4. Fourier Analysis

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[3] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019, Chapter 15.

Fourier Transform

- Fourier Transform
  - Basic
    - Definition (Sec. 4-1)
    - Properties (Sec. 4-3)
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4.1 Definition of the Fourier Transform

Fourier transform

$$\Im\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)e^{-j2\pi fx}dx=G(f)$$

$\Im$ 代表 Fourier transform

inverse Fourier transform

$$\Im^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df=g(x)$$

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019, Sections 15-2.

Review: Fourier Series of the Complex Form

(Compared to pages 267, 270)

If $g(x) = g(x+T)$, then

$$g(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi}{T}nx\right)$$

where

$$c_n = \frac{\int_{-T/2}^{T/2} g(x)\, conj\left(\exp\left(j\frac{2\pi}{T}nx\right)\right) dx}{\int_{-T/2}^{T/2} \exp\left(j\frac{2\pi}{T}nx\right) conj\left(\exp\left(j\frac{2\pi}{T}nx\right)\right) dx}$$

$$c_n = \frac{\int_{-T/2}^{T/2} g(x) \exp\left(-j\frac{2\pi}{T}nx\right) dx}{T}$$

4.1.1 Derivation and Physical Meaning

Fourier transform can be viewed as the Fourier series where

$$T \to \infty$$

Note that, if

$$g_n = \int_{-T/2}^{T/2} g(x)\exp\left(-j\frac{2\pi}{T}nx\right)dx \qquad g(x) = \sum_{n=-\infty}^{\infty} \frac{g_n}{T}\exp\left(j\frac{2\pi}{T}nx\right)$$

$$g_n = \int_{-T/2}^{T/2} g(x)\exp\left(-j2\pi n\Delta_f x\right)dx \qquad g(x) = \sum_{n=-\infty}^{\infty} g_n\Delta_f \exp\left(j2\pi n\Delta_f x\right)$$

where $\Delta_f = 1/T$

$$G(f) = \int_{-T/2}^{T/2} g(x)\exp\left(-j2\pi fx\right)dx \qquad g(x) = \sum_{n=-\infty}^{\infty} G(f)\exp\left(j2\pi fx\right)\Delta_f$$

where $f = n\Delta_f$, $G(f) = g_n$

If $p \to \infty$,

$$G(f) = \int_{-\infty}^{\infty} g(x)\exp\left(-j2\pi fx\right)dx \qquad g(x) = \int_{-\infty}^{\infty} G(f)\exp\left(j2\pi fx\right)df$$

Physical Meaning of the Fourier Transform:

expanding a signal as a combination of $\exp(j2\pi fx)$

$\exp(j2\pi fx)$ period: 1/*f*, frequency: *f*

G(*f*): the expansion coefficient for $\exp(j2\pi fx)$

$$g(x)=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df$$

When will the Fourier transform exist?

Sufficient Conditions:

(1) $\int_{-\infty}^{\infty} \left| g(x) \right| dx < \infty$

(2) $g(x)$ is of bounded variations (It means that $g(x)$ can be represented by a curve of finite length in any finite interval of x).

4.1.2 Transform Pair

[Example 1] Find the Fourier transform of

$$g(x) = \exp(-3|x|)$$

(Solution):

$$\mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} e^{-3|x|} e^{-j2\pi fx} dx = \int_{-\infty}^{0} e^{3x} e^{j2\pi fx} dx + \int_{0}^{\infty} e^{-3x} e^{-j2\pi fx} dx$$

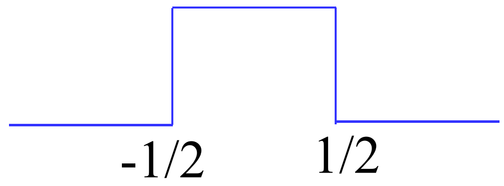
$$= \left.\frac{e^{3x} e^{-j2\pi fx}}{3 - j2\pi f}\right|_{-\infty}^{0} + \left.\frac{e^{-3x} e^{-j2\pi fx}}{-3 - j2\pi f}\right|_{0}^{\infty} = \frac{1}{3 - j2\pi f} - \frac{1}{-3 - j2\pi f}$$

$$= \frac{6}{9 + (2\pi f)^2}$$

[Example 2]

Find the Fourier transform of the rectangular function $\Pi(x)$ where

$$\Pi(x) = \begin{cases} 1 & \text{for } -1/2 < x < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$



(Solution):

$$\mathcal{F}\{g(x)\} = \int_{-1/2}^{1/2} e^{-j2\pi fx}\,dx = \left.\frac{e^{-j2\pi fx}}{-j2\pi f}\right|_{-1/2}^{1/2} = \frac{\sin(\pi f)}{\pi f}$$

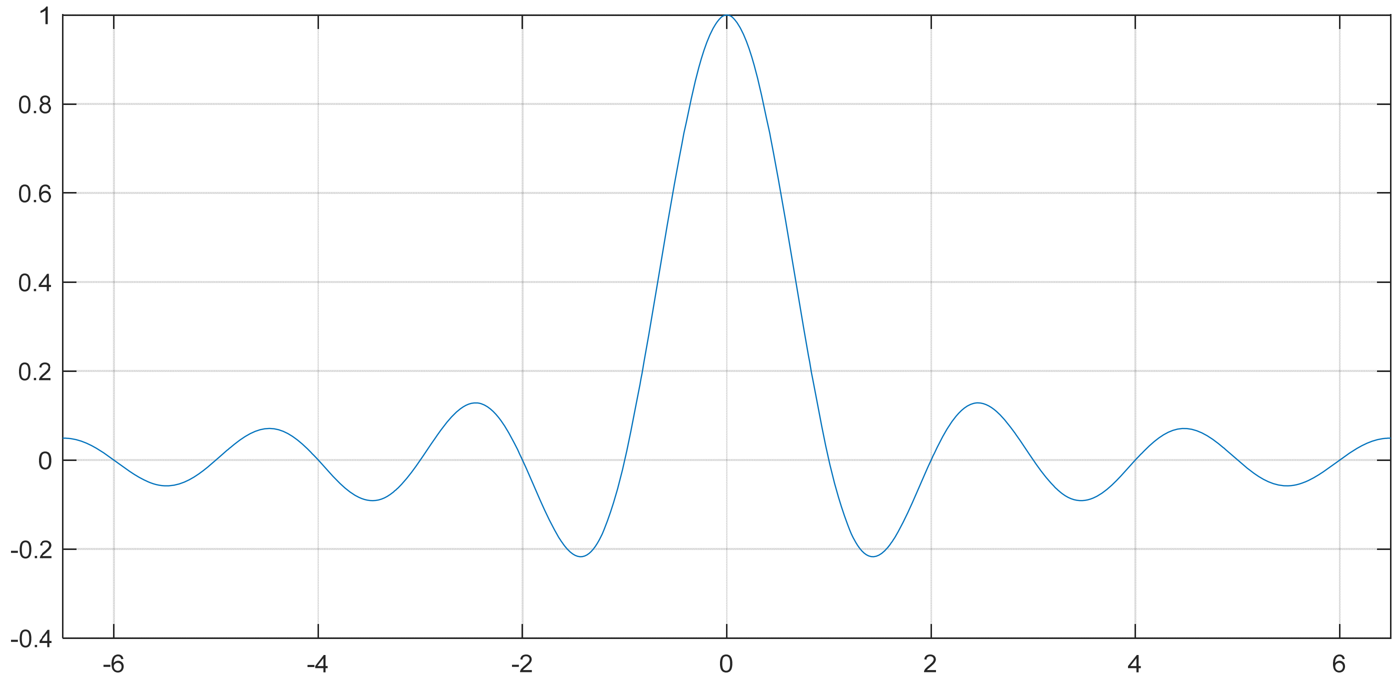
$$\mathcal{F}\{g(x)\} = \operatorname{sinc} f$$

[Sinc Function] $\operatorname{sinc} x = \dfrac{\sin(\pi x)}{\pi x}$

$\operatorname{sinc} 0 = 1,\qquad \operatorname{sinc} n = 0\quad$ if n is a nonzero integer,

$\operatorname{sinc} x = \operatorname{sinc}(-x)$

Applications: sampling theorem; ideal filters



[Example 3]

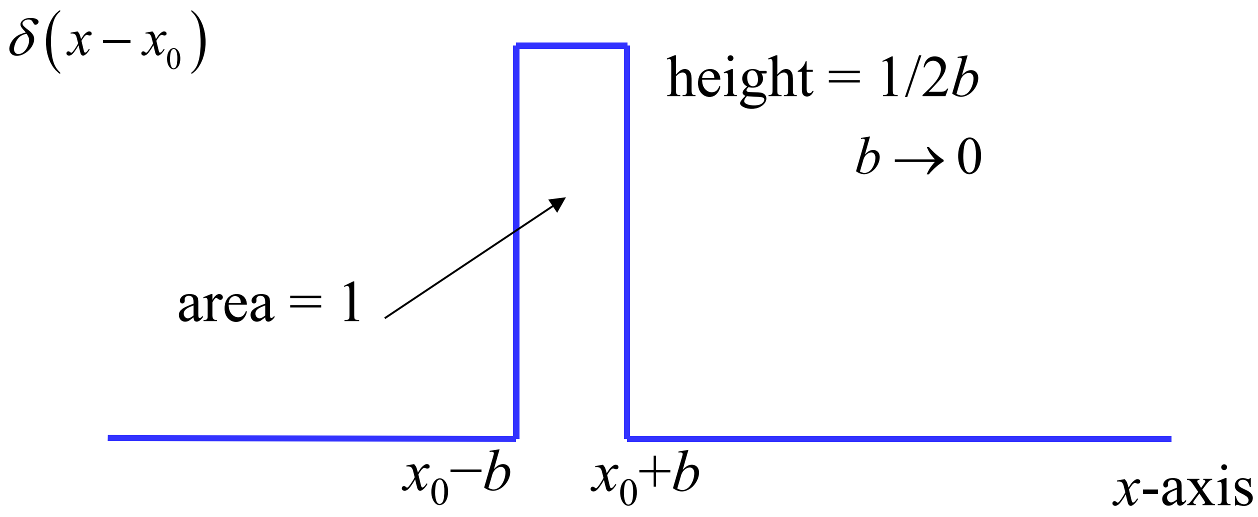
Find the Fourier transform of the Dirac delta function $\delta(x)$

(Solution): From the sifting property of $\delta(x)$:

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, y(x)\, dx = y(x_0)$$

we have

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\, e^{-j2\pi fx}\, dx = e^{-j2\pi f \cdot 0} = 1$$



Note:

(1) More generally,

$$\mathcal{F}\left\{\delta\left(x-x_0\right)\right\}=\int_{-\infty}^{\infty}\delta\left(x-x_0\right)e^{-j2\pi fx}dx=e^{-j2\pi x_0 f}$$

(2) Although $\delta(x - x_0)$ does not satisfy the sufficient condition on page 314, its Fourier transform exists.

Linearity Property of the Fourier Transform

If

$$\Im\left[g_1(x)\right] = G_1(f) \qquad \Im\left[g_2(x)\right] = G_2(f)$$

then

$$\Im\left[\alpha g_1(x) + \beta g_2(x)\right] = \alpha G_1(f) + \beta G_2(f)$$

Duality Property of the Fourier Transform

If $\Im[g(x)] = G(f)$

then $\Im[G(x)] = g(-f)$

(Proof): Since

$$\Im^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f)e^{j2\pi fx}df = g(x)$$

$$\int_{-\infty}^{\infty} G(x)e^{j2\pi xf}dx = g(f)$$

$$\int_{-\infty}^{\infty} G(x)e^{-j2\pi fx}dx = g(-f) \Longrightarrow \Im[G(x)] = g(-f)$$

[Example 4] Find the Fourier transform of sinc(x) where

(Solution): Since

$$\Im\left[\Pi(x)\right] = \text{sinc}(f)$$

from the duality property, we have

$$\Im\left[\text{sinc}(x)\right] = \Pi(-f) = \Pi(f)$$

[Example 5] Find the Fourier transform of exp($j2\pi k x$)

(Solution): Since

$$\mathcal{F}\left\{\delta\left(x-k\right)\right\}=e^{-j2\pi k f}, \qquad \mathcal{F}\left\{\delta\left(x+k\right)\right\}=e^{j2\pi k f}$$

from the duality property, we have

$$\mathcal{F}\left\{e^{j2\pi kx}\right\}=\delta\left(-f+k\right)=\delta\left(f-k\right)$$

(Here we apply the fact that $\delta(x) = \delta(-x)$).

