3. Function Approximation

[1] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[2] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

Function Approximation

- Function Approximation
  - Theorems for Orthogonal Basis (Sec. 3.1)
  - by Polynomials
    - Legendre Polynomials (Sec. 3.2)
    - Other Polynomials (Sec. 3.4)
  - by Other Function Set (Sec. 3.3)

Sec. 3.1 Review for Orthogonal Basis Expansion

(1) inner product on an interval [a, b]

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx \qquad (f_1, f_2 \text{ 為 real 時})$$

or $$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx \qquad \text{(more standard definition)}$$

(2) orthogonal on an interval [a, b]

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0 \qquad (f_1, f_2 \text{ 為 real 時})$$

$$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx = 0 \qquad \text{(more standard definition)}$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Sections 11-1~11-3.

注意：任何 even function 和任何 odd function 在 [-*b*, *b*] 之間必為 orthogonal, $(a = -b)$

$$\int_{-b}^{b} f_1(x) f_2(x)\, dx = 0$$

even odd

(3) orthogonal set

For a set of functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ………., if

$$\int_{a}^{b} \phi_m(x) \phi_n^*(x)\, dx = 0 \quad \text{for any } m \neq n$$

then $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. is an orthogonal set on the interval $[a, b]$

(4) square norm

$$\|f(x)\|^2 = (f(x), f(x)) = \int_a^b f(x) f^*(x)\,dx = \int_a^b |f(x)|^2\,dx$$

(5) norm

$$\|f(x)\| = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x) f^*(x)\,dx}$$

(6) orthonormal set

對一個 orthogonal set, 若更進一步的滿足

$$\int_a^b \phi_n(x)\phi_n^*(x)\,dx = 1 \qquad \text{for all } n$$

則被稱為 orthonormal set

(7) normalize

$$\psi(x) \xrightarrow{\text{normalize}} v(x) = \frac{\psi(x)}{\|\psi(x)\|}$$

將 norm 變為 1

(8) complete

若在 interval [a, b] 之間，任何一個 function $f(x)$ 都可以表示成 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ………..的 linear combination

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots\cdots\cdots = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

則 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. 被稱作 complete

(9) orthogonal series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \qquad \text{where} \qquad c_n = \frac{\int_a^b f(x)\phi_n^*(x)dx}{\int_a^b \phi_n(x)\phi_n^*(x)dx}$$

Specially, if $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. are orthonormal

$$c_n = \int_a^b f(x)\phi_n^*(x)dx$$

(10) weight

inner product with weight function

$$\left(f_1(x), f_2(x)\right) = \int_a^b w(x) f_1(x) f_2^*(x)\, dx$$

$w(x)$ is called the weight function

With the weight function

(10-1) orthogonal 的定義改成

$$\left(f_m, f_n\right) = \int_a^b w(x) f_m(x) f_n^*(x)\, dx = 0 \quad \text{for } m \neq n$$

(10-2) square norm 的定義改成 $\|f(x)\|^2 = \int_a^b w(x) f(x) f^*(x)\, dx$

(10-3) norm 的定義改成 $\|f(x)\| = \sqrt{\int_a^b w(x) f(x) f^*(x)\, dx}$

(10-4) orthonormal 的定義改成

$$\int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

$$\int_a^b w(x) f_n(x) f_n^*(x) dx = 1$$

(10-5) normalize 的算法改成

$$v(x) = \frac{\psi(x)}{\|\psi(x)\|} = \frac{\psi(x)}{\sqrt{\int_a^b w(x)\psi(x)\psi^*(x) dx}}$$

(10-6) orthogonal series expansion of $f(x)$ 的算法改成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) \phi_n(x) \phi_n^*(x) dx}$$

orthonormal case: $c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$

trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots\ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\ldots\right\}$$

form a complete and orthogonal set on the interval of $[-p\ ,\ p]$

$\cos\frac{n\pi}{p}x,\quad \sin\frac{n\pi}{p}x$ 週期：$\frac{2p}{n}$ 頻率：$\frac{n}{2p}$

Fourier Series (expanded by trigonometric functions)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

Q:

Why should we use the orthogonal basis?

