[Example 2]

$$\mathbf{A}=\begin{bmatrix}4&0&1&2&1\\0&4&1&-2&1\\0&0&3&0&1\\0&0&0&4&0\\0&0&1&0&3\end{bmatrix}$$

(Solution): Since

$$\det(\mathbf{A}-\lambda\mathbf{I})=-(\lambda-2)(\lambda-4)^4$$

the eigenvalues are 2, 4, 4, 4, 4.

The eigenvector corresponding to $\lambda=2$ is

$$(\mathbf{A}-2\mathbf{I})\mathbf{e}=\mathbf{0},\quad \begin{bmatrix}2&0&1&2&1\\0&2&1&-2&1\\0&0&1&0&1\\0&0&0&2&0\\0&0&1&0&1\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\\k_4\\k_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

One of the solution is $\begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix}$

The eigenvectors corresponding to $\lambda = 4$ are

$$(\mathbf{A}-4\mathbf{I})\mathbf{e}=\mathbf{0}, \qquad \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} k_3 + 2k_4 + k_5 = 0 \\ k_3 - 2k_4 + k_5 = 0 \\ -k_3 + k_5 = 0 \\ k_3 - k_5 = 0 \end{cases} \Rightarrow \begin{cases} k_3 + 2k_4 + k_5 = 0 \\ k_4 = 0 \\ k_3 = k_5 \end{cases} \Rightarrow k_3 = k_4 = k_5 = 0$$

Therefore, there are only two linearly independent solutions:

$$\mathbf{e}_{\mathbf{1,a}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T \quad \text{and} \quad \mathbf{e}_{\mathbf{1,b}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$$

If we set

$$\mathbf{e}_{1,a} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}^T$$

then

$$(\mathbf{A}-4\mathbf{I})\mathbf{e}_{2,a} = \mathbf{e}_{1,a}, \quad \begin{bmatrix}0 & 0 & 1 & 2 & 1\\ 0 & 0 & 1 & -2 & 1\\ 0 & 0 & -1 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ k_3\\ k_4\\ k_5\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{cases} k_3 + 2k_4 + k_5 = 1 \\ k_3 - 2k_4 + k_5 = 0 \\ -k_3 + k_5 = 0 \\ k_3 - k_5 = 0 \end{cases} \Rightarrow \begin{cases} k_3 - 2k_4 + k_5 = 0 \\ 4k_4 = 1 \\ k_3 = k_5 \end{cases} \Rightarrow k_3 = k_4 = k_5 = 1/4$$

One of the solution is $\mathbf{e}_{2,a} = \begin{bmatrix}0 & 0 & 1/4 & 1/4 & 1/4\end{bmatrix}^T$

If we set

$$\mathbf{e}_{1,b} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$$

then

$$(\mathbf{A} - 4\mathbf{I})\mathbf{e}_{2,b} = \mathbf{e}_{1,b}, \quad \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} k_3 + 2k_4 + k_5 = 0 \\ k_3 - 2k_4 + k_5 = 1 \\ -k_3 + k_5 = 0 \\ k_3 - k_5 = 0 \end{cases} \Rightarrow \begin{cases} k_3 + 2k_4 + k_5 = 0 \\ 4k_4 = -1 \\ k_3 = k_5 \end{cases} \Rightarrow k_3 = -k_4 = k_5 = 1/4$$

One of the solution is $\mathbf{e}_{2,b} = \begin{bmatrix} 0 & 0 & 1/4 & -1/4 & 1/4 \end{bmatrix}^T$

Therefore, the Jordan-Canonical form of **A** is

$$\mathbf{A} = \mathbf{E}\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\mathbf{E}^{-1}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 1 \\ 0 & 1/4 & 0 & -1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & -1 \end{bmatrix} \qquad \mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1/2 & 0 & -1/2 \end{bmatrix}$$

6.3.2 Power of the Jordan Canonical Form

If

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

then

$$\mathbf{A}^{\alpha} = \mathbf{E}\mathbf{D}^{\alpha}\mathbf{E}^{-1}$$

Problem: How do we determine $\mathbf{D}^{\alpha}$ for the Jordan-canonical form?

$$\mathbf{D} = \begin{bmatrix} \mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_K} \end{bmatrix}$$

where $\mathbf{D_k} = \lambda_k \mathbf{I}$

or

$$\mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_K} \end{bmatrix} \qquad \mathbf{D}^{\alpha} = \begin{bmatrix} \mathbf{D}_1^{\alpha} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2^{\alpha} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_{\mathbf{K}}^{\alpha} \end{bmatrix}$$

$$\mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix} = \lambda_k \mathbf{I} + \mathbf{U} \quad \text{where} \quad \mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Therefore,

$$\mathbf{D}_{\mathbf{k}}^{\alpha} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \lambda_k^{\alpha-\beta} \mathbf{U}^{\beta}$$

Note:

If $\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ then

$$\mathbf{U}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{U}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{U}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U}^\beta = \mathbf{0} \quad for\ \beta \geq 5$$

In general, if size of $\mathbf{D}_k$ is MxM then

$$\mathbf{U}^{\beta}[m,n]=1 \quad when \quad n-m=\beta$$
$$\mathbf{U}^{\beta}[m,n]=0 \quad otherwise$$

Also,

$$\mathbf{U}^{\beta}=\mathbf{0} \quad when\ \beta \geq M$$

Therefore,

$$\mathbf{D}_{\mathbf{k}}^{\alpha}[n,n]=\lambda_k^{\alpha},$$
$$\mathbf{D}_{\mathbf{k}}^{\alpha}[m,n]=\binom{\alpha}{n-m}\lambda_k^{\alpha-n+m} \quad if\ n>m\ \ and\ \ \alpha \geq n-m$$
$$\mathbf{D}_{\mathbf{k}}^{\alpha}[m,n]=0 \quad otherwise$$

$$\mathbf{D}_{\mathbf{k}}^{\alpha} = \begin{bmatrix} \lambda_k^{\alpha} & C_1^{\alpha}\lambda_k^{\alpha-1} & C_2^{\alpha}\lambda_k^{\alpha-2} & \cdots & C_{\alpha}^{\alpha}\lambda_k^{0} & 0 & \cdots & 0 \\ 0 & \lambda_k^{\alpha} & C_1^{\alpha}\lambda_k^{\alpha-1} & C_2^{\alpha}\lambda_k^{\alpha-2} & \ddots & C_{\alpha}^{\alpha}\lambda_k^{0} & \ddots & \vdots \\ 0 & 0 & \lambda_k^{\alpha} & C_1^{\alpha}\lambda_k^{\alpha-1} & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k^{\alpha} & \ddots & C_2^{\alpha}\lambda_k^{\alpha-2} & \ddots & C_{\alpha}^{\alpha}\lambda_k^{0} \\ 0 & 0 & 0 & 0 & \ddots & C_1^{\alpha}\lambda_k^{\alpha-1} & C_2^{\alpha}\lambda_k^{\alpha-2} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \lambda_k^{\alpha} & C_1^{\alpha}\lambda_k^{\alpha-1} & C_2^{\alpha}\lambda_k^{\alpha-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_k^{\alpha} & C_1^{\alpha}\lambda_k^{\alpha-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_k^{\alpha} \end{bmatrix}$$

