6. Advanced Linear Algebra

[1] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019.

[2] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

Linear Algebra

- Linear Algebra
  - Operations
    - Addition, Multiplication, Scalar Multi. Transpose, Hermitian, Permutation, Determinant, Rank, Inverse (Sec. 6-1)
    - Kroneker Product (Sec. 6-2)
    - Element-Wise Product (Sec. 6-2)
    - Functions of Matrices (Sec. 6-4)
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    - Jordan Canonical Form (Sec. 6-3)
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  - Applications
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    - Markov Model (Sec. 6-6)
    - Discrete Basis Approximate (Chap. 7)
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6.1 Review of Linear Algebra

6.1.1 Matrix

Scalar: x_1

Vector: $\begin{bmatrix} x_1 & x_2 & \cdots & \cdots & x_N \end{bmatrix}$

Matrix:
$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & \cdots & a_{M,N} \end{bmatrix}$$

$a_{m,n}$: entry (also called an element or a scalar)

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & \cdots & a_{M,N} \end{bmatrix}$$

$M = N$: square matrix

$M = 1$: row vector, $N = 1$: column vector, $M = N = 1$: scalar

For a square matrix,

if $a_{m,n} = 0$ for $m \neq n$ $\rightarrow$ diagonal matrix

if $a_{m,n} = 0$ for $m > n$ $\rightarrow$ upper triangular matrix

if $a_{m,n} = 0$ for $m < n$ $\rightarrow$ lower triangular matrix

A linear system can be expressed as a matrix operation

$$\begin{cases} 2x+3y=1 \\ x+4y=2 \end{cases} \Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,N}x_N = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,N}x_N = b_2 \\ \vdots \\ a_{N,1}x_1 + a_{N,2}x_2 + \cdots + a_{N,N}x_N = b_N \end{cases}$$

$$\Rightarrow \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

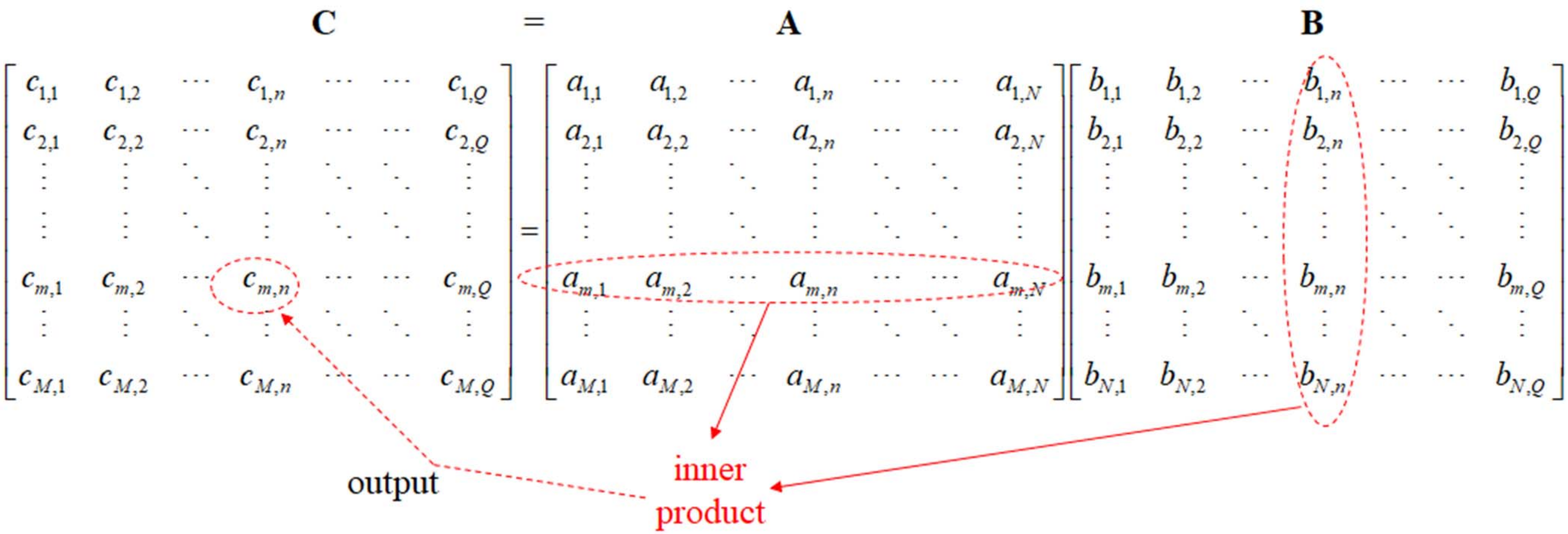
6.1.2 Matrix Operations

(1) Addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{where } c_{m,n} = a_{m,n} + b_{m,n}.$$

