# 5. Sampling and Discrete Fourier Transform



[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

# Sampling and Discrete Fourier Transform

- Sampling and Discrete Fourier Transform
  - Sampling (Sec. 5-1-2)
    - Varying the Sampling Rate (Sec. 5-1-4)
  - Reconstruction (Sec. 5-1-3)
  - Discrete Fourier Transform (Sec. 5-2-1)
    - Implementing the Continuous FT (Sec. 5-2-2)
    - Properties (Sec. 5-2-3)
    - Complexity (Sec. 5-2-4)
    - 2D Version (Sec. 5-2-5)

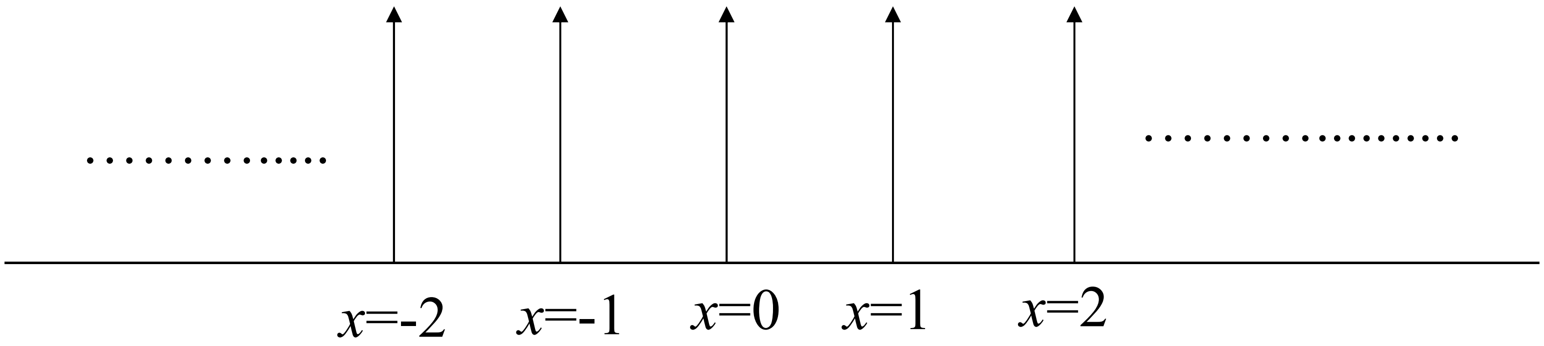
# 5.1 Sampling

## 5.1.1 Impulse Train

Impulse Train

$$p(x)=\sum_{n}\delta(x-n)=\cdots+\delta(x+1)+\delta(x)+\delta(x-1)+\delta(x-2)+\cdots\cdots$$

It is also called the comb function.



Signal sampling can be express in terms of the impulse train

$$g(x) \xrightarrow{sampling} g_s(x) = g(x)\sum_n \delta(x - n\Delta_x)$$

$$= \sum_n g_n \delta(x - n\Delta_x)$$

where $g_n = g(n\Delta_x)$

Since

$$\sum_n \delta(x - n\Delta_x) = \frac{1}{\Delta_x}\sum_n \delta\left(\frac{x}{\Delta_x} - n\right) = \frac{1}{\Delta_x} p\left(\frac{x}{\Delta_x} - n\right)$$

the sampled signal $g_s(x)$ can be expressed in terms of

$$g_s(x) = \frac{1}{\Delta_x} g(x)\, p\left(\frac{x}{\Delta_x} - n\right)$$

[**Theorem 5.1.1**] The impulse train is also an eigenfunction of the Fourier transform, i.e.,

$$P(f) = \Im\{p(x)\} = \sum_n \delta(f-n)$$

(Proof): Note that the impulse train is a periodic function

$$p(x) = p(x+1)$$

Therefore, it can be expanded by the Fourier series of the complex form with $T = 1$

$$p(x) = \sum_n c_n \exp(j2\pi nx)$$

where

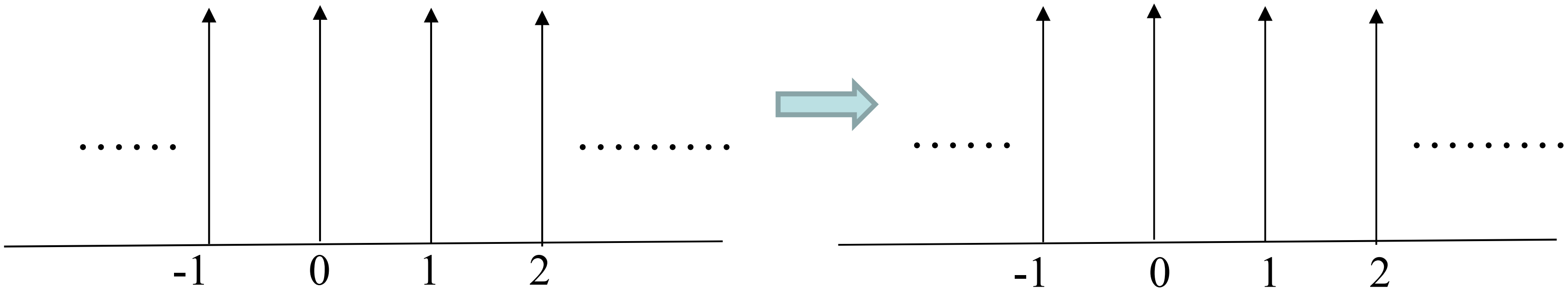
$$c_n = \frac{1}{1}\int_{-1/2}^{1/2} p(x)\exp(-j2\pi nx)\,dx = \int_{-1/2}^{1/2} \delta(x)\exp(-j2\pi nx)\,dx = 1$$

Therefore,

$$p(x) = \sum_n \exp(j2\pi nx)$$

$$\mathcal{F}[p(x)] = \sum_n \mathcal{F}[\exp(j2\pi nx)]$$
$$= \sum_n \delta(f - n)$$

$$\Im\left\{\sum_n \delta(x-n)\right\} = \sum_n \delta(f-n)$$

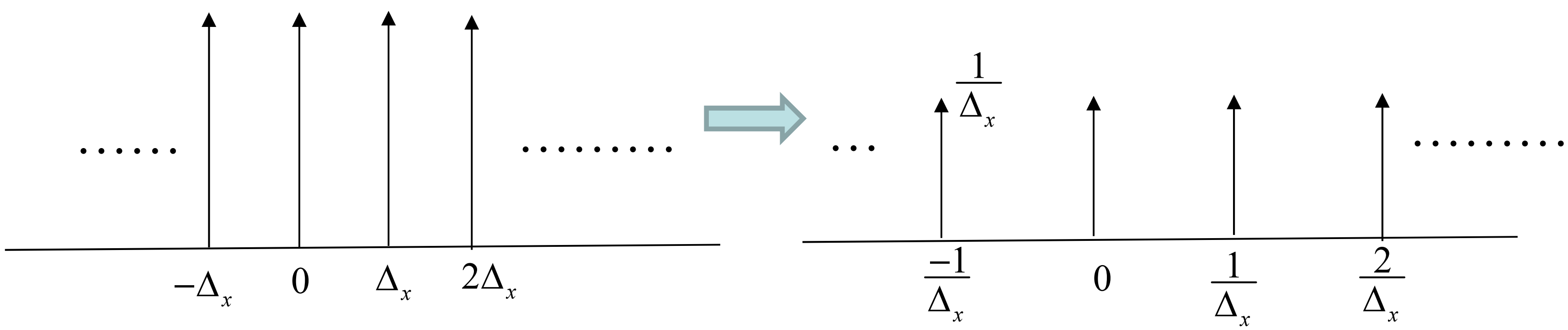


# Varying the interval of the impulse train

$$\sum_n \delta(x - n\Delta_x) = \frac{1}{\Delta_x}\sum_n \delta\left(\frac{x}{\Delta_x} - n\right) = \frac{1}{\Delta_x} p\left(\frac{x}{\Delta_x}\right)$$

$$\mathcal{F}\left[\sum_n \delta(x - n\Delta_x)\right] = \mathcal{F}\left[\frac{1}{\Delta_x} p\left(\frac{x}{\Delta_x}\right)\right] = p(\Delta_x f)$$

$$= \sum_n \delta(\Delta_x f - n) = \frac{1}{\Delta_x}\sum_n \delta\left(f - \frac{n}{\Delta_x}\right)$$



# 5.1.2 Sampling Theory

[**Theorem 5.1.2**] Suppose that we perform sampling for a continuous signal with sampling interval $\Delta_x$

$$g(x) \xrightarrow{sampling} g_s(x) = g(x)\sum_n \delta(x - n\Delta_x)$$

$$= \sum_n g_n \delta(x - n\Delta_x)$$

where

$$g_n = g(n\Delta_x)$$

then

$$G_s(f) = \frac{1}{\Delta_x}\sum_n G\left(f - \frac{n}{\Delta_x}\right)$$

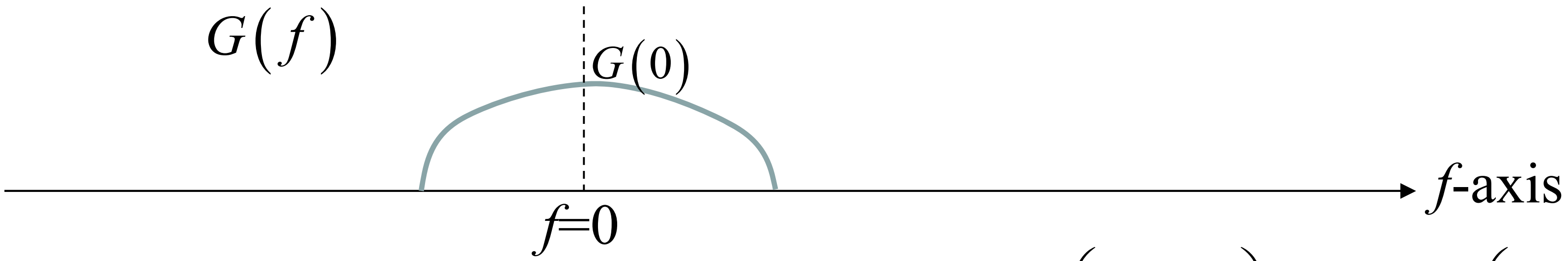
(Proof): Since
$$g_s(x) = g(x)\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)$$

$$\mathcal{F}[g_s(x)] = \mathcal{F}[g(x)] * \mathcal{F}\left[\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)\right]$$

$$G_s(f) = G(f) * p(\Delta_x f) = \frac{1}{\Delta_x}\sum_n G(f) * \delta\left(f - \frac{n}{\Delta_x}\right) = \frac{1}{\Delta_x}\sum_n G\left(f - \frac{n}{\Delta_x}\right)$$

