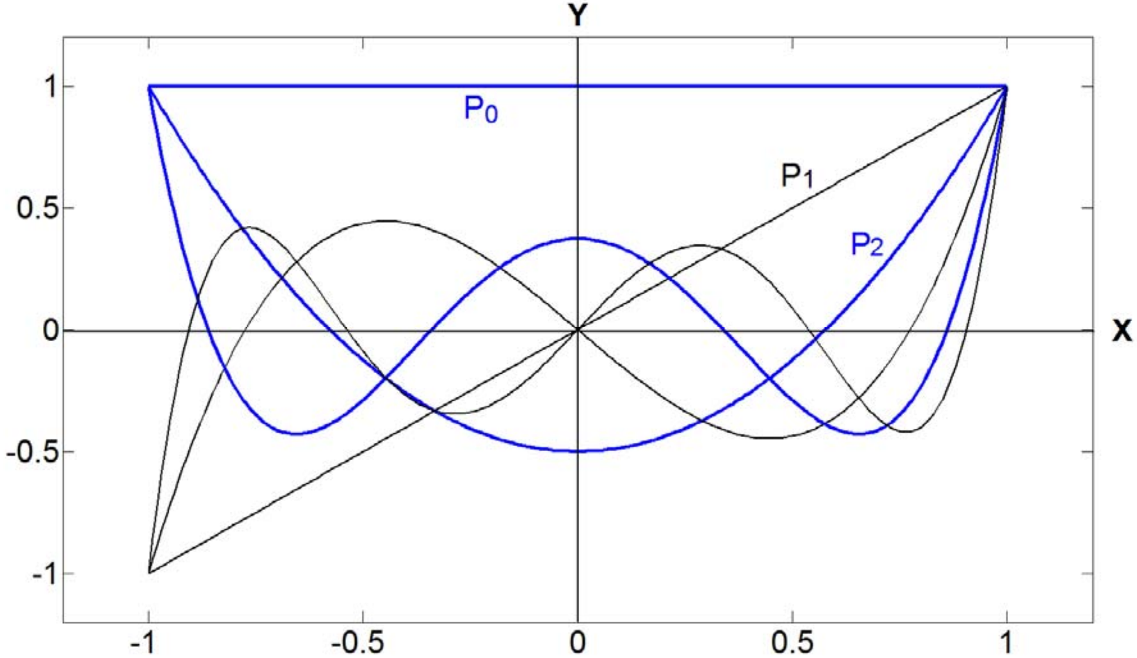
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1}\delta_{n,m}$$

# Legendre polynomials




Interval:

$x \in [-1, 1]$



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

## 2.8.2 PDE for Cylindrical Coordinates

### Laplacian in Cylindrical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

For the case of **radial symmetry** $\frac{\partial u}{\partial \theta} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}.$$

# [Example 1] Steady Temperatures in a Circular Cylinder

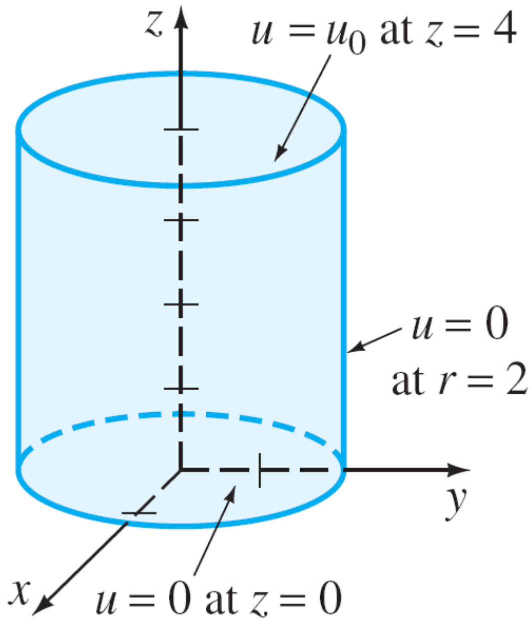



FIGURE 13.2.5 Circular cylinder in Example 2

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2.$$

It is a problem of the 2D heat equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

steady temperatures ↓

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

radial symmetry ↓ (since $u = 0$ when $r = 2$)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

(Step 1) $u(r,z)=R(r)Z(z)$

$$R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0$$

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{Z''(z)}{Z(z)}=0$$

(Step 2) $$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$$

$$rR''+R'+\lambda rR=0 \qquad Z''-\lambda Z=0$$

From $u(2,z)=0,\quad u(r,0)=0,$

$$R(2)=0 \qquad Z(0)=0$$

$$rR''+R'+\lambda rR=0 \qquad R(2)=0$$

$$Z''-\lambda Z=0 \qquad Z(0)=0$$

(Step 3)

Case 1: $\lambda=0$ $\quad rR''+R'=0$

auxiliary: $m(m\text{-}1)+m=0,\quad m=0,0$

$$R(r)=c_1+c_2\ln r$$

Since $\ln 0\to-\infty$, $c_2=0$

Since $R(2)=0\to c_1+c_2\ln 2=0\to c_1=0$

$R(r)=0$ (trivial)

Case 2: $\lambda = -\alpha^2 < 0$ $\quad rR'' + R' - \alpha^2 rR = 0$

$$r^2R'' + rR' - \alpha^2 r^2 R = 0$$

compared to modified Bessel function

$x^2y'' + xy' - (x^2 + v^2)y = 0$ solution：$c_1I_v(x) + c_2K_v(x)$

$x^2y'' + xy' - (\alpha^2x^2 + v^2)y = 0$ solution：$c_1I_v(\alpha x) + c_2K_v(\alpha x)$