(1)

(2) No Interference

(3) More basis functions $\rightarrow$ Less error

Section 3-2 Polynomial Approximation Using Legendre Polynomials

- **Legendre's equation of order *n***

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

One of the solution: Legendre polynomials

$n=0$ $P_0(x)=1$

$n=1$ $P_1(x)=x$

$n=2$ $P_2(x)=\frac{1}{2}(3x^2-1)$

$n=3$ $P_3(x)=\frac{1}{2}(5x^3-3x)$

$n=4$ $P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$

$n=5$ $P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Sections 6-4, 11-5.

3.2.1 Legendre Polynomial

Legendre's Equation $(1-x^2)y''-2xy'+n(n+1)y=0$

$$y(x)=\sum_{k=0}^{\infty}c_k x^k$$ 代入，得出

Two linearly independent solutions are

$$y_1(x)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4 -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\cdots\cdots\right]$$

$$y_2(x)=c_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\cdots\cdots\right]$$

(a) When n is not an integer, both the two solutions have infinite number of terms.

(b) When n is an even integer, $y_1(x)$ has finite number of terms.

In $y_1(x)$, the coefficient of x^k is zero when $k > n$.

(c) When n is an odd integer, $y_2(x)$ has finite number of terms.

In $y_2(x)$, the coefficient of x^k is zero when $k > n$.

$y_1(x)$ when n is an even integer and $y_2(x)$ when n is an odd integer are called the Legendre polynomials (denoted by $P_n(x)$).

通常選

$$c_0 = (-1)^{n/2}\frac{1\cdot 3\cdots\cdots(n-1)}{2\cdot 4\cdots\cdots n} \qquad c_1 = (-1)^{(n-1)/2}\frac{1\cdot 3\cdots\cdots n}{2\cdot 4\cdots\cdots(n-1)}$$

(讓 $P_n(1)$ 一律等於 1)

由 $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

由 $y_2(x)$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

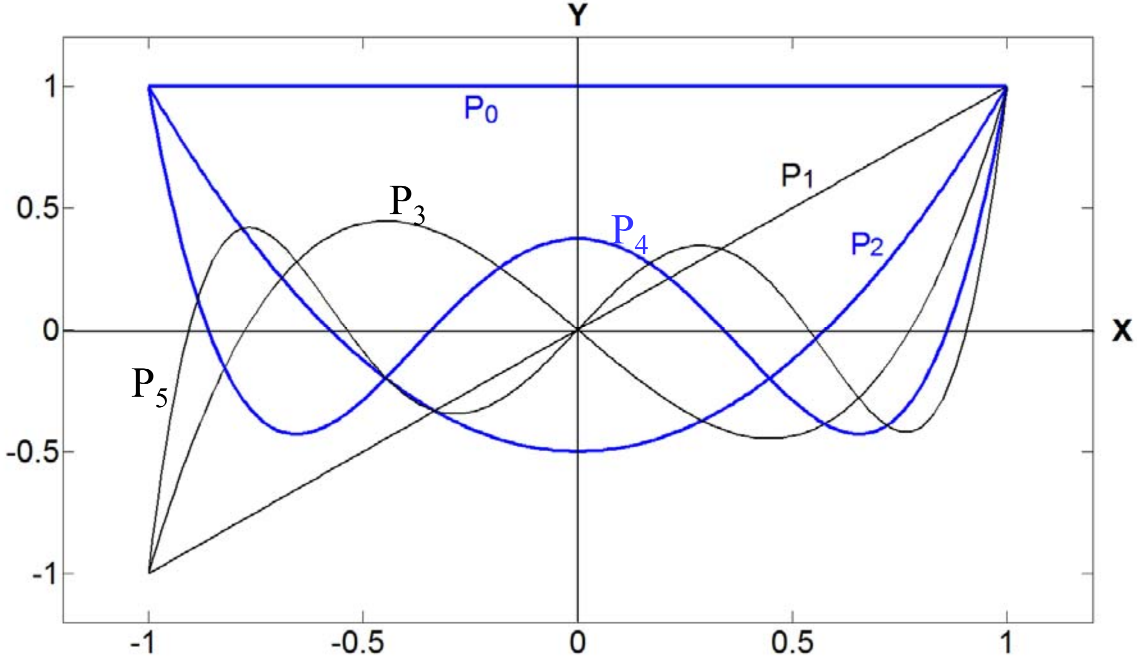
$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