[Example 3] If

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & -2 \\ 0 & 4 & -2 & 0 \\ -1 & -1 & 4 & 0 \\ -1 & -1 & 0 & 4 \end{bmatrix}$$

then determine $\mathbf{A}^5$

(Solution): From Example 1,

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1/2 & -1 \\ 1 & 0 & -1/2 & -1 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{E}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\mathbf{E}^{-1}$$

$$\mathbf{E}^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1/4 & 1/4 & -1/4 & -1/4 \end{bmatrix}$$

$$\mathbf{A}^5 = \mathbf{E}\begin{bmatrix} 2^5 & 0 & 0 & 0 \\ 0 & 4^5 & 4^4\binom{5}{1} & 0 \\ 0 & 0 & 4^5 & 0 \\ 0 & 0 & 0 & 6^5 \end{bmatrix}\mathbf{E}^{-1} = \mathbf{E}\begin{bmatrix} 32 & 0 & 0 & 0 \\ 0 & 1024 & 1280 & 0 \\ 0 & 0 & 1024 & 0 \\ 0 & 0 & 0 & 7776 \end{bmatrix}\mathbf{E}^{-1}$$

$$\mathbf{A}^5 = \begin{bmatrix} 2464 & 1440 & -656 & -3216 \\ 1440 & 2464 & -3216 & -656 \\ -1936 & -1936 & 2464 & 1440 \\ -1936 & -1936 & 1440 & 2464 \end{bmatrix}$$

6.4 Functions of Matrices

Functions of Matrices are also called spectral mapping.

6.4.1 Exponential of a Matrix

From the Taylor series

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

therefore,

$$\exp(\mathbf{A}) = 1 + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots\cdots$$

If

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

then

$$\exp(\mathbf{A}) = \mathbf{E}\left(\mathbf{I} + \frac{\mathbf{D}}{1!} + \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^3}{3!} + \cdots\cdots\right)\mathbf{E}^{-1} = \mathbf{E}\exp(\mathbf{D})\mathbf{E}^{-1}$$

$$\exp(\mathbf{D}) = \mathbf{I} + \frac{\mathbf{D}}{1!} + \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^3}{3!} + \cdots\cdots$$

(Case 1): If **A** has a complete eigenvector set, then

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

$$\exp(\mathbf{D}) = \mathbf{I} + \frac{\mathbf{D}}{1!} + \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^3}{3!} + \cdots\cdots = \begin{bmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2!}+\cdots & 0 & \cdots & 0 \\ 0 & 1+\lambda_2+\frac{\lambda_2^2}{2!}+\cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+\lambda_N+\frac{\lambda_N^2}{2!}+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \exp(\lambda_1) & 0 & \cdots & 0 \\ 0 & \exp(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp(\lambda_N) \end{bmatrix}$$

(Case 2): If **A** does not have a complete eigenvector set, then

$$\mathbf{D} = \begin{bmatrix} \mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_K} \end{bmatrix} \qquad \mathbf{D_k} = \lambda_k \mathbf{I} \ \text{ or } \ \mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

$$\mathbf{I} + \frac{\mathbf{D}}{1!} + \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^3}{3!} + \cdots\cdots = \begin{bmatrix} \mathbf{S_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{S_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S_K} \end{bmatrix}$$

where $\quad \mathbf{S_k} = \exp(\mathbf{D_k}) = \mathbf{I} + \frac{\mathbf{D_k}}{1!} + \frac{\mathbf{D_k}^2}{2!} + \frac{\mathbf{D_k}^3}{3!} + \cdots\cdots$

$$\mathbf{S_k} = \mathbf{I} + \frac{\mathbf{D_k}}{1!} + \frac{\mathbf{D_k^2}}{2!} + \frac{\mathbf{D_k^3}}{3!} + \cdots\cdots \quad \text{If} \quad \mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

then $\mathbf{S_k}[m,n] = 0 \qquad \text{if } m > n,$

$$\mathbf{S_k}[n,n] = 1 + \frac{\lambda_k}{1!} + \frac{\lambda_k^2}{2!} + \frac{\lambda_k^3}{3!} + \cdots = \exp(\lambda_k)$$

$$\mathbf{S_k}[m,n] = \sum_{\alpha=n-m}^{\infty} \frac{1}{\alpha!} \binom{\alpha}{n-m} \lambda_k^{\alpha-n+m}$$

$$= \sum_{\alpha=n-m}^{\infty} \frac{1}{(\alpha-n+m)!} \frac{1}{(n-m)!} \lambda_k^{\alpha-n+m} = \frac{1}{(n-m)!} \sum_{\beta=0}^{\infty} \frac{1}{\beta!} \lambda_k^{\beta}$$

$$= \frac{1}{(n-m)!} \exp(\lambda_k)$$

Therefore,

$$\mathbf{S_k} = \begin{bmatrix} e^{\lambda_k} & \frac{1}{1!}e^{\lambda_k} & \frac{1}{2!}e^{\lambda_k} & \cdots & \frac{1}{(M-1)!}e^{\lambda_k} \\ 0 & e^{\lambda_k} & \frac{1}{1!}e^{\lambda_k} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{2!}e^{\lambda_k} \\ 0 & 0 & \cdots & e^{\lambda_k} & \frac{1}{1!}e^{\lambda_k} \\ 0 & 0 & \cdots & 0 & e^{\lambda_k} \end{bmatrix}$$

[Example 1] If

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

determine $\exp(\mathbf{A})$

(Solution): The eigenvalues of **A** are 2, 2, 2.
The Jordan-Canonical form of **A** is

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1} \qquad \mathbf{D} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{E}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\exp(\mathbf{A}) = \mathbf{E}\exp(\mathbf{D})\mathbf{E}^{-1}$$

where $$\exp(\mathbf{D}) = \begin{bmatrix} e^2 & e^2 & e^2/2 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{bmatrix}$$

Therefore,

$$\exp(\mathbf{A}) = \begin{bmatrix} 2e^2 & 2e^2 & -e^2 \\ e^2 & e^2 & -e^2 \\ e^2 & 2e^2 & 0 \end{bmatrix}$$

6.4.2 Functions Using Taylor Series

(這個 subsection 只教不考)

(1) cos(A)