$I_v(x)$: modified Bessel function of the 1st kind

$K_v(x)$: modified Bessel function of the 2nd kind

The solution of $rR'' + R' - \alpha^2 rR = 0$ is

$$R(r) = c_1I_0(\alpha r) + c_2K_0(\alpha r)$$

Since $K_0(0) \to \infty$, $c_2 = 0$

Since $R(2) = 0$, $c_1I_0(2\alpha) = 0$ ⟹ $R(r) = 0$ (trivial)

(page 199) From $I_0(x) \neq 0$ for all $x \to c_1 = 0$

Case 3: $\lambda = \alpha^2 > 0$

$$rR'' + R' + \alpha^2 rR = 0$$

$$r^2R'' + rR' + \alpha^2 r^2 R = 0$$

compared to modified Bessel function

$$x^2y'' + xy' + (\alpha^2x^2 - v^2)y = 0$$

solution：$c_1J_v(\alpha x) + c_2Y_v(\alpha x)$

$$R(r) = c_1J_0(\alpha r) + c_2Y_0(\alpha r)$$

Since $Y_0(0) \to -\infty$, $c_2 = 0$ $\quad R(r) = c_1J_0(\alpha r)$

Since $R(2) = 0$ $\quad c_1J_0(2\alpha) = 0$

Therefore,

$$\boxed{R(r) = c_1J_0(\alpha_n r)} \quad \text{where} \quad \alpha_n = x_n / 2$$

$x_n$ are the zeros of $J_0(x)$, i.e., $\quad J_0(x_n) = 0$

$$\lambda_n = \alpha_n^2, \quad \lambda_n = x_n^2 / 4$$

(Step 4) Try to solve

$$Z'' - \lambda Z = 0 \qquad Z(0) = 0$$

Since $\lambda_n = \alpha_n^2,$

$$Z'' - \alpha_n^2 Z = 0$$

$$Z(z) = c_3 \cosh(\alpha_n z) + c_4 \sinh(\alpha_n z)$$

From $Z(0) = 0, \quad c_3 = 0$

$$\boxed{Z(z) = c_4 \sinh(\alpha_n z)}$$

(Step 5) $u(r,z) = R(r)Z(z) = A_n J_0(\alpha_n r)\sinh(\alpha_n z)$

(Step 6) $u(r,z) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n z J_0(\alpha_n r).$

where $\alpha_n = x_n / 2 \qquad J_0(x_n) = 0$

(Step 7) From $u(r,4)=u_0,$

$$u_0=\sum_{n=1}^{\infty}A_n\sinh(4\alpha_n)J_0(\alpha_n r).$$

From page 511 in

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2

$$A_n=\frac{u_0}{2\sinh\left(4\alpha_n\right)J_1^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr$$

(Solution):

$$u(r,z)=u_0\sum_{n=1}^{\infty}\frac{1}{\alpha_n\sinh(4\alpha_n)J_1(2\alpha_n)}\sinh(\alpha_n z)J_0(\alpha_n r)$$

where $\alpha_n=x_n/2$ $\quad J_0(x_n)=0$

**[Example 2]** Steady Temperatures in a Circular Cylinder

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

$$u(1,z)=1-z,\quad 0<z<1$$

$$u(r,0)=0,\quad u(r,1)=0\quad 0<r<1$$

(Solution):

(Step 1) $u(r,z)=R(r)Z(z)$

$$R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0$$

(Step 2) $$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$$

$$rR''+R'+\lambda rR=0 \qquad Z''-\lambda Z=0$$

From $u(r,0)=0,\quad u(r,1)=0$

$$Z(0)=0,\quad Z(1)=0$$

(Step 3): Solve

$$Z''-\lambda Z=0 \qquad Z(0)=0,\ \ Z(1)=0$$

There is no non-trivial solution for $\lambda=0$ and $\lambda>0$

When $\lambda<0$, set $\lambda=-\alpha^2$,

$$Z''+\alpha^2 Z=0$$

$$Z(z)=c_1\cos\alpha z+c_2\sin\alpha z$$

From $Z(0)=0,\ \ Z(1)=0$

$$\boxed{Z(z)=c_2\sin\alpha_n z} \quad \text{where} \quad \alpha_n=n\pi \qquad \lambda_n=-\alpha_n^2=-n^2\pi^2$$

(Step 4): Solve

$$rR''+R'+\lambda rR=0$$

$$rR''+R'-n^2\pi^2 rR=0$$

$$rR''+R'-n^2\pi^2 rR=0 \qquad r^2R''+rR'-n^2\pi^2r^2R=0$$

Since $x^2y''+xy'-(\alpha^2x^2+v^2)y=0$ solution：$c_1I_v(\alpha x)+c_2K_v(\alpha x)$

the solution of $r^2R''+rR'-n^2\pi^2r^2R=0$ is

$$R(r)=c_3I_0(n\pi r)+c_4K_0(n\pi r)$$

$I_v(x)$: modified Bessel function of the 1st kind

$K_v(x)$: modified Bessel function of the 2nd kind

Since $K_0(0)\to\infty$

$$R(r)=c_3I_0(n\pi r)$$

(Step 5): $u(r,z)=R(r)Z(z)$ $\qquad u_n(r,z)=A_nI_0(n\pi r)\sin(n\pi z)$

(Step 6): $$u(r,z)=\sum_{n=1}^{\infty}A_nI_0(n\pi r)\sin(n\pi z)$$

$$u(r,z)=\sum_{n=1}^{\infty}A_n I_0(n\pi r)\sin(n\pi z)$$

(Step 7): From $u(1,z)=1-z$

$$\sum_{n=1}^{\infty}A_n I_0(n\pi)\sin(n\pi z)=1-z$$

From the Fourier sine series

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

we have

$$A_n I_0(n\pi)=2\int_0^1(1-z)\sin(n\pi z)dz=\frac{2}{n\pi}(z-1)\cos(n\pi z)\Big|_0^1-\int_0^1\frac{2}{n\pi}\cos(n\pi z)dx$$