Specially,

$$\Im\{1\}=\delta\left(f\right)$$

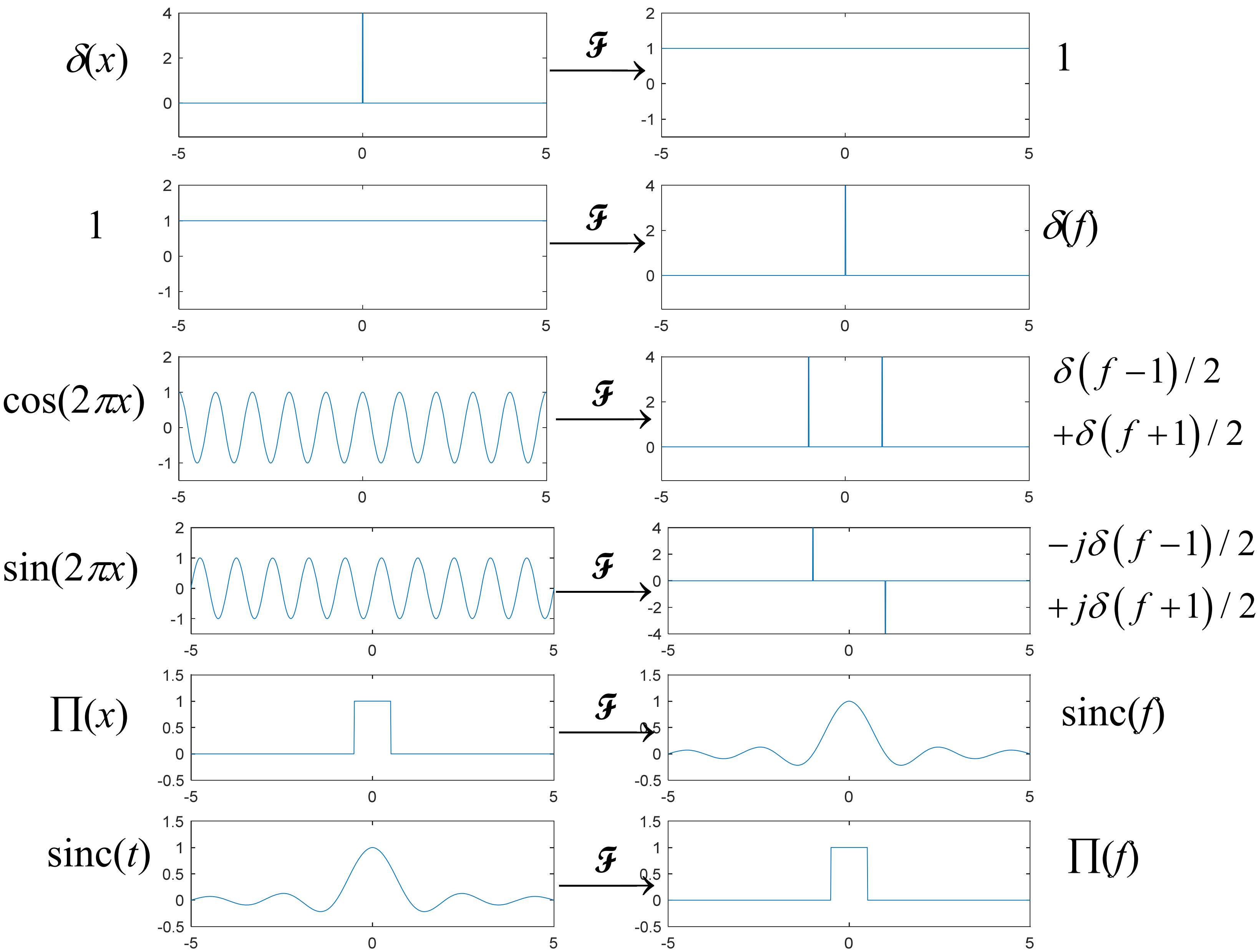
[Example 6] Find the Fourier transform of $\cos(2\pi k x)$

(Solution):

Some basic Fourier transform pairs

$g(x)$	$G(f) = \mathcal{F}\{g(x)\}$
(1) $\delta(x)$	1
(2) 1	$\delta(f)$
(3) $\delta(x-k)$	$\exp(-j2\pi kf)$
(4) $\exp(j2\pi kx)$	$\delta(f-k)$
(5) $\cos(2\pi kx)$	$\frac{1}{2}\delta(f-k)+\frac{1}{2}\delta(f+k)$
(6) $\sin(2\pi kx)$	$\frac{-j}{2}\delta(f-k)+\frac{j}{2}\delta(f+k)$
(7) $\prod(x)$	$\text{sinc}(f)$
(8) $\text{sinc}(x)$	$\prod(f)$
(9) $\exp(-kx)U(x)$ $(k>0)$	$\frac{1}{k+j2\pi f}$
(10) $\exp(-k\|x\|)$ $(k>0)$	$\frac{2k}{k^2+4\pi^2 f^2}$

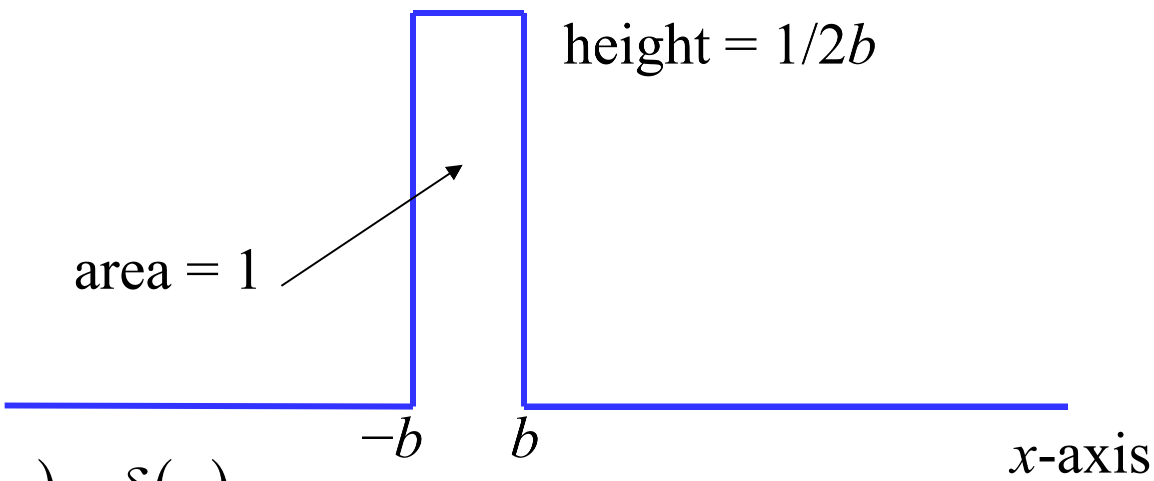
Some basic Fourier transform pairs



4.2 Dirac Delta Functions

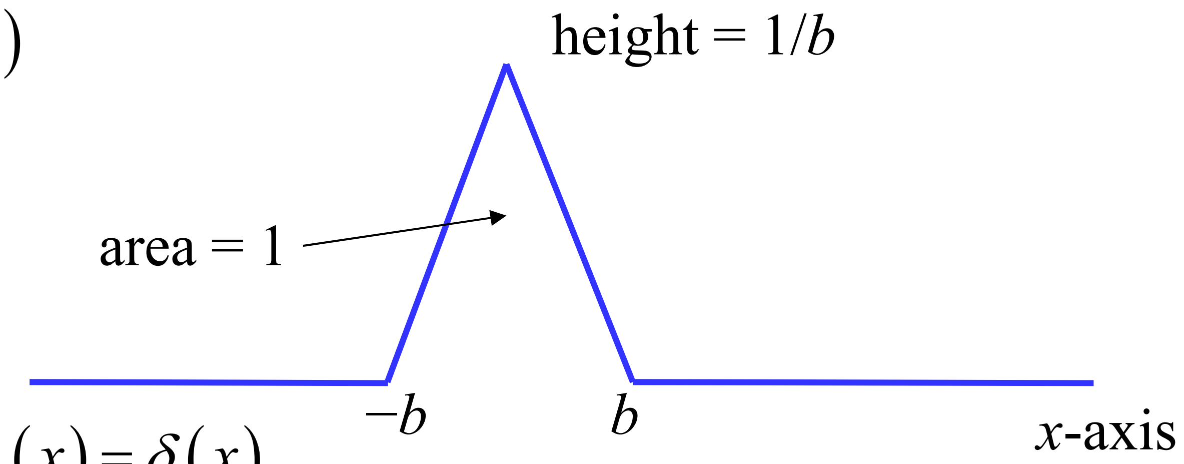
The Dirac delta function does not have a fixed definition. It is in fact the limitation of a distribution.

(1) $r_b\left(x\right)$



$$\lim_{b\to 0} r_b\left(x\right) = \delta\left(x\right)$$

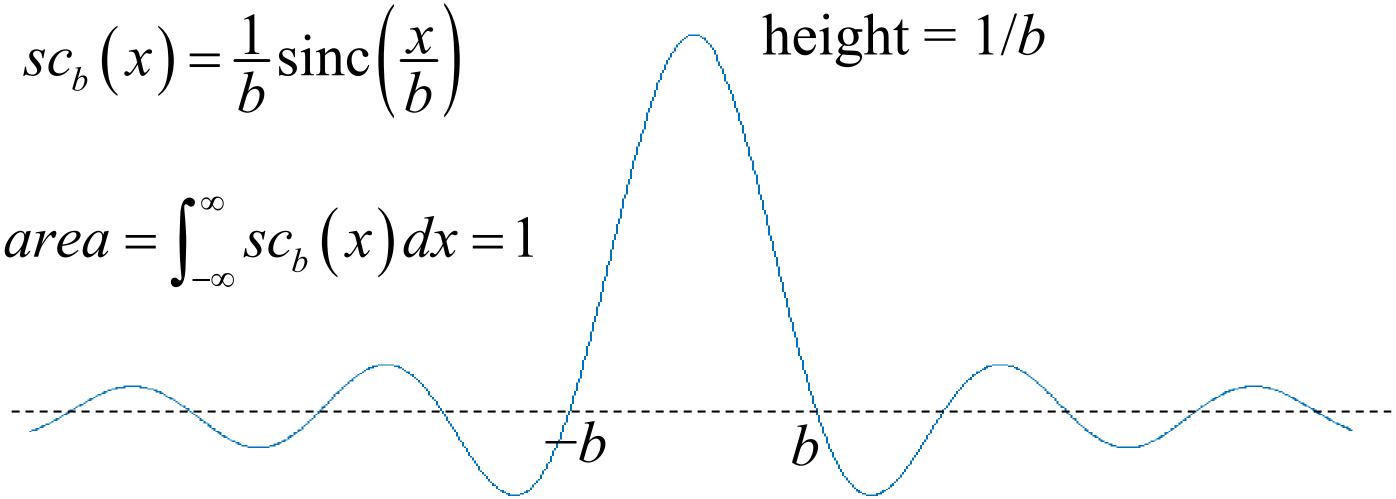
(2) $tri_b(x)$



$$\lim_{b\to 0} tri_b(x) = \delta(x)$$

(3) $sc_b(x) = \frac{1}{b}\,\text{sinc}\left(\frac{x}{b}\right)$

$$area = \int_{-\infty}^{\infty} sc_b(x)\,dx = 1$$



$$\lim_{b\to 0} sc_b(x) = \delta(x)$$

Definition of the Dirac delta function:

(1) $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

(2) $\delta(x) = 0 \qquad \text{if } x \neq 0$

(3) $\delta(x) = \delta(-x)$

Properties of the Dirac delta function:

(1) $\delta(x)=\frac{d}{dx}U(x)$ $U(x)$: unit step function

$$\delta(x-k)=\frac{d}{dx}U(x-k)$$

(2) Sifting property

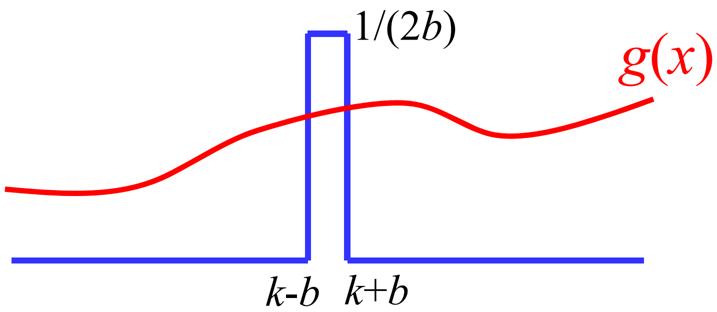
$$\int_{-\infty}^{\infty}\delta(x-k)g(x)dx=g(k)$$

(Proof):

$$\int_{-\infty}^{\infty}\delta(x-k)g(x)dx$$

$$=\lim_{b\to 0}\frac{1}{2b}\int_{k-b}^{k+b}g(x)dx$$

$$=\lim_{b\to 0}\frac{1}{2b}g(k)\int_{k-b}^{k+b}dx=g(k)$$



(3) Sifting property (without integral)

$$\delta(x-k)g(x)=\delta(x-k)g(k)$$

(4) Scaling property

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

(to balance the integral)

for $a>0$ $\quad \int_{-\infty}^{\infty}\delta(ax)dx=\int_{-\infty}^{\infty}\delta(x')\frac{dx'}{a}=\frac{1}{a}\int_{-\infty}^{\infty}\delta(x)dx$

for $a<0$ $\quad \int_{-\infty}^{\infty}\delta(ax)dx=\int_{\infty}^{-\infty}\delta(x')\frac{dx'}{a}=\frac{-1}{a}\int_{-\infty}^{\infty}\delta(x)dx$

(5) Convolution property

$$g(x)*\delta(x)=g(x)$$

Specially, $\delta(x)*\delta(x)=\delta(x)$

(6) Integral for exponential functions

$$\int_{-\infty}^{\infty} e^{j2\pi fx} df = \delta(x)$$

It is directly from the fact that

$$\Im\{\delta(x)\} = 1, \qquad \Im^{-1}\{1\} = \delta(x)$$

(7) Generalization of the integral for exponential functions

$$\int_{-\infty}^{\infty} e^{j2\pi f\, g(x)} df = \delta(g(x))$$

How do we define it?

(8) $\delta(g(x))$

If $g(x) = 0$ only at $x = x_0$, then

$$\delta\left(g(x)\right) = \frac{\delta(x-x_0)}{|\,g'(x_0)\,|}$$

(Proof):

$$g\left(x\right) \cong g'(x_0)(x-x_0) \qquad \text{when} \quad x \cong x_0$$

$$\delta\left(g(x)\right) = \delta\left(g'(x_0)(x-x_0)\right) = \frac{\delta(x-x_0)}{|\,g'(x_0)\,|}$$

In general, if $g(x) = 0$ only at $x = x_1, x_2, \ldots, x_N$, then

$$\delta\left(g(x)\right) = \sum_{n=1}^{N} \frac{\delta(x-x_n)}{|\,g'(x_n)\,|}$$

(9) Derivative of $\delta(x)$ $\delta'(x) = \frac{d}{dx}\delta(x)$

$$\delta'(x) * g(x) = g'(x)$$

(Proof):

$$\delta'(x) * g(x) = \int_{-\infty}^{\infty} \delta'(\tau) g(x-\tau) d\tau$$

$$= \delta(\tau) g(x-\tau)\Big|_{\tau\to-\infty}^{\tau\to\infty} - \int_{-\infty}^{\infty} \delta(\tau) \frac{d}{d\tau} g(x-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \delta(\tau) g'(x-\tau) d\tau$$ (from the sifting property)

$$= g'(x)$$

(10) Properties related to derivative of $\delta(x)$

(i) $\delta'(x) = -\delta'(-x)$

(ii) $\int_{-\infty}^{\infty} \delta'(x-x_0)\, g(x)\, dx = -g'(x_0)$

(iii) $\delta'(x-x_0)\, g(x) = \delta'(x-x_0)\, g(x_0) - \delta(x-x_0)\, g'(x_0)$

(Proof): Since

$$\delta(x-x_0)\, g(x) = \delta(x-x_0)\, g(x_0)$$

$$\frac{d}{dx}\delta(x-x_0)\, g(x) = \frac{d}{dx}\delta(x-x_0)\, g(x_0)$$

$$\delta'(x-x_0)\, g(x) + \delta(x-x_0)\, g'(x) = \delta'(x-x_0)\, g(x_0)$$

$$\delta'(x-x_0)\, g(x) = \delta'(x-x_0)\, g(x_0) - \delta(x-x_0)\, g'(x_0)$$

(11) Higher order derivative of $\delta(x)$

$$\delta^{(n)}(x) * g(x) = g^{(n)}(x)$$

$$\int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) g(x) dx = (-1)^n g^{(n)}(x_0)$$

$$\int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) dx = 0 \quad \text{when } n > 0$$

$$\delta^{(n)}(x) = 0 \qquad \text{when } x \neq 0$$

$$\delta^{(n)}(x) = (-1)^n \delta^{(n)}(-x)$$

4.3 Properties of the Fourier Transform

4.3.1 List of Properties

$$G(f)=\mathcal{F}[g(x)]=\int_{-\infty}^{\infty} g(x)\exp(-j2\pi f x)\,dx$$

(1) Recovery (inverse Fourier transform)	$g(x)=\int_{-\infty}^{\infty} G(f)\exp(j2\pi f x)\,dx$
(2) Integration	$G(0)=\int_{-\infty}^{\infty} g(x)\,dx \qquad g(0)=\int_{-\infty}^{\infty} G(f)\,df$
(3) Modulation	$\mathcal{F}\left[g(x)e^{j2\pi f_0 x}\right]=G(f-f_0)$
(4) Time Shifting	$\mathcal{F}[g(x-x_0)]=G(f)e^{-j2\pi f x_0}$
(5) Scaling	$\mathcal{F}[g(ax)]=\frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$
(6) Time Reverse	$\mathcal{F}[g(-x)]=G(-f)$

(7) Real / Imaginary Input	If $g(x)$ is real, then $G(f) = G^*(-f)$; If $g(x)$ is pure imaginary, then $G(f) = -G^*(-f)$
(8) Even / Odd Input	If $g(x) = g(-x)$, then $G(f) = G(-f)$; If $g(x) = -g(-x)$, then $G(f) = -G(-f)$;
(9) Conjugation	$\mathcal{F}\left[g^*(x)\right] = G^*(-f) \qquad \mathcal{F}\left[g^*(-x)\right] = G^*(f)$
(10) Differentiation	$\mathcal{F}\left[g'(x)\right] = j2\pi f\, G(f)$
(11) Multiplication by x	$\mathcal{F}\left[xg(x)\right] = \frac{j}{2\pi} G'(f)$
(12) Division by x	$\mathcal{F}\left[\frac{g(x)}{x}\right] = -j2\pi \int_{-\infty}^{f} G(\mu)\, d\mu$
(13) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} \lvert g(x)\rvert^2 dx = \int_{-\infty}^{\infty} \lvert G(f)\rvert^2 df$
(14) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} g(x) h^*(x) dx = \int_{-\infty}^{\infty} G(f) H^*(f) df$

(15) Linearity	$\mathcal{F}\left[ag(x)+bh(x)\right]=aG(f)+bH(f)$
(16) Convolution	If $z(x)=g(x)*h(x)=\int_{-\infty}^{\infty}g(\tau)h(x-\tau)d\tau$, then $Z(f)=G(f)H(f)$
(17) Multiplication	If $z(x)=g(x)h(x)$, then $Z(f)=G(f)*H(f)=\int_{-\infty}^{\infty}G(\mu)H(f-\mu)d\mu$
(18) Correlation	If $z(x)=\int_{-\infty}^{\infty}g(\tau)h^{*}(\tau-x)d\tau$, then $Z(f)=G(f)H^{*}(f)$
(19) Two Times of Fourier Transforms	$\mathcal{F}\left\{\mathcal{F}\left[g(x)\right]\right\}=g(-x)$
(20) Four Times of Fourier Transforms	$\mathcal{F}\left[\mathcal{F}\left(\mathcal{F}\left\{\mathcal{F}\left[g(x)\right]\right\}\right)\right]=g(x)$

(Proof of (2) Integration Property)

$$G(f)=\int_{-\infty}^{\infty}\exp(-j2\pi f\,x)\,g(x)\,dx$$

$$G(0)=\int_{-\infty}^{\infty}\exp(-j2\pi 0\,x)\,g(x)\,dx=\int_{-\infty}^{\infty}g(x)\,dx$$

(Proof of (5) Scaling Property)

$$\mathcal{F}[g(ax)]=\int_{-\infty}^{\infty}\exp(-j2\pi f\,x)\,g(ax)\,dx$$

$$=\int_{-\infty}^{\infty}\exp\left(-j2\pi f\,\frac{x'}{a}\right)g(x')\frac{dx'}{|a|}$$

$$=\frac{1}{|a|}\int_{-\infty}^{\infty}\exp\left(-j2\pi\frac{f}{a}x'\right)g(x')\,dx'=\frac{1}{|a|}G\left(\frac{f}{a}\right)$$

Property (6) is a special case of Property (5) where a = -1.

(Proof of (7) and (9))

$$G^*(-f) = \int_{-\infty}^{\infty} \overline{\exp(j2\pi f x)g(x)}dx$$

$$= \int_{-\infty}^{\infty} \exp(-j2\pi f x)g^*(x)dx = \mathcal{F}\left[g^*(x)\right] \quad \text{((9) is proven)}$$

If $g(x)$ is real, then

$$G^*(-f) = \mathcal{F}\left[g^*(x)\right] = \mathcal{F}\left[g(x)\right] = G(f)$$

(Proof of (10) Differentiation Property)

$$g(x) = \mathcal{F}^{-1}\left[G(f)\right] = \int_{-\infty}^{\infty} \exp(j2\pi f x)G(f)df$$

$$\frac{d}{dx}g(x) = \int_{-\infty}^{\infty} j2\pi f \exp(j2\pi f x)G(f)df = \mathcal{F}^{-1}\left[j2\pi f\, G(f)\right]$$

[Example 1] Determine the Fourier transform of the following signal.