If we set $f_s = \dfrac{1}{\Delta_x}$ ($f_s$ is call the sampling frequency)

then $$G_s(f) = f_s \sum_n G(f - nf_s)$$

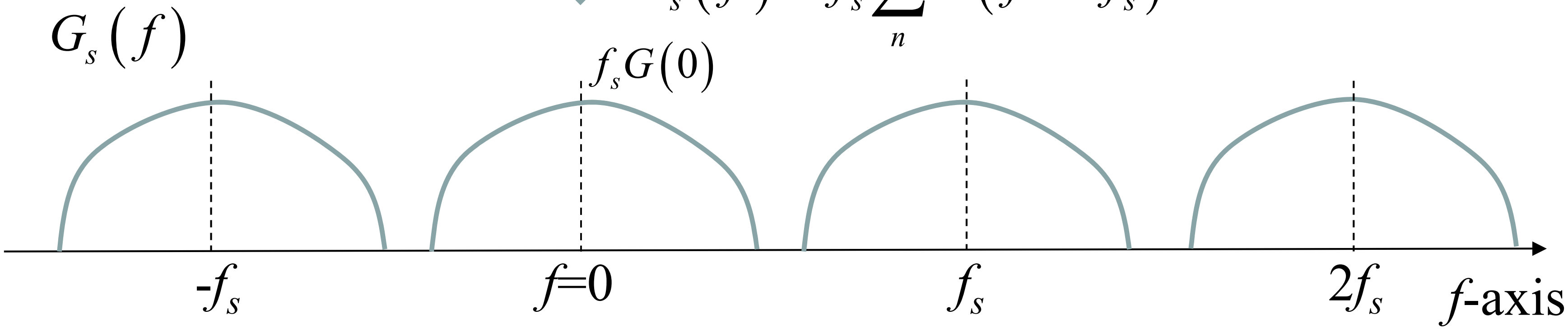



$g_n = g(n\Delta_x)$

sampling

$$g_s(x) = g(x)\sum_n \delta\left(x - \frac{n}{f_s}\right) = \sum_n g_n \delta\left(x - \frac{n}{f_s}\right)$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$



**[Sampling Theory]**

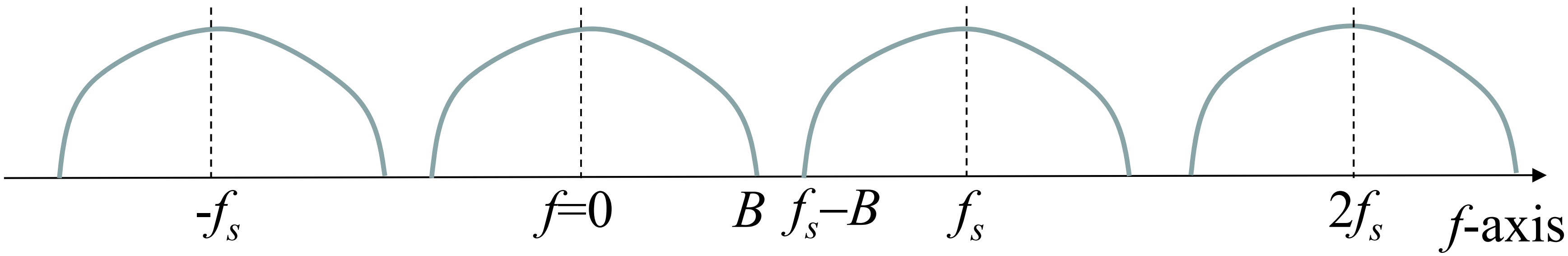
The sampling frequency $f_s = \frac{1}{\Delta_x}$ should be larger than twice of the bandwidth of the original continuous function:

$$f_s > 2B \qquad \text{(Nyquist criterion)}$$

where

$$G(f) = 0 \qquad \text{when } f > B.$$

Otherwise, the original function cannot be reconstructed and the aliasing effect is led.



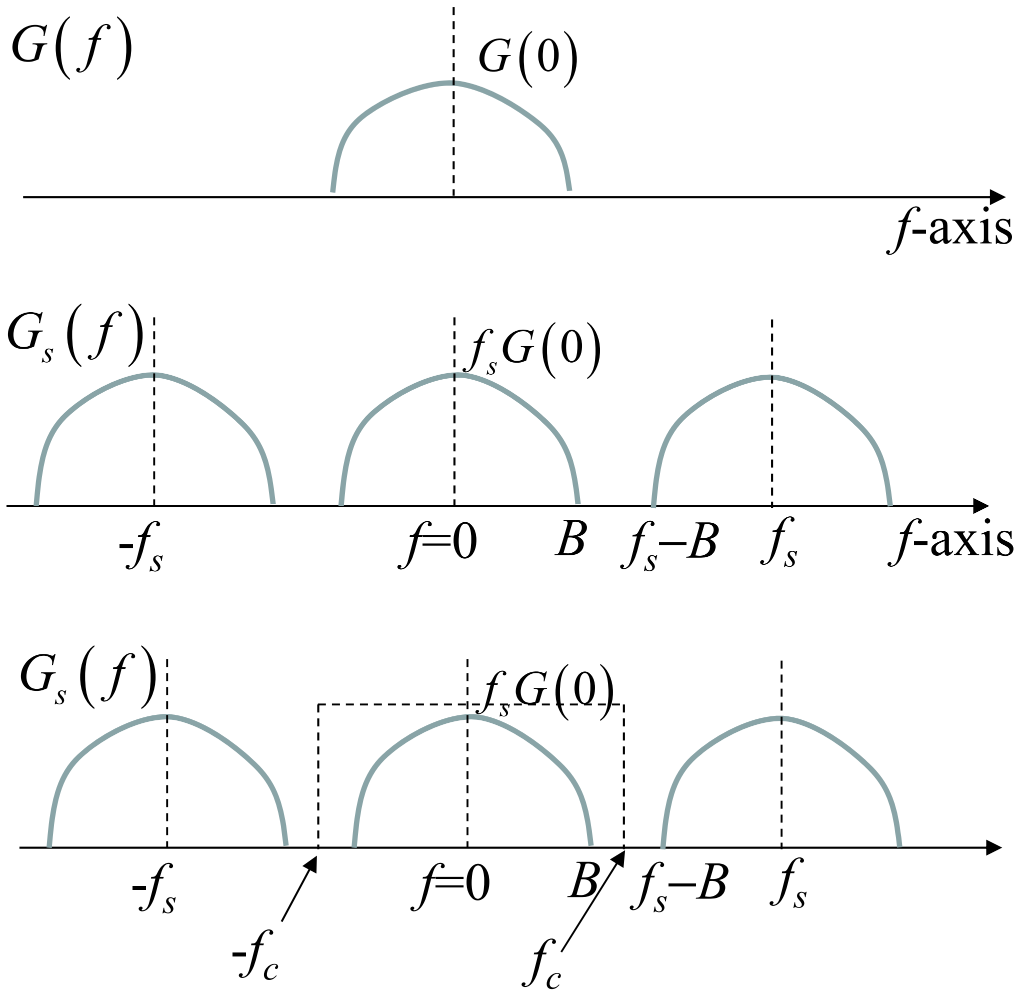
Q: What will happen if $f_s = 2B$?

Even component with frequency $f = \pm B \rightarrow$ preserved

Odd component with frequency $f = \pm B \rightarrow$ destroyed

## 5.1.3 Reconstruction (Digital to Analogous)

When the Nyquist criterion is satisfied, one can apply the lowpass filter to reconstruct the original signal.




Frequency Domain

$$G(f)$$

$$\downarrow$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$

$$\downarrow$$

$$G(f) = \frac{1}{f_s} \Pi\left(\frac{f}{2f_c}\right) G_s(f)$$

where $B < f_c < f_s - B$

Time Domain

$$g(x)$$

sampling

$$g_n = g(n\Delta_x)$$

$$g_s(x) = \sum_n g_n \delta\left(x - \frac{n}{f_s}\right)$$

reconstruction

$$g(x) = g_s(x) * \frac{2f_c}{f_s}\operatorname{sinc}(2f_c x)$$

Frequency Domain

$$G(f)$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$

$$G(f) = \frac{1}{f_s}\Pi\left(\frac{f}{2f_c}\right)G_s(f)$$

where $B < f_c < f_s - B$

$$g(x) = \frac{2f_c}{f_s} \int g_s(\tau) \operatorname{sinc}(2f_c(x-\tau))\, d\tau$$

$$= \frac{2f_c}{f_s} \int \sum_n g_n \delta\left(\tau - \frac{n}{f_s}\right) \operatorname{sinc}(2f_c(x-\tau))\, d\tau$$

$$g(x) = \frac{2f_c}{f_s} \sum_n g_n \operatorname{sinc}\left(2f_c(x - \frac{n}{f_s})\right)$$

Specially, when $f_c = f_s / 2$

$$g(x) = \sum_n g_n \operatorname{sinc}(f_s x - n)$$

Signal Reconstruction Formula:

$$g(x) = \sum_n g_n \operatorname{sinc}\left(\frac{x}{\Delta_x} - n\right) \quad \text{where} \quad g_n = g(n\Delta_x)$$