$$A_n=\frac{2}{n\pi I_0(n\pi)}$$

$$u(r,z)=2\sum_{n=1}^{\infty}\frac{I_0(n\pi r)}{n\pi I_0(n\pi)}\sin(n\pi z)$$

# 2.9 SPHERICAL COORDINATES

## Spherical Coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

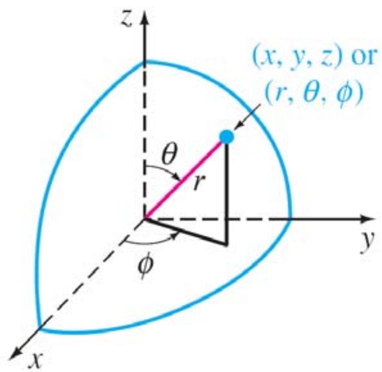



**FIGURE 13.3.1** Spherical coordinates of a point $(x, y, z)$ are $(r, \theta, \phi)$.

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

## Laplacian in Spherical Coordinates

The 3D Laplacian is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\Downarrow$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.$$

$\Downarrow$ Suppose that $u$ is independent of $\phi$.

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.$$

## [Example 1] Steady Temperatures in a Sphere

Find the steady-state temperature $u(r, \theta)$ within the sphere shown in Figure 13.3.2.

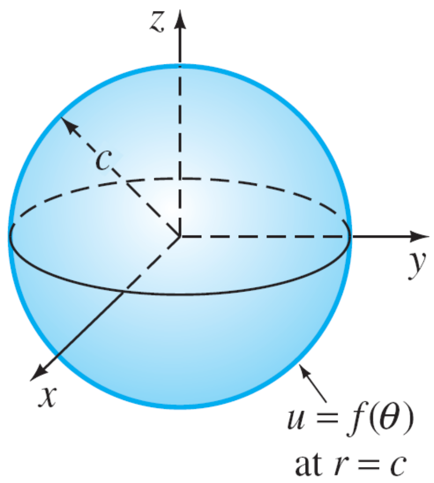



**FIGURE 13.3.2** Dirichlet problem for a sphere in Example 1

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3.

**SOLUTION**

Steady Temperatures

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\Downarrow$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$$

With $u(c,\theta) = f(\theta), \quad 0 < \theta < \pi$

(Step 1): $u = R(r)\Theta(\theta),$

$$R''(r)\Theta(\theta) + \frac{2}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) + \frac{\cot\theta}{r^2}R(r)\Theta'(\theta) = 0$$

$$\frac{R''(r)}{R(r)} + \frac{2}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{\cot\theta}{r^2}\frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

$$r^2\frac{R''(r)}{R(r)} + 2r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} + \cot\theta\frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

(Step 2): $\dfrac{r^2 R'' + 2rR'}{R} = -\dfrac{\Theta'' + \cot\theta\Theta'}{\Theta} = \lambda.$

$$r^2 R'' + 2rR' - \lambda R = 0 \qquad \sin\theta\Theta'' + \cos\theta\Theta' + \lambda\sin\theta\Theta = 0.$$

(Step 3): Try to solve

$$\sin\theta\Theta''(\theta) + \cos\theta\Theta'(\theta) + \lambda\sin\theta\Theta(\theta) = 0. \qquad 0 \le \theta \le \pi$$

Set $x = \cos(\theta)$ $\quad -1 \le x \le 1$

$$\frac{d}{d\theta}\Theta = \frac{dx}{d\theta}\frac{d}{dx}\Theta = -\sin\theta\frac{d}{dx}\Theta$$

$$\frac{d^2}{d\theta^2}\Theta = \frac{d}{d\theta}\frac{d}{d\theta}\Theta = -\frac{d}{d\theta}\left(\sin\theta\frac{d}{dx}\Theta\right) = -\cos\theta\frac{d}{dx}\Theta - \sin\theta\frac{d}{d\theta}\frac{d}{dx}\Theta$$

$$= -\cos\theta\frac{d}{dx}\Theta - \sin\theta\frac{dx}{d\theta}\frac{d^2}{dx^2}\Theta = -\cos\theta\frac{d}{dx}\Theta + \sin^2\theta\frac{d^2}{dx^2}\Theta$$

$$\sin^3\theta\frac{d^2\Theta}{dx^2} - 2\sin\theta\cos\theta\frac{d\Theta}{dx} + \lambda\sin\theta\Theta = 0$$

$$\sin^3\theta \frac{d^2\Theta}{dx^2} - 2\sin\theta\cos\theta \frac{d\Theta}{dx} + \lambda \sin\theta\Theta = 0$$

$x = \cos(\theta)$

$$\sin^2\theta \frac{d^2\Theta}{dx^2} - 2\cos\theta \frac{d\Theta}{dx} + \lambda\Theta = 0$$

$$\left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0 \qquad -1 \leq x \leq 1$$

When $\lambda = n(n+1)$, it becomes Legendre's equation of order $n$

This is one of the applications of the Legendre equation

The solution is

$$\Theta(x) = cP_n(x)$$

$P_n(x)$: the Legendre polynomial of order $n$

$$\Theta(\theta) = cP_n(\cos\theta)$$

$n = 0, 1, 2, \ldots.$

Note: The other linearly independent solution may not have finite derivatives at $x = \pm 1$.