In general,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

Rodrigues' formula

Legendre polynomials



Interval:

$x \in [-1, 1]$

(1) $P_n(-x) = (-1)^n P_n(x)$ even / odd symmetry

(2) $P_n(1) = 1$ $P_n(-1) = (-1)^n$

(3) $P_n(0) = 0$ when n is odd

(4) $P_n'(0) = 0$ when n is even

(5) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ recursive relation

(6) $\int_{-1}^{1} P_n(x) P_n(x) dx = \frac{2}{2n+1}$

(7) $\int_{-1}^{1} P_m(x) P_n(x) dx = 0$ If $m \neq n$ orthogonality property

Orthogonality property 才是 Legendre polynomials 最重要的性質

3.2.2 Expansion by Legendre Polynomials

若任何在 $x \in [-1, 1]$ 區間為 continuous 的函式 $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

由於

$$\int_{-1}^{1} f(x) P_m(x)\,dx = \underbrace{\sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\,dx = a_m \int_{-1}^{1} P_m(x) P_m(x)\,dx}_{\text{根據 orthogonality property}}$$

所以

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\,dx}{\int_{-1}^{1} P_n(x) P_n(x)\,dx} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\,dx$$

Q: How do we use the N^{th} order polynomial to approximate a function $f(x)$ for $x \in [-1, 1]$?

Answer:

note

$$f(x) = \sum_{n=0}^{N} a_n P_n(x)$$

where $$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx$$

The detail of proof can be seen from Section 3.3.

[Example 1] Suppose that

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad \text{when } -1 \le x \le 1.$$

Try to approximate $f(x)$ by a 2nd order polynomial

$$f(x) \cong f_2(x) = a_0 + a_1 x + a_2 x^2$$

to minimize $\int_{-1}^{1} \left(f(x) - f_2(x)\right)^2 dx$

(Solution):

Note that $1, x, x^2$ are not a orthonormal set with $x \in [-1, 1]$.
Therefore, instead, we may adopt the Legendre polynomials.

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

Then $$f(x) \cong f_2(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$$

where $$c_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2}\int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2}\int_{-1}^{1} x\cos\left(\frac{\pi}{2}x\right) dx = 0$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)\, dx = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}\left(3x^2 - 1\right)\cos\left(\frac{\pi}{2}x\right) dx$$

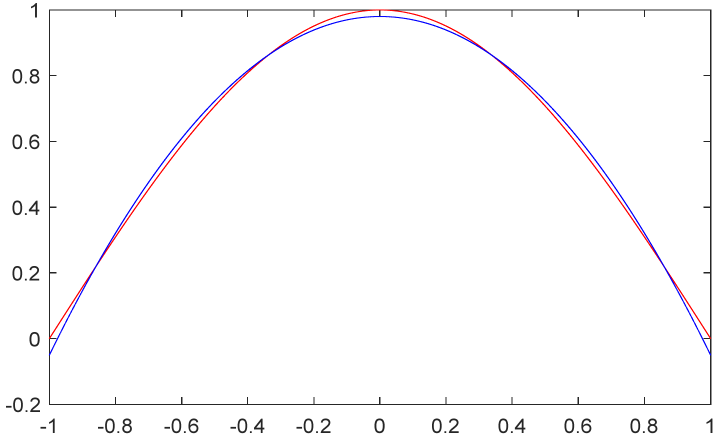
$$= \frac{1}{2}\frac{5}{2}\left(\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\left(3x^2 - 1\right)\Big|_{-1}^{1} - \int_{-1}^{1} \frac{12}{\pi} x \sin\left(\frac{\pi}{2}x\right) dx \right)$$

$$= \frac{5}{4}\left(\frac{8}{\pi} - \frac{96}{\pi^3}\right)$$

$$f_2(x) = \frac{2}{\pi}(1) + 0 \cdot x + \frac{5}{4}\left(\frac{8}{\pi} - \frac{96}{\pi^3}\right)\frac{1}{2}\left(3x^2 - 1\right)$$