**[Example 1]** Suppose that

$$g_n = g\left(\frac{n}{2}\right)$$

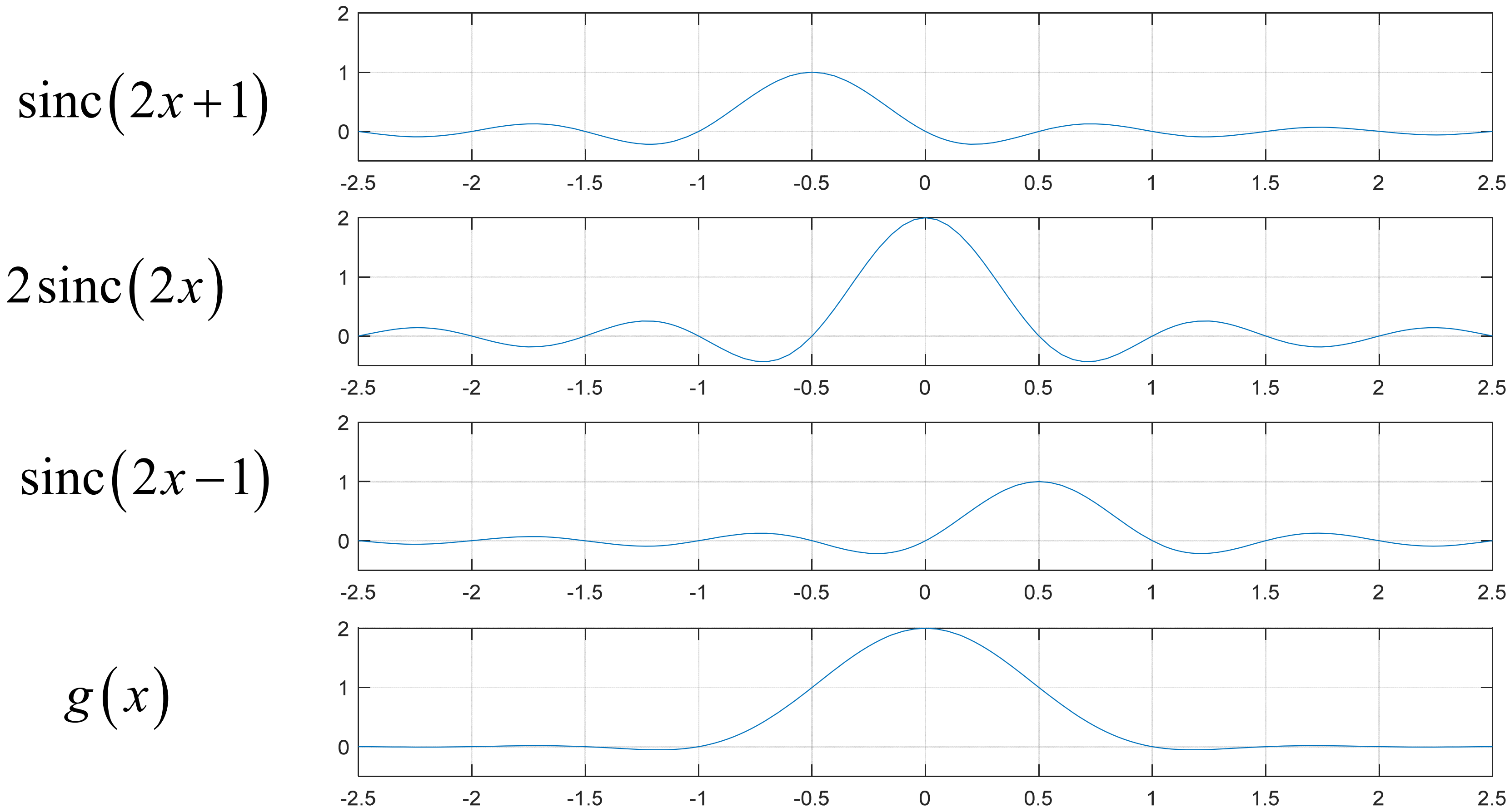
$$g_{-1} = g_1 = 1, \quad g_0 = 2, \qquad\qquad g_n = 0 \quad \text{otherwise}$$

$$G(f) = 0 \qquad \text{for } f \geq 1$$

Try to reconstruct $g(x)$.

(Solution): $\Delta_x = 1/2$

$$g(x) = \operatorname{sinc}(2x+1) + 2\operatorname{sinc} 2x + \operatorname{sinc}(2x-1)$$


sinc(2x+1)
2sinc(2x)
sinc(2x−1)
g(x)

# 5.1.4 Varying the Sampling Rate

(i) D/A conversion

$$g(x) = \sum_n g_n \operatorname{sinc}\left( \frac{x}{\Delta_x} - n \right)$$

(ii) Re-sampling

$$\hat{g}_n = g(n\Delta_{new}) = \sum_m g_m \operatorname{sinc}\left( \frac{n\Delta_{new}}{\Delta_x} - m \right)$$

Note: When $\Delta_{new} = k\Delta_x$ and $k$ is an integer

$$\hat{g}_n = g_{kn}$$

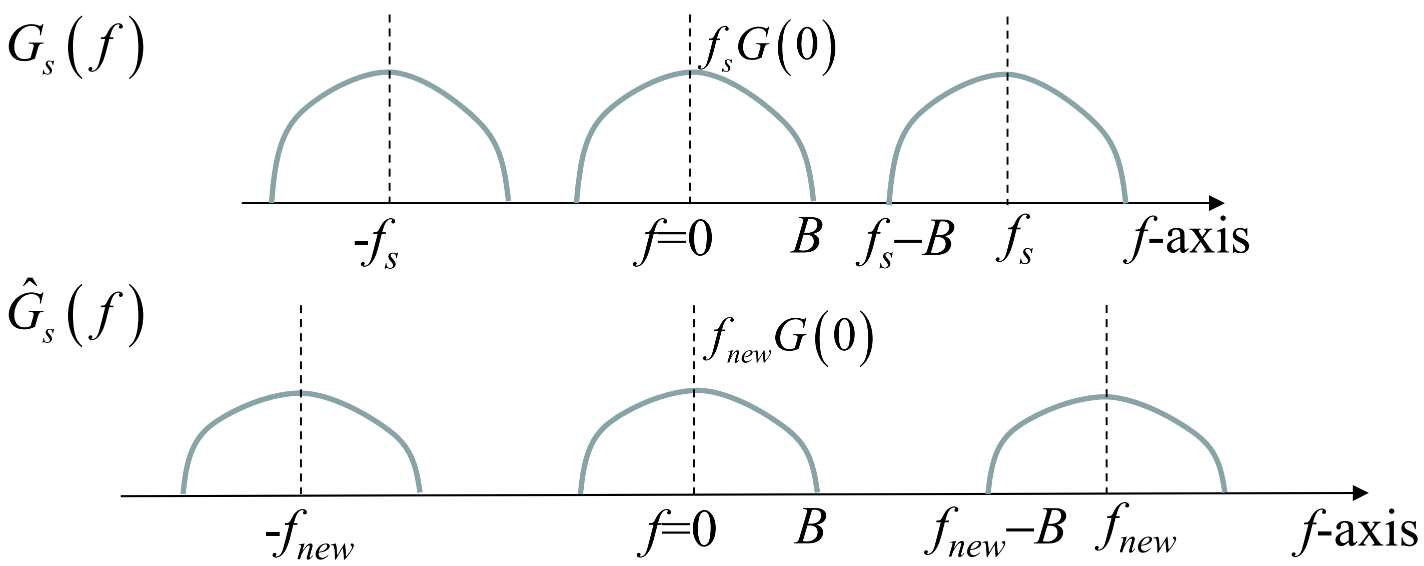
From the view point of the spectrum, if

$$g_s(x) = \sum_n g_n \delta(x - n\Delta_x) \qquad \hat{g}_s(x) = \sum_n \hat{g}_n \delta(x - n\Delta_{new})$$

then

$$G_s(f) = f_s \sum_n G(f - nf_s) \qquad \hat{G}_s(f) = f_{new} \sum_n G(f - nf_{new})$$

where $f_s = 1/\Delta_x, \quad f_{new} = 1/\Delta_{new}$



# 5.2 Discrete Fourier Transform

## 5.2.1 Derivation and Definitions of the Discrete Fourier Transform

To process discrete functions, the continuous Fourier transform should be converted into the discrete version.

Continuous Fourier transform:

$$G(f)=\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx$$

If we set

$$f=m\Delta_f, \quad x=n\Delta_x$$

then

$$G(m\Delta_f)=\sum_n g(n\Delta_x)e^{-j2\pi mn\Delta_f\Delta_x}\Delta_x$$

$$G\left(m\Delta_f\right) = \sum_n g\left(n\Delta_x\right) e^{-j2\pi mn\Delta_f\Delta_x} \Delta_x$$

Specially, if

$$\Delta_f \Delta_x = 1/N$$

then

$$G\left(m\Delta_f\right) = \sum_n g\left(n\Delta_x\right) e^{-j\frac{2\pi mn}{N}} \Delta_x$$

Discrete Fourier Transform (DFT)

$$G[m] = DFT\{g[n]\} = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

Inverse Discrete Fourier Transform (IDFT)

$$g[n] = IDFT\{G[m]\} = \frac{1}{N}\sum_{m=0}^{N-1} G[m] e^{j\frac{2\pi mn}{N}}$$

Note that the output of $g[n]$ is periodic

$$G[m] = G[m+N]$$

Also note that, on page 424,

$$G(m\Delta_f) = DFT(g(n\Delta_x))\Delta_x$$

The DFT and the IDFT form a transform pair since

$$\frac{1}{N}\sum_{m=0}^{N-1} G[m]e^{j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} g[k]e^{-j\frac{2\pi mk}{N}}e^{j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} g[k]\left(\sum_{m=0}^{N-1} e^{-j\frac{2\pi mk}{N}}e^{j\frac{2\pi mn}{N}}\right)$$

Because

$$\sum_{m=0}^{N-1} e^{j\frac{2\pi a}{N}m} = \frac{1-e^{j\frac{2\pi a}{N}N}}{1-e^{j\frac{2\pi a}{N}}} = \frac{1-e^{j2\pi a}}{1-e^{j\frac{2\pi a}{N}}} = 0 \qquad \text{if } a \neq 0,$$

$$\sum_{m=0}^{N-1} e^{j\frac{2\pi a}{N}m} = N \qquad \text{if } a = 0,$$

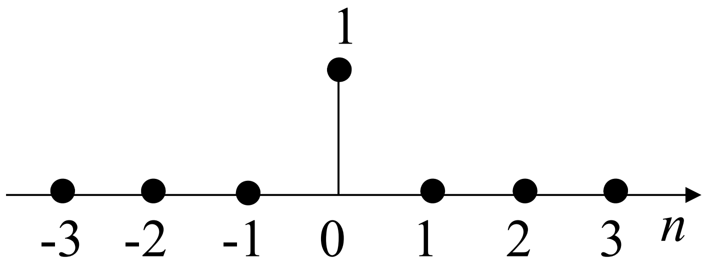
$$\sum_{m=0}^{N-1} e^{j\frac{2\pi m}{N}(n-k)} = N\delta_d[n-k]$$

unit impulse function
(discrete Dirac delta function)

$$\frac{1}{N}\sum_{m=0}^{N-1} G[m]e^{j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} g[k]N\delta_d[n-k] = g[n]$$

Unit Impulse Function (discrete Dirac delta function)

$$\delta_d[n] = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n \neq 0 \end{cases}$$




The unit impulse function has an explicit form. It does not a limitation of a distribution.