(2) Multiplication

$$\mathbf{C} = \mathbf{AB} \qquad \text{where} \qquad c_{m,n} = \sum_{v=1}^{N} a_{m,v} b_{v,n}$$



Multiplication (with the Sub-Matrices)

If $\mathbf{A} = \begin{bmatrix} \mathbf{A_{1,1}} & \mathbf{A_{1,2}} & \cdots & \mathbf{A_{1,V}} \\ \mathbf{A_{2,1}} & \mathbf{A_{2,2}} & \cdots & \mathbf{A_{2,V}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A_{U,1}} & \mathbf{A_{U,2}} & \cdots & \mathbf{A_{U,V}} \end{bmatrix}$ $\quad \mathbf{B} = \begin{bmatrix} \mathbf{B_{1,1}} & \mathbf{B_{1,2}} & \cdots & \mathbf{B_{1,X}} \\ \mathbf{B_{2,1}} & \mathbf{B_{2,2}} & \cdots & \mathbf{B_{2,X}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B_{W,1}} & \mathbf{B_{W,2}} & \cdots & \mathbf{B_{W,X}} \end{bmatrix}$

and the columns of $\mathbf{A_{m,v}}$ is equal to the rows of $\mathbf{B_{v,x}}$

then $\mathbf{AB} = \begin{bmatrix} \mathbf{C_{1,1}} & \mathbf{C_{1,2}} & \cdots & \mathbf{C_{1,X}} \\ \mathbf{C_{2,1}} & \mathbf{C_{2,2}} & \cdots & \mathbf{C_{2,X}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C_{U,1}} & \mathbf{C_{U,2}} & \cdots & \mathbf{C_{U,X}} \end{bmatrix}$ where $\mathbf{C_{u,x}} = \sum_{v=1}^{V} \mathbf{A_{u,v}} \mathbf{B_{v,x}}$

[Example 1]

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}$$

where $\mathbf{C}_{1,1} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$$\mathbf{C}_{1,2} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} 1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\mathbf{C}_{2,1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix} \qquad \mathbf{C}_{2,2} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \cdot 1 = 1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(3) Scalar Multiplication

$$k\mathbf{A} = k\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & \cdots & a_{M,N} \end{bmatrix} = \begin{bmatrix} ka_{1,1} & ka_{1,2} & \cdots & \cdots & ka_{1,N} \\ ka_{2,1} & ka_{2,2} & \cdots & \cdots & ka_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ ka_{M,1} & ka_{M,2} & \cdots & \cdots & ka_{M,N} \end{bmatrix}$$

(4) Transpose

$$\mathbf{B} = \mathbf{A}^{\mathbf{T}} \quad \textit{if } b_{m,n} = a_{n,m}$$

(5) Hermitian

$$\mathbf{B} = \mathbf{A}^{\mathbf{H}} \quad \textit{if } b_{m,n} = a^*_{n,m}$$

* Means conjugation

In fact,

$$\mathbf{B} = \left(\mathbf{A}^T\right)^* = \left(\mathbf{A}^*\right)^T$$

* Definition related to transpose and Hermitian

If $\mathbf{A} = \mathbf{A}^{\mathbf{T}}$ $\Longrightarrow$ Symmetric Matrix

If $\mathbf{A} = -\mathbf{A}^{\mathbf{T}}$ $\Longrightarrow$ Skew Symmetric Matrix

If $\mathbf{A} = \mathbf{A}^{\mathbf{H}}$ $\Longrightarrow$ Symmetric Hermitian Matrix

If $\mathbf{A} = -\mathbf{A}^{\mathbf{H}}$ $\Longrightarrow$ Skew Symmetric Hermitian Matrix

(6) Permutation: exchanging rows or exchanging columns

Examples: Exchange the 2nd and the 4th rows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

permuting matrix

Exchange the 2nd and the 3rd columns:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 7 & 6 & 8 \\ 9 & 11 & 10 & 12 \\ 13 & 15 & 14 & 16 \end{bmatrix}$$

