1. (a) $$\frac{y''(x)}{(y'(x))^3} = 1$$, integrate both side

$$\int \frac{y''(x)}{(y'(x))^3}\,dx = \frac{1}{-2}\cdot(y'(x))^{-2} + C_1 = x$$

$$\Rightarrow (y'(x))^{-2} = -2x + C_1$$

$$\Rightarrow y'(x) = (-2x+C_1)^{-\frac{1}{2}}$$, integrate both side

$$\Rightarrow y(x) = -(-2x+C_1)^{\frac{1}{2}} + C_2 \quad \#$$

(b) integrate both side

$$\int y''(x)\,dx = \int y'(x)e^{y(x)}\,dx$$

$$\Rightarrow y'(x) + C_1 = e^{y(x)}$$, substitute i.c.

$$1 + C_1 = 1 \quad , \quad \therefore C_1 = 0$$

integrate both side again

$$\int y'(x)e^{-y(x)}\,dx = \int 1\,dx$$

$$\Rightarrow -e^{-y(x)} = x + C_2$$; substitute i.c.

$$C_2 = -1$$

$$\therefore e^{-y(x)} = -x+1$$

$$y(x) = -\ln|-x+1| \quad \#$$

(c) integrate both side

$$\int y''(x)\cdot y'(x)\,dx = \int 2\sin(2x)\,dx$$

$$\frac{1}{2}(y'(x))^2 + c_1 = -\cos(2x)\quad,\ \text{substitute I.C.}$$

$$c_1 = -1$$

$$y'(x) = \pm\left[2(1-\cos(2x))\right]^{\frac{1}{2}}$$

$$= \pm(4\sin^2 x)^{\frac{1}{2}} = 2\sin x$$

$$y = \pm(2\cos x + c_2)\quad,\ \text{substitute I.C.}$$

$$c_2 = \mp 2$$

$$\therefore\ y = \pm 2(\cos x - 1)\ \#$$

2. (a) Assume $u(x,y) = X(x)\cdot Y(y)$

$$X'Y = 2XY + XY'\quad,\ \text{divide } XY$$

$$\Rightarrow \frac{X'}{X} = 2 + \frac{Y'}{Y}$$

assume $\frac{Y'}{Y} = \lambda$, so $\frac{X'}{X} = \lambda + 2$, where λ is arbitrary real number.

solution of two variables are respectively

$$c_1 e^{\lambda y}\ \&\ c_2 e^{(\lambda+2)x}$$

$u(x,y)$ will be all combination of above two sols

$$u(x,y) = \sum_{\lambda} C_\lambda \cdot e^{\lambda y + (\lambda+2)x}$$

(b) Assume $u(x,y) = X(x) \cdot Y(y)$

$\Rightarrow \; yX'Y = x^2 XY'$ divide XY

$\Rightarrow \; y\dfrac{X'}{X} = x^2 \dfrac{Y'}{Y} \;\Rightarrow\; \dfrac{X'}{x^2 X} = \dfrac{Y'}{yY}$

assume $\dfrac{X'}{x^2 X} = \dfrac{Y'}{yY} = \lambda$

solve $\dfrac{X'}{X} = x^2\lambda$ and $\dfrac{Y'}{Y} = y\lambda$, where $\lambda \in \mathbb{R}$

obtain sol : $c_1 e^{\lambda \frac{x^3}{3}}$ and $c_2 e^{\lambda \frac{y^2}{2}}$

$u(x,y)$ will be all combination of λ

$= \sum_{\lambda} c_\lambda e^{\lambda\left(\frac{x^3}{3} + \frac{y^2}{2}\right)}$ #

(c) Assume $u(x,y) = X(x) \cdot Y(y)$

$X'' Y = XY'$, divide XY

$\Rightarrow \; \dfrac{X''}{X} = \dfrac{Y'}{Y}$

assume $\dfrac{X''}{X} = \dfrac{Y'}{Y} = -\lambda$, discuss λ with 3 cases

case 1 : $\lambda = 0$, sol for $X = c_1 + c_2 x$

consider B.C. $u(0,y) = u(1,y) = 0$

ignore trivial sol that $Y(y) = 0$ will lead

$X(1) = X(0) = 0$, with this condition

$c_1 = c_2 = 0$

$X(x) = 0$, is a trivial sol.