# Other possible definitions of the DFT

DFT $$G[m] = \sum_{n=n_0}^{n_0+N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

IDFT $$g[n] = \frac{1}{N} \sum_{m=m_0}^{m_0+N-1} G[m] e^{j\frac{2\pi mn}{N}}$$

DFT $$G[m] = \sqrt{\frac{1}{N}} \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

IDFT $$g[n] = \sqrt{\frac{1}{N}} \sum_{m=0}^{0\,N-1} G[m] e^{j\frac{2\pi mn}{N}}$$

## 5.2.2 Implementing Continuous FT by the DFT

Suppose that we want to calculate the continuous FT of $g(x)$ digitally and

$$g(x) = 0 \qquad \text{for } x \notin [x_1, x_1+T]$$

(i) Shifting

$$g_1(x) = g(x + x_1)$$

Note: $g_1(x) = 0 \qquad$ for $x \notin [0, T]$

(ii) Sampling

$$g_d[n] = g_1(n\Delta_x)$$

(iii) DFT

$$G_d[m] = \sum_{n=0}^{N-1} g_d[n] e^{-j\frac{2\pi mn}{N}}$$

(iv) Mapping to the true frequency

$$G_1\left(m\Delta_f\right) = G_d\left[m\right]\Delta_x \qquad \text{(from pages 424 and 425)}$$

Since

$$\Delta_f \Delta_x = 1/N \qquad \Delta_f = \frac{1}{N\Delta_x} = \frac{f_s}{N}$$

Therefore,

$$G_1\left(m\frac{f_s}{N}\right) = G_d\left[m\right]\Delta_x \qquad \text{if } 0 \le m \le N/2,$$

$$G_1\left(m\frac{f_s}{N} - f_s\right) = G_d\left[m\right]\Delta_x \qquad \text{if } N/2 \le m \le N\text{-}1$$

(v) Using the modulation property

$$G\left(f\right) = e^{-j2\pi x_1 f} G_1\left(f\right)$$

**[Example 1]**：Suppose that

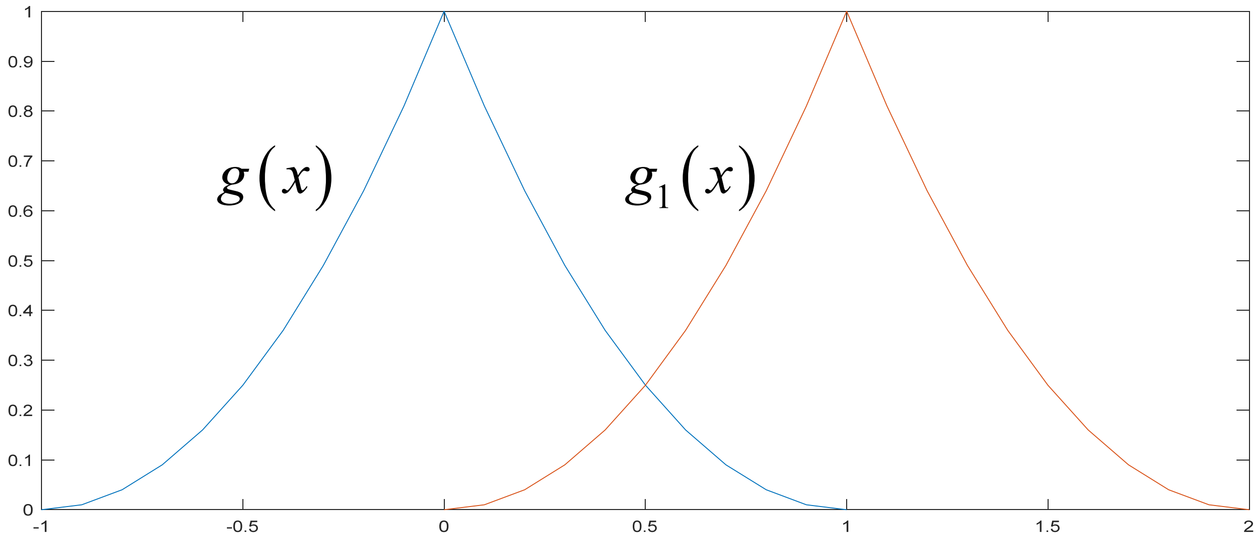
$g(x) = (1-|x|)^2$ for $-1 \le x \le 1$ $g(x) = 0$ otherwise

Sampling interval： $\Delta_x = 0.1$

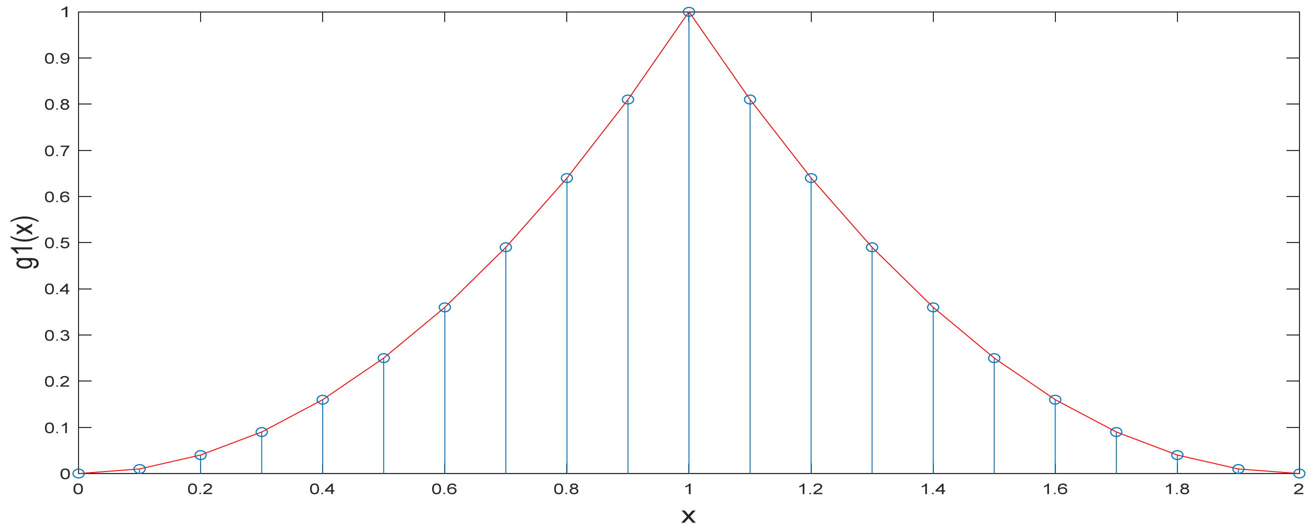
How do we obtain the FT of $g(x)$ by the DFT?

(Solution):