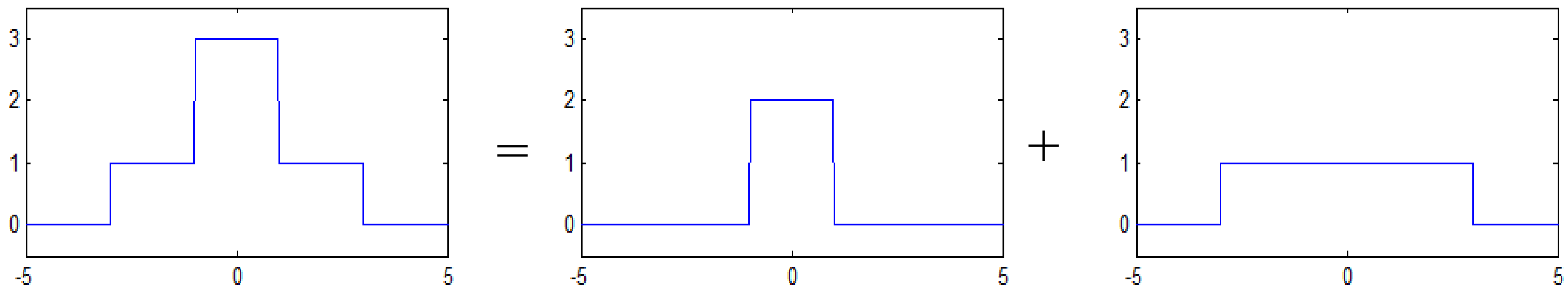
$$g(x) = 3 \quad \text{for } |x| < 1,$$

$$g(x) = 1 \quad \text{for } 1 < |x| < 3, \qquad g(x) = 0 \quad \text{for } |x| > 3$$

(Solution): Note that

$$g(x) = 2\Pi\left(\frac{x}{2}\right) + \Pi\left(\frac{x}{6}\right)$$

$g(x)$ $\qquad$ $2\Pi(x/2)$ $\qquad$ $\Pi(x/6)$



Therefore, $\quad G(f) = 2 \cdot 2\,\text{sinc}(2f) + 6\,\text{sinc}(6f)$

$$= 4\,\text{sinc}(2f) + 6\,\text{sinc}(6f)$$

[Example 2] Determine the Fourier transform of the following signal.

$$g(x) = x\exp(-|x|)$$

(Solution): From page 325, we have

$$\mathcal{F}\left[\exp(-|x|)\right] = \frac{2}{1+4\pi^2 f^2}$$

Then, from the differentiation property $\mathcal{F}\left[xg(x)\right] = \frac{j}{2\pi}G'(f)$

$$\mathcal{F}\left[x\exp(-|x|)\right] = \frac{j}{2\pi}\frac{d}{df}\frac{2}{1+4\pi^2 f^2}$$

$$= -j\frac{8\pi f}{(1+4\pi^2 f^2)^2}$$

[Example 3] Determine the Fourier transform of the following signal.

$$g(x) = \exp(-3|x-1| + j6\pi x)$$

(Solution): Since

$$\mathcal{F}\left[\exp(-3|x|)\right] = \frac{6}{9+4\pi^2 f^2}$$

$$\mathcal{F}\left[\exp(-3|x-1|)\right] = \frac{6}{9+4\pi^2 f^2} e^{-j2\pi f}$$ time shifting property

$$\mathcal{F}\left[\exp(-3|x-1| + j6\pi x)\right] = \frac{6}{9+4\pi^2 (f-3)^2} e^{-j2\pi (f-3)}$$

modulation property

[Example 4] Determine the Fourier transform of the following signal.

$$g(x) = \cos(6\pi x) \qquad \text{for } 0 < x < 8,$$

$$g(x) = 0 \qquad \text{otherwise}$$

(Solution): Note that

$$g(x) = \cos(6\pi x)\Pi\left(\frac{x-4}{8}\right)$$

$$= \frac{1}{2}\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right) + \frac{1}{2}\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)$$

Since

$$\mathcal{F}[\Pi(x)] = \text{sinc}(f)$$

$$\mathcal{F}\left[\Pi\left(\frac{x}{8}\right)\right] = 8\,\text{sinc}(8f) \qquad \text{(scaling)}$$

$$\mathcal{F}\left[\Pi\left(\frac{x-4}{8}\right)\right] = 8e^{-j8\pi f}\,\text{sinc}(8f) \qquad \text{(time shifting)}$$

$$\mathcal{F}\left[\Pi\left(\frac{x-4}{8}\right)\right]=8e^{-j8\pi f}\operatorname{sinc}(8f)$$

$$\mathcal{F}\left[\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]=8e^{-j8\pi(f-3)}\operatorname{sinc}(8(f-3)) \quad \text{(modulation)}$$

$$\mathcal{F}\left[\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]=8e^{-j8\pi(f+3)}\operatorname{sinc}(8(f+3))$$

Therefore,

$$\begin{aligned}
&\mathcal{F}[g(x)] \\
&=\frac{1}{2}\mathcal{F}\left[\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]+\frac{1}{2}\mathcal{F}\left[\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right] \\
&=4e^{-j8\pi(f-3)}\operatorname{sinc}(8(f-3))+4e^{-j8\pi(f+3)}\operatorname{sinc}(8(f+3))
\end{aligned}$$

4.3.2 Real, Imaginary, Even, and Odd Parts

Moreover, from Properties (7), (8), (9), we can conclude that

(i) $\Im\left[\mathcal{Re}\{g(x)\}\right]=\frac{1}{2}\left(G(f)+G^{*}(-f)\right)$

(ii) $\Im\left[j\mathcal{Im}\{g(x)\}\right]=\frac{1}{2}\left(G(f)-G^{*}(-f)\right)$

(Practice to prove them)

Also, any function can be decomposed into

(1) $$g(x) = g_e(x) + g_o(x)$$

where $$g_e(x) = \frac{1}{2}\big(g(x) + g(-x)\big)$$

$$g_o(x) = \frac{1}{2}\big(g(x) - g(-x)\big)$$

(2) $$g(x) = g_{e,r}(x) + g_{e,i}(x) + g_{o,r}(x) + g_{o,i}(x)$$

where $$g_{e,r}(x) = \mathcal{R}e\left\{\frac{1}{2}\big(g(x) + g(-x)\big)\right\}$$

$$g_{e,i}(x) = j\mathcal{I}m\left\{\frac{1}{2}\big(g(x) + g(-x)\big)\right\}$$

$$g_{o,r}(x) = \mathcal{R}e\left\{\frac{1}{2}\big(g(x) - g(-x)\big)\right\}$$

$$g_{o,i}(x) = j\mathcal{I}m\left\{\frac{1}{2}\big(g(x) - g(-x)\big)\right\}$$

One can prove that

$$\Im[g_e(x)] = G_e(f) \qquad \Im[g_o(x)] = G_o(f)$$

$$\Im[g_{e,r}(x)] = G_{e,r}(f) \qquad \Im[g_{e,i}(x)] = G_{e,i}(f)$$

$$\Im[g_{o,r}(x)] = G_{o,i}(f) \qquad \Im[g_{o,i}(x)] = G_{o,r}(f)$$

note

where

$$G_e(f) = \frac{1}{2}(G(f) + G(-f)) \qquad G_o(f) = \frac{1}{2}(G(f) - G(-f))$$

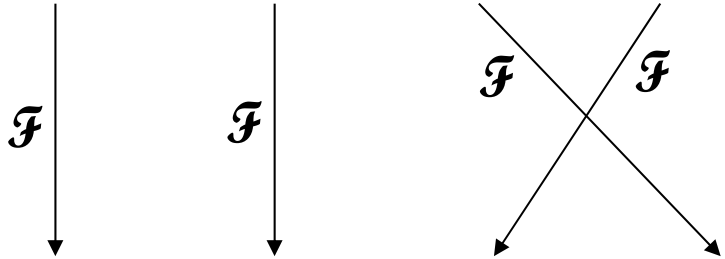
$$G_{e,r}(f) = \mathcal{R}e\left\{\frac{1}{2}(G(f) + G(-f))\right\}$$

$$G_{e,i}(f) = j\mathcal{I}m\left\{\frac{1}{2}(G(f) + G(-f))\right\}$$

$$G_{o,r}(f) = \mathcal{R}e\left\{\frac{1}{2}(G(f) - G(-f))\right\}$$

$$G_{o,i}(f) = j\mathcal{I}m\left\{\frac{1}{2}(G(f) - G(-f))\right\}$$

$$g(x) = g_{e,r}(x) + g_{e,i}(x) + g_{o,r}(x) + g_{o,i}(x)$$



$$G(f) = G_{e,r}(f) + G_{e,i}(f) + G_{o,r}(f) + G_{o,i}(f)$$

4.3.3 Parseval's Theorem

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Parseval's theorem is also called the energy preservation property, Rayleigh's Theorem, or Plancheral's Theorem.

(Proof):

$$\int_{-\infty}^{\infty} G(f) G^*(f) df = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx \overline{\int_{-\infty}^{\infty} g(\tau) e^{-j2\pi f\tau} d\tau} df$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) g^*(\tau) \left[\int_{-\infty}^{\infty} e^{j2\pi f(\tau - x)} df \right] dx d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) g^*(\tau) \delta(\tau - x) d\tau dx$$ (from page 332)

$$= \int_{-\infty}^{\infty} g(x) g^*(x) dx$$ (from the sifting property)

$$= \int_{-\infty}^{\infty} |g(x)|^2 dx$$

Generalized Parseval's Theorem

$$\int_{-\infty}^{\infty} g(x)h^*(x)\,dx = \int_{-\infty}^{\infty} G(f)H^*(f)\,df$$

It also called the power theorem.

[Example 5] Determine the following integral:

$$\int_{-\infty}^{\infty} \mathrm{sinc}^2(x)\,dx$$

[Example 6] Determine the following integral:

$$\int_{-\infty}^{\infty} \cos(8\pi x)\,\mathrm{sinc}(3x)\,dx$$

(Solution): Since

$$\mathcal{F}\left[\cos(8\pi x)\right] = \frac{1}{2}\left(\delta(f-4)+\delta(f+4)\right)$$

$$\mathcal{F}\left[\mathrm{sinc}(3x)\right] = \frac{1}{3}\Pi\left(\frac{f}{3}\right)$$

$$\int_{-\infty}^{\infty} \cos(8\pi x)\,\mathrm{sinc}(3x)\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\left(\delta(f-4)+\delta(f+4)\right)\frac{1}{3}\Pi\left(\frac{f}{3}\right)df$$

$$= \frac{1}{6}\int_{-3/2}^{3/2}\left(\delta(f-4)+\delta(f+4)\right)df = 0$$

4.4 Uncertainty Principles

4.4.1 Uncertainty Principles from Different Views

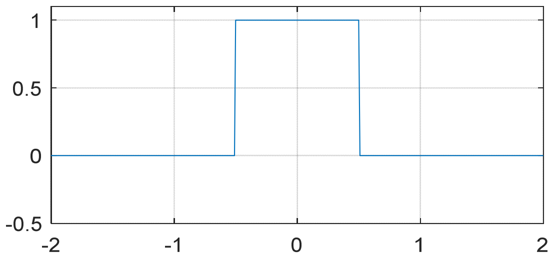
(1) From the Point of View of the Scaling Property

$$\mathcal{F}\left[g(ax)\right] = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

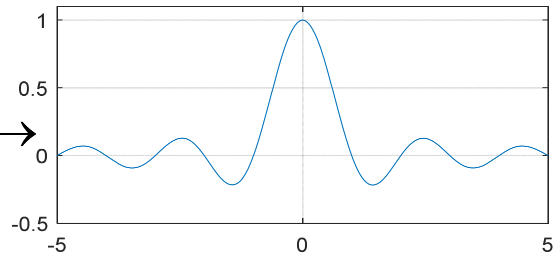
wide in the time domain $\rightarrow$ narrow in the frequency domain

narrow in the time domain $\rightarrow$ wide in the frequency domain

$\Pi(x)$

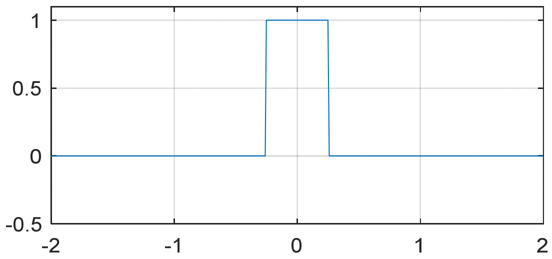


$\xrightarrow{\mathcal{F}}$

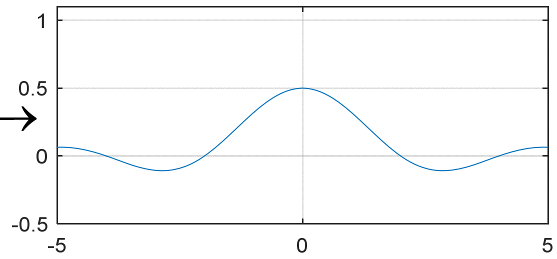


$\mathrm{sinc}(f)$

$\Pi(2x)$



$\xrightarrow{\mathcal{F}}$



$\frac{1}{2}\mathrm{sinc}\left(\frac{f}{2}\right)$

(2) From the Point of View of Equivalent Width

Equivalent Width in the Time Domain:

$$W_g = \frac{\int_{-\infty}^{\infty} g(x)\,dx}{g(0)}$$

from the integration property $W_g = \dfrac{G(0)}{g(0)}$

Equivalent Width in the Frequency Domain:

$$W_G = \frac{\int_{-\infty}^{\infty} G(f)\,df}{G(0)} = \frac{g(0)}{G(0)}$$

Product of the Two Equivalent Widths:

$$W_g W_G = \frac{G(0)}{g(0)} \frac{g(0)}{G(0)} = 1$$

(3) Heisenberg's Uncertainty Principle

For a signal $g(x)$, if $\sqrt{x}\, g(x) = 0$ when $|x| \longrightarrow \infty$, then

$$\sigma_x \sigma_f \geq 1/4\pi$$

where $\sigma_x^2 = \int (x-\mu_x)^2 P_g(x)\, dx,$ $\sigma_f^2 = \int (f-\mu_f)^2 P_G(f)\, df,$