(Step 4): Try to solve

$$r^2 R'' + 2rR' - \lambda R = 0$$

Since $\lambda = n(n+1)$,

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

Auxiliary: $m(m-1) + 2m - n(n+1) = 0$

$$m = n, \quad -(n+1)$$

$$R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

Since $0^{-(n+1)} \to \infty$ $\qquad c_2 = 0$

$$\boxed{R(r) = c_1 r^n}$$

(Step 5): $u_n(r,\theta) = A_n r^n P_n(\cos\theta) \qquad n = 0, 1, 2, \ldots.$

(Step 6): $\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)}$

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$$

(Step 7): From $u(c,\theta)=f(\theta)$

$$\sum_{n=0}^{\infty}A_n c^n P_n(\cos\theta)=f(\theta)$$

We have known that

$$\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{n,m} \qquad \delta_{n,n}=1$$

$$\delta_{n,m}=0 \quad \text{if } n\neq m$$

Set $x=\cos(\theta)$ $\qquad \frac{dx}{d\theta}=-\sin\theta$

$$\int_{\pi}^{0}P_n(\cos\theta)P_m(\cos\theta)(-\sin\theta)d\theta=\frac{2}{2n+1}\delta_{n,m}$$

$$\int_{0}^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{2}{2n+1}\delta_{n,m}$$

(orthogonal with the weight function $\sin\theta$)

$$f(\theta)=\sum_{m=0}^{\infty}A_m c^m P_m(\cos\theta) \qquad \int_0^{\pi}P_n\left(\cos\theta\right)P_m\left(\cos\theta\right)\sin\theta d\theta=\frac{2}{2n+1}\delta_{n,m}$$

$$\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta=\sum_{m=0}^{\infty}A_m c^m\int_0^{\pi}P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta=A_n c^n\frac{2}{2n+1}$$

$$A_n=\frac{2n+1}{2c^n}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta\right)\left(\frac{r}{c}\right)^n P_n(\cos\theta)$$

# Another Method for Solving PDEs: Method of Characteristics (只教不考)

## Method of Characteristics

The method is suitable for the 1st Order PDE.

Suppose that there is a 1st Order PDE as follows:

$$a_1(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_1}+a_2(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_2}$$

$$+\cdots\cdots+a_k(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_k}=g(x_1,x_2,\cdots,x_k,u)$$

# Method of Characteristics

$$a_1(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_1}+a_2(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_2}$$

$$+\cdots\cdots+a_k(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_k}=g(x_1,x_2,\cdots,x_k,u)$$

We set a variable $s$ and suppose that

$$\frac{\partial x_1}{\partial s}=a_1(x_1,x_2,\cdots,x_k,u),\quad \frac{\partial x_2}{\partial s}=a_2(x_1,x_2,\cdots,x_k,u),$$

$$\cdots\cdots,\quad \frac{\partial x_k}{\partial s}=a_k(x_1,x_2,\cdots,x_k,u).$$

Then, the original equation can be expressed as

$$\frac{\partial x_1}{\partial s}\frac{\partial u}{\partial x_1}+\frac{\partial x_2}{\partial s}\frac{\partial u}{\partial x_2}+\cdots\cdots+\frac{\partial x_k}{\partial s}\frac{\partial u}{\partial x_k}=g(x_1,x_2,\cdots,x_k,u)$$

$$\frac{du}{ds}=g(x_1,x_2,\cdots,x_k,u)$$

Then, we apply the process as follows to solve $u$:

(1) Solve each of the following ODEs:

$$\frac{\partial x_1}{\partial s} = a_1\left(x_1, x_2, \cdots, x_k, u\right)$$

$$\frac{\partial x_2}{\partial s} = a_2\left(x_1, x_2, \cdots, x_k, u\right),$$

$$\vdots$$

$$\frac{\partial x_k}{\partial s} = a_k\left(x_1, x_2, \cdots, x_k, u\right).$$

(2) Try to find the general equation such that

$b\left(x_1, x_2, \cdots, x_k\right) = c_1$ where $c_1$ is a constant

(3) Solve the following ODE

$$\frac{du}{ds} = g\left(x_1(s), x_2(s), \cdots, x_k(s), u\right)$$

The solution has the form of $u = u_1\left(s, c_2\right)$

where $c_2$ is some unknown constant.

(Specially, for the homogeneous case, $u = c_2$)

(4) Replace $c_2$ by

$$c_2 = f\left(c_1\right) = f\left(b\left(x_1, x_2, \cdots, x_k, u\right)\right)$$

where $f$ is any function, then the solution is

$$u = u_1\left(s\left(x_1\right), f\left(c_1\right)\right)$$ Here, we express $s$ as a function of $x_1$.

(Specially, for the homogeneous case, $u = f(c_1)$)

**[Example 1]** Solve

$$\frac{\partial u(x,y)}{\partial x} = 2\frac{\partial u(x,y)}{\partial y}$$

(Solution): $$\frac{\partial u(x,y)}{\partial x} - 2\frac{\partial u(x,y)}{\partial y} = 0$$

(1) $$\frac{\partial x}{\partial s} = 1, \quad x = s + c_x$$

$$\frac{\partial y}{\partial s} = -2, \quad y = -2s + c_y$$

(2) $$2x + y = c_1$$

(3) $$\frac{du}{ds} = 0, \quad u = c_2 = f(c_1)$$

$u = f(2x + y)$ where $f$ is any function.