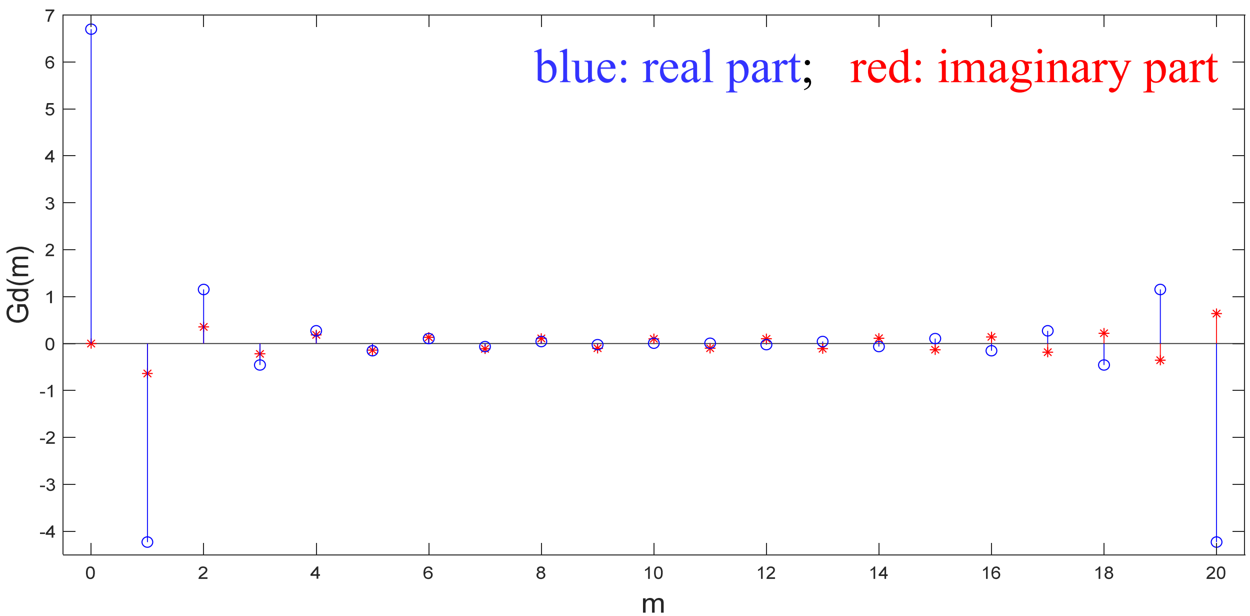
(i) $g_1(x) = g(x-1)$



(ii) $g_d[n] = g_1(n\Delta_x)$

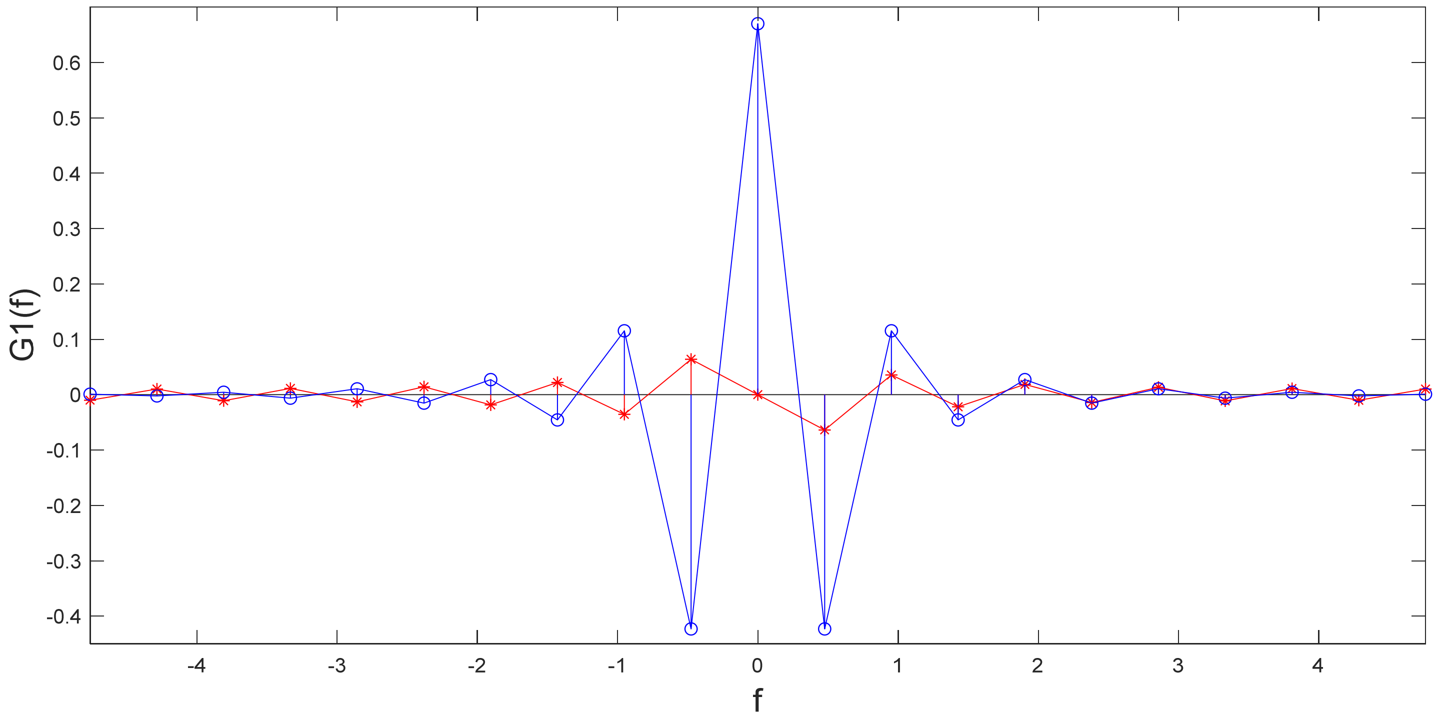



(iii) $G_d[m] = DFT(g_d[n])$



(iv) Mapping

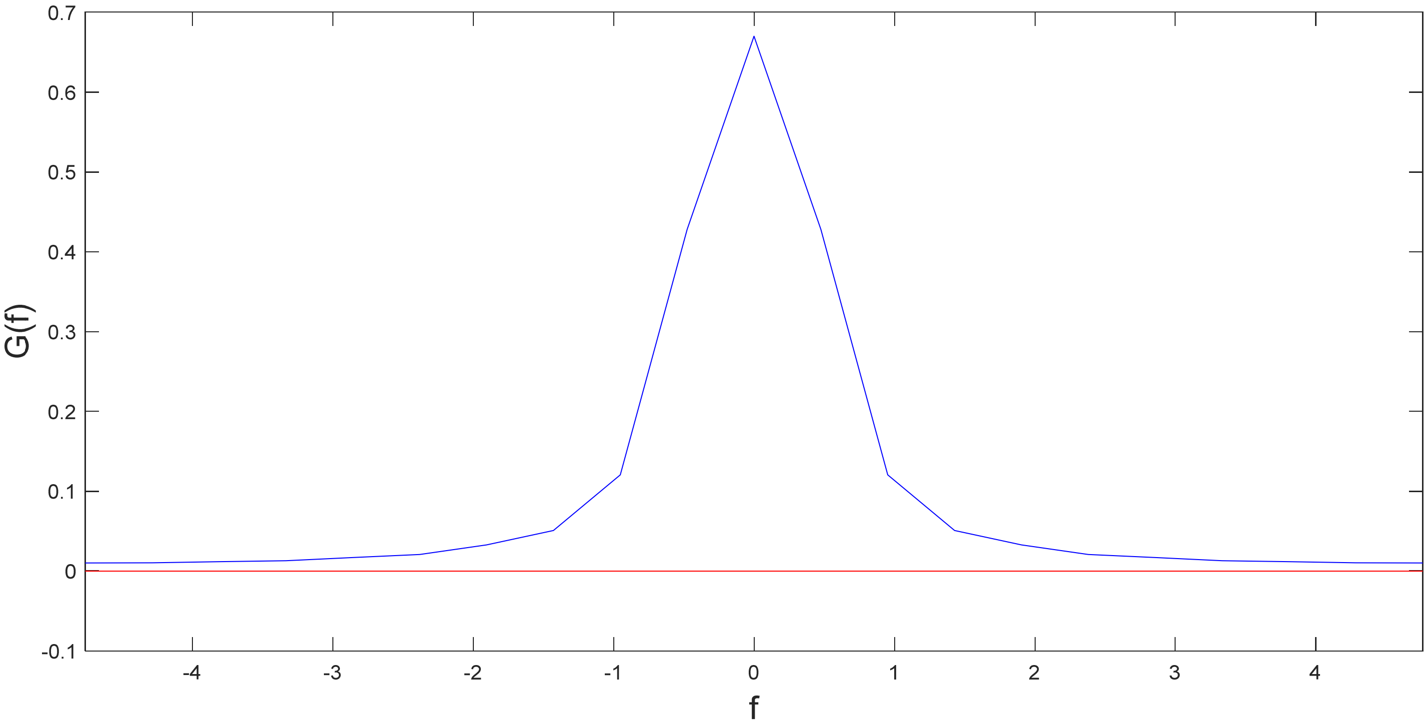
blue: real part; red: imaginary part




$$G_1\left(m\frac{f_s}{N}\right) = G_d[m]\Delta_x \qquad \text{if } 0 \le m \le N/2,$$

$$G_1\left(m\frac{f_s}{N} - f_s\right) = G_d[m]\Delta_x \qquad \text{if } N/2 \le m \le N\text{-}1$$

(v) $$G(f) = e^{j2\pi f} G_1(f)$$



# 5.2.3 Transform Pairs and Properties

**[Duality Property]**

If $G[m] = DFT\{g[n]\}$

then $g[-m] = \frac{1}{N} DFT\{G[n]\}$ $DFT\{G[n]\} = Ng[-m]$

(Proof):
$$\frac{1}{N} DFT\{G[n]\} = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi mn}{N}} G[n]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi mn}{N}} \sum_{k=0}^{N-1} e^{-j\frac{2\pi nk}{N}} g[k] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} e^{-j\frac{2\pi mn}{N}} e^{-j\frac{2\pi nk}{N}}\right) g[k]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} N\delta_d[-m-k]g[k] = \sum_{k=0}^{N-1}\delta_d[m+k]g[k]$$

$$= g[-m]$$

**[Determine the IDFT by the DFT]**

$$g_1[m] = DFT\{G[n]\}$$

$$g[n] = \frac{1}{N} g_1[-n]$$

Note:

(i) Computation loading of the IDFT = Computation loading of the DFT

(ii) In industry, only the chip of the DFT is required.

**[Transform Pair]**

| $g[n]$ | $G[m]$ |
|---|---|
| (1) $\delta_d[n]$ | $1$ |
| (2) $1$ | $N\delta_d[m]$ |
| (3) $\delta_d[n-k]$ | $\exp[-j2\pi km/N]$ |
| (4) $\exp[j2\pi kn/N]$ | $N\delta_d[m-k]$ |
| (5) $\cos[2\pi kn/N]$ | $\frac{N}{2}\delta_d[m-k]+\frac{N}{2}\delta_d[m-(N-k)]$ |
| (6) $\sin[2\pi kn/N]$ | $-j\frac{N}{2}\delta_d[m-k]+j\frac{N}{2}\delta_d[m-(N-k)]$ |
| (7) $g[n] = 1$ for $0 \le n \le W$<br>$g[n] = 0$ otherwise | $e^{-j\frac{\pi W}{N}m}\frac{\sin(\pi m(W+1)/N)}{\sin(\pi m/N)}$ $\quad for\ m \neq 0,$<br>$W+1 \quad for\ m = 0$ |
| (8) $\exp[-kn]$, $k \neq 0$ | $\frac{1-e^{-Nk}}{1-e^{-k-j2\pi m/N}}$ |

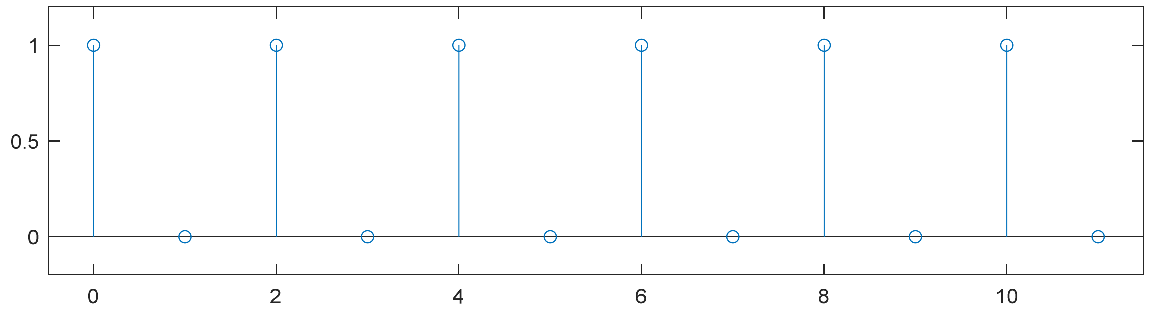
# [Discrete Impulse Train]

$$p_c[n] = \begin{cases} 1 & if\ n\ is\ a\ multiple\ of\ c \\ 0 & otherwise \end{cases}$$