permuting matrix

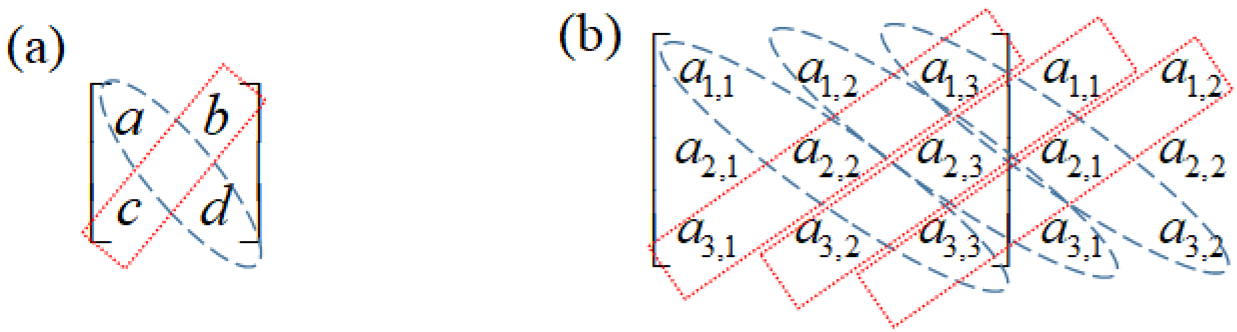
(7) Determinant

denoted by det(**A**) or |**A**|

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}$$



In general, if

$$\mathbf{A}=\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & \cdots & a_{N,N} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,N}C_{i,N}$$
$$= a_{k,1}C_{k,1} + a_{k,2}C_{k,2} + \cdots + a_{k,N}C_{k,N}$$

We call $C_{i,j}$ the cofactor of $a_{i,j}$ and call $M_{i,j}$ a minor determinant.

where $\quad C_{i,j} = (-1)^{i+j} M_{i,j}$

$$M_{i,j} = \det\begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,N} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,N} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & \cdots & a_{N,j-1} & a_{N,j} & a_{N,j+1} & \cdots & a_{N,N} \end{bmatrix}$$

(8) Rank

The number of linearly independent rows in a matrix.
(It is equivalent to the number of linearly independent columns in a matrix).

Example:

$$rank\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\right) = 2$$

In a linearly independent set, any vector cannot be expressed by a linearly combination of other vectors.

That is, the solution of

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_N\mathbf{v_N} = \mathbf{0}$$

is

$$c_1 = c_2 = \cdots = c_N = \mathbf{0}$$

Note: For an NxN square matrix $\mathbf{A}$, if

$$rank\left(\mathbf{A}\right) < N$$

then

$$\det\left(\mathbf{A}\right) = 0$$

(9) Inverse

If **A** is a square matrix and det(**A**) ≠ 0, then the inverse of **A** (denoted by $\mathbf{A}^{-1}$) is a matrix that satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

Application: For a linear system that can be expressed by

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

its solution is:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

Determining the Inverse (Method 1: by Adjoint Matrix)

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \operatorname{adj} \mathbf{A}$$

where

$$\operatorname{adj} \mathbf{A} = \begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & \cdots & C_{N,1} \\ C_{1,2} & C_{2,2} & \cdots & \cdots & C_{N,2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ C_{1,N} & C_{2,N} & \cdots & \cdots & C_{N,N} \end{bmatrix} = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & \cdots & C_{1,N} \\ C_{2,1} & C_{2,2} & \cdots & \cdots & C_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ C_{N,1} & C_{N,2} & \cdots & \cdots & C_{N,N} \end{bmatrix}^T$$

$C_{i,j}$ the cofactor of $a_{i,}$

Determining the Inverse (Method 2: by Row Elimination)

$$\left[\mathbf{A}\middle|\mathbf{I}\right] \xrightarrow{\text{elementary row operations}} \left[\mathbf{I}\middle|\mathbf{A}^{-1}\right] \qquad \text{(preferred)}$$

[Example 2]: Determine the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ -1 & -1 & 2 & 3 \\ 1 & 3 & 3 & 4 \end{bmatrix}$$

(Solution):

$$\left[\mathbf{A}\middle|\mathbf{I}\right] = \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{E}_1\left[\mathbf{A}|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&2&1&0&0&0\\0&-1&-1&-3&-2&1&0&0\\0&1&3&5&1&0&1&0\\0&1&2&2&-1&0&0&1\end{array}\right]$$

where $\mathbf{E}_1=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\1&0&1&0\\-1&0&0&1\end{bmatrix}$

$$\mathbf{E}_2\mathbf{E}_1\left[\mathbf{A}|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&2&1&0&0&0\\0&1&1&3&2&-1&0&0\\0&1&3&5&1&0&1&0\\0&1&2&2&-1&0&0&1\end{array}\right]$$

where $\mathbf{E}_2=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\left[\mathbf{A}|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&2&1&0&0&0\\0&1&1&3&2&-1&0&0\\0&0&2&2&-1&1&1&0\\0&0&1&-1&-3&1&0&1\end{array}\right]$$

where $\mathbf{E}_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{bmatrix}$

$$\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\left[\mathbf{A}|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&2&1&0&0&0\\0&1&1&3&2&-1&0&0\\0&0&1&1&-1/2&1/2&1/2&0\\0&0&1&-1&-3&1&0&1\end{array}\right]$$

where $\mathbf{E}_4=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1/2&0\\0&0&0&1\end{bmatrix}$