**[Example 2]** Solve

$$\frac{\partial u(x,y,z)}{\partial x}=\frac{\partial u(x,y,z)}{\partial y}-\frac{\partial u(x,y,z)}{\partial z}+u(x,y,z)$$

(Solution):

$$\frac{\partial u(x,y,z)}{\partial x}-\frac{\partial u(x,y,z)}{\partial y}+\frac{\partial u(x,y,z)}{\partial z}=u(x,y,z)$$

(1) $\frac{\partial x}{\partial s}=1, \quad \frac{\partial y}{\partial s}=-1, \quad \frac{\partial z}{\partial s}=1$

$$x=s+c_x, \quad y=-s+c_y, \quad z=s+c_z$$

(2) Note that

$$x+y=c_x+c_y \quad \text{and} \quad y+z=c_y+c_z$$

can all generate a constant. Although $x$-$z$ can also generate a constant, it is dependent on $x$+$y$ and $y$+$z$. Therefore, a general way to obtain a constant is

$$f(x+y,\, y+z)=c$$

where $f$ is any function with two independent variables.

(3) $$\frac{\partial x}{\partial s}\frac{\partial u(x,y,z)}{\partial x}+\frac{\partial y}{\partial s}\frac{\partial u(x,y,z)}{\partial y}+\frac{\partial z}{\partial s}\frac{\partial u(x,y,z)}{\partial z}=u(x,y,z)$$

$$\frac{du}{ds}=u,$$

$u=c_1e^s=c_1e^{x-c_x}=ce^x$ Here we set $c=c_1e^{-c_x}$

$$u=f(x+y,\ y+z)e^x$$

# 附錄五 Review for Laplace Transforms

Laplace transform (1-sided form) $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Differentiation Property for the Laplace transform

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

| $f(t)$ | $F(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $\exp(at)$ | $\frac{1}{s-a}$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ |
| $\sinh(kt)$ | $\frac{k}{s^2-k^2}$ |
| $\cosh(kt)$ | $\frac{s}{s^2-k^2}$ |

# Seven Important Properties of the Laplace Transform

| input | Laplace transform |
|---|---|
| (1) Differentiation<br>$f^{(n)}(t)$ | $s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots$<br>$-sf^{(n-2)}(0) - f^{(n-1)}(0)$ |
| (2) Multiplication by $t$<br>$t^n f(t)$ | $(-1)^n \dfrac{d^n}{ds^n} F(s)$ |
| (3) Integration<br>$\int_0^t f(\tau) d\tau$ | $\dfrac{F(s)}{s}$ |

(續)

| input | Laplace transform |
| --- | --- |
| (4) Multiplication by exp<br>$e^{at}f(t)$ | $F(s-a)$ |
| (5.1) Translation $u(t)$: unit step<br>$f(t-a)u(t-a)$ | $e^{-as}F(s)$ |
| (5.2) Translation<br>$g(t)u(t-a)$ | $e^{-as}\mathcal{L}\{g(t+a)\}$ |
| (6) Convolution<br>$y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$ | $Y(s)=F(s)G(s)$ |
| (7) Periodic Input<br>$f(t)=f(t+T)$ | $\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$ |

# 2.10 Solving PDEs by Laplace Transforms

In this section, we see that a linear PDE with constant coefficients is transformed into an ODE using the 1-sided Laplace transform.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.2

## Transform of a Function of Two Variables

$$U(x,s)=\mathcal{L}\{u(x,t)\}=\int_0^\infty e^{-st}u(x,t)dt$$

To avoid confusing, we denote it as

$$U(x,s)=\mathcal{L}_{t\to s}\{u(x,t)\}=\int_0^\infty e^{-st}u(x,t)dt$$

## Transform of Partial Derivatives

$$\mathcal{L}_{t\to s}\left\{\frac{\partial u}{\partial t}\right\}=sU(x,s)-u(x,0),$$

$$\mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}=s^2U(x,s)-su(x,0)-u_t(x,0).$$

where

$$u_t(x,t)=\frac{\partial}{\partial t}u(x,t)$$

Because we are transforming with respect to *t*, (not to *x*)

Be careful

$$\mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial x^2}\right\}=\frac{d^2U}{dx^2}.$$

**[Example 1]** Laplace Transform of a PDE

Find the Laplace transform of the wave equation $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \ t > 0.$
(respect to *t*)

**SOLUTION**

$$\mathcal{L}_{t\to s}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

where $u_t = \frac{\partial}{\partial t} u$

$$a^2 \frac{d^2}{dx^2} \mathcal{L}_{t\to s}\{u(x,t)\} = s^2 \mathcal{L}_{t\to s}\{u(x,t)\} - su(x,0) - u_t(x,0)$$

$$a^2 \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -su(x,0) - u_t(x,0).$$

The process for solving the BVP or IVP of a partial differential equation (PDE) by the Laplace transform

The range of the independent variable should be [0, ∞)

**(Step 1)** Apply the Laplace transform for one independent variable to change the PDE into an ordinary differential equation (ODE) with another independent variable.

**(Step 2)** Solve the ODE obtain in Step 1.

**(Step 3)** Solution in Step 2 contains some constants. Find these constants by transforming the initial conditions.

**(Step 4)** Inverse transform.

**[Example 2]** Using the Laplace Transform to Solve a BVP

Solve $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

Subject to $$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \pi x, \quad 0 < x < 1.$$

**SOLUTION**

(Step 1) From Example 1 and the given initial conditions,

$$\frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -\sin \pi x,$$

where $U(x,s) = \mathcal{L}_{t\to s}\{u(x,t)\}$.