To derive $\cos(\mathbf{A})$

we can apply $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$

$$\cos(\mathbf{A}) = \mathbf{I} - \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^4}{4!} - \frac{\mathbf{A}^6}{6!} + \cdots\cdots$$

$$= \mathbf{E}\left(\mathbf{I} - \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^4}{4!} - \frac{\mathbf{D}^6}{6!} + \cdots\cdots\right)\mathbf{E}^{-1} = \mathbf{E}\cos(\mathbf{D})\mathbf{E}^{-1}$$

where $\cos(\mathbf{D}) = \left(\mathbf{I} - \frac{\mathbf{D}^2}{2!} + \frac{\mathbf{D}^4}{4!} - \frac{\mathbf{D}^6}{6!} + \cdots\cdots\right)$

if $\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$

(Case 1) When **A** has a complete eigenvector set, then

$$\mathbf{D}=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_N\end{bmatrix}$$

$$\mathbf{I}-\frac{\mathbf{D}^2}{2!}+\frac{\mathbf{D}^4}{4!}-\cdots\cdots=\begin{bmatrix}1-\frac{\lambda_1^2}{2!}+\frac{\lambda_1^2}{4!}-\cdots & 0 & \cdots & 0\\ 0 & 1-\frac{\lambda_2^2}{2!}+\frac{\lambda_2^2}{4!}-\cdots & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1-\frac{\lambda_N^2}{2!}+\frac{\lambda_N^2}{4!}-\cdots\end{bmatrix}$$

$$\cos(\mathbf{D})=\begin{bmatrix}\cos(\lambda_1) & 0 & \cdots & 0\\ 0 & \cos(\lambda_2) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \cos(\lambda_N)\end{bmatrix}$$

(Case 2) When **A** does not have a complete eigenvector set, then

$$\mathbf{D}=\begin{bmatrix}\mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_K}\end{bmatrix} \qquad \mathbf{D_k}=\begin{bmatrix}\lambda_k & 1 & 0 & \cdots & 0\\ 0 & \lambda_k & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_k & 1\\ 0 & 0 & \cdots & 0 & \lambda_k\end{bmatrix}$$

$$\mathbf{I}-\frac{\mathbf{D}^2}{2!}+\frac{\mathbf{D}^4}{4!}-\frac{\mathbf{D}^6}{6!}+\cdots\cdots=\begin{bmatrix}\mathbf{S_1} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{S_2} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S_K}\end{bmatrix}$$

where $\quad \mathbf{S_k}=\cos(\mathbf{D_k})=\mathbf{I}-\frac{\mathbf{D_k}^2}{2!}+\frac{\mathbf{D_k}^4}{4!}-\frac{\mathbf{D_k}^6}{6!}+\cdots\cdots$

$$\mathbf{S_k}\left[m,n\right]=0 \qquad \text{if } m>n,$$

$$\mathbf{S_k}\left[n,n\right]=1-\frac{\lambda_k^2}{2!}+\frac{\lambda_k^4}{4!}-\frac{\lambda_k^6}{6!}+\cdots=\cos\left(\lambda_k\right)$$

If $n-m$ is positive and even,

$$\mathbf{S_k}\left[m,n\right]=\sum_{\alpha=(n-m)/2}^{\infty}\frac{(-1)^{\alpha}}{(2\alpha)!}\binom{2\alpha}{n-m}\lambda_k^{2\alpha-n+m}$$

$$=\sum_{\alpha=(n-m)/2}^{\infty}\frac{(-1)^{\alpha}}{(2\alpha-n+m)!(n-m)!}\lambda_k^{2\alpha-n+m}$$

$$=\sum_{\beta=0}^{\infty}\frac{(-1)^{\beta+(n-m)/2}}{(2\beta)!(n-m)!}\lambda_k^{2\beta} \qquad \beta=\alpha-\frac{n-m}{2}$$

$$=\frac{(-1)^{(n-m)/2}}{(n-m)!}\cos\left(\lambda_k\right)$$

If $n - m$ is positive and odd,

$$\mathbf{S_k}[m,n] = \sum_{\alpha=(n-m+1)/2}^{\infty} \frac{(-1)^{\alpha}}{(2\alpha)!}\binom{2\alpha}{n-m}\lambda_k^{2\alpha-n+m}$$

$$= \sum_{\alpha=(n-m+1)/2}^{\infty} \frac{(-1)^{\alpha}}{(2\alpha-n+m)!(n-m)!}\lambda_k^{2\alpha-n+m}$$

$$= \sum_{\beta=0}^{\infty} \frac{(-1)^{\beta+(n-m+1)/2}}{(2\beta+1)!(n-m)!}\lambda_k^{2\beta+1} = \frac{(-1)^{(n-m+1)/2}}{(n-m)!}\sin(\lambda_k)$$

$$\beta = \alpha - \frac{n-m+1}{2}$$

$$\mathbf{S_k} = \begin{bmatrix} \cos\lambda_k & \frac{-1}{1!}\sin\lambda_k & \frac{-1}{2!}\cos\lambda_k & \cdots & \cdots \\ 0 & \cos\lambda_k & \frac{-1}{1!}\sin\lambda_k & \ddots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \frac{-1}{2!}\cos\lambda_k \\ 0 & 0 & \cdots & \cos\lambda_k & \frac{-1}{1!}\sin\lambda_k \\ 0 & 0 & \cdots & 0 & \cos\lambda_k \end{bmatrix}$$

(2) sin(**A**)

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$$

(Case 1) When **A** has a complete eigenvector set and

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1} \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

then

$$\sin(\mathbf{A}) = \mathbf{E}\mathbf{S}\mathbf{E}^{-1} \qquad \mathbf{S} = \sin(\mathbf{D}) = \begin{bmatrix} \sin\lambda_1 & 0 & \cdots & 0 \\ 0 & \sin\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sin\lambda_N \end{bmatrix}$$

(Case 2) When **A** does not have a complete eigenvector set and

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{D_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D_K} \end{bmatrix} \qquad \mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

then

$$\sin(\mathbf{A}) = \mathbf{E}\mathbf{S}\mathbf{E}^{-1} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{S_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{S_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S_K} \end{bmatrix}$$

$$\sin(\mathbf{A}) = \mathbf{E}\mathbf{S}\mathbf{E}^{-1} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{S_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{S_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{S_K} \end{bmatrix}$$

$$\mathbf{S_k}[m,n] = 0 \qquad \text{if } m > n,$$

$$\mathbf{S_k}[n,n] = \sin(\lambda_k)$$

$$\mathbf{S_k}[m,n] = \frac{(-1)^{(n-m-1)/2}}{(n-m)!}\cos(\lambda_k) \qquad \text{if } n - m \text{ is positive and odd,}$$

$$\mathbf{S_k}[m,n] = \frac{(-1)^{(n-m)/2}}{(n-m)!}\sin(\lambda_k) \qquad \text{if } n - m \text{ is positive and even}$$

(3) In general,

if

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots\cdots\cdots$$

and

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

then

$$f(\mathbf{A}) = \mathbf{E}f(\mathbf{D})\mathbf{E}^{-1}$$

where $f(\mathbf{D}) = f(0) + \frac{f'(0)}{1!}\mathbf{D} + \frac{f''(0)}{2!}\mathbf{D}^2 + \frac{f'''(0)}{3!}\mathbf{D}^3 + \cdots\cdots\cdots$

$$f(\mathbf{A}) = \mathbf{E} f(\mathbf{D}) \mathbf{E}^{-1}$$

where $$f(\mathbf{D}) = f(0) + \frac{f'(0)}{1!}\mathbf{D} + \frac{f''(0)}{2!}\mathbf{D}^2 + \frac{f'''(0)}{3!}\mathbf{D}^3 + \cdots\cdots\cdots$$

(i) When **A** has a complete eigenvector set and

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

then

$$f(\mathbf{D}) = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_N) \end{bmatrix}$$

(ii) However, the form of $f(\mathbf{D})$ is rather complicated if **A** does not have a complete eigenvector set.