$$\mathbf{E_5E_4E_3E_2E_1}\left[\mathbf{A}\middle|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&2&1&0&0&0\\0&1&1&3&2&-1&0&0\\0&0&1&1&-1/2&1/2&1/2&0\\0&0&0&-2&-5/2&1/2&-1/2&1\end{array}\right]\quad\text{where}\quad\mathbf{E_5}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}$$

$$\mathbf{E_6E_5E_4E_3E_2E_1}\left[\mathbf{A}\middle|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&2&1&0&0&0\\0&1&1&3&2&-1&0&0\\0&0&1&1&-1/2&1/2&1/2&0\\0&0&0&1&5/4&-1/4&1/4&-1/2\end{array}\right]\qquad\mathbf{E_6}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1/2\end{bmatrix}$$

$$\mathbf{E_7E_6E_5E_4E_3E_2E_1}\left[\mathbf{A}\middle|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc}1&2&1&0&-3/2&1/2&-1/2&1\\0&1&1&0&-7/4&-1/4&-3/4&3/2\\0&0&1&0&-7/4&3/4&1/4&1/2\\0&0&0&1&5/4&-1/4&1/4&-1/2\end{array}\right]\qquad\mathbf{E_7}=\begin{bmatrix}1&0&0&-2\\0&1&0&-3\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

$$\mathbf{E_9E_8E_7E_6E_5E_4E_3E_2E_1}\left[\mathbf{A}\middle|\mathbf{I}\right]=\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & 7/4 & 5/4 & -3/2 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -7/4 & 3/4 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 & 5/4 & -1/4 & 1/4 & -1/2 \end{array}\right]$$

where

$$\mathbf{E_8}=\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E_9}=\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\mathbf{A}^{-1}=\begin{bmatrix} 1/4 & 7/4 & 5/4 & -3/2 \\ 0 & -1 & -1 & 1 \\ -7/4 & 3/4 & 1/4 & 1/2 \\ 5/4 & -1/4 & 1/4 & -1/2 \end{bmatrix}$$

6.1.3 Eigenvalues and Eigenvectors

[Definitions]

For a square matrix **A**, if

$$\mathbf{A}\mathbf{e} = \lambda\mathbf{e}$$

then λ is called the eigenvalue
e is called the eigenvector corresponding to λ

[Determining Eigenvalues]

Eigenvalues are the solutions of

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

characteristic equation

[Determining Eigenvectors]

Solve **e** from

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{e}=\mathbf{0}$$

[Eigenspace]

Suppose that $\mathbf{e_1}$, $\mathbf{e_2}$, …, $\mathbf{e_k}$ are linearly independent eigenvectors corresponding to λ. Then the eigenspace corresponding to λ is

$$span\{\mathbf{e_1},\mathbf{e_2},\cdots,\mathbf{e_k}\}$$

$$=c_1\mathbf{e_1}+c_2\mathbf{e_2}+\cdots+c_k\mathbf{e_k}$$

c_1, c_2, …, c_k can be any constants

[Eigenvalue-Eigenvector Deconposition]

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \cdots & \mathbf{e_{N-1}} & \mathbf{e_N} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}$$

$$\mathbf{A}\mathbf{e_n} = \lambda_n \mathbf{e_n} \qquad n = 1, 2, \ldots, N$$

$\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_{N-1}}, \mathbf{e_N}$ are linearly independent

Specially, if

$$\mathbf{A} = \mathbf{A}^{\mathbf{H}}$$

the eigenvectors form a complete and orthogonal set (after normalization, it becomes a complete and orthonormal set), then

$$\mathbf{E}^{-1} = \mathbf{E}^{\mathbf{H}}$$

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{\mathbf{H}}$$

and **A** can be decomposed into

$$\mathbf{A} = \lambda_1 \mathbf{e_1}\mathbf{e_1^H} + \lambda_2 \mathbf{e_2}\mathbf{e_2^H} + \cdots + \lambda_{N-1} \mathbf{e_{N-1}}\mathbf{e_{N-1}^H} + \lambda_N \mathbf{e_N}\mathbf{e_N^H}$$

6.1.4 Properties

(A) Geometric Operation Formula

(1) Rotation	Clockwise $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Clockwise rotation with respect to (x_0, y_0) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$
(2) Scaling	$\begin{bmatrix} x_1 - x_0 \\ y_1 - x_0 \end{bmatrix} = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - x_0 \end{bmatrix}$
(3) Shearing	$\begin{bmatrix} x_1 - x_0 \\ y_1 - x_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - x_0 \end{bmatrix}$ or $\begin{bmatrix} x_1 - x_0 \\ y_1 - x_0 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - x_0 \end{bmatrix}$