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t\to s}\{u(1,t)\} = U(1,s) = 0.$$

(Step 2) $$\frac{d^2U(x,s)}{dx^2} - s^2U(x,s) = -\sin \pi x,$$

Particular Solution

Complementary Function

$$a \sin \pi x + b \cos \pi x.$$

$$U_c(x,s) = c_1 \cosh sx + c_2 \sinh sx.$$

$$a = \frac{1}{s^2 + \pi^2}, \quad b = 0$$

$$U(x,s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

$$U(x,s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2+\pi^2} \sin \pi x.$$

(Step 3) From

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t\to s}\{u(1,t)\} = U(1,s) = 0.$$

$$c_1 = 0. \qquad c_1 \cosh s + c_2 \sinh s = 0$$

We have $c_1 = 0$ and $c_2 = 0$.

$$U(x,s) = \frac{1}{s^2+\pi^2} \sin \pi x$$

(Step 4) $$u(x,t) = \mathcal{L}^{-1}_{s\to t}\left\{\frac{1}{s^2+\pi^2}\sin \pi x\right\} = \frac{1}{\pi}\sin \pi x \mathcal{L}^{-1}_{s\to t}\left\{\frac{\pi}{s^2+\pi^2}\right\}.$$

$$u(x,t) = \frac{1}{\pi}\sin \pi x \sin \pi t.$$

**[Example 3]** The Wave Equation with Gravity

Solve $$a^2\frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0$$

Subject to $$u(0,t) = 0, \quad \lim_{x\to\infty}\frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0$$

**Solution:**

(Step 1) $$\mathcal{L}_{t\to s}\left\{a^2\frac{\partial^2 u}{\partial x^2}\right\} - \mathcal{L}_{t\to s}\{g\} = \mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

$$a^2\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{g}{s} = s^2U(x,s) - su(x,0) - u_t(x,0) = s^2U(x,s)$$

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{a^2}U(x,s) = \frac{g}{a^2 s}$$

(Step 2) $$\underbrace{\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{a^2} U(x,s)}_{\text{Complementary Function}} = \underbrace{\frac{g}{a^2 s}}_{\text{Particular Solution}}$$

Complementary Function

Particular Solution

$$U_c(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right)$$

$$U_p(x,y) = c_3$$

$$c_3 = -\frac{g}{s^3}$$

$$U(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right) - \frac{g}{s^3}$$

$$U(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right) - \frac{g}{s^3}$$

(Step 3) From initial conditions

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t\to s}\{\lim_{x\to\infty}\frac{\partial u(x,t)}{\partial x}\} = \lim_{x\to\infty}\frac{\partial}{\partial x}U(x,s) = 0.$$

$$c_1 - \frac{g}{s^3} = 0$$

$$c_1 = \frac{g}{s^3}$$

$$\lim_{x\to\infty} c_1 \frac{s}{a}\sinh\left(\frac{s}{a}x\right) + c_2 \frac{s}{a}\cosh\left(\frac{s}{a}x\right) = 0$$

$$\lim_{x\to\infty} c_1 \frac{s}{2a}\exp\left(\frac{s}{a}x\right) + c_2 \frac{s}{2a}\exp\left(\frac{s}{a}x\right) = 0$$

$$c_2 = -c_1 = -\frac{g}{s^3}$$

Note:

$$\lim_{x\to\infty}\sinh(x) = \lim_{x\to\infty}\cosh(x) = \exp(x)/2$$

$$U(x,s) = \frac{g}{s^3}\left(\cosh\left(\frac{s}{a}x\right) - \sinh\left(\frac{s}{a}x\right)\right) - \frac{g}{s^3} = \frac{g}{s^3}\exp\left(-\frac{s}{a}x\right) - \frac{g}{s^3}$$

$$U(x,s)=\frac{g}{s^3}\exp\left(-\frac{s}{a}x\right)-\frac{g}{s^3}$$

(Step 4)

$$u(x,t)=\mathcal{L}^{-1}_{s\to t}\left\{\frac{g}{s^3}\exp\left(-\frac{s}{a}x\right)-\frac{g}{s^3}\right\}$$

$$u(x,t)=\frac{1}{2}g\left(t-\frac{x}{a}\right)^2\boldsymbol{u}\left(t-\frac{x}{a}\right)-\frac{1}{2}gt^2$$

where $\boldsymbol{u}(t)$ is the unit step function

We have applied the translation property:

$$\mathcal{L}\{f(t-a)u(t-a)\}=\exp(-as)F(s)$$

# 附錄六 Review for Fourier Transforms

Fourier transform

$$\Im\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)e^{-j2\pi fx}dx=G(f)$$

$\Im$ 代表 Fourier transform

inverse Fourier transform

$$\Im^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df=g(x)$$

文獻上其他 Fourier transform 的定義

$$\Im\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}\left[G(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者 $$\Im\left[g(x)\right] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}\left[G(\omega)\right] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者 $$\Im\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) e^{j\alpha x} dx = G(\alpha)$$

$$\Im^{-1}\left[G(\alpha)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha) e^{-j\alpha x} d\alpha = g(x)$$