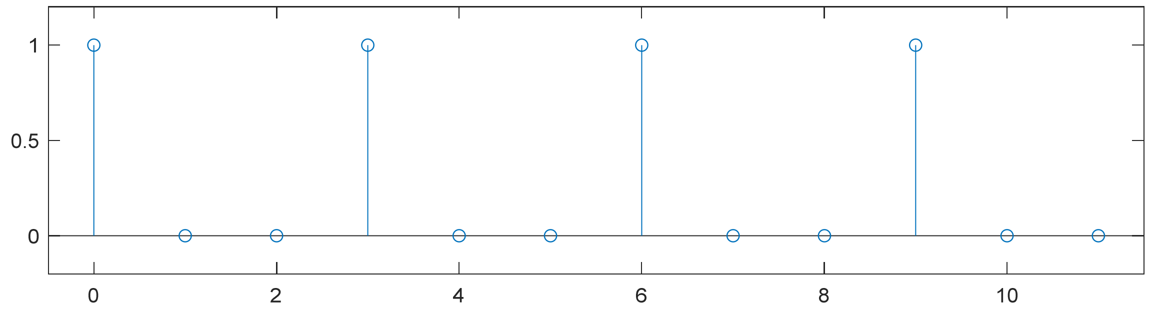
$c$ is a factor of $N$

$N = 12$

$p_2[n]$




$p_3[n]$



**[Example 2]** Determine the DFT of $p_c[n]$

$c$ is a factor of $N$

$n = ck$

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi m}{N}n} p_c[n] = \sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N}ck} = \sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k}$$

$$\sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k} = \frac{1-e^{-j\frac{2\pi m}{N/c}\frac{N}{c}}}{1-e^{-j\frac{2\pi m}{N/c}}} = 0$$

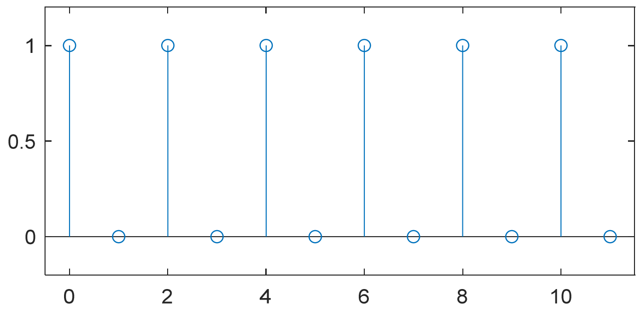
if $m$ is not a multiple *of* $N/c$

$$\sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k} = \frac{N}{c}$$

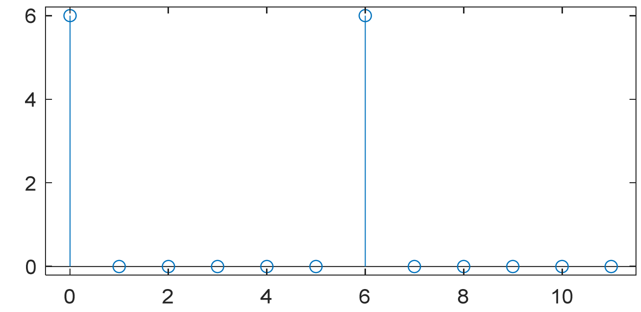
if $m$ is a multiple *of* $N/c$

Therefore, $$DFT\{p_c[n]\} = \frac{N}{c} p_{N/c}[m]$$

$p_2[n]$

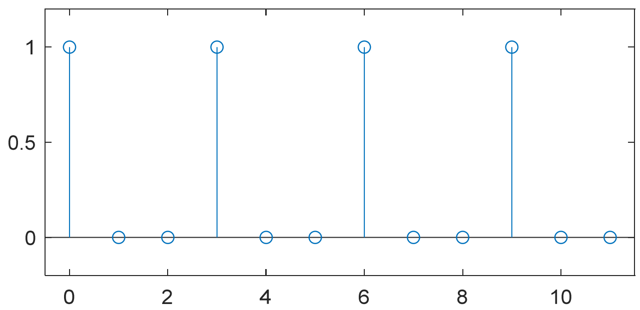



DFT $\longrightarrow$

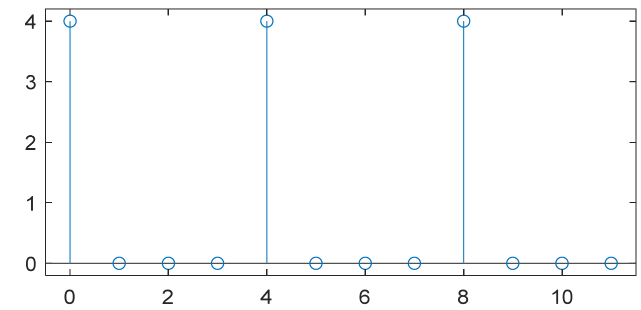



$6p_6[m]$

$p_3[n]$




DFT $\longrightarrow$




$4p_4[m]$

## Properties



| | |
|---|---|
| (1) Linear | $DFT\{ax[n]+by[n]\} = aX[m]+bY[m]$ |
| (2) DC Values | $G[0]=\sum_{n=0}^{N-1} g[n], \quad g[0]=\frac{1}{N}\sum_{n=0}^{N-1} G[m]$ |
| (3) Shifting | $DFT\{g[((n-k))_N]\} = W^{km}G[m]$<br>where $((n))_N = n$ if $0 \le n \le N\text{-}1$<br>$((n))_N = n+N$ if $-N \le n \le -1$<br>$((n))_N = n-N$ if $N \le n \le 2N\text{-}1$<br>$W = \exp(-j2\pi/N)$ |
| (4) Modulation | $DFT\{W^{kn}g[n]\} = G[((m+k))_N]$ |
| (5) Time Reverse | $DFT\{g[N-n]\} = G[N-m]$ |
| (6) Even /Odd Input | If $g[n] = g[N\text{-}n]$, then $G[m] = G[N\text{-}m]$;<br>If $g[n] = -g[N\text{-}n]$, then $G[m] = -G[N\text{-}m]$ |

| | |
|---|---|
| (7) Conjugate | $DFT\{g^*[n]\} = G^*[N\text{-}m]$ |
| (8) Real/Imaginary Input | If $g[n]$ is real, then $G[m] = G^*[N\text{-}m]$;<br>If $g[n]$ is pure imaginary, then<br>$G[m] = -G^*[N\text{-}m]$; |
| (9) Circular Convolution | If $y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$<br>then<br>$Y[m] = G[m]H[m]$ |
| (10) Circular Correlation | If<br>$y[n] = g[n] *_c h^*[-n] = \sum_{k=0}^{N-1} g[((k+n))_N]h^*[k]$<br>then<br>$Y[m] = G[m]H^*[m]$ |
| (11) Parseval's Theorem (Energy Preservation) | $N\sum_{n=0}^{N-1}\lvert g[n]\rvert^2 = \sum_{m=0}^{N-1}\lvert G[m]\rvert^2$ |
| (12) Generalized Parseval's Theorem | $N\sum_{n=0}^{N-1} g[n]h^*[n] = \sum_{m=0}^{N-1} G[m]H^*[m]$ |

# 5.2.4 Discrete Circular Convolution



**[Discrete Circular Convolution]** $g[n]*_c h[n]=\sum_{k=0}^{N-1} g[k]h[((n-k))_N]$

(Proof of the convolution property)

$$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[m]e^{j\frac{2\pi m}{N}n}=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} g[k]e^{-j\frac{2\pi m}{N}k}\sum_{s=0}^{N-1} h[s]e^{-j\frac{2\pi m}{N}s}e^{j\frac{2\pi m}{N}n}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{s=0}^{N-1} g[k]h[s]\sum_{m=0}^{N-1} e^{j\frac{2\pi(n-s-k)}{N}m}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{s=0}^{N-1} g[k]h[s]N\delta_d[((n-s-k))_N]$$

$$=\sum_{k=0}^{N-1}\sum_{s=0}^{N-1} g[k]h[((n-k))_N]$$

Here we apply $\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n}=N\delta_d[((a))_N]$

$((a))_N$: the remainder of $a$ after divided by $N$

(proved on the next page)

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = N\delta_d\left[((a))_N\right]$$

$((a))_N$: the remainder of $a$ after divided by $N$

When $a \neq bN$ where $b$ is some integer

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = \frac{1-e^{j\frac{2\pi a}{N}N}}{1-e^{j\frac{2\pi a}{N}}} = \frac{1-1}{1-e^{j\frac{2\pi a}{N}}} = 0$$

When $a = bN$ where $b$ is some integer (i.e., $((a))_N = 0$)

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = \sum_{n=0}^{N-1} e^{j2\pi bn} = \sum_{n=0}^{N-1} 1 = N$$

**[Discrete Circular Convolution and Discrete Linear Convolution]**

A discrete linear time-invariant (LTI) system can always be expressed a discrete linear convolution:

$$y_1[n] = g[n] * h[n] = \sum_{k=0}^{N-1} g[k]h[n-k]$$

However, the convolution implemented by the DFT is the discrete circular convolution:

If

$$y[n] = IDFT\left(DFT\{g[n]\}\, DFT\{h[n]\}\right) = IDFT\left(G[m]H[m]\right)$$

then

$$y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$$

$((a))_N$: the remainder of $a$ after divided by $N$

linear convolution: $y_1[n] = g[n] * h[n] = \sum_{k=0}^{N-1} g[k]h[n-k]$

circular convolution: $y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$

For example,

$$y_1[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[-1] + g[4]h[-2] + \cdots\cdots$$

$$y[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[N-1] + g[4]h[N-2] + \cdots\cdots + g[N-1]h[3]$$