(4) Reflection	with respect to (x_0, y_0) $$\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$ with respect to (x_0, y_0), only reflect on x-axis $$\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$
(5) Affine Transformation	$$\begin{bmatrix} x_1 - x_0 \\ y_1 - x_0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - x_0 \end{bmatrix}$$
(6) Projection	on the x-axis $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ on the axis of $c(\cos\theta, \sin\theta)$: $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(B) Properties of Determinants

(1) Transpose	$\det(\mathbf{A}) = \det(\mathbf{A}^{\mathrm{T}})$
(2) Multiplication	$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A})\det(\mathbf{B})$
(3) Zero row / column	$\det(\mathbf{A}) = 0$ if all the entries of a row (or a column) are 0.
(4) Row exchange	If **B** is the same as **A** but two of the rows are exchanged $$\det(\mathbf{B}) = -\det(\mathbf{A})$$
(5) Scaling	If $b_{i,j} = a_{i,j}$ when $i \neq k$, and $b_{k,j} = \tau a_{k,j}$, then $$\det(\mathbf{B}) = \tau \det(\mathbf{A})$$
(6) Row addition	If $b_{i,j} = a_{i,j}$ when $i \neq k$, and $b_{k,j} = a_{k,j} + \tau a_{h,j}$, then $$\det(\mathbf{B}) = \det(\mathbf{A})$$

(C) Properties of Ranks

(i)	rank($\mathbf{0}$) = 1
(ii)	rank($\mathbf{I}$) = N if $\mathbf{I}$ is an $N \times N$ identify matrix
(iii)	rank($\mathbf{P}$) = N if $\mathbf{P}$ is an $N \times N$ permutation matrix
(iv)	rank($\mathbf{D}$) = N_1 if $\mathbf{D}$ is a diagonal matrix and there are N_1 nonzero entries in the diagonal line
(v)	rank($\mathbf{A}$) = N if $\mathbf{A}$ is an $N \times N$ triangular matrix and all of the entries in the diagonal line are nonzero
(vi)	rank($\mathbf{A}$) = rank($\mathbf{A}^{\mathbf{T}}$)
(vii)	rank($\mathbf{A}$) $\leq$ min(M, N) if $\mathbf{A}$ is an $M \times N$ matrix
(viii)	rank($\mathbf{DA}$) = rank($\mathbf{A}$) if $\mathbf{A}$ is an $M \times N$ and $\mathbf{D}$ is an $N \times N$ matrix whose entries in the diagonal line are all nonzero
(ix)	rank($\mathbf{BA}$) $\leq$ min(rank($\mathbf{A}$), rank($\mathbf{B}$))
(x)	rank($\mathbf{BA}$) = rank($\mathbf{A}$) if $\mathbf{A}$ is an $M \times N$ matrix and rank($\mathbf{B}$) = M

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(D) Properties of Matrix Inverse

(i)	$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
(ii)	$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
(iii)	$(\mathbf{A_1A_2}\cdots\mathbf{A_h})^{-1} = \mathbf{A_h}^{-1}\cdots\mathbf{A_2}^{-1}\,\mathbf{A_1}^{-1}$
(iv)	$(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}, \quad (\mathbf{A}^{\mathrm{H}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{H}}$
(v)	$\mathbf{I}^{-1} = \mathbf{I}$
(vi)	If $\mathbf{D}$ is a diagonal matrix, $\mathbf{D}^{-1} = \mathbf{F}$ where $f_{m,n} = 0$ if $m \neq n$, $f_{n,n} = 1/d_{n,n}$
(vii)	$\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$ if $\mathbf{P}$ is a permutation matrix
(viii)	$\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2^{-1} \end{bmatrix}$
(ix)	$\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$

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(E) Properties of Eigenvectors and Eigenvalues

(1) sum of eigenvalues	$\sum_{m=1}^{N} \lambda_m = \sum_{n=1}^{N} A[n,n]$
(2) product of eigenvalues	$\prod_{m=1}^{N} \lambda_m = \det(\mathbf{A})$
(3) eigenvectors / eigenvalues for $\mathbf{A} = \mathbf{A}^{\mathbf{H}}$	If $\mathbf{A} = \mathbf{A}^{\mathbf{H}}$, then (i) the eigenvalues are real, (ii) if $\mathbf{A}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1, \mathbf{A}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2$, $\lambda_1 \neq \lambda_2$, then $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 0$ Eigenvectors with different eigenvalues are orthogonal.
(4) eigenvectors / eigenvalues for $\mathbf{A}^{-1}$	If $\mathbf{A}\mathbf{e} = \lambda \mathbf{e}$, then $\mathbf{A}^{-1}\mathbf{e} = \lambda^{-1}\mathbf{e}$
(5) similar matrix	$\mathbf{S}(\mathbf{B}\mathbf{e}) = \lambda \mathbf{B}\mathbf{e}$ if $\mathbf{A}\mathbf{e} = \lambda \mathbf{e}$ and $\mathbf{S} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$