When $g(x)$ is even

Fourier transform ⟶ Fourier cosine transform

$$\Im_c\left[g(x)\right] = \int_0^\infty g(x)\cos(2\pi fx)dx = G_c(f)$$

$$\Im_c^{-1}\left[G_c(f)\right] = 4\int_0^\infty G_c(f)\cos(2\pi fx)df = g(x)$$

When $g(x)$ is odd

Fourier transform ⟶ Fourier sine transform

$$\Im_s\left[g(x)\right] = \int_0^\infty g(x)\sin(2\pi fx)dx = G_s(f)$$

$$\Im_s^{-1}\left[G_s(f)\right] = 4\int_0^\infty G_s(f)\sin(2\pi fx)df = g(x)$$

# Fourier, Fourier Cosine / Sine Transforms 的微分性質

(1) Fourier transform 的微分性質

$$\Im[g'(x)] = \int_{-\infty}^{\infty} g'(x) e^{-j2\pi fx} dx = g(x) e^{-j2\pi fx}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx$$

$$= j2\pi f\, \Im[g(x)]$$

微分性質做了一些假設： $g(x) = 0$ when $x \to \infty$ and $x \to -\infty$

以此類推 $\Im\left[g^{(n)}(x)\right] = (j2\pi f)^{(n)} G(f)$

比較：對 Laplace transform

$$L\{f'(x)\} = sL\{f(x)\} - f(0) \qquad \int_0^{\infty} f(x) e^{-sx} dx$$

對 Fourier transform

$s \to -j\alpha$, without initial conditions

(2) Fourier cosine transform 的微分性質

$$\Im_c\left[g'(x)\right]=\int_0^\infty g'(x)\cos(2\pi fx)\,dx$$

$$=g(x)\cos(2\pi fx)\Big|_0^\infty+2\pi f\int_0^\infty f(x)\sin(2\pi fx)\,dx$$

$$=2\pi f\,\Im_s\left[g(x)\right]-g(0)$$

(3) Fourier sine transform 的微分性質

$$\Im_s\left[g'(x)\right]=\int_0^\infty g'(x)\sin(2\pi fx)\,dx$$

$$=g(x)\sin(2\pi fx)\Big|_0^\infty-2\pi f\int_0^\infty g(x)\cos(2\pi fx)\,dx$$

$$=-2\pi f\,\Im_c\left[g(x)\right]$$

注意： (1) Fourier sine, cosine transforms 互換

(2) $\alpha$ 正負號不同

(3) Fourier cosine transform 要考慮 initial condition

$$\Im_c\left[g'(x)\right] = 2\pi f\,\Im_s\left[g(x)\right] - g(0)$$

$$\Im_s\left[g'(x)\right] = -2\pi f\,\Im_c\left[g(x)\right]$$

$$\Im_c\left[g''(x)\right] = 2\pi f\,\Im_s\left[g'(x)\right] - g'(0) = -4\pi^2 f^2\,\Im_c\left[g(x)\right] - g'(0)$$

$$\Im_s\left[g''(x)\right] = -2\pi f\,\Im_c\left[g'(x)\right] = -4\pi^2 f^2\,\Im_s\left[g(x)\right] + 2\pi f\,g(0)$$

# 2.11 Solving PDEs by Fourier Transforms

Differentiation → Multiplication

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.4

# Method

(Condition 1) interval 為 $-\infty < v < \infty$ 時:

用 Fourier transform

(Condition 2) interval 為 $0 < v < \infty$，

有 "$u(v, \ldots..) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier sine transform

(Condition 3) interval 為 $0 < v < \infty$，

有 "$\frac{\partial}{\partial v} u(v, \cdots\cdots) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier cosine transform

**使用 Fourier transform, Fourier cosine transform, Fourier sine transform 來解 partial differential equation (PDE) 的 BVP 或IVP 的解法流程**

**(Step 1)** 以 page 253 的規則，來決定要針對 **哪一個** independent variable ，做**什麼** transform (Fourier, Fourier cosine, 或Fourier sine transform)

**(Step 2)** 對 PDE 做 Step 1 所決定的 transform, 則原本的 PDE 變成針對另外一個 independent variable 的 ordinary differential equation (ODE)

**(Step 3)** 將 Step 2 所得出的 ODE 的解算出來

**(Step 4)** Step 3 所得出來的解會有一些 constants，可以對 initial conditions (或 boundary conditions) 做 transform 將 constants 解出

(※ 和 Step 1 所做的 transform 一樣，只是 transform 的對象變成是 initial 或 boundary conditions，見 pages 256, 259 的例子)

**(Step 5)** 最後，別忘了做 inverse transform (畫龍點睛)

**[Example 1]**

heat equation: $k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$ $\quad -\infty<x<\infty \quad t>0$

subject to $u(x,0)=g(x)$ where $g(x)=\begin{cases} u_0, & |x|<1 \\ 0, & |x|>1 \end{cases}$

Step 1 決定針對 $x$ 做 Fourier transform

$$\Im_{x\to f}\{u(x,t)\}=\int_{-\infty}^{\infty}u(x,t)e^{-i2\pi fx}dx=U(f,t)$$

Step 2 $\Im_{x\to f}\left\{k\frac{\partial^2 u}{\partial x^2}\right\}=\Im_{x\to f}\left\{\frac{\partial u}{\partial t}\right\}$

原本對 $x, t$ 兩個變數做偏微分

$$-k4\pi^2 f^2 U(f,t)=\frac{\partial U(f,t)}{\partial t}$$

經過 Fourier transform 之後，
只剩下對 $t$ 做偏微分

$$\frac{dU(f,t)}{dt} + 4k\pi^2 f^2 U(f,t) = 0$$

對於 $t$ 而言，是 1st order ODE

Step 3 $U(f,t) = c\, e^{-4k\pi^2 f^2 t}$ 這邊的 $c$ 值，對 $t$ 而言是 constant ，

但是可能會 **dependent on *f*** (特別注意)