6.5 Generalized Norm

For a vector

$$\mathbf{x} = \begin{bmatrix} x[1] & x[2] & \cdots & x[N-1] & x[N] \end{bmatrix}$$

(1) Norm (L_α norm): $\|\mathbf{x}\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$

(2) L_2 norm (conventional norm):

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$$

(Physical meaning: Distance)

$$\|\mathbf{x}\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$$

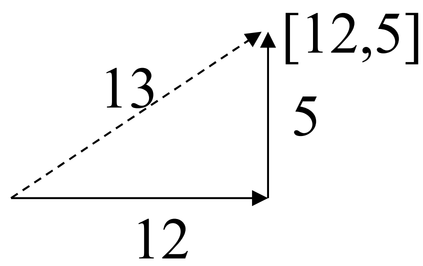
(3) L_1 norm: $$\|\mathbf{x}\|_1 = \sum_{n=0}^{N-1} |x[n]|$$

(Physical meaning: Sum of Amplitudes)

[**Example 1**]

$$\|[12,5]\|_2 = \sqrt{12^2+5^2} = 13$$

$$\|[12,5]\|_1 = 12+5 = 17$$



$$\|\mathbf{x}\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$$

(3) L_∞ norm: $\|\mathbf{x}\|_\infty = \text{Max}\{|x[n]|\}$

(Physical meaning: The maximal amplitude)

Note: $$\lim_{\alpha\to\infty} \sum_{n=0}^{N-1} |x[n]|^\alpha \cong \left(Max\{|x[n]|\}\right)^\alpha$$

$$\lim_{\alpha\to\infty} \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha} \cong Max\{|x[n]|\}$$

If there are M entries of $|x[n]|$ that equals to Max$|x[n]|$

$$\lim_{\alpha\to\infty} \sum_{n=0}^{N-1} |x[n]|^\alpha \cong M\left(Max\{|x[n]|\}\right)^\alpha$$

$$\lim_{\alpha\to\infty} \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha} \cong M^{1/\alpha} Max\{|x[n]|\} = Max\{|x[n]|\}$$

$$\|\mathbf{x}\|_{\alpha} = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^{\alpha}}$$

(5) $\lim_{\alpha \to 0} (L_{\alpha}\text{ norm})^{\alpha}$ (Also call as the L_0 norm)

$$\lim_{\alpha \to 0} \left(\|\mathbf{x}\|_{\alpha}\right)^{\alpha} = K$$

where K is the number of points such that $x[n] \neq 0$

(Physical meaning: The number of nonzero points)

The L_2 norm is easier for optimization, but it often happens that using the L_0 or the L_1 norm can obtain even better optimization results.

For the matrix case

$$\|\mathbf{A}\|_\alpha = \sqrt[\alpha]{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|A[m,n]\right|^\alpha}$$

L_2 norm: (also call the Frobenius norm):

$$\|\mathbf{A}\|_2 = \sqrt{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|A[m,n]\right|^2}$$

L_1 norm:

$$\|\mathbf{A}\|_1 = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left|A[m,n]\right|$$

L_∞ norm: (also call the max norm):

$$\|\mathbf{A}\|_\infty = \underset{m,n}{Max}\left|A[m,n]\right|$$

L_0 norm:

$$\lim_{\alpha\to 0}\left(\|\mathbf{A}\|_\alpha\right)^\alpha = K$$

where K is the number of points s that satisfy $A[m, n] \neq 0$

[Example 2]

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

L_2 norm: $\sqrt{4(1^2) + 3(2^2) + 3^2} = 5$

L_1 norm: $4 \cdot 1 + 3 \cdot 2 + 3 = 13$

L_∞ norm: 3

L_0 norm: 8

Note: In fact, a more standard definition for the norm of a matrix is:

$$\|\mathbf{A}\|_{\alpha} = \sup_{\mathbf{x}\neq 0} \frac{\|\mathbf{Ax}\|_{\alpha}}{\|\mathbf{x}\|_{\alpha}}$$

where sup (supremum) means the least upper bound.

The norm with this definition is called the operator norm. One possible application of the operator norm is to determine the passivity of electrical components.

For image processing and machine learning applications, it is more often to use the same definition of the norms of a vector to define the norms of a matrix, as on page 544.

The norms on page 544 are also called "Entry-wise" matrix norms or vector-based norms.

6.6 Markov Model

6.6.1 Definitions

State Vector:

$$\mathbf{x}(t)=\left[x_1(t),x_2(t),\cdots,x_N(t)\right]^T$$

In a Markov model,

(1) $x_1(t)$, $x_2(t)$, …, $x_N(t)$ can be expressed as linear combinations of $\{x_1(t-1]$, $x_2(t-1]$, …, $x_N(t-1]\}$

$$x_m(t)=a_{m,1}x_1(t-1)+a_{m,2}x_2(t-1)+\cdots+a_{m,N}x_N(t-1)$$

$m = 1, 2, \ldots, N$:

(2) $0\le a_{m,n}\le 1$

(3) $\sum_{m=1}^{N} a_{m,n}=1$ for $n = 1, 2, \ldots, N$:

Note that, if $\sum_{m=1}^{N} a_{m,n} = 1$

then

$$\sum_{m=1}^{N} x_m(t) = \sum_{m=1}^{N}\left(a_{m,1}x_1(t-1) + a_{m,2}x_2(t-1) + \cdots + a_{m,N}x_N(t-1)\right)$$

$$= \sum_{m=1}^{N} a_{m,1}x_1(t-1) + \sum_{m=1}^{N} a_{m,2}x_2(t-1) + \cdots + \sum_{m=1}^{N} a_{m,N}x_N(t-1)$$

$$\sum_{m=1}^{N} x_m(t) = \sum_{m=1}^{N} x_m(t-1)$$

Many problems in physics, environment, and social science can be expressed by the Markov model.