$$f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \qquad f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$



red line: $f(x)$

blue line: $f_2(x)$

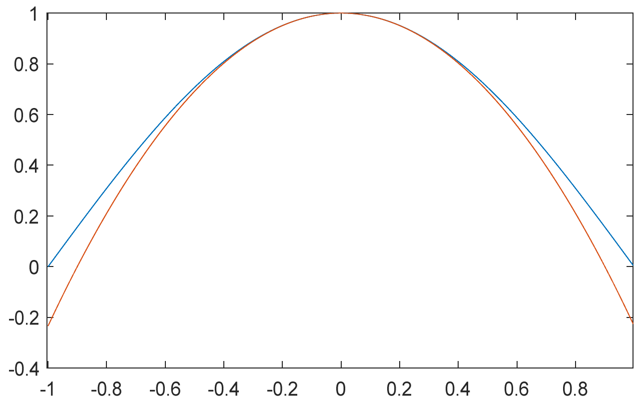
Approximation error is

$$\int_{-1}^{1}\left(f(x) - f_2(x)\right)^2 dx = 0.00059606$$

Comparison: When using the Taylor series,

$$f(x) \cong T(x) = 1 - \frac{\pi^2}{8}x^2$$

$$\int_{-1}^{1}\left(f(x) - T(x)\right)^2 dx = 0.0125$$



[Example 2] (Zill page 456)

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0\le x<1.\end{cases}$$

SOLUTION

Using $$c_n=\frac{\int_a^b f(x)P_n^*(x)\,dx}{\int_a^b P_n(x)P_n^*(x)\,dx}$$

$$c_0=\frac{1}{2}\int_{-1}^{1} f(x)P_0(x)dx=\frac{1}{2}\int_0^1 1\cdot 1dx=\frac{1}{2}$$

$$c_1=\frac{3}{2}\int_{-1}^{1} f(x)P_1(x)dx=\frac{3}{2}\int_0^1 1\cdot xdx=\frac{3}{4}$$

$$c_2=\frac{5}{2}\int_{-1}^{1} f(x)P_2(x)dx=\frac{5}{2}\int_0^1 1\cdot \frac{1}{2}(3x^2-1)dx=0$$

$$c_3=-\frac{7}{16},\quad c_4=0,\quad c_5=\frac{11}{32},\quad \cdots\cdots\cdots$$

$$f(x)=\frac{1}{2}P_0(x)+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)+\cdots.$$

If we want to approximate $f(x)$ by the 1st order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate $f(x)$ by the 2nd order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate $f(x)$ by the 3rd order polynomial,

$$\begin{aligned} f(x) &\cong c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) \\ &= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) - \frac{7}{16} P_3(x) = \frac{1}{2} + \frac{45}{32} x - \frac{35}{32} x^3 \end{aligned}$$

3.2.3 Generalization for the Interval

Problem :

How do we expand $f(x)$ where the range of x is $[a, b]$ and $a \neq -1$, $b \neq 1$?

(1) $$g(x) = f\left(\frac{a-b}{2}x + \frac{a+b}{2}\right)$$ Note: The range of x is changed into [-1, 1]

(2) Expand $g(x)$ by Legendre polynomials

$$g(x) \cong \sum_{n=0}^{N} c_n P_n(x)$$ where $$c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} g(x) P_n(x)\,dx$$

(3) $$f(x) = g\left(\frac{2}{a-b}\left(x - \frac{a+b}{2}\right)\right) \cong \sum_{n=0}^{N} c_n P_n\left(\frac{2}{a-b}\left(x - \frac{a+b}{2}\right)\right)$$

Section 3.3 Generalization for Function Set Approximation

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. are a set of independent functions where $x \in [a, b]$ and the weight function is $w(x)$.

Problem :

How do we expand $f(x)$ by $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ …. $\phi_N(x)$

$$f_N(x) = \sum_{n=0}^{N} c_n \phi_n(x)$$

where $$error = \left\| f(x) - f_N(x) \right\| = \sqrt{\int_a^b w(x)\left(f(x) - f_N(x)\right)^2 dx}$$

is minimized.