$\mu_x = \int x P_g(x)\, dx,$ $\mu_f = \int f\, P_G(f)^2\, df$

$$P_g(x) = \frac{|g(x)|^2}{\int |g(x)|^2\, dx}, \qquad P_G(f) = \frac{|G(f)|^2}{\int |G(f)|^2\, df},$$

(Proof of Henseinberg's uncertainty principle):

For simplification, we consider the case where $\mu_x = \mu_f = 0$

Then, use Parseval's theorem

$$\sigma_x^2 \sigma_f^2 = \frac{\int x^2 |g(x)|^2 dx}{\int |g(x)|^2 dx} \frac{\int f^2 |G(f)|^2 df}{\int |G(f)|^2 df}$$

$$= \frac{1}{4\pi^2} \frac{\int x^2 |g(x)|^2 dx}{\int |g(x)|^2 dx} \frac{\int |g'(x)|^2 dx}{\int |g(x)|^2 dx}$$

Here, we apply the fact that

$$\int |g(x)|^2 dx = \int |G(f)|^2 df$$

$$\mathcal{F}\left[g'(x)\right] = j2\pi f\, G(f) \qquad \int |g'(x)|^2 dx = 4\pi^2 \int f^2 |G(f)|^2 df$$

From Schwarz's inequality $\langle g(x), g(x)\rangle\langle h(x), h(x)\rangle \geq \left|\langle g(x), h(x)\rangle\right|^2$

$$\int x^2 \,|\, g(x)\,|^2\, dx \int |\, g'(x)\,|^2\, dx \geq \left(\left|\int x g^*(x)\frac{d}{dx}g(x)dx\right|^2 + \left|\int x g(x)\frac{d}{dx}g^*(x)dx\right|^2\right)/2$$

$$\geq \left|\int\left(x g^*(x)\frac{d}{dx}g(x) + x g(x)\frac{d}{dx}g^*(x)\right)dx\right|^2 / 4$$ (using $|a+b|^2 + |a-b|^2 \geq 2|a|^2$)

$$= \left|\int x\frac{d}{dx}\left[g(x)g^*(x)\right]dx\right|^2 / 4 = \left|x g(x)g^*(x)\Big|_{-\infty}^{\infty} - \int g^*(x)g(x)\,dx\right|^2 / 4$$

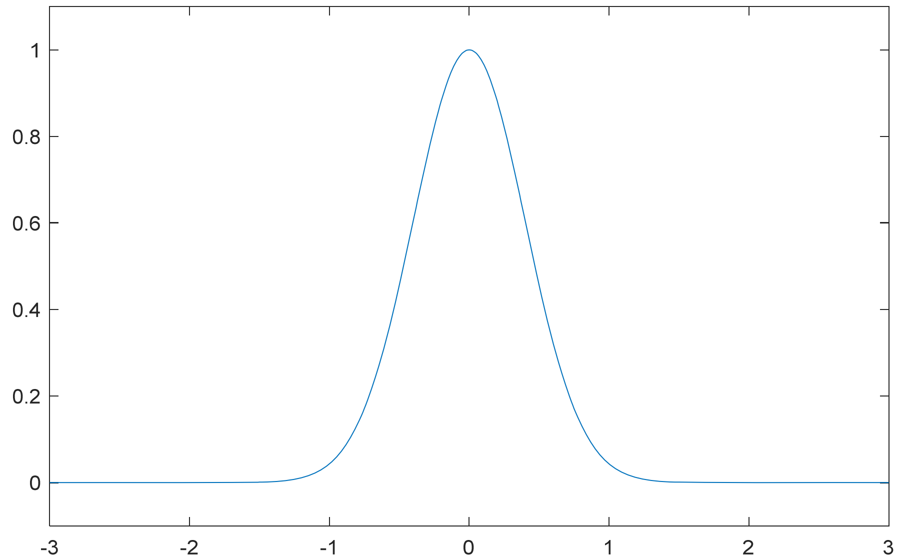
$$= \left|\left[x g(x)g^*(x)\Big|_{x\to\infty} - x g(x)g^*(x)\Big|_{x\to-\infty}\right] - \int g^*(x)g(x)\,dx\right|^2 / 4$$

$$= \left|\int |g(x)|^2\, dx\right|^2 / 4$$

$$\sigma_t^2\sigma_f^2 \geq \frac{1}{16\pi^2} \implies \sigma_t\sigma_f \geq \frac{1}{4\pi}$$

4.4.2 Gaussian and Hermite-Gaussian Functions

Gaussian function: $\exp\left(-\pi x^2\right)$



The Gaussian function is an eigenfunction of the Fourier transform with eigenvalue = 1:

$$\Im\left[\exp\left(-\pi x^2\right)\right] = \exp\left(-\pi f^2\right)$$

$$\Im\left[\exp\left(-\pi x^2\right)\right] = \exp\left(-\pi f^2\right)$$

(Proof): From the fact that

$$\int_{-\infty}^{\infty} e^{-(at^2+bt)}dt = \sqrt{\pi / a}\cdot e^{b^2/4a}$$

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

we have

$$\mathcal{F}\left\{e^{-\pi x^2}\right\} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-j2\pi fx} df = \sqrt{\frac{\pi}{\pi}} e^{\frac{(j2\pi f)^2}{4\pi}} = e^{-\pi f^2}$$

The Gaussian function is not the only eigenfunction of the Fourier transform.

$$\Im\left[H_n\left(\sqrt{2\pi}x\right)\exp\left(-\pi x^2\right)\right]=(-j)^n\exp\left(-\pi f^2\right)H_n\left(\sqrt{2\pi}f\right)$$

$H_n(x)$: The Hermite polynomial of order n (see page 304).

$$H_0(x)=1 \qquad H_1(x)=2x$$

$$H_2(x)=4x^2-2 \qquad H_3(x)=8x^3-12x$$

$$H_4(x)=16x^4-48x^2+12 \qquad H_5(x)=32x^5-160x^3+120x$$

The Gaussian function satisfies the lower bound of Heisenberg's uncertainty principle.

$$\mathcal{F}\left[e^{-\pi x^2}\right] = e^{-\pi f^2}$$

$$\int_{-\infty}^{\infty} |g(x)|^2\, dx = \int_{-\infty}^{\infty} e^{-2\pi x^2}\, dx = \sqrt{1/2}$$

use $\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi / a} \cdot e^{b^2/4a}$

$$\int_{-\infty}^{\infty} x^2 |g(x)|^2\, dx = \int_{-\infty}^{\infty} x^2 e^{-2\pi x^2}\, dx = 2\int_{0}^{\infty} x^2 e^{-2\pi x^2}\, dt =$$

$$= \frac{2\Gamma[3/2]}{2(2\pi)^{3/2}} = \frac{\sqrt{\pi}/2}{(2\pi)^{3/2}} = \frac{1}{4\sqrt{2}\pi}$$

use $\int_{0}^{\infty} x^m e^{-ax^2}\, dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

$\Gamma(1/2) = \sqrt{\pi}$ $\quad\quad$ $\Gamma(n+1) = n\Gamma(n)$

$$\sigma_x^2 = \frac{\int_{-\infty}^{\infty} x^2 |g(x)|^2\, dx}{\int_{-\infty}^{\infty} |g(x)|^2\, dx} = \frac{1}{4\pi}$$

$$\sigma_x^2 = \frac{1}{4\pi} \qquad \sigma_x = \sqrt{\frac{1}{4\pi}}$$

Since $G(f) = g(f)$,

$$\sigma_f = \sqrt{\frac{1}{4\pi}}$$

Therefore,

$$\sigma_x \sigma_f = \frac{1}{4\pi}$$

Note: Other Hermite Gaussian functions do not satisfy the lower bound of Heisenberg's uncertainty principle.

附錄八 Convolution

Convolution: $g(x)*h(x)=\int_{-\infty}^{\infty} g(x-\tau)h(\tau)d\tau=\int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau$

Specially, if $g(x) = 0$ for $x < 0$ and $h(x) = 0$ for $x < 0$, then

Convolution (causal form):

$$g(x)*h(x)=\int_{0}^{\infty} g(x-\tau)h(\tau)d\tau=\int_{0}^{\infty} g(\tau)h(x-\tau)d\tau$$

Physical meaning: The effect of the input on the output is determined by their time difference.

$$y(x)=\int_{0}^{\infty} g(\tau)h(x-\tau)d\tau$$

output input effect of $g(\tau)$ on $y(x)$

$$y(x) = \int_0^{\infty} g(\tau) h(x-\tau) d\tau$$

output — input — effect of $g(\tau)$ on $y(x)$

$$y(x) = g(0)h(x)\Delta + g(\Delta)h(x-\Delta)\Delta + g(2\Delta)h(x-2\Delta)\Delta + \cdots g(x)h(0)\Delta$$

Any linear time-invariant system can be expressed as the convolution form.

$$z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau$$

[Support Theorem]:

If the support of $g(x)$ is $x \in [x_1, x_2]$

(i.e., $g(x) = 0$ for $x < x_1$ and $x > x_2$)

and the support of $h(x)$ is $x \in [x_3, x_4]$,

then the support of $z(x)$ is

$x \in [x_1 + x_3,\ x_2 + x_4]$

(Proof):

$$z(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau = \int_{x_1}^{x_2} g(\tau) h(x-\tau) d\tau$$

$h(x-\tau) \neq 0$ when $x - \tau \in [x_3, x_4]$,

$x \in [x_3 + \min(\tau), x_4 + \max(\tau)] = [x_1 + x_3,\ x_2 + x_4]$

附錄九 Change of Independent Variables for Integrals

$$\int\int \cdots\cdots\cdots dxdy = \int\int \cdots\cdots\cdots C^{-1} dwdv$$

where $C = \det\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix}^{-1}$

For the definite integral case

$$C = \left| \det\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| = \left| \det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix} \right|^{-1}$$

4.5 Convolution and Correlation

4.5.1 Convolution Property

If $z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau,$

then $Z(f) = G(f) H(f)$

convolution ⟹ multiplication

(Proof): If $z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau,$

$$\mathcal{F}^{-1}\left[G(f)H(f)\right] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-j2\pi f\tau} g(\tau) d\tau \int_{-\infty}^{\infty} e^{-j2\pi ft} h(t) dt\right] e^{j2\pi f x} df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f\tau} g(\tau) e^{-j2\pi f t} h(t) e^{j2\pi f x} d\tau dt df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-j2\pi f\tau} e^{-j2\pi f t} e^{j2\pi f x} df\right] g(\tau) h(t) d\tau dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\tau-t) g(\tau) h(t) d\tau dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \delta(t+\tau-x) h(t) dt\right] g(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(x-\tau) g(\tau) d\tau = z(x)$$

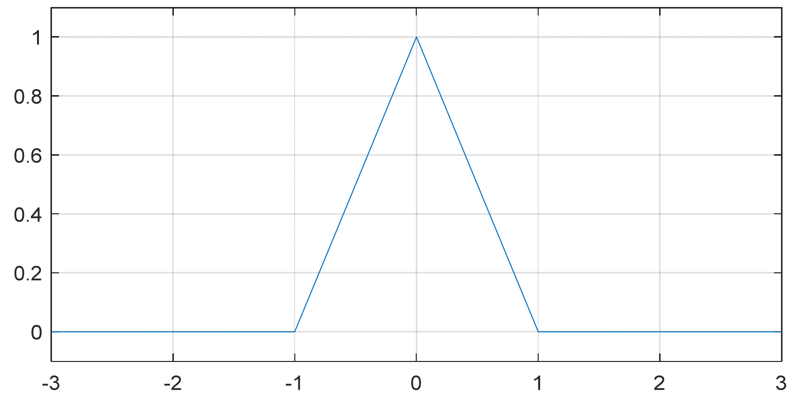
Therefore,

$$\mathcal{F}\left[z(x)\right] = G(f)H(f) \qquad\qquad Z(f) = G(f)H(f)$$

[Example 1] Determine the Fourier transform of

$$\Lambda(x)=\begin{cases} x+1 & for\ -1\le x\le 0 \\ 1-x & for\ 0\le x\le 1 \\ 0 & otherwise \end{cases}$$

We call it the triangular function.