(6) quadratic form	If $f(x,y)=ax^2+bxy+cy^2=\begin{bmatrix}x & y\end{bmatrix}\mathbf{M}\begin{bmatrix}x\\y\end{bmatrix}$, where $\mathbf{M}=\begin{bmatrix}a & b/2\\ b/2 & c\end{bmatrix}$, then $f(x_1,y_1)=\lambda_1 x_1^2+\lambda_2 y_1^2$ where $\begin{bmatrix}x_1 & y_1\end{bmatrix}=\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}\mathbf{e_1} & \mathbf{e_2}\end{bmatrix}$ $\mathbf{M}=\begin{bmatrix}\mathbf{e_1} & \mathbf{e_2}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}\mathbf{e_1^T}\\ \mathbf{e_2^T}\end{bmatrix}$
(7) shape for the quadratic form	(i) ellipse: $\lambda_1>0$ and $\lambda_2>0$ or $\lambda_1<0$ and $\lambda_2<0$ (ii) hyperbola: $\lambda_1>0$ and $\lambda_2<0$ or $\lambda_1<0$ and $\lambda_2>0$ (iii) parabola: $\lambda_1=0$ or $\lambda_2=0$ (but are not all zero) (iv) line: $\lambda_1=\lambda_2=0$

6.2 Kronecker and Element-Wise Products

6.2.1 Kronecker Product

Suppose that

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & \cdots & a_{M,N} \end{bmatrix}$$

then

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \cdots & \cdots & a_{1,N}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \cdots & \cdots & a_{2,N}\mathbf{B} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1}\mathbf{B} & a_{M,2}\mathbf{B} & \cdots & \cdots & a_{M,N}\mathbf{B} \end{bmatrix}$$

Kronecker Product

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \cdots & \cdots & a_{1,N}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \cdots & \cdots & a_{2,N}\mathbf{B} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1}\mathbf{B} & a_{M,2}\mathbf{B} & \cdots & \cdots & a_{M,N}\mathbf{B} \end{bmatrix}$$

Notation of the Kronecker product: $\otimes$

A: Global information
B: Local information

If the size of **A** is $M_1 \times N_1$
the size of **B** is $M_2 \times N_2$
then the size of $\mathbf{A}\otimes\mathbf{B}$ is $(M_1 M_2)\times(N_1 N_2)$

[Example 1]

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -2 & -2 & 0 & 2 & 2 & 0 \\ 0 & -4 & 0 & 0 & 4 & 0 \\ 0 & -2 & -2 & 0 & 2 & 2 \end{bmatrix}$$

[Addition Property]

If size(**A**) = size(**C**)

$$\mathbf{A}\otimes\mathbf{B}+\mathbf{C}\otimes\mathbf{B}=\left(\mathbf{A}+\mathbf{C}\right)\otimes\mathbf{B}$$

If size(**B**) = size(**D**)

$$\mathbf{A}\otimes\mathbf{B}+\mathbf{A}\otimes\mathbf{D}=\mathbf{A}\otimes\left(\mathbf{B}+\mathbf{D}\right)$$

[Multiplication Property]

If size(**A**) = $M_1 \times N_1$, size(**B**) = $M_2 \times N_2$, size(**C**) = $M_3 \times N_3$, size(**D**) = $M_4 \times N_4$, and

$$N_1 = M_3, \quad N_2 = M_4$$

then

$$\left(\mathbf{A}\otimes\mathbf{B}\right)\left(\mathbf{C}\otimes\mathbf{D}\right)=\left(\mathbf{AC}\right)\otimes\left(\mathbf{BD}\right)$$

(Proof of the Multiplication Property)

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \cdots & \cdots & a_{1,N}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \cdots & \cdots & a_{2,N}\mathbf{B} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1}\mathbf{B} & a_{M,2}\mathbf{B} & \cdots & \cdots & a_{M,N}\mathbf{B} \end{bmatrix} \qquad \mathbf{C}\otimes\mathbf{D}=\begin{bmatrix} c_{1,1}\mathbf{D} & c_{1,2}\mathbf{D} & \cdots & \cdots & c_{1,K}\mathbf{D} \\ c_{2,1}\mathbf{D} & c_{2,2}\mathbf{D} & \cdots & \cdots & c_{2,K}\mathbf{D} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ c_{N,1}\mathbf{D} & c_{M,2}\mathbf{D} & \cdots & \cdots & c_{N,K}\mathbf{D} \end{bmatrix}$$