(Case 1): The function set is complete and orthogonal / orthonormal

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) \left|\phi_n(x)\right|^2 dx}$$

(orthogonal case)

$$c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

(orthonormal case)

$$error = 0$$

(Case 2): The function set is incomplete and orthonormal

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x) \qquad d_n = \int_a^b w(x) f(x) \phi_n^*(x)\,dx$$

$$\text{error} = \left\| f(x) - f_N(x) \right\| = \sqrt{\int_a^b w(x) f^2(x)\,dx - \sum_{n=0}^{N} |d_n|^2}$$

(Case 3): The function set is incomplete and orthogonal

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x) \qquad d_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x)\,dx}{\int_a^b w(x) |\phi_n(x)|^2\,dx}$$

$$\text{error} = \left\| f(x) - f_N(x) \right\|$$

$$= \sqrt{\int_a^b w(x) f^2(x)\,dx - \sum_{n=0}^{N} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2\,dx}$$

(Case 4): The function set is not orthogonal

Using the Gram-Schmidt method to convert it into an orthonormal set.

(page 295)

3.3.1 Orthonormal / Orthogonal Function Set Expansion

(Case 1)

[Theorem 3.3.1] Parseval's Theorem (Energy Preservation Theorem)

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. are a complete and orthonormal function set for $x \in [a, b]$ and

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad \text{where} \qquad c_n = \left(f(x), \phi_n(x)\right) = \int_a^b w(x) f(x) \phi_n^*(x)\, dx$$

Then

$$\left(f(x), f(x)\right) = \sum_{n=1}^{\infty} |c_n|^2 \qquad \int_a^b w(x) |f(x)|^2\, dx = \sum_{n=1}^{\infty} |c_n|^2$$

(Proof):

$$\int_a^b w(x)\left|f(x)\right|^2 dx = \int_a^b w(x) f(x) f^*(x)\,dx$$

$$= \int_a^b w(x) \sum_{n=0}^{\infty} c_n \phi_n(x) \sum_{m=0}^{\infty} c_m^* \phi_m^*(x)\,dx$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x) \phi_n(x) \phi_m^*(x)\,dx$$

Since $\int_a^b w(x)\phi_n(x)\phi_m^*(x)\,dx = 0$ if $n \neq m$

$$\int_a^b w(x)\phi_n(x)\phi_n^*(x)\,dx = 1$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x) \phi_n(x) \phi_m^*(x)\,dx = \sum_{n=0}^{\infty} c_n c_n^* = \sum_{n=0}^{\infty} \left|c_n\right|^2$$

$$\int_a^b w(x)\left|f(x)\right|^2 dx = \sum_{n=0}^{\infty} \left|c_n\right|^2$$

(Case 2)

[Theorem 3.3.2] Error When Expanded by an Incomplete Orthonormal Set

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ………., $\phi_N(x)$ are an incomplete and orthonormal function set for $x \in [a, b]$. Now, we want to approximate $f(x)$ by a linear combination of $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ………., $\phi_N(x)$:

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x)$$

To minimize the approximation error, d_n should be calculated from

$$d_n = \int_a^b w(x) f(x) \phi_n^*(x)\, dx$$

Then, the approximation error is:

$$\text{approximation error} = \left\| f(x) - f_N(x) \right\| = \sqrt{\int_a^b w(x) \left(f(x) - f_N(x) \right)^2 dx}$$

$$= \sqrt{\int_a^b w(x) f^2(x)\, dx - \sum_{n=0}^{N} \left| d_n \right|^2}$$

(Proof):

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……….., $\phi_N(x)$ is a subset of the complete and orthonormal function set $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……..., $\phi_N(x)$, $\phi_{N+1}(x)$, ……..……. Then, $f(x)$ can be expressed as

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x) \quad \text{where} \quad c_n=\int_a^b w(x)f(x)\phi_n^*(x)\,dx$$

Therefore,

$$f(x)-f_N(x)=\sum_{n=0}^{N}(c_n-d_n)\phi_n(x)+\sum_{n=N+1}^{\infty}c_n\phi_n(x)$$