The condition where the circular convolution is equal to the linear convolution:

(i) $g[n] = 0 \quad \mathit{for}\ n < 0 \ \ \mathit{or} \ \ n \geq M$

(ii) $h[n] = 0 \quad \mathit{for}\ n < 0 \ \ \mathit{or} \ \ n \geq L$

(iii) $N \geq M + L - 1$

The condition where the circular convolution is equal to the linear convolution:

(i) $g[n]=0 \quad for\ n<0\ \ or\ \ n\geq M$

(ii) $h[n]=0 \quad for\ n<0\ \ or\ \ n\geq L$

(iii) $N\geq M+L-1$

(Proof): $y[n]=\sum_{k=0}^{N-1} g[k]h[((n-k))_N]$

$$y[n]=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]+g[n+1]h[N-1]+$$
$$g[n+2]h[N-2]+\cdots+g[N+n+1-L]h[L-1]+\cdots+g[N-1]h[n+1]$$
$$=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]$$
$$+g[N+n+1-L]h[L-1]+\cdots+g[N-1]h[n+1]$$
$$=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]=y_1[n]$$

(Since $N+n+1-L \geq N+1-L \geq M$)

# 5.2.5 Complexity

$$G[m]=\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

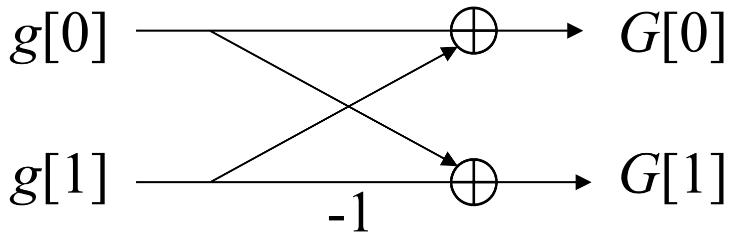
Direct implementation: Complexity = $O(N^2)$

With the fast algorithm: Complexity = $O(N\log_2 N)$

$$G[m]=\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

2-point DFT

$$\begin{bmatrix} G[0] \\ G[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \end{bmatrix}$$



When $N = 2^k$

$$G[m] = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

$$= \sum_{n=0}^{N/2-1} g[2n] e^{-j\frac{2\pi m(2n)}{N}} + \sum_{n=0}^{N/2-1} g[2n+1] e^{-j\frac{2\pi m(2n+1)}{N}}$$

$$= \sum_{n=0}^{N/2-1} g_1[n] e^{-j\frac{2\pi mn}{N/2}} + e^{-j\frac{2\pi m}{N}} \sum_{n=0}^{N/2-1} g_2[n] e^{-j\frac{2\pi mn}{N/2}}$$

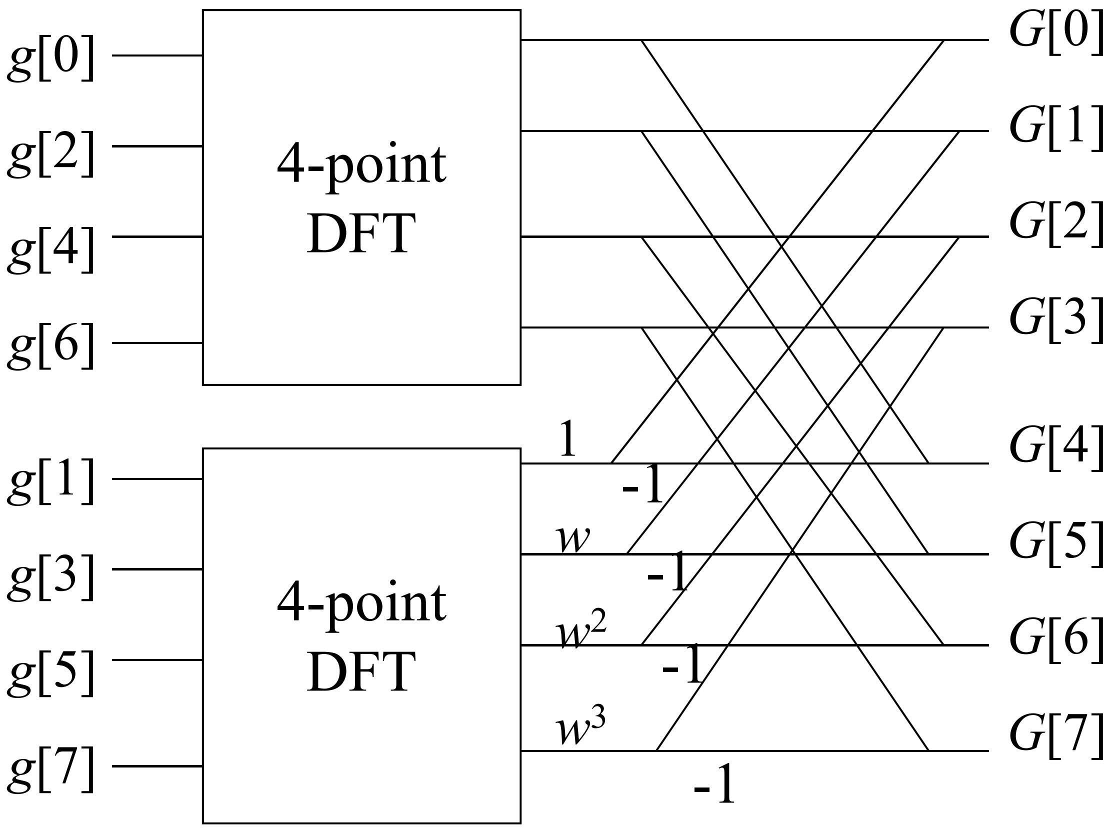
twiddle factors

$$g_1[n] = g[2n], \quad g_2[n] = g[2n+1]$$

Therefore,

one $N$-point DFT = two ($N$/2)-point DFT + twiddle factors

8-point DFT




When $N = 8$

$$w = e^{-j\frac{2\pi}{N}}$$

$$w = e^{-j\frac{2\pi}{8}}$$

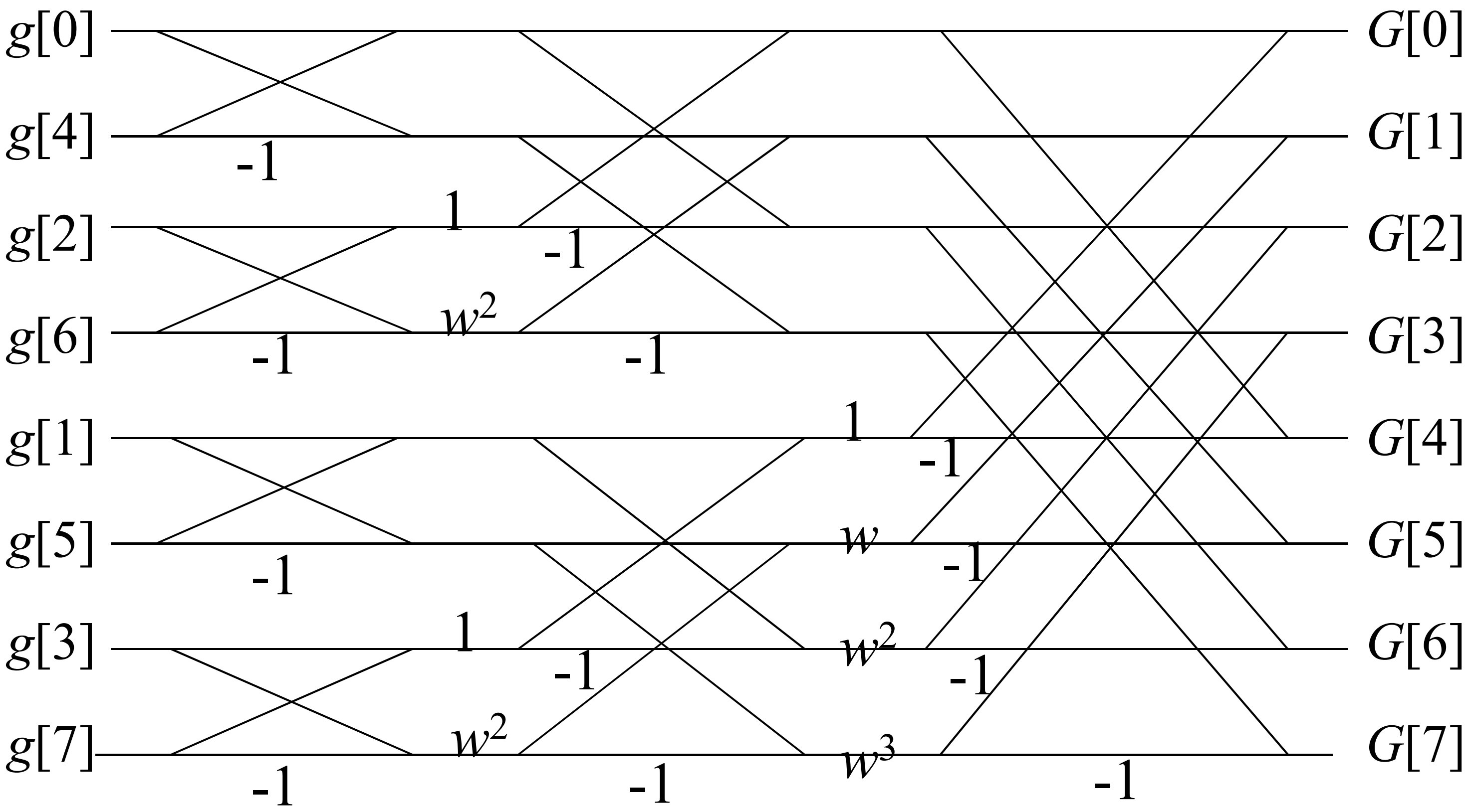
$$w^4 = -1$$

$$w^8 = 1$$

$$w^{(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}m} e^{-j\frac{2\pi}{N}\frac{N}{2}} = -e^{-j\frac{2\pi}{N}m} = -w^m$$