case 2: $\lambda = -\alpha^2$, sol for $X = c_3\cosh(\alpha x) + c_4\sinh(\alpha x)$

similarly consider B.C. will lead to

conlusion that $c_3 = c_4 = 0$

case 3: $\lambda = \alpha^2$, sol for $X = c_5\cos(\alpha x) + c_6\sin(\alpha x)$

$\because$ B.C.: $X(1) = X(0) = 0$

$\therefore$ $c_5 = 0$, $\alpha = n\pi$, $\lambda = n^2\pi^2$, $n \in N$

now solve $\frac{Y'}{Y} = -n^2\pi^2$, sol $= c_7 e^{-n^2\pi^2 y}$

$\therefore$ $u(x,y) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 y} \cdot \sin(n\pi x)$, now consider B.C.

$u(x,0) = x(1-x)$

$$\sum_{n=1}^{\infty} A_n \cdot \sin(n\pi x) = x(1-x)$$

From Fourier Series

$$A_n = 2\int_0^1 (-x^2+x)\sin(n\pi x)\,dx$$

The calculation can be done by "Integration by Parts"

The result is $\frac{4(1-(-1)^n)}{n^3\pi^3}$

$$\therefore\ u(x,y) = \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^3\pi^3} \sin(n\pi x)\, e^{-n^2\pi^2 y}$$

#

(d) Assume $u(x,y) = X(x) \cdot Y(y)$

$$X''Y + XY'' = 0 \quad , \text{ divide } XY$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} \quad , \text{ assume } \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

case 1: $\lambda = 0$

$$X = C_1 + C_2 x$$

$$Y = C_3 + C_4 y$$

But by B.C. $Y(0)$ & $Y(1) = 0$

$$C_3 = C_4 = 0$$

$X(0) = 0$ & $X(1)$ should be const

$$\Rightarrow C_1 = 0 \, , \, C_2 = 0$$

case 2: $\lambda = -\alpha^2$

$$X = c_5 \cosh(\alpha x) + C_6 \sinh(\alpha x)$$

$$Y = C_7 \cosh(\alpha y) + C_8 \sinh(\alpha y)$$

By B.C. $C_7 = C_8 = 0$
trivial sol

case 3: $\lambda = \alpha^2$

$$X = C_9 \cosh(\alpha x) + C_{10} \sinh(\alpha x)$$

$$Y = C_{11} \cos(\alpha y) + C_{12} \sin(\alpha y)$$

By B.C.

$C_{11} = 0$, $\alpha = n\pi$, where $n \in N$

then $X = C_9 \cosh(n\pi x) + C_{10} \sinh(n\pi x)$

By B.C. $C_9 = 0$,

$$\therefore u(x,y) = \sum_{n=1}^{\infty} A_n \sinh(n\pi x) \sin(n\pi y)$$

$$\because u(1,y) = \sin(\pi y)$$

$$A_n = \frac{2}{\sinh(n\pi)} \int_0^1 \sin(\pi y)\sin(n\pi y)\,dy$$

$$= \begin{cases} \int_0^1 \sin^2(\pi y)\,dy = \int_0^1 \frac{1-\cancelto{0}{\cos(2\pi y)}}{2}\,dy = \frac{1}{2}, & n=1 \\ \int_0^1 -\frac{1}{2}\cdot\left(\cos((n+1)\pi y) - \cos((n-1)\pi y)\right) dy & \end{cases}$$

$$= -\frac{1}{2}\left[\frac{\sin(n+1)\pi y}{(n+1)\pi}\Big|_0^1 - \frac{\sin(n-1)\pi y}{(n-1)\pi}\Big|_0^1 \right] = 0$$

for $\forall n \geq 2$

$$\therefore u(x,y) = \frac{\sinh(\pi x)}{\sinh(\pi)} \sin(\pi y) \quad \#$$