Then, from Theorem A1,

$$\left\|f(x)-f_N(x)\right\|^2=\sum_{n=0}^{N}\left|c_n-d_n\right|^2+\sum_{n=N+1}^{\infty}\left|c_n\right|^2$$

To minimize $\left\|f(x)-f_N(x)\right\|^2$, we should set

$$d_n=c_n=\int_a^b w(x)f(x)\phi_n^*(x)\,dx \qquad \text{for } n=0,\ 1,\ \ldots,\ N$$

Note that, when $d_n = c_n$, then

$$\left\| f(x) - f_N(x) \right\|^2 = \sum_{n=N+1}^{\infty} \left| c_n \right|^2$$

$$= \left\| f(x) \right\|^2 - \sum_{n=0}^{N} \left| c_n \right|^2$$

$$= \int_a^b w(x) \left| f(x) \right|^2 dx - \sum_{n=0}^{N} \left| c_n \right|^2$$

Theorem 3.3.2 is hence proved.

(Case 3)

[Theorem 3.3.3] Error When Expanded by an Incomplete Orthogonal Set

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_N(x)$ are an incomplete and orthogonal function set for $x \in [a, b]$. Now, we want to approximate $f(x)$ by a linear combination of $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_N(x)$:

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x)$$

To minimize the approximation error, d_n should be calculated from

$$d_n = \int_a^b w(x) f(x) \phi_n^*(x) dx \Big/ \int_a^b w(x) |\phi_n(x)|^2 dx$$

Then, the approximation error is:

$$\text{approximation error} = \|f(x) - f_N(x)\| = \sqrt{\int_a^b w(x)\left(f(x) - f_N(x)\right)^2 dx}$$

$$= \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^{N} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx}$$

3.3.2 Non-Orthogonal Function Set Expansion

(Case 4)

How do we generate an orthonormal function set if $\{p_0(x),\ p_1(x),\ p_2(x),\ p_3(x)\ ,\ \ldots\ldots\ldots\}$ is not an orthogonal function set?

Gram-Schmidt process

$$\phi_0(x)=\frac{p_0(x)}{\left\|p_0(x)\right\|} \qquad n=0$$

$$\Downarrow$$

$$g_n(x)=p_n(x)-\sum_{m=0}^{n-1}\left(p_n(x),\phi_m(x)\right)\phi_m(x)$$

$$\Downarrow$$

$$\phi_n(x)=\frac{g_n(x)}{\left\|g_n(x)\right\|}$$

$$\Downarrow$$

$$n=n+1$$

Proof for orthogonality of the function set generated from the Gram-Schmidt process

If $k < n$,

$$\left(\phi_n(x),\phi_k(x)\right)=\left(\frac{g_n(x)}{\left\|g_n(x)\right\|},\phi_k(x)\right)$$

$$=\frac{1}{\left\|g_n(x)\right\|}\left(p_n(x)-\sum_{m=0}^{n-1}\left(p_n(x),\phi_m(x)\right)\phi_m(x),\quad \phi_k(x)\right)$$

$$=\frac{1}{\left\|g_n(x)\right\|}\left[\left(p_n(x),\phi_k(x)\right)-\sum_{m=0}^{n-1}\left(p_n(x),\phi_m(x)\right)\left(\phi_m(x),\phi_k(x)\right)\right]$$

$$=\frac{1}{\left\|g_n(x)\right\|}\left[\left(p_n(x),\phi_k(x)\right)-\left(p_n(x),\phi_k(x)\right)\right]=0$$

[Example 1]