(Solution):

Since

$$\Lambda(x)=\Pi(x)*\Pi(x)$$

$$\mathscr{F}[\Lambda(x)]=\mathscr{F}[\Pi(x)]\mathscr{F}[\Pi(x)]$$
$$=\text{sinc}^2 x$$

[Example 2] Determine the inverse Fourier transform of

$$G(f)=\frac{1}{1+\pi^2 f^2}\frac{1}{j\pi f+1}$$

(Solution): Note that

$$G(f)=2\frac{2\cdot 2}{4+4\pi^2 f^2}\frac{1}{j2\pi f+2}$$

From page 325

$$\mathcal{F}^{-1}\left[\frac{2\cdot 2}{4+4\pi^2 f^2}\right]=\exp(-2|x|) \qquad \mathcal{F}^{-1}\left[\frac{1}{j2\pi f+2}\right]=\exp(-2x)U(x)$$

$$\begin{aligned} g(x) &= 2\exp(-2|x|)*\exp(-2x)U(x) \\ &= 2\int_{-\infty}^{\infty}\exp(-2|\tau|)\exp(-2x+2\tau)U(x-\tau)\,d\tau \\ &= 2\int_{-\infty}^{x}\exp(-2|\tau|)\exp(-2x+2\tau)\,d\tau \end{aligned}$$

$$g(x) = 2\int_{-\infty}^{x} \exp(-2|\tau|)\exp(-2x+2\tau)\,d\tau$$

When $x \le 0$,

$$g(x) = 2\exp(-2x)\int_{-\infty}^{x} \exp(4\tau)\,d\tau = \frac{\exp(2x)}{2}$$

When $x > 0$,

$$g(x) = 2\exp(-2x)\left[\int_{-\infty}^{0} \exp(4\tau)\,d\tau + \int_{0}^{x} d\tau\right]$$

$$= \exp(-2x)\left[\frac{1}{2} + 2x\right]$$

[Example 3] Determine

$$\mathrm{sinc}(t) * \mathrm{sinc}(t)$$

[Example 4] Determine

$$\mathrm{sinc}(t) * \mathrm{sinc}(2t) * \mathrm{sinc}(3t)$$

[Example 5] Determine

$$\delta(t) * \delta(t)$$

[Example 6] Determine

$$\mathrm{sinc}(4t) * \sin(2\pi t)$$

[Theorem 4.5.1]

$$g(x)*\delta(x-x_0)=g(x-x_0)$$

Specially,

$$g(x)*\delta(x)=g(x)$$

(Proof):

$$\mathcal{F}\left[g(x)*\delta(x-x_0)\right]=\mathcal{F}\left[g(x)\right]\mathcal{F}\left[\delta(x-x_0)\right]$$

$$=G(f)\exp(-j2\pi x_0 f)$$

$$\mathcal{F}^{-1}\left\{\mathcal{F}\left[g(x)*\delta(x-x_0)\right]\right\}=\mathcal{F}^{-1}\left[G(f)\exp(-j2\pi x_0 f)\right]$$

$$g(x)*\delta(x-x_0)=g(x-x_0)$$

(from the time-shifting property)

4.5.2 Multiplication Property

If $z(x) = g(x)h(x)$

then $Z(f) = G(f) * H(f) = \int_{-\infty}^{\infty} G(s)H(f-s)\,ds$

multiplication $\Longrightarrow$ convolution

(Proof): $\mathcal{F}\left[z(x)\right]=\mathcal{F}\left[g(x)h(x)\right]$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{j2\pi sx}G(s)ds\int_{-\infty}^{\infty}e^{j2\pi ux}H(u)du\right]e^{-j2\pi fx}dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j2\pi sx}G(s)e^{j2\pi ux}H(u)e^{-j2\pi fx}dsdudx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{j2\pi sx}e^{j2\pi ux}e^{-j2\pi fx}dx\right]G(s)H(u)dsdu$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(s+u-f)G(s)H(u)dsdu$$

$$=\int_{-\infty}^{\infty}G(s)H(f-s)ds$$

Therefore,

$$Z(f)=G(f)*H(f)=\int_{-\infty}^{\infty}G(s)H(f-s)ds$$

[Example 7] Determine the Fourier transform of

$$g(x) = \cos(4\pi x)\,\mathrm{rect}\left(\frac{x}{6}\right)$$

(Solution):

$$\mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{6}\right)\right\} = 6\,\mathrm{sinc}(6f)$$

$$\mathcal{F}\left\{\cos(4\pi x)\right\} = \frac{1}{2}\left[\delta(f-2)+\delta(f+2)\right]$$

Therefore,

$$G(f) = 6\,\mathrm{sinc}(6f) * \frac{1}{2}\left[\delta(f-2)+\delta(f+2)\right]$$

$$G(f) = 3\,\mathrm{sinc}\big(6(f-2)\big) + 3\,\mathrm{sinc}\big(6(f+2)\big)$$

4.5.3 Correlation

Correlation

$$z(x) = corr\left(g(x), h(x)\right) = \int_{-\infty}^{\infty} g(\tau + x) h^*(\tau) d\tau$$

Auto-Correlation

$$a_g(x) = corr\left(g(x), g(x)\right) = \int_{-\infty}^{\infty} g(\tau + x) g^*(\tau) d\tau$$

Applications: Matched filter, communication, pattern recognition, signal detection ……

[Theorem 4.5.2] In fact, correlation is equivalent to convolution with the conjugate + time reverse of a signal.

$$corr\left(g(x),h(x)\right)=g(x)*h^*(-x)$$

(Proof):

$$g(x)*h^*(-x)=\int_{-\infty}^{\infty}g(x-\tau)h_1(\tau)d\tau \quad \text{where} \quad h_1(x)=h^*(-x)$$

$$=\int_{-\infty}^{\infty}g(x-\tau)h^*(-\tau)d\tau$$

$$=-\int_{\infty}^{-\infty}g(x+\tau)h^*(\tau)d\tau \qquad \tau_{new}=-\tau_{old}$$

$$=\int_{-\infty}^{\infty}g(x+\tau)h^*(\tau)d\tau$$

$$=corr\left(g(x),h(x)\right)$$

Since

$$corr\left(g(x),h(x)\right)=g(x)*h^*(-x)$$

we have

$$\mathcal{F}\left[corr\left(g(x),h(x)\right)\right]=G(f)H^*(f)$$

Specially, if $a_g(x)$ is the auto-correlation of $g(x)$:

$$a_g(x)=corr\left(g(x),g(x)\right)$$

then

$$\mathcal{F}\left[a_g(x)\right]=\left|G(f)\right|^2$$

4.6 Two-Dimensional Fourier Transform

4.6.1 Rectangular Coordinate

Two Dimensional Fourier Transform

$$\Im_{2D}\left[g(x,y)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)e^{-j2\pi fx}e^{-j2\pi hy}dxdy=G(f,h)$$

Two Dimensional Inverse Fourier Transform

$$\Im_{2D}^{-1}\left[G(f,h)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(f,h)e^{j2\pi fx}e^{j2\pi hy}dfdh=g(x,y)$$

Possible Applications: Image processing, optics, electromagnet wave propagation analysis, ….

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

Physical meaning: Express a signal by a linear combination of

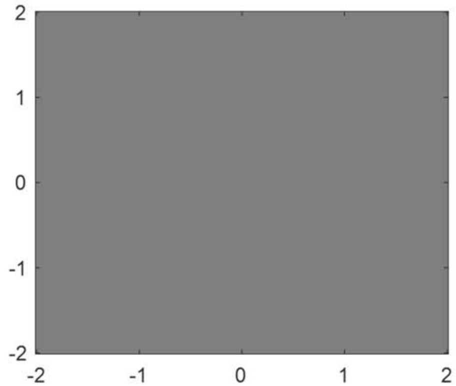
$$e^{j2\pi fx}e^{j2\pi hy}$$

f is the number of periods per unit of x

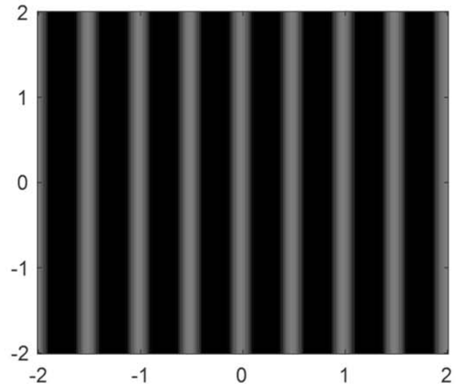
h is the number of periods per unit of y

real part of $e^{j2\pi fx}e^{j2\pi hy}$

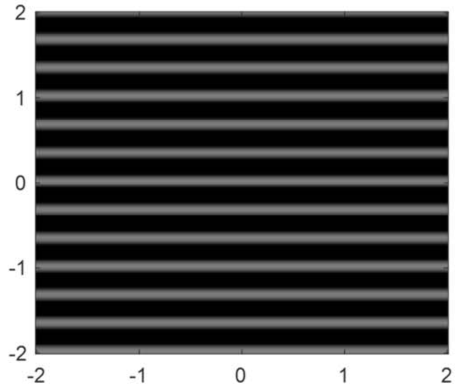
Bright colors mean higher values and dark colors mean lower values.