If

$$\mathbf{G}=(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\begin{bmatrix} \mathbf{G}_{1,1} & \mathbf{G}_{1,2} & \mathbf{G}_{1,3} & \cdots & \mathbf{G}_{1,K} \\ \mathbf{G}_{2,1} & \mathbf{G}_{2,2} & \mathbf{G}_{2,3} & \cdots & \mathbf{G}_{2,K} \\ \mathbf{G}_{3,1} & \mathbf{G}_{3,2} & \mathbf{G}_{3,3} & \cdots & \mathbf{G}_{3,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{M,1} & \mathbf{G}_{M,2} & \mathbf{G}_{M,3} & \cdots & \mathbf{G}_{M,K} \end{bmatrix}$$

$$\mathbf{G}_{m,k}=\sum_{n=1}^{N} a_{m,n}\mathbf{B}c_{n,k}\mathbf{D}=\left(\sum_{n=1}^{N} a_{m,n}c_{n,k}\right)\mathbf{BD}=E_{m,k}\mathbf{BD}$$

where $E_{m,k}$ is an entry of $\mathbf{E}=\mathbf{AC}$

[Example 2]

$$\mathbf{A}=\begin{bmatrix}1&0\\2&3\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}2&1\\1&0\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}0&1\\2&1\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}1&1\\0&2\end{bmatrix}$$

$$\mathbf{A}\otimes\mathbf{B}=\begin{bmatrix}2&1&0&0\\1&0&0&0\\4&2&6&3\\2&0&3&0\end{bmatrix}\quad \mathbf{C}\otimes\mathbf{D}=\begin{bmatrix}0&0&1&1\\0&0&0&2\\2&2&1&1\\0&4&0&2\end{bmatrix}$$

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D})=\begin{bmatrix}0&0&2&4\\0&0&1&1\\12&24&10&20\\6&6&5&5\end{bmatrix}$$

$$\mathbf{AC}=\begin{bmatrix}0&1\\6&5\end{bmatrix}\quad \mathbf{BD}=\begin{bmatrix}2&4\\1&1\end{bmatrix}\quad \mathbf{AC}\otimes\mathbf{BD}=\begin{bmatrix}0&0&2&4\\0&0&1&1\\12&24&10&20\\6&6&5&5\end{bmatrix}$$

[Inverse Property]

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$$

[Eigenvector Property]

If

$$\mathbf{A}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1, \qquad \mathbf{B}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2$$

then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{e}_1 \otimes \mathbf{e}_2) = \lambda_1 \lambda_2 (\mathbf{e}_1 \otimes \mathbf{e}_2)$$

That is, $\mathbf{e}_1 \otimes \mathbf{e}_2$ is an eigenvector of **A⊗B** and $\lambda_1 \lambda_2$ is the corresponding eigenvalue.

[Orthogonal Property]

If

$$\mathbf{A}\mathbf{A}^{\mathbf{H}} = \mathbf{I}, \quad \mathbf{B}\mathbf{B}^{\mathbf{H}} = \mathbf{I}, \quad \mathbf{C} = \left(\mathbf{A} \otimes \mathbf{B}\right)$$

then

$$\mathbf{C}\mathbf{C}^{\mathbf{H}} = \mathbf{I}$$

[Rank Property]

If rank($\mathbf{A}$) = c_1, rank($\mathbf{B}$) = c_2, then

rank($\mathbf{A}\otimes\mathbf{B}$) = $c_1 c_2$.

[Determinant Property]

Suppose that size(**A**) = $M \times M$, size(**B**) = $N \times N$,
then

$$\det(\mathbf{A} \otimes \mathbf{B}) = \det(\mathbf{A})^{N} \det(\mathbf{B})^{M}$$

(Proof can be done by eigenvector-eigenvalue decomposition)

6.2.2 Element-Wise Product

Suppose that **A** and **B** are two matrices and their sizes are the same ($M \times N$). Then

$$\mathbf{A} \circ \mathbf{B} = \left(\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & \cdots & a_{M,N} \end{bmatrix} \circ \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & \cdots & b_{1,N} \\ b_{2,1} & b_{2,2} & \cdots & \cdots & b_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ b_{M,1} & b_{M,2} & \cdots & \cdots & b_{M,N} \end{bmatrix} \right)$$

$$= \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & \cdots & \cdots & a_{1,N}b_{1,N} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & \cdots & \cdots & a_{2,N}b_{2,N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{M,1}b_{M,1} & a_{M,2}b_{M,2} & \cdots & \cdots & a_{M,N}b_{M,N} \end{bmatrix}$$

[Example 3]

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} 1\cdot 1 & 1\cdot 2 & 1\cdot 1 \\ 1\cdot 1 & 2\cdot(-2) & 4\cdot 1 \\ 1\cdot 2 & 3\cdot(-1) & 5\cdot(-1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 4 \\ 2 & -3 & -5 \end{bmatrix}$$

[Properties of the Element-Wise Product]