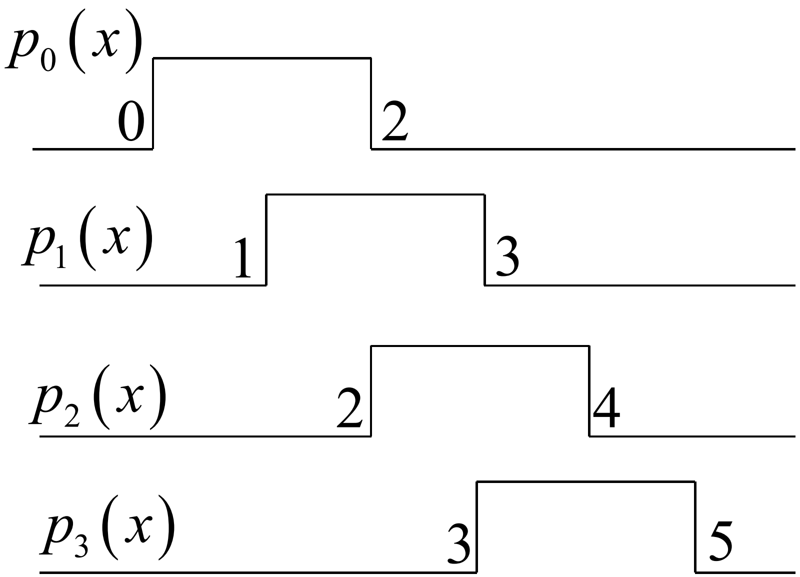
Derive $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……….. such that

$$\phi_n(x) = \sum_{k=0}^{N} \tau_{n,k} p_k(x), \qquad p_k(x) = u(x-k) - u(x-k-2)$$

$u(x)$: unit step function

and

$$\int_0^\infty \phi_n(x)\phi_m(x)\,dx = 0 \quad \text{when } m \neq n \qquad\qquad \int_0^\infty \phi_n(x)\phi_n(x)\,dx = 1$$



⟹ orthonormalization

Answer:

Set $p_k(x) = u(x-k) - u(x-k-2)$

$$\phi_0(x) = \frac{p_0(x)}{\|p_0(x)\|} = \frac{u(x)-u(x-2)}{\sqrt{2}}$$

$$g_1(x) = p_1(x) - (p_1(x), \phi_0(x))\phi_0(x)$$

$$= u(x-1) - u(x-3) - \frac{1}{\sqrt{2}}\left(\frac{u(x)-u(x-2)}{\sqrt{2}}\right)$$

$$\phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|} = \frac{u(x-1)-u(x-3)-\frac{1}{2}(u(x)-u(x-2))}{\sqrt{6}/2}$$

$$g_2(x) = p_2(x) - \left(p_2(x), \phi_0(x)\right)\phi_0(x) - \left(p_2(x), \phi_1(x)\right)\phi_1(x)$$

$$= u(x-2) - u(x-4) - \frac{2}{\sqrt{6}} \frac{u(x-1) - u(x-3) - \frac{1}{2}\left(u(x) - u(x-2)\right)}{\sqrt{6}/2}$$

$$= u(x-2) - u(x-4) - \frac{2}{3}u(x-1) + \frac{2}{3}u(x-3) + \frac{1}{3}\left(u(x) - u(x-2)\right)$$

$$\phi_2(x) = \frac{g_2(x)}{\left\| g_2(x) \right\|}$$

$$= \frac{u(x-2) - u(x-4) - \frac{2}{3}\left(u(x-1) - u(x-3)\right) + \frac{1}{3}\left(u(x) - u(x-2)\right)}{2/\sqrt{3}}$$

…………………

…………………

附錄七 Compressive Sensing

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ ………….. form an over-complete and non-orthogonal spanning set.

(Problem 1) We want to minimize $||c||_0$ ($||\ ||_0$ 是 zero-order norm， $||c||_0$ 意指 c_m 的值不為 0 的個數） such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_0$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_0$ is limited to M, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Examples for the spanning set used for compressive sensing (Since they are not linearly independent, they are not basis).

3-atom form

$$b_m(t) = \exp(j2\pi f_m t)\exp(-\frac{\pi(t-t_m)^2}{\sigma_m^2})$$

4-atom form

$$b_m(t) = \exp(j2\pi(f_m t + \eta_m t^2))\exp(-\frac{\pi(t-t_m)^2}{\sigma_m^2})$$

Problems: over-complete, non-orthogonal, too many functions in the sets

Section 3.4 Other Orthogonal Polynomials

In addition to Legendre Polynomials, there are many other orthogonal polynomials. However, their weight functions and intervals are different.

[1] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

[2] M. R. Spiegel, Mathematical Handbook, Schaum, 1990.