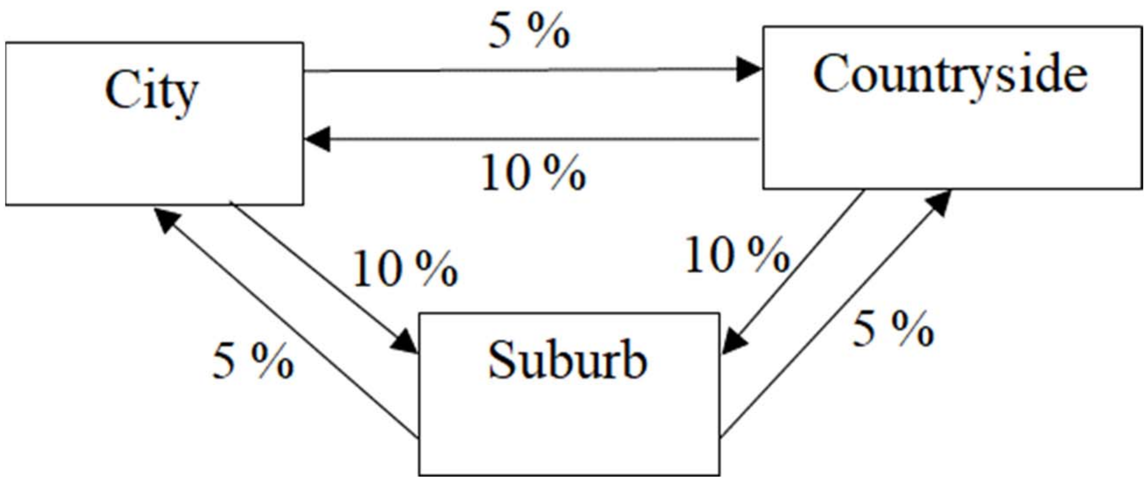
Markov Model (Matrix Form)

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1)$$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N-1} & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N-1} & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} & a_{N-1,N} \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N-1} & a_{N,N} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N-1}(t) \\ x_N(t) \end{bmatrix} \qquad \mathbf{x}(t-1) = \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ \vdots \\ x_{N-1}(t-1) \\ x_N(t-1) \end{bmatrix}$$

[Example 1] Migration Model

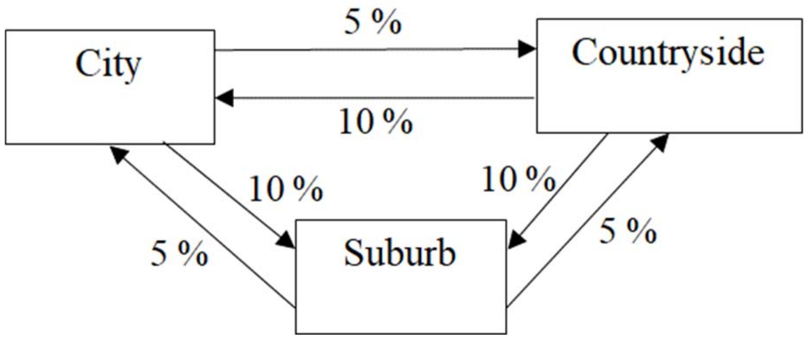


Suppose that, every year,

(1) 10% of the people lived in the city move to the suburb and 5% of the people lived in the city move to the countryside every year

(2) In the suburb, 5% of the people move to the city and 5% of the people move to the countryside every year.

(3) 10% of the people lived in the countryside move to the city and 10% of the people lived in the countryside move to the suburb.

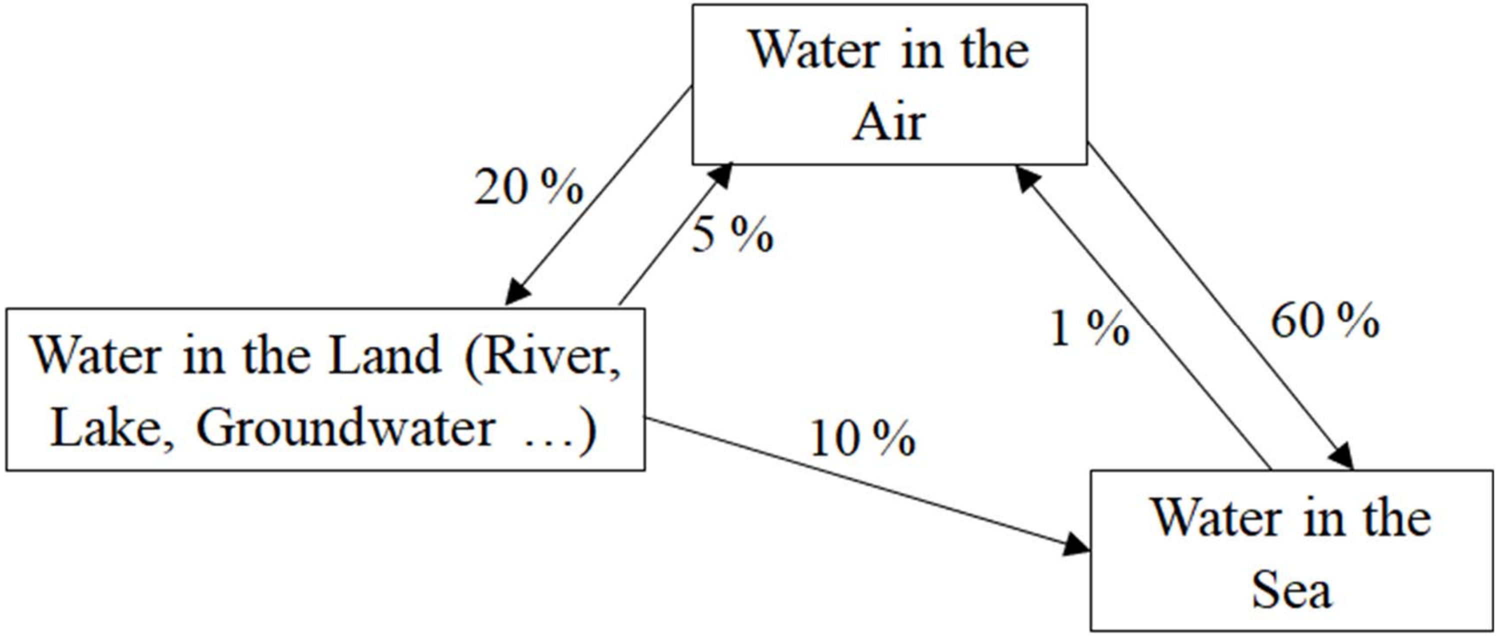


$$\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t-1) \qquad \text{where}$$