Commutative Property

$$\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A}$$

Associative Property

$$(\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} = \mathbf{A} \circ (\mathbf{B} \circ \mathbf{C})$$

Distributive Property

$$\mathbf{A} \circ (\mathbf{B}+\mathbf{C}) = \mathbf{A} \circ \mathbf{B} + \mathbf{A} \circ \mathbf{C}$$

$$(\mathbf{A}+\mathbf{B}) \circ \mathbf{C} = \mathbf{A} \circ \mathbf{C} + \mathbf{B} \circ \mathbf{C}$$

6.3 Jordan-Canonical Form

6.3.1 Generalization for Eigenvector-Eigenvalue Decomposition

[Eigenvalue-Eigenvector Deconposition]

$$\mathbf{A} = \mathbf{EDE^{-1}}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \cdots & \mathbf{e_{N-1}} & \mathbf{e_N} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}$$

However, not all the matrices have a complete eigenvector set.

Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = (2-\lambda)^3 \qquad \text{eigenvalues: } 2, 2, 2$$

$$(\mathbf{A} - 2\mathbf{I})\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5k_3 = 0, \quad k_2 + 6k_3 = 0 \quad \Longrightarrow \quad k_2 = k_3 = 0$$

only one independent solution: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

How do we perform eigenvector-eigenvalue decomposition for it?

We try to solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{e_2}=\mathbf{e_1}, \qquad \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$5k_3=0, \quad k_2+6k_3=1$$

One of the solution is $\mathbf{e_2}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The, we try to solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{e_3}=\mathbf{e_2}, \qquad \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$5k_3=1, \quad k_2+6k_3=0$$

One of the solution is $\mathbf{e_3}=\begin{bmatrix} 0 \\ -6/5 \\ 1/5 \end{bmatrix}$

Since

$$\mathbf{Ae_1} = \lambda \mathbf{e_1}$$

$$\mathbf{Ae_2} = \lambda \mathbf{e_2} + \mathbf{e_1}$$

$$\mathbf{Ae_3} = \lambda \mathbf{e_3} + \mathbf{e_2}$$

$$\mathbf{A}\begin{bmatrix}\mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix} = \begin{bmatrix}\mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix}\begin{bmatrix}\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda\end{bmatrix}$$

$$\mathbf{A} = \mathbf{E}\begin{bmatrix}\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda\end{bmatrix}\mathbf{E}^{-1} \quad \text{where} \quad \mathbf{E} = \begin{bmatrix}\mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & -6/5 \\ 0 & 0 & 1/5\end{bmatrix}\begin{bmatrix}2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & -6/5 \\ 0 & 0 & 1/5\end{bmatrix}^{-1}$$

[Jordan-Canonical Form]

It try to decompose a matrix into

$$\mathbf{A} = \mathbf{E}\begin{bmatrix} \mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_K} \end{bmatrix}\mathbf{E}^{-1}$$

where $\mathbf{D_k}$ has the form of

$$\mathbf{D_k} = \lambda_k \qquad \text{or} \qquad \mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

[Example 1]

$$\mathbf{A}=\begin{bmatrix}4&0&0&-2\\0&4&-2&0\\-1&-1&4&0\\-1&-1&0&4\end{bmatrix}$$

$\lambda=2,4,4,6$

For $\lambda=2$, $\mathbf{e}=\begin{bmatrix}1&1&1&1\end{bmatrix}^T$

For $\lambda=6$, $\mathbf{e}=\begin{bmatrix}1&1&-1&-1\end{bmatrix}^T$

For $\lambda=4$,

$$(\mathbf{A}-4\mathbf{I})\mathbf{e}=0,\quad \begin{bmatrix}0&0&0&-2\\0&0&-2&0\\-1&-1&0&0\\-1&-1&0&0\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\\k_4\end{bmatrix}=0$$

$$-2k_4=0$$
$$-2k_3=0$$
$$-k_1-k_2=0$$

For $\lambda = 4$, there is only one linearly independent eigenvector

$$\mathbf{e_1} = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$$

Then, we try to solve

$$(\mathbf{A} - 4\mathbf{I})\mathbf{e_2} = \mathbf{e_1}, \quad \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -2k_4 &= 1 \\ -2k_3 &= -1 \\ -k_1 - k_2 &= 0 \end{aligned}$$

One of the solution is

$$\mathbf{e_2} = \begin{bmatrix} 0 & 0 & 1/2 & -1/2 \end{bmatrix}^T$$

Therefore,

$$\mathbf{A} = \mathbf{E}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\mathbf{E}^{-1}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1/2 & -1 \\ 1 & 0 & -1/2 & -1 \end{bmatrix}$$

$$\mathbf{E}^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1/4 & 1/4 & -1/4 & -1/4 \end{bmatrix}$$