# 8-point DFT






$$w = e^{-j\frac{2\pi}{8}}$$

$$N \log_2 N$$

*N*/2 twiddle factors are required between two stages

Number of stages


- J. W. Cooley and J. W. Tukey, "An algorithm for the machine computation of complex Fourier series," *Mathematics of Computation*, vol. 19, pp. 297-301, Apr. 1965. (Cooley-Tukey)
- C. S. Burrus, "Index Mappings for multidimensional formulation of the DFT and convolution," *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol. 25, pp. 1239-242, June 1977. (Prime factor)

# 5.2.6 2D DFTs

2-D Discrete Fourier Transform (2-D DFT)

$$G[p,q]=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g[m,n]e^{-j\frac{2\pi mp}{M}}e^{-j\frac{2\pi nq}{N}}$$

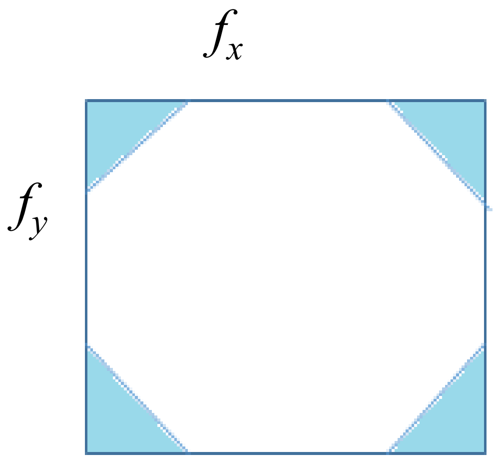
2-D Inverse Discrete Fourier Transform (2-D IDFT)

$$G[m,n]=\frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} g[p,q]e^{j\frac{2\pi mp}{M}}e^{j\frac{2\pi nq}{N}}$$

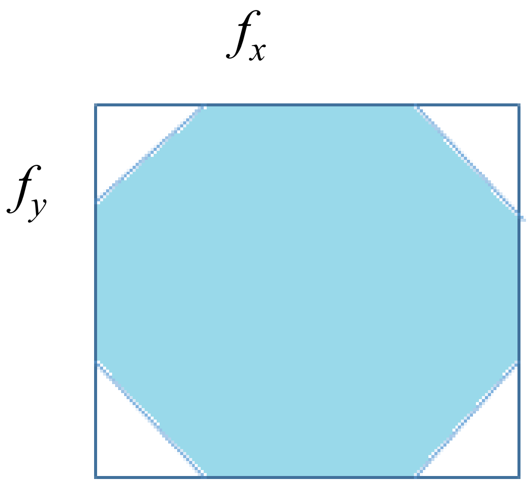
Low Frequency Part → Mild Variation → Plane

High Frequency Part → Large Variation → Edge and Noise

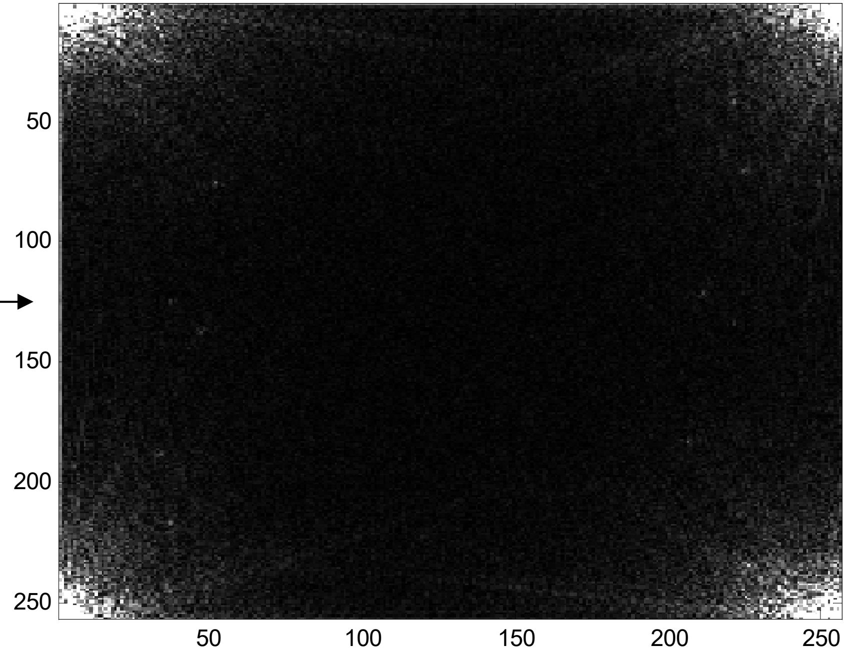
$G[m,n]$ low frequency part




high frequency part



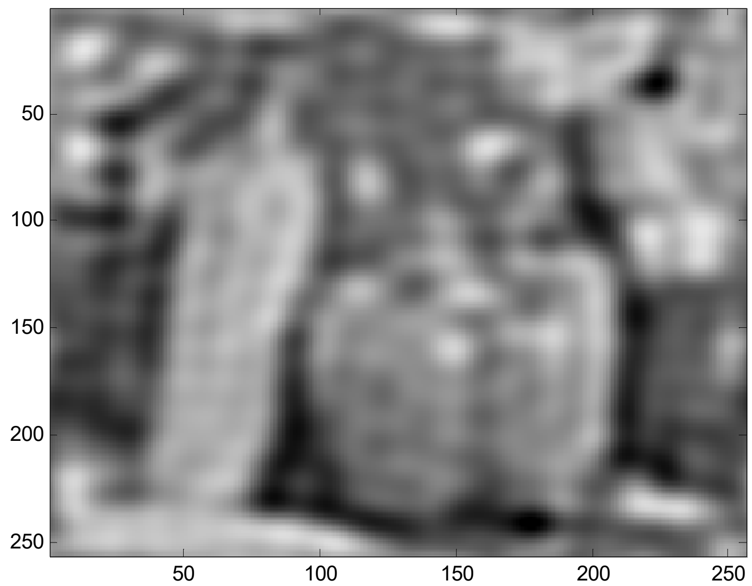
2D
DFT



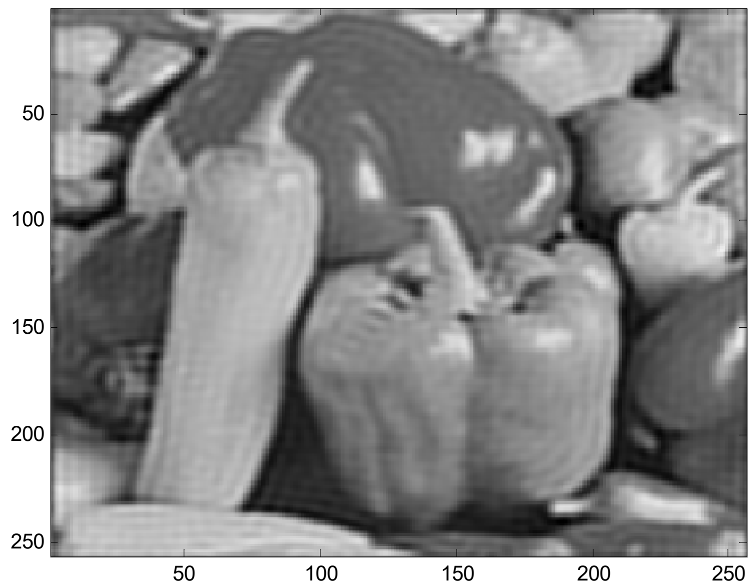

Using the gray level to show the intensity

image(…….)
colormap(gray(256))

Low Frequency Part (similar of the blurred version of the input image)

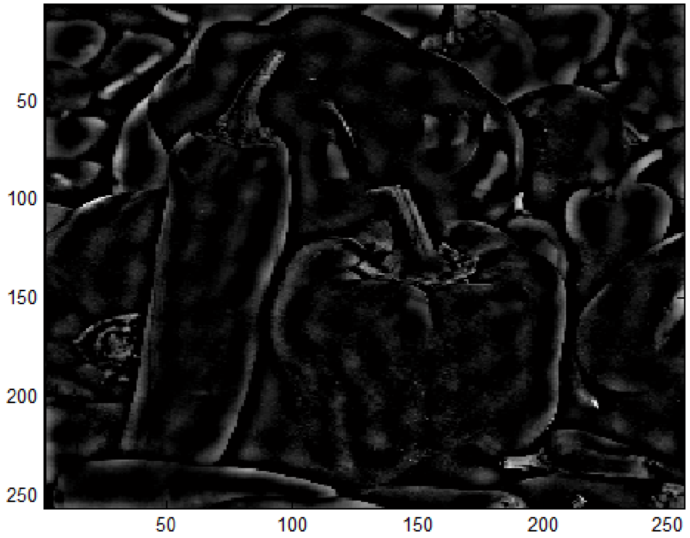



Passband: $|f_x| + |f_y| \leq N/30$

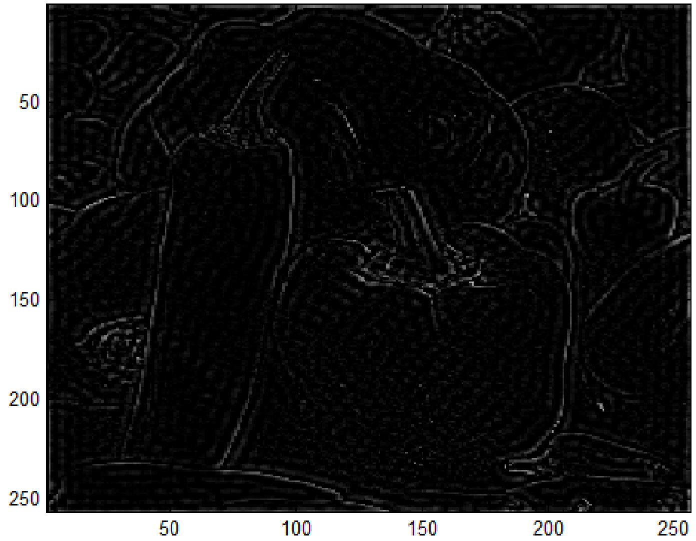



Passband: $|f_x| + |f_y| \leq N/10$

High Frequency Part (similar of the edges)




Passband: $|f_x| + |f_y| > N/30$




Passband: $|f_x| + |f_y| > N/10$