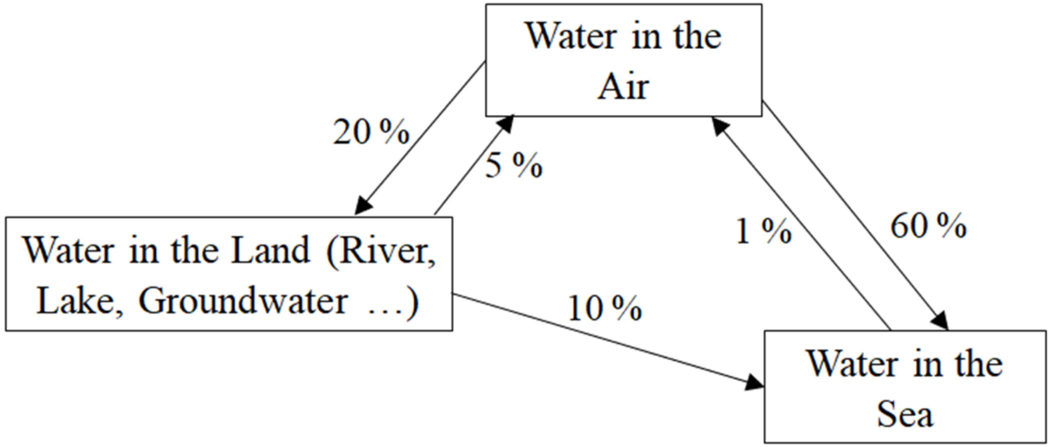
$$\mathbf{A} = \begin{bmatrix} 0.85 & 0.05 & 0.1 \\ 0.1 & 0.9 & 0.1 \\ 0.05 & 0.05 & 0.8 \end{bmatrix} \qquad \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \begin{matrix} \text{city} \\ \text{suburb} \\ \text{countryside} \end{matrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.05 & 0.1 \\ 0.1 & 0.9 & 0.1 \\ 0.05 & 0.05 & 0.8 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix}$$

[Example 2] Water Cycle System



Suppose that, every year,
(1) 20% of the water in the air will precipitate to the land through rain or snow, 60% of the water in the air will precipitate to the sea.
(2) 10% of the water in the land will flow into the sea and 5% of the water will evaporate into the air.
(3) Also, 1% of the water in the see will evaporate to the air every year.



$$\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t-1) \qquad \text{where}$$

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.05 & 0.01 \\ 0.2 & 0.85 & 0 \\ 0.6 & 0.1 & 0.99 \end{bmatrix} \qquad \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \begin{matrix} \text{air} \\ \text{land} \\ \text{sea} \end{matrix}$$

6.6.2 Analysis

In Example 1, suppose that, initially, the populations of the city, the suburb, and the country are 50,000, 50,000, and 100,000.
(1) Determine the populations of the three places 1 year, 2 years, and 10 years later.
(2) Also, determine what will the populations of the three places converge to.

$$\mathbf{A} = \begin{bmatrix} 0.85 & 0.05 & 0.1 \\ 0.1 & 0.9 & 0.1 \\ 0.05 & 0.05 & 0.8 \end{bmatrix}$$

Eigenvector-eigenvalue decomposition

$$\mathbf{A} = \mathbf{EDE^{-1}} \qquad \text{where}$$

$$\mathbf{E} = \begin{bmatrix} 3 & 1 & 1 \\ 5 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.75 \end{bmatrix} \qquad \mathbf{E}^{-1} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.5 & -0.5 & 0.5 \\ 0.2 & 0.2 & -0.8 \end{bmatrix}$$

Initial $\mathbf{X}(0) = \begin{bmatrix} 50000 \\ 50000 \\ 100000 \end{bmatrix}$

After one year $\mathbf{X}(1) = \mathbf{A}\mathbf{X}(0) = \begin{bmatrix} 55000 \\ 60000 \\ 85000 \end{bmatrix}$

After two years $\mathbf{X}(2) = \mathbf{A}\mathbf{X}(1) = \begin{bmatrix} 58250 \\ 68000 \\ 73750 \end{bmatrix}$

After ten years $\mathbf{X}(10) = \mathbf{A}^{10}\mathbf{X}(0) = \mathbf{E}\mathbf{D}^{\mathbf{10}}\mathbf{E}^{\mathbf{-1}}\mathbf{X}(0) = \begin{bmatrix} 61990 \\ 94631 \\ 43379 \end{bmatrix}$

$$\mathbf{D}^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (0.8)^{10} & 0 \\ 0 & 0 & (0.75)^{10} \end{bmatrix}$$

To determine what will the populations of the three places converge to, we can apply

$$\lim_{t\to\infty}\mathbf{X}(t)=\lim_{t\to\infty}\mathbf{A}^t\mathbf{X}(0)=\lim_{t\to\infty}\mathbf{E}\mathbf{D}^t\mathbf{E}^{-1}\mathbf{X}(0)$$

Since

$$\lim_{t\to\infty}\mathbf{D}^t=\begin{bmatrix}1^t & 0 & 0\\ 0 & (0.8)^t & 0\\ 0 & 0 & (0.75)^t\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\lim_{t\to\infty}\mathbf{X}(t)=\begin{bmatrix}3 & 1 & 1\\ 5 & -1 & 0\\ 2 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0.1 & 0.1 & 0.1\\ 0.5 & -0.5 & 0.5\\ 0.2 & 0.2 & -0.8\end{bmatrix}\begin{bmatrix}50000\\ 50000\\ 100000\end{bmatrix}$$

$$\lim_{t\to\infty}\mathbf{X}(t)=\begin{bmatrix}60000\\ 100000\\ 40000\end{bmatrix}$$

In Example 2, suppose that, initially, the amounts water in the air, in the land, and in the sea are $x_1(0)$, $x_2(0)$, and $x_3(0)$, respectively.

(1) Determine the amounts of water in the air, the land, and the sea after 10 years.

(2) Also determine what will the amounts of the water in the three places converge to.

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.05 & 0.01 \\ 0.2 & 0.85 & 0 \\ 0.6 & 0.1 & 0.99 \end{bmatrix}$$

Eigenvector-eigenvalue decomposition $\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$

$$\mathbf{E} = \begin{bmatrix} 3 & 1 & 3.3595 \\ 4 & 16.7977 & -1 \\ 220 & -17.9799 & -2.3595 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8619 & 0 \\ 0 & 0 & 0.1781 \end{bmatrix}$$

$$\mathbf{E}^{-1} = \begin{bmatrix} 1/227 & 1/227 & 1/227 \\ 0.01615 & 0.05723 & -0.00126 \\ 0.28892 & -0.02097 & -0.00356 \end{bmatrix}$$

After ten years

$$\mathbf{X}(10)=\mathbf{A}^{10}\mathbf{X}(0)=\mathbf{E}\mathbf{D}^{10}\mathbf{E}^{-1}\mathbf{X}(0)=\mathbf{E}\begin{bmatrix}1 & 0 & 0\\ 0 & 0.2263 & 0\\ 0 & 0 & 3.2098\cdot 10^{-8}\end{bmatrix}\mathbf{E}^{-1}\mathbf{X}(0)$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & (0.8619)^{10} & 0\\ 0 & 0 & (0.1781)^{10}\end{bmatrix}$$

$$\begin{bmatrix}x_1(10)\\ x_2(10)\\ x_3(10)\end{bmatrix}=\begin{bmatrix}0.01687x_1(0)+0.02617x_2(0)+0.01293x_3(0)\\ 0.07900x_1(0)+0.23515x_2(0)+0.01283x_3(0)\\ 0.90413x_1(0)+0.73869x_2(0)+0.97424x_3(0)\end{bmatrix}$$