Step 4 根據 $u(x, 0) = g(x)$ 將 $c$ 解出

和 Step 1 一樣，也是針對 $x$ 做 Fourier transform

只是對象改成 initial condition

$$\Im_{x\to f}\left\{u(x,0)\right\} = \int_{-\infty}^{\infty} g(x) e^{-i2\pi fx} dx = \int_{-1}^{1} u_0 e^{-i2\pi fx} dx$$

$$= u_0 \frac{e^{-i2\pi f} - e^{i2\pi f}}{-i2\pi f} = u_0 \frac{\sin(2\pi f)}{\pi f}$$

因為 $\Im_{x\to f}\left\{u(x,0)\right\} = U(f,0)$

$$U(f,0) = u_0 \frac{\sin(2\pi f)}{\pi f}$$

$$U(f,t)=c\,e^{-4k\pi^2 f^2 t} \xleftrightarrow{\text{比較係數}} U(f,0)=u_0\frac{\sin(2\pi f)}{\pi f}$$

解出 $c=u_0\dfrac{\sin(2\pi f)}{\pi f}$

$$U(f,t)=u_0\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}$$

Step 5 未完待續，別忘了最後要做 inverse Fourier transform

$$u(x,t)=\mathfrak{I}^{-1}_{f\to x}\left[U(f,t)\right]=\int_{-\infty}^{\infty}u_0\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}e^{j2\pi fx}df$$

不易化簡，課本僅依據 $\dfrac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}$ 對 $f$ 而言是 even function 將 $u(x, t)$ 化簡為

$$u(x,t)=u_0\int_{-\infty}^{\infty}\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}(\cos(2\pi fx)+j\sin(2\pi fx))df$$

$$=u_0\int_{-\infty}^{\infty}\frac{\sin(2\pi f)\cos(2\pi fx)}{\pi f}e^{-4k\pi^2 f^2 t}df$$

**[Example 2]** Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 1 \qquad y > 0$$

$$u(0,y) = 0 \qquad u(1,y) = e^{-y} \qquad y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \qquad 0 < x < 1$$

Step 1 決定針對 $y$ 做 Fourier cosine transform

$$\Im_{c,y\to f}\{u(x,y)\} = \int_0^\infty u(x,y)\cos(2\pi f y)dy = U(x,f)$$

Step 2 $\Im_{c,y\to f}\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \Im_{c,y\to f}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \Im_{c,y\to f}\{0\}$

from $\Im_c\left[g''(y)\right] = -4\pi^2 f^2 \Im_c\left[g(y)\right] - g'(0)$

$$\frac{d^2 U(x,f)}{dx^2} - 4\pi^2 f^2 U(x,f) = 0$$ 對於 $x$ 的 $2^{nd}$ order ODE

Step 3 $\dfrac{d^2U(x,f)}{dx^2} - 4\pi^2 f^2 U(x,f) = 0$

$\Rightarrow$ $U(x,f) = c_1 \cosh(2\pi fx) + c_2 \sinh(2\pi fx)$

Step 4 由 $u(0,y) = 0$ $u(\pi,y) = e^{-y}$ 來解 $c_1$, $c_2$ (may be dependent on $f$)

和 Step 1 一樣，也是針對 $y$ 做 Fourier cosine transform

只是對象改成 boundary conditions

(1) $U(0,f) = \Im_{c,y\to f}\{u(0,y)\} = \int_0^\infty 0\cdot\cos(2\pi fy)dy = 0$

(2) $U(1,f) = \Im_{c,y\to f}\{u(1,y)\} = \int_0^\infty e^{-y}\cdot\cos(2\pi fy)dy = \dfrac{1}{1+(2\pi f)^2}$

(可以用Laplace transform 的「取巧法」)

$$U(x,f)=c_1\cosh(2\pi fx)+c_2\sinh(2\pi fx)$$

分別代入 $U(0,f)=0$ $\quad U(1,f)=\dfrac{1}{1+(2\pi f)^2}$

$$c_1=0 \qquad c_1\cosh(2\pi f)+c_2\sinh(2\pi f)=\frac{1}{1+(2\pi f)^2}$$

$$\Longrightarrow \quad c_1=0 \qquad c_2=\frac{1}{(1+4\pi^2 f^2)\sinh(2\pi f)}$$

$$U(x,f)=\frac{\sinh(2\pi fx)}{(1+4\pi^2 f^2)\sinh(2\pi f)}$$

Step 5 inverse cosine transform

$$u(x,y)=\mathfrak{I}^{-1}_{c,f\to y}\left[U(x,f)\right]=4\int_0^{\infty}\frac{\sinh(2\pi fx)}{(1+4\pi^2 f^2)\sinh(2\pi f)}\cos(2\pi fy)\,df$$

(算到這裡即可，難以繼續化簡)

## 本節需要注意的地方

(1) 微分公式當中，Fourier cosine transform 和 Fourier sine transform 會有互換的情形。(See page 250)

(2) 在解 boundary value problem 時，要了解

何時用 Fourier transform，

何時用Fourier cosine transform,

何時用 Fourier sine transform (see page 253)

(3) 在解 partial differential equation 時，往往只針對一個 independent variable 做 Fourier transform, 另一個 independent variable 不受影響，如 Examples 1 and 2 的例子

計算過程中，自己要清楚是對哪一個 independent variable 做Fourier transform

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