- Associated Legendre Functions

$$P_{n,m}(x) = \frac{d^m}{dx^m} P_n(x)$$

$P_n(x)$: the Legendre polynomial of order n, in fact, $P_n(x) = P_{n,0}(x)$

They are orthogonal for $x \in [-1, 1]$, $w(x) = \left(1 - x^2\right)^m$

$$\int_{-1}^{1} \left(1 - x^2\right)^m P_{n,m}(x) P_{k,m}(x) dx = 0 \qquad \text{if } n \neq k$$

$$\int_{-1}^{1} \left(1 - x^2\right)^m \left(P_{n,m}(x)\right)^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

Ex: When $m = 1$,

$$P_{1,1}(x) = 1 \qquad P_{2,1}(x) = 2x$$

$$P_{3,1}(x) = \frac{15}{2} x^2 - \frac{3}{2} \qquad P_{4,1}(x) = \frac{35}{2} x^3 - \frac{15}{2} x$$

- Hermite polynomials 電磁波、光學、頻譜分析常用

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

They are orthogonal for $x \in (-\infty, \infty)$ $w(x) = e^{-x^2}$

$$\int_{-\infty}^{\infty} e^{-x^2} P_m(x) P_n(x)\,dx = 0 \qquad \int_{-\infty}^{\infty} e^{-x^2} \left(P_n(x)\right)^2 dx = 2^n n! \sqrt{\pi}$$

They are the solutions of $P_n''(x) - xP_n'(x) + nP_n(x) = 0$

$P_0(x) = 1$ $P_1(x) = 2x$

$P_2(x) = 4x^2 - 2$ $P_3(x) = 8x^3 - 12x$

$P_4(x) = 16x^4 - 48x^2 + 12$ $P_5(x) = 32x^5 - 160x^3 + 120x$

- Laguerre polynomials

$$P_n(x) = e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

They are orthogonal for $x \in [0, \infty)$ $w(x) = e^{-x}$

$$\int_0^\infty e^{-x} P_m(x) P_n(x)\, dx = 0 \qquad \int_0^\infty e^{-x} \left(P_n(x)\right)^2 dx = (n!)^2$$

They are the solutions of $xP_n''(x) + (1-x)P_n'(x) + nP_n(x) = 0$

$P_0(x) = 1$ $P_1(x) = -x + 1$

$P_2(x) = x^2 - 4x + 2$ $P_3(x) = -x^3 + 9x^2 - 18x + 6$

- Associated Laguerre polynomials

$$P_{n.m}(x)=\frac{d^m}{dx^m}\left(P_n(x)\right)$$

where $P_n(x)$ is the nth order Laguarre polynomial

They are orthogonal for $x \in [0, \infty)$ $w(x)=x^m e^{-x}$

$$\int_0^\infty x^m e^{-x} P_m(x) P_n(x)\,dx=0 \qquad \int_0^\infty x^m e^{-x}\left(P_n(x)\right)^2 dx=\frac{(n!)^2}{(n-m)!}$$

They are the solutions of $xP_n''(x)+(m+1-x)P_n'(x)+(n-m)P_n(x)=0$

$$P_{1,1}(x)=-1 \qquad P_{2,1}(x)=2x-4$$

$$P_{3,1}(x)=-3x^2+18x-18 \qquad P_{4,1}(x)=4x^3-48x^2+144x-96$$

- Chebychev polynomials 電子學和 filter design 常用

$$P_n(\cos\theta) = \cos n\theta$$

$$P_n(x) = \sum_{k=0}^{n/2} \binom{n}{2k} x^{n-2k} \left(1-x^2\right)^k$$

They are orthogonal for $x \in [-1, 1]$ $w(x) = \frac{1}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} P_m(x) P_n(x)\, dx = 0$$

$$\int_{-1}^{1} \frac{P_n^2(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & \text{if } n = 0, \\ \pi/2 & \text{otherwise} \end{cases}$$

They are the solutions of $(1-x^2)P_n''(x) - xP_n'(x) + n^2 P_n(x) = 0$

$P_0(x) = 1$ $P_1(x) = x$

$P_2(x) = 2x^2 - 1$ $P_3(x) = 4x^3 - 3x$