To determine what will the amounts of the water in the three places converge to,

$$\lim_{t\to\infty}\mathbf{X}(t)=\lim_{t\to\infty}\mathbf{A}^t\mathbf{X}(0)=\lim_{t\to\infty}\mathbf{E}\mathbf{D}^t\mathbf{E}^{-1}\mathbf{X}(0)$$

Since

$$\lim_{t\to\infty}\mathbf{D}^t=\begin{bmatrix}1^t & 0 & 0\\ 0 & (0.8619)^t & 0\\ 0 & 0 & (0.1781)^t\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\lim_{t\to\infty}\mathbf{X}(t)=\mathbf{E}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\mathbf{E}^{-1}\mathbf{X}(0)$$

$$\lim_{t\to\infty}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}=\begin{bmatrix}\frac{3}{227}\left(x_1(0)+x_2(0)+x_3(0)\right)\\ \frac{4}{227}\left(x_1(0)+x_2(0)+x_3(0)\right)\\ \frac{220}{227}\left(x_1(0)+x_2(0)+x_3(0)\right)\end{bmatrix}$$

[**Theorem 1**] For a Markov model matrix, at least one of the eigenvalue is $\lambda = 1$.

Other eigenvalues are no larger than 1.

Moreover, if the multiplicity of $\lambda = 1$ is 1, then the eigenvector corresponding to $\lambda = 1$ determines the ratio of $x_1(t) : x_2(t) : \ldots : x_N(t)$ in the convergence case.

(Proof): Suppose that **A** is the transfer matrix of a Markov model.

(i) To show 1 must be an eigenvalue of **A**:

If $\mathbf{A_1} = \mathbf{LAL^{-1}}$ where

$$\mathbf{L} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad \mathbf{L^{-1}} = \begin{bmatrix} 1 & -1 & \cdots & -1 & -1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\mathbf{LA} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N-1} & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} & a_{N-1,N} \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N-1} & a_{N,N} \end{bmatrix}$$

$$\mathbf{A_1} = \mathbf{LAL}^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2}-a_{2,1} & \cdots & a_{2,N-1}-a_{2,1} & a_{2,N}-a_{2,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N-1,1} & a_{N-1,2}-a_{N-1,1} & \cdots & a_{N-1,N-1}-a_{N-1,1} & a_{N-1,N}-a_{N-1,1} \\ a_{N,1} & a_{N,2}-a_{N,1} & \cdots & a_{N,N-1}-a_{N,1} & a_{N,N}-a_{N,1} \end{bmatrix}$$

$$\det\left(\mathbf{A_1} - \lambda \mathbf{I}\right) = \left(1-\lambda\right)\det\left(\begin{bmatrix} a_{2,2}-a_{2,1}-\lambda & \cdots & a_{2,N-1}-a_{2,1} & a_{2,N}-a_{2,1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{N-1,2}-a_{N-1,1} & \cdots & a_{N-1,N-1}-a_{N-1,1}-\lambda & a_{N-1,N}-a_{N-1,1} \\ a_{N,2}-a_{N,1} & \cdots & a_{N,N-1}-a_{N,1} & a_{N,N}-a_{N,1}-\lambda \end{bmatrix}\right)$$

Therefore, $\det\left(\mathbf{A_1} - \lambda \mathbf{I}\right) = 0 \quad for\ \lambda = 1$

$\lambda = 1$ should be one of the eigenvalues of $\mathbf{A_1}$ and hence $\mathbf{A}$ (note that $\mathbf{A}$ and $\mathbf{A_1}$ are similar) .

(ii) To show 1 is the largest eigenvalue of $\mathbf{A}$:

If

$$\mathbf{A}_2 = \mathbf{A}^{\mathbf{T}}$$

then $\mathbf{A_2}$ and $\mathbf{A}$ have the same eigenvalues since

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A_2} - \lambda\mathbf{I}).$$

If $\mathbf{g} = [g_1, g_2, \ldots, g_N]^T$ is an eigenvector of $\mathbf{A_2}$ and the corresponding eigenvalue is λ. Without the loss of generalization, we suppose that

$$|g_p| \geq |g_n| \text{ where } n = 1, 2, \ldots, N.$$

Compare the p^{th} entry on the both sides of $\mathbf{A_2 g} = \lambda \mathbf{g}$, we have

$$a_{1,p}g_1 + a_{2,p}g_2 + \cdots + a_{N,p}g_N = \lambda g_p$$

$$a_{1,p}g_1 + a_{2,p}g_2 + \cdots + a_{N,p}g_N = \lambda g_p$$

$$\left|\lambda g_p\right| = \left|a_{1,p}g_1 + a_{2,p}g_2 + \cdots + a_{N,p}g_N\right| \le a_{1,p}\left|g_1\right| + a_{2,p}\left|g_2\right| + \cdots + a_{N,p}\left|g_N\right|$$

$$\le (a_{1,p} + a_{2,p} + \cdots + a_{N,p})\left|g_p\right| = \left|g_p\right|$$

$$\left|\lambda\right| \le 1$$

(iii) To show that, if the multiplicity of $\lambda = 1$ is 1, then its corresponding eigenvector determines the ratio of $x_1(t)$: $x_2(t)$: … : $x_N(t)$ in the convergence case, suppose that

$$\mathbf{E} = \begin{bmatrix} \mathbf{e_1} & \mathbf{e_2} & \cdots & \mathbf{e_N} \end{bmatrix}, \quad \mathbf{E}^{-1} = \begin{bmatrix} \mathbf{f_1} & \mathbf{f_2} & \cdots & \mathbf{f_N} \end{bmatrix}^T$$

then

$$\mathbf{A} = \mathbf{EDE^{-1}}$$

$$\mathbf{A} = \lambda_1\mathbf{e_1}\mathbf{f_1^T} + \lambda_2\mathbf{e_2}\mathbf{f_2^T} + \cdots + \lambda_{N-1}\mathbf{e_{N-1}}\mathbf{f_{N-1}^T} + \lambda_N\mathbf{e_N}\mathbf{f_N^T}$$

$$\mathbf{A} = \lambda_1 \mathbf{e_1 f_1^T} + \lambda_2 \mathbf{e_2 f_2^T} + \cdots + \lambda_{N-1} \mathbf{e_{N-1} f_{N-1}^T} + \lambda_N \mathbf{e_N f_N^T}$$

$$\lim_{t\to\infty} \mathbf{A}^t = \lim_{t\to\infty} \lambda_1^t \mathbf{e_1 f_1^T} + \lambda_2^t \mathbf{e_2 f_2^T} + \cdots + \lambda_{N-1}^t \mathbf{e_{N-1} f_{N-1}^T} + \lambda_N^t \mathbf{e_N f_N^T}$$

If $\lambda_1 = 1$ and $|\lambda_m| < 1$ for $m \neq 1$, then

$$\lim_{t\to\infty} \mathbf{A}^t = \lim_{t\to\infty} \mathbf{e_1 f_1^T}$$

$$\lim_{t\to\infty} \mathbf{X}(t) = \lim_{t\to\infty} \mathbf{A}^t \mathbf{X}(0) = c\mathbf{e_1} \qquad \text{where} \qquad c = \mathbf{f_1^T X}(0)$$