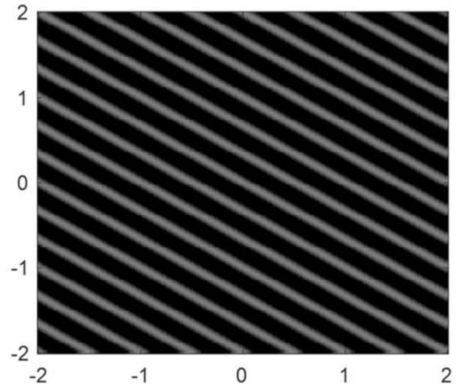
(a) $f = h = 0$



(b) $f = 2$, $h = 0$



(c) $f = 0$, $h = 3$



(d) $f = 2$, $h = 3$

[**Example 1**] Find the 2D Fourier transform of

$$g(x,y)=\text{sinc}(x)\text{sinc}(y)$$

(Solution):

$$G(f,h)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j2\pi fx}e^{-j2\pi hy}\,\text{sinc}(x)\text{sinc}(y)\,dxdy$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{-j2\pi fx}\,\text{sinc}(x)\,dx\right]e^{-j2\pi hy}\,\text{sinc}(y)\,dy$$

$$=\int_{-\infty}^{\infty}\Pi(f)e^{-j2\pi hy}\,\text{sinc}(y)\,dy$$

$$=\Pi(f)\int_{-\infty}^{\infty}e^{-j2\pi hy}\,\text{sinc}(y)\,dy=\Pi(f)\Pi(h)$$

[**Example 2**] Find the 2D Fourier transform of

$$g(x,y)=\Pi(x)$$

(Solution): $G(f,h)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j2\pi fx}e^{-j2\pi hy}\Pi(x)dxdy$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{-j2\pi fx}\Pi(x)dx\right]e^{-j2\pi hy}dy=\text{sinc}(f)\int_{-\infty}^{\infty}e^{-j2\pi hy}dy$$

$$=\text{sinc}(f)\delta(h)$$

[**Example 3**] Find the 2D Fourier transform of

$$g(x,y)=\sin(2\pi(x+2y))$$

4.6.2 Circular Coordinate Conversion

$$g(x,y) \Longrightarrow g(r,\theta) \qquad x = r\cos\theta, \quad y = r\sin\theta$$

$$G(f,h) \Longrightarrow G(s,\phi) \qquad f = s\cos\phi, \quad h = s\sin\phi$$

$$G(f,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi hy} g(x,y)\,dxdy$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$G(s,\phi) = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-j2\pi sr\cos\phi\cos\theta} e^{-j2\pi sr\sin\phi\sin\theta} g(r,\theta)\,Cd\theta dr$$

where $C = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$

$$G(s,\phi) = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-j2\pi sr\cos(\phi-\theta)} g(r,\theta)\,rd\theta dr$$

$$G(s,\phi)=\int_0^{\infty}\int_0^{2\pi}e^{-j2\pi sr\cos(\phi-\theta)}g(r,\theta)rd\theta dr$$

Specially, if $g(r, \theta)$ is independent of θ

$$g(r,\theta)=g(r)$$

$$G(s,\phi)=\int_0^{\infty}\left[\int_0^{2\pi}e^{-j2\pi sr\cos(\phi-\theta)}d\theta\right]rg(r)dr$$

From the fact that

$$\int_0^{2\pi}e^{-jx\cos(\phi-\theta)}d\theta=\int_0^{2\pi}e^{-jx\cos(\theta)}d\theta=2\pi J_0(x)$$

Bessel function of the 1st kind of zero order,
See page 194

we have

$$G(s,\phi)=2\pi\int_0^{\infty}J_0(2\pi sr)rg(r)dr$$

(Note that it is independent of ϕ)

$$G(s)=2\pi\int_0^{\infty}J_0(2\pi sr)rg(r)dr$$

Hankel Transform

$$G(s) = 2\pi \int_0^\infty J_0(2\pi sr) r g(r)\, dr$$

It is in fact the 2D Fourier transform for a rotationally symmetric signal.

$$g(r,\theta) = g(r)$$

Inverse Hankel Transform

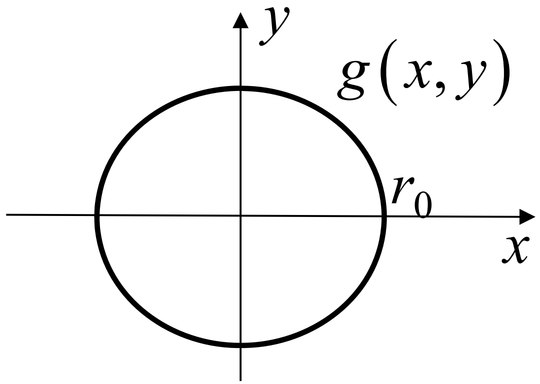
$$g(r) = 2\pi \int_0^\infty J_0(2\pi sr) s G(s)\, ds$$

It has the same form as the forward transform.

Several Hankel Transform Pairs

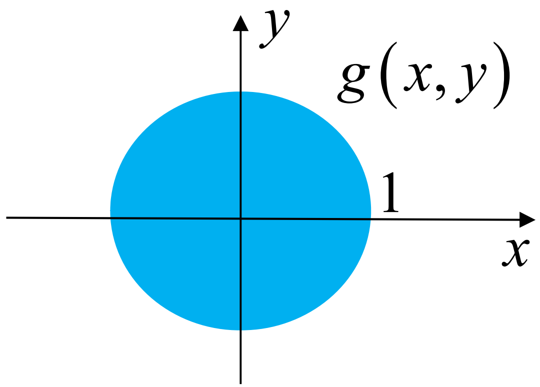
(i) If $g(r) = \delta(r - r_0)$

then $G(s) = 2\pi r_0 J_0(2\pi r_0 s)$



(ii) If $g(r) = \text{circ}(r)$

$$\text{circ}(r) = \begin{cases} 1 & for\ r < 1 \\ 0 & for\ r > 1 \end{cases}$$



then $G(s) = \dfrac{J_1(2\pi s)}{s}$

Bessel function of the 1st kind of 1st order,
See page 194.

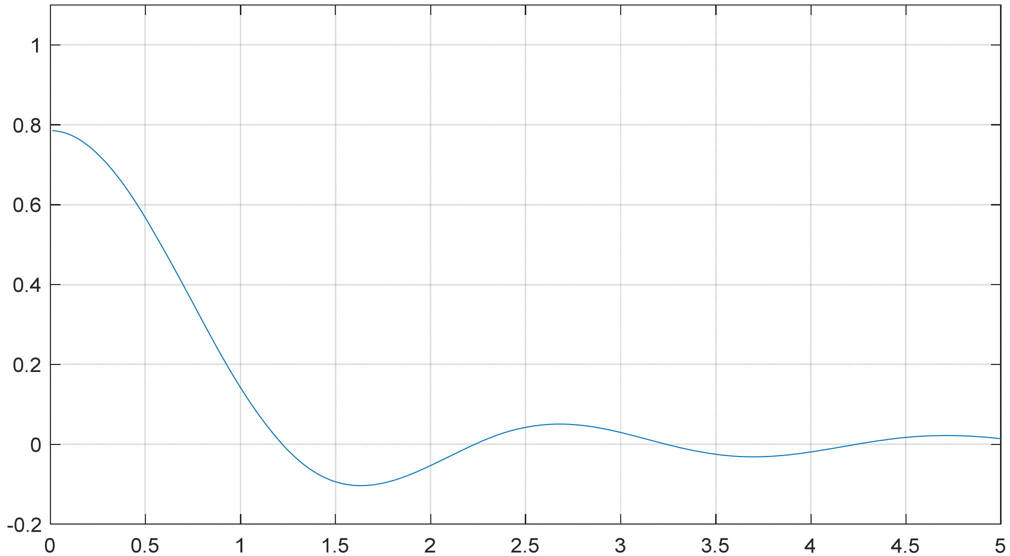
Jinc function (also called the Besinc function).

$$\mathrm{jinc}(s) = \frac{J_1(\pi s)}{2s}$$

If $\quad g(r) = \mathrm{circ}(r) \qquad G(s) = \dfrac{J_1(2\pi s)}{s} = 4\,\mathrm{jinc}(2s)$

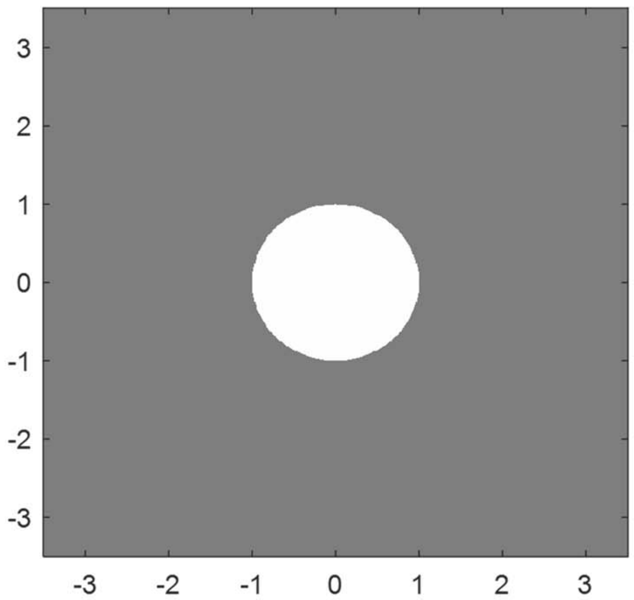
It plays a similar role as the sinc function.

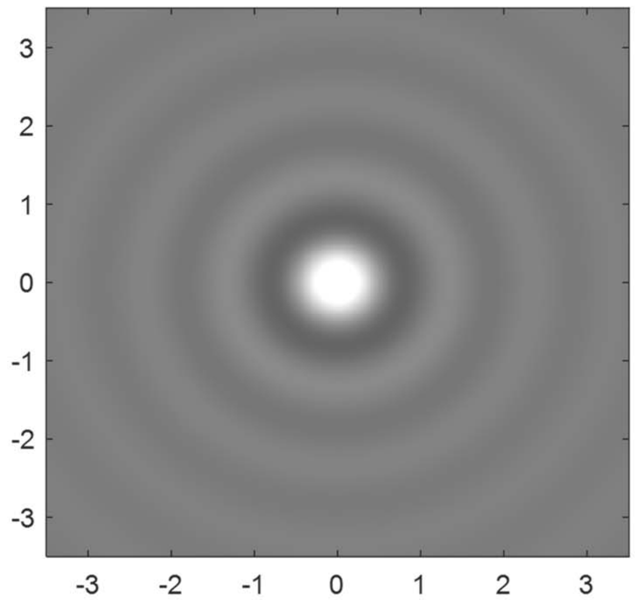
$\mathrm{jinc}(s)$



Hankel transform

$$g(r) = \operatorname{circ}(r) \Longrightarrow G(s) = 4\,\operatorname{jinc}(2s)$$





4.7 The Operations Closely Related to the Fourier Transform (只教不考)

(1) Two-Sided Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

Note that when

$$s = j2\pi f$$

it is reduced to the Fourier transform. When

$$s = \sigma + j2\pi f$$

it is equivalent to the Fourier transform of $\exp(-\sigma t)f(t)$.

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \mathfrak{I}\left[e^{-\sigma t} f(t)\right]$$

(2) One-Sided Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

When

$$s = \sigma + j2\pi f$$

it is equivalent to the Fourier transform of $\exp(-\sigma t) f(t) U(t)$.

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \Im\left[e^{-\sigma t} f(t) U(t)\right]$$

It has less physical meaning, but the probability that the transform exists is higher.

FT $$\Im\left[g(x)\right]=\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx=G(f)$$

inverse FT $$\Im^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty} G(f)e^{j2\pi fx}df=g(x)$$

(3) Fourier Cosine Transform

When $g(x)$ is even, the FT is reduced to the Fourier cosine transform.

$$\Im_c\left[g(x)\right]=\int_{0}^{\infty} g(x)\cos(2\pi fx)dx=G_c(f)$$

$$\Im_c^{-1}\left[G_c(f)\right]=4\int_{0}^{\infty} G_c(f)\cos(2\pi fx)df=g(x)$$

(4) Fourier Sine Transform

When $g(x)$ is odd, the FT is reduced to the Fourier sine transform.

$$\Im_s\left[g(x)\right]=\int_{0}^{\infty} g(x)\sin(2\pi fx)dx=G_s(f)$$

$$\Im_s^{-1}\left[G_s(f)\right]=4\int_{0}^{\infty} G_s(f)\sin(2\pi fx)df=g(x)$$

(5) Hartley Transform

$$\Im_{ha}\left[g(x)\right] = \int_{-\infty}^{\infty} g(x)\,\text{cas}(2\pi fx)\,dx = G_{ha}(f)$$

where cas(x) = cos(x) + sin(x)

$$\Im_{ha}^{-1}\left[G_{ha}(f)\right] = \int_{-\infty}^{\infty} G_{ha}(f)\,\text{cas}(2\pi fx)\,dx = g(x)$$

real input $\rightarrow$ real output

(6) Mellin Transform

$$\Im_{M}\left[g(x)\right] = \int_{0}^{\infty} g(x)\,x^{s-1}\,dx = G_{M}(s)$$

If we set x = exp(-t), $\dfrac{dx}{dt} = -x, \quad dt = -x^{-1}dx$, then

$$G_M(s) = \int_{-\infty}^{\infty} g\left(e^{-t}\right)e^{-st}\,dt$$

It is the two-sided Laplace transform of $g\left(e^{-t}\right)$

(7) Hilbert Transform

$$g_H(x) = \Im^{-1}\left\{\Im\left[g(x)\right]H(f)\right\}$$

where

$$H(f) = \begin{cases} -j & if\ f > 0 \\ 0 & if\ f = 0 \\ j & if\ f < 0 \end{cases}$$