Note that c is a constant. In other words, when $t \to \infty$,

$$x_1(t) : x_2(t) : \cdots : x_N(t) = e_1[1] : e_1[2] : \cdots : e_1[N]$$

6.7 Discrete Transforms (只教不考)

(1) Discrete Fourier Transform (DFT)

$$G[m]=\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

inverse:

$$g[n]=\frac{1}{N}\sum_{m=0}^{N-1} G[m]e^{j\frac{2\pi mn}{N}}$$

When $N=2$

$$\begin{bmatrix} G[0] \\ G[1] \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} g[0] \\ g[1] \end{bmatrix}$$

When $N=3$

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{-1-j\sqrt{3}}{2} & \frac{-1+j\sqrt{3}}{2} \\ 1 & \frac{-1+j\sqrt{3}}{2} & \frac{-1-j\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} G[0] \\ G[1] \\ G[2] \end{bmatrix}$$

(2) Discrete Cosine Transform (DCT)

$$\mathbf{y} = \mathbf{C_N x}$$

where

$$C_N[m,n] = k_m \cos\left(\pi \frac{m(n+1/2)}{N}\right)$$

$m = 0, 1, \ldots, N\text{-}1, \qquad n = 0, 1, \ldots, N\text{-}1,$

$$k_0 = \sqrt{1/N}$$

$$k_m = \sqrt{2/N} \qquad \text{when } m \neq 0$$

Inverse:

$$\mathbf{x} = \mathbf{C_N^T y} \qquad \mathbf{C_N^{-1}} = \mathbf{C_N^T}$$

Application: Data Compression

When $N = 8$

8-Point DCT

$$\mathbf{C_8} = \begin{bmatrix} 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\ 0.4904 & 0.4157 & 0.2778 & 0.0975 & -0.0975 & -0.2778 & -0.4157 & -0.4904 \\ 0.4619 & 0.1913 & -0.1913 & -0.4619 & -0.4619 & -0.1913 & 0.1913 & 0.4619 \\ 0.4157 & -0.0975 & -0.4904 & -0.2778 & 0.2778 & 0.4904 & 0.0975 & -0.4157 \\ 0.3536 & -0.3536 & -0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & 0.3536 \\ 0.2778 & -0.4904 & 0.0975 & 0.4157 & -0.4157 & -0.0975 & 0.4904 & -0.2778 \\ 0.1913 & -0.4619 & 0.4619 & -0.1913 & -0.1913 & 0.4619 & -0.4619 & 0.1913 \\ 0.0975 & -0.2778 & 0.4157 & -0.4904 & 0.4904 & -0.4157 & 0.2778 & -0.0975 \end{bmatrix}$$

Zero Crossings of the 8-Point DCT

0.3536 0.3536 0.3536 0.3536 0.3536 0.3536 0.3536 0.3536
0.4904 0.4157 0.2778 0.0975 * -0.0975 -0.2778 -0.4157 -0.4904
0.4619 0.1913 * -0.1913 -0.4619 -0.4619 -0.1913 * 0.1913 0.4619
0.4157 * -0.0975 -0.4904 -0.2778 * 0.2778 0.4904 0.0975 * -0.4157
0.3536 * -0.3536 -0.3536 * 0.3536 0.3536 * -0.3536 -0.3536 * 0.3536
0.2778 * -0.4904 * 0.0975 0.4157 * -0.4157 -0.0975 * 0.4904 * -0.2778
0.1913 * -0.4619 * 0.4619 * -0.1913 -0.1913 * 0.4619 * -0.4619 * 0.1913
0.0975 * -0.2778 * 0.4157 * -0.4904 * 0.4904 * -0.4157 * 0.2778 * -0.0975

more zero crossings → high frequency component

(3) Walsh Transform (Hadamard Transform)

$$\mathbf{y} = \mathbf{W_N}\mathbf{x}$$

where $\mathbf{W_N}$ is an $N \times N$ matrix and the entries of $\mathbf{W_N}$ is either 1 or -1, N is limited to 2^k.

Inverse: $(\mathbf{W_N})^{-1} = \frac{1}{N}\mathbf{W_N^T}$

Applications:

(i) simplification for computation

(ii) analysis step-like functions

(iii) code division multiple access (CDMA)

2-point Walsh transform

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

4-point Walsh transform

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

To obtain the 2^{k+1}-point Walsh transform from the 2^k-point Walsh transform,

Step 1

$$\mathbf{V}_{2^{k+1}} = \begin{bmatrix} \mathbf{W}_{2^k} & \mathbf{W}_{2^k} \\ \mathbf{W}_{2^k} & -\mathbf{W}_{2^k} \end{bmatrix}$$

Step 2 Reorder according to sign changes

$$\mathbf{V}_{2^{k+1}} \xrightarrow{\textit{permutation}} \mathbf{W}_{2^{k+1}}$$

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{V}_4 = \begin{bmatrix} \mathbf{W}_2 & \mathbf{W}_2 \\ \mathbf{W}_2 & -\mathbf{W}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

sign changes

0

3

1

2

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{V}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \quad \begin{matrix} \text{sign changes} \\ 0 \\ 3 \\ 4 \\ 7 \\ 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$\mathbf{W}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

(4) Haar Transform

$$\mathbf{y} = \mathbf{H_N x}$$

where $\mathbf{H_N}$ is an $N \times N$ matrix and the entries of $\mathbf{H_N}$ is 1 0, or -1,

N is limited to 2^k.

Inverse: $(\mathbf{H_N})^{-1} = \mathbf{D_N H_N^T}$ $\mathbf{D_N}$ is diagonal and

$$\mathbf{D_N}[0,0] = \mathbf{D_N}[1,1] = 1/N$$

$$\mathbf{D_N}[2,2] = \mathbf{D_N}[3,3] = 2/N$$

$$\mathbf{D_N}[4,4] = \mathbf{D_N}[5,5] = \mathbf{D_N}[6,6] = \mathbf{D_N}[7,7] = 4/N$$

$$\mathbf{D_N}[n,n] = 8/N \quad \textit{for}\ 8 \le n \le 15$$

$$\vdots \qquad \vdots \qquad \vdots$$

Applications:

(i) simplification for computation

(ii) extract local features

$N = 2$

$$\mathbf{H_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$N = 4$

$$\mathbf{H_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$N = 8$

$$\mathbf{H_8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

General way to generate the Haar transform:

$$\mathbf{H}_{\mathbf{2N}} = \begin{bmatrix} \mathbf{H}_{\mathbf{N}} \otimes [1,1] \\ \mathbf{I}_{\mathbf{N}} \otimes [1,-1] \end{bmatrix}$$ where $\otimes$ means the Kronecker product

$$\mathbf{I}_{\mathbf{N}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$N = 16$

$$\mathbf{H} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$$