Note that $\mathcal{F}^{-1}\left[H(f)\right] = j\int_{-\infty}^{0} e^{j2\pi fx}df - j\int_{0}^{\infty} e^{j2\pi fx}df$

$$= \lim_{\sigma\to 0}\left[j\int_{-\infty}^{0} e^{\sigma f}e^{j2\pi fx}df - j\int_{0}^{\infty} e^{-\sigma f}e^{j2\pi fx}df \right]$$

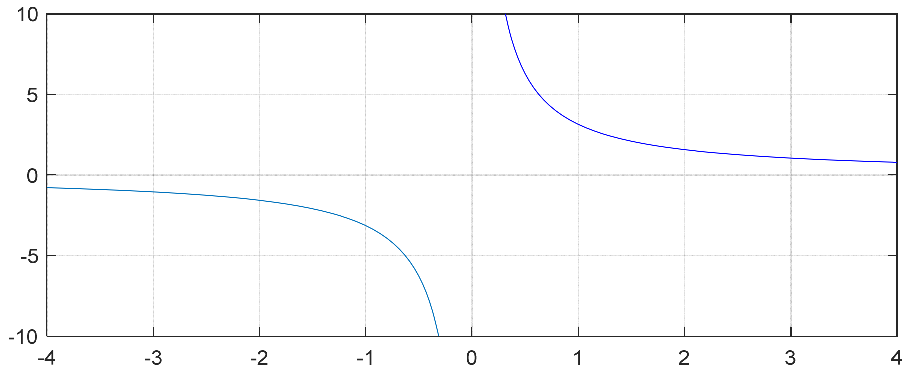
$$= \lim_{\sigma\to 0}\left[j\frac{e^{\sigma f}e^{j2\pi xf}}{\sigma + j2\pi x}\Bigg|_{-\infty}^{0} - j\frac{e^{-\sigma f}e^{j2\pi xf}}{-\sigma + j2\pi x}\Bigg|_{0}^{\infty} \right]$$

$$= \lim_{\sigma\to 0}\left[j\frac{1}{\sigma + j2\pi x} + j\frac{1}{-\sigma + j2\pi x} \right] = \lim_{\sigma\to 0}\frac{4\pi x}{\sigma^2 + 4\pi^2 x^2} = \frac{1}{\pi x}$$

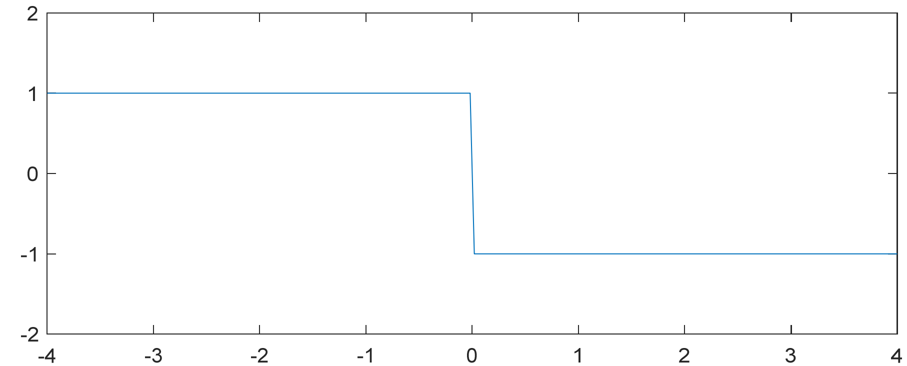
Therefore,

$$g_H(x) = g(x) * \overbrace{\frac{1}{\pi x}}^{h(x)} = \int_{-\infty}^{\infty} \frac{g(\tau)}{\pi(x-\tau)}\,d\tau$$

$h(x)$



$\mathcal{Im}\left[H(f)\right]$



$$g(x) = \cos(2\pi kx) \xrightarrow{\text{Hilbert}} g_H(x) = \sin(2\pi kx)$$

$$k \neq 0$$

$$g(x) = \sin(2\pi kx) \xrightarrow{\text{Hilbert}} g_H(x) = -\cos(2\pi kx)$$

$$k \neq 0$$

(Proof): If $g(x) = \cos(2\pi kx)$

then $G(f) = \frac{1}{2}\delta(f-k) + \frac{1}{2}\delta(f+k)$

$$H(f)G(f) = \frac{-j}{2}\delta(f-k) + \frac{j}{2}\delta(f+k)$$

$$g_H(x) = \Im^{-1}\{G(f)H(f)\} = \sin(2\pi kx)$$

(8) Analytic Signal

$$g_a(x) = g(x) + jg_H(x)$$

reconstruction:

$$g(x) = \mathcal{Re}\{g_a(x)\}$$

if $g(x)$ is real

Since

$$\mathcal{F}[g_a(x)] = \mathcal{F}[g(x)] + j\mathcal{F}[g_H(x)]$$

$$G_a(f) = G(f) + jH(f)G(f) = (1 + jH(f))G(f)$$

$$1 + jH(f) = \begin{cases} 2 & if\ f > 0 \\ 1 & if\ f = 0 \\ 0 & if\ f < 0 \end{cases}$$

we have

$$G_a(f) = \begin{cases} 2G(f) & if\ f > 0 \\ G(f) & if\ f = 0 \\ 0 & if\ f < 0 \end{cases}$$

(It is called the 'single sided band signal')

(halve the bandwidth)

(9) Fractional Fourier Transform

$$X_\phi(u) = \sqrt{1 - j\cot\phi}\, e^{j\pi\cot\phi\cdot u^2} \int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t} e^{j\pi\cdot\cot\phi\cdot t^2} x(t)\,dt$$

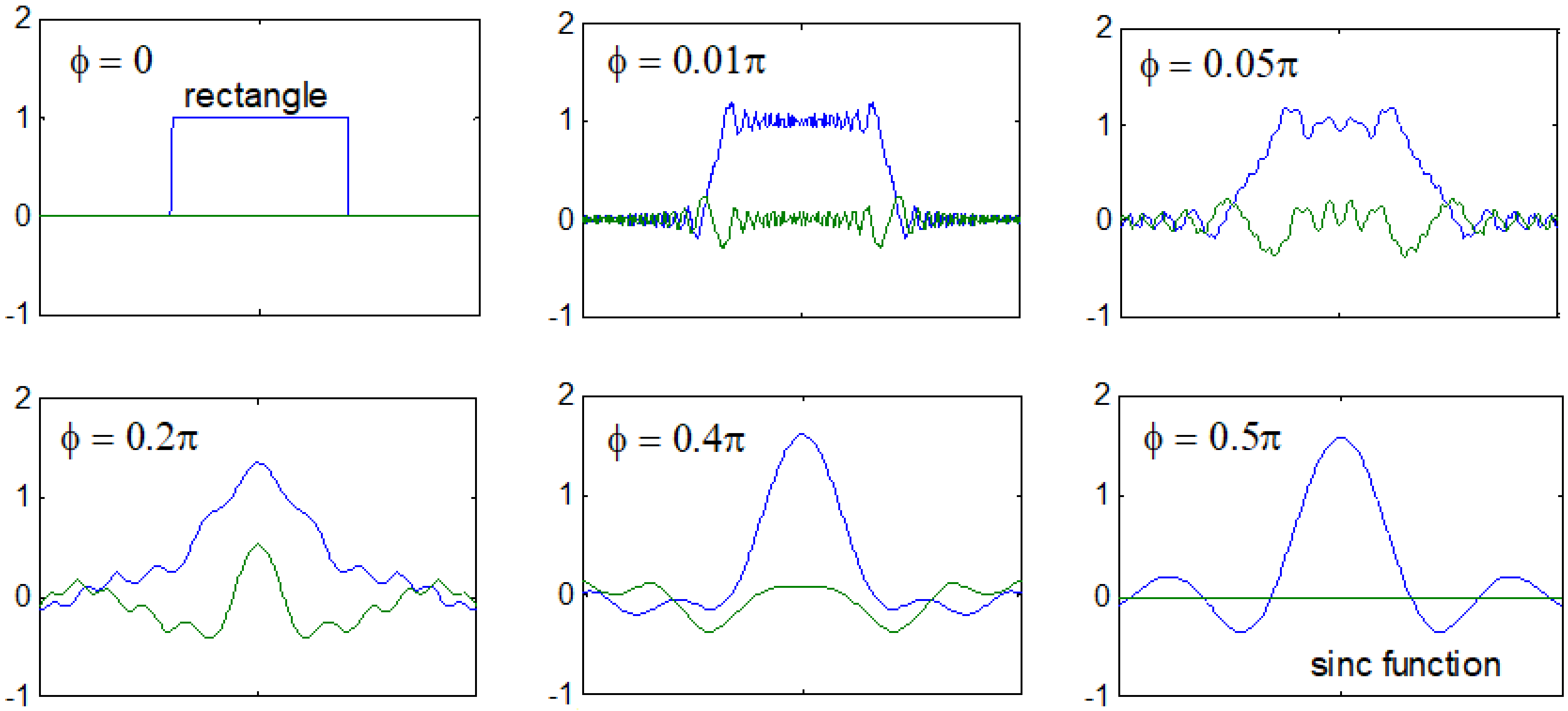
When $\phi = 0.5\pi$, the FRFT becomes the FT.

Physical meaning: Performing the FT *a* times, $\phi = 0.5a\pi$

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

Fractional Fourier transforms for a rectangular function



blue lines: real parts; green lines: imaginary part

(10) Linear Canonical Transform

$$X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{jb}}\,e^{j\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-j2\pi\frac{1}{b}ut}\,e^{j\pi\frac{a}{b}t^2}\,x(t)dt$$

where $ad-bc=1$

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\Longrightarrow$ Fourier transform

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ $\Longrightarrow$ fractional Fourier transform

[Ref] K. B. Wolf, "*Integral Transforms in Science and Engineering*," Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

Summary of Transforms

Fourier Transform	$G(f)=\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx$
2D Fourier Transform	$G(f,h)=$ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi fx}e^{-j2\pi hy}dxdy$
Hankel Transform	$G(s)=2\pi\int_{0}^{\infty} J_0(2\pi sr)rg(r)dr$
Two-Sided Laplace Transform	$G(s)=\int_{-\infty}^{\infty} e^{-st}g(t)dt$
One-Sided Laplace Transform	$G(s)=\int_{0}^{\infty} e^{-st}g(t)dt$
Fourier Cosine Transform	$G_c(f)=\int_{0}^{\infty} g(x)\cos(2\pi fx)dx$
Fourier Sine Transform	$G_s(f)=\int_{0}^{\infty} g(x)\sin(2\pi fx)dx$

Hartley Transform	$G_{ha}(f)=\int_{-\infty}^{\infty} g(x)\operatorname{cas}(2\pi fx)\,dx$
Mellin Transform	$G_M(s)=\int_0^{\infty} g(x)x^{s-1}dx$
Hilbert Transform	$g_H(x)=\Im^{-1}\{\Im[g(x)]H(f)\}$ $H(f)=-j \;\; for \;\; f>0,$ $H(f)=j \;\; for \;\; f<0, \quad H(0)=0$
Analytic Signal	$g_a(x)=g(x)+jg_H(x)$
Fractional Fourier Transform	$X_\phi(u)=\sqrt{1-j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2}$ $\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot ut} e^{j\pi\cdot\cot\phi\cdot t^2}\; x(t)dt$
Linear Canonical Transform	$X_{(a,b,c,d)}(u)=\sqrt{1/jb}\; e^{j\pi d u^2/b}$ $\int_{-\infty}^{\infty} e^{-j2\pi\, ut/b} e^{j\pi a t^2/b} x(t)dt$