1. (a) assume $u(x,y) = v(x,y) + \psi(y)$

$$\Rightarrow \begin{cases} x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} + v = 0 \\ y\psi'(y) + \psi(y) = y \end{cases}$$

assume $v(x,y) = X(x)Y(y)$

$$\Rightarrow x\frac{X'}{X} + y\frac{Y'}{Y} + 1 = 0 \Rightarrow \begin{cases} xX' + \lambda X = 0 \\ -yY' + (\lambda-1)Y = 0 \end{cases}$$

solve $X, Y = Ax^{-\lambda}$ and $By^{(\lambda-1)}$ respectively

As for $\psi(y)$, assume it as $ay+b$

$$ya + ay + b = y, \quad a = \frac{1}{2}, \quad b = 0$$

$$\psi(y) = \frac{y}{2}$$

$$\therefore u(x,y) = \sum_{\lambda} c x^{-\lambda} y^{(\lambda-1)} + \frac{1}{2}y$$

(b) assume $u(x,y,z,t) = XYZT$

we have $\dfrac{xX'}{X} + \dfrac{yY'}{Y} + \dfrac{zZ'}{Z} = \dfrac{T'}{T}$

let left side be $\lambda_1, \lambda_2, \lambda_3$ respectively

we have $X = Ax^{\lambda_1}$, $Y = By^{\lambda_2}$, $Z = Cz^{\lambda_3}$

by Cauchy - Euler

as for T, we have $T' - (\lambda_1+\lambda_2+\lambda_3)T = 0$

$$\therefore T = De^{(\lambda_1+\lambda_2+\lambda_3)t}$$

then $u(x,y,z,t) = \displaystyle\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\lambda_3} E_\lambda x^{\lambda_1} y^{\lambda_2} z^{\lambda_3} e^{(\lambda_1+\lambda_2+\lambda_3)t}$

(c) assume $u = XYZ + \varphi(x)$

we have $\varphi''(x) = x$, $\varphi(x) = \frac{x^3}{6} - \frac{x}{6}$ by given B.C.

as for $\frac{X''}{X} + \frac{Y''}{Y} = \frac{Z''}{Z}$

let LHS be λ_1, λ_2 respectively

we have $\begin{cases} X = \sin(m\pi x) \\ Y = \sin(n\pi y) \end{cases}$ by the given B.C.s where $m, n \in \mathbb{N}$

and Z will only have non-trivial sol when $\sqrt{m^2+n^2} \in \mathbb{Z}$

$Z = \sin\sqrt{m^2+n^2}\,\pi z$

$$\therefore u(x,y,z) = \sum_m \sum_n \sin(m\pi x)\sin(n\pi y)\sin(\sqrt{m^2+n^2}\,\pi z) + \frac{x^3}{6} - \frac{x}{6}$$

(d) assume $u = RZ$

we have $\begin{cases} rR'' + R' + \lambda r R = 0 \\ -Z'' + \lambda Z = 0 \end{cases}$

if $\lambda = -\alpha^2$, the sol $c_1 I_v(\alpha x) + c_2 K_v(\alpha x)$ will lead the conclusion that $R = 0$

if $\lambda = \alpha^2$ the sol will be $c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

$c_2 = 0$ to avoid infinite

$\therefore R = c_1 J_0(\alpha_n r)$, where $\alpha_n = x_n$

now solve $z'' - \lambda z = 0$, where $\lambda_n = \alpha_n^2$

$Z = A_1\cosh(\alpha_n z) + A_2 \sinh(\alpha_n z)$

$\because Z'(0) = 0 \quad , \therefore Z = A_1 \cosh(\alpha_n z)$

$$u = RZ = \sum_{n=1}^{\infty} A_n \cosh(\alpha_n) J_0(\alpha_n r)$$

$$\because u(r,1) = 1 \quad , \int_0^1 r J_0(\alpha_m r)\, u(r,1)\, dr$$

$$= \sum_{n=1}^{\infty} A_n \cosh(\alpha_n) \int_0^1 r J_0(\alpha_n r) J_0(\alpha_m r)\, dr$$

$$= A_m \cosh(\alpha_m) J_1^2(\alpha_m) / 2$$

$$A_n = \frac{2u(r,1)}{\cosh(\alpha_n) J_1^2(\alpha_n)} \int_0^1 r J_0(\alpha_n r)\, dr$$

$$= \frac{2}{\alpha_n \cosh(\alpha_n) J_1(\alpha_n)}$$

$$\therefore u(r,z) = \sum_{n=1}^{\infty} \frac{2\cosh(\alpha_n z) J_0(\alpha_n r)}{\alpha_n \cosh(\alpha_n) J_1(\alpha_n)}$$

where $\alpha_n = \chi_n$

2. set $v(r,\theta) = u(r, \theta - \frac{\pi}{3})$

$\Rightarrow v(1, \theta + \frac{\pi}{3}) = 1 \quad , \quad 0 < \theta < \frac{2}{3}\pi$

$v(r,0) = 0 \quad , \quad v(r, \frac{2}{3}\pi) = 0 \quad , \quad 0 < r < 1$

assume $v = R\Theta$

we have $\dfrac{r^2 R'' + rR'}{R} = \dfrac{-\Theta''}{\Theta} = \lambda$

only when $\lambda = \alpha^2 > 0$, we'll have non-trivial sol

where $\Theta = A_1 \cos\alpha\theta + A_2 \sin\alpha\theta$

given $\Theta(0)$, $A_1 = 0$, $\Theta(\frac{2}{3}\pi) = A_2 \sin(\frac{2}{3}\pi\alpha) = 0$

$\alpha = \frac{3}{2}n$, where $n \in N$

$\Theta = A_2 \sin(\frac{3}{2}n\theta)$

as for R, we have $R = A_3 r^{\frac{3}{2}n} + A_4 r^{-\frac{3}{2}n}$

$\because R(0)$ should be finite, $A_4 = 0$

$$v = R\Theta = \sum_{n=1}^{\infty} A_n r^{\frac{3}{2}n} \sin(\frac{3}{2}n\theta)$$

By $v(1, \theta + \frac{\pi}{3}) = 1$, $A_n = \frac{3}{\pi}\int_0^{\frac{3}{2}\pi} \sin(\frac{3}{2}n\theta)\, d\theta$

$$= -\frac{3}{\pi} \cdot \frac{2}{3n} \cos(\frac{3}{2}n\theta)\Big|_0^{\frac{2}{3}\pi} = \frac{-2}{\pi n}[(-1)^n - 1] = \frac{2}{n\pi}(1-(-1)^n)$$

$$u = v(r, \theta + \frac{\pi}{3}) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n] r^{\frac{3}{2}n} \sin(\frac{3}{2}n\theta + \frac{\pi n}{2})$$

3. $\mathcal{L}_{t\to s}\{\frac{\partial u}{\partial x}\} = \mathcal{L}_{t\to s}\{\frac{\partial u}{\partial t}\}$

let $\mathcal{L}_{t\to s}\{\frac{\partial u}{\partial x}\} = U(x, s)$

we have $\frac{\partial U(x,s)}{\partial x} = s \cdot U(x,s) - u(x,0) = sU - 1$

$$\frac{\partial U}{\partial x} - sU = -1$$

$U_c = Ae^{sx}$, $U_p = \frac{1}{s}$

$\therefore \quad U = Ae^{sx} + \frac{1}{s}$

By $u(0,t) = 1+t$, $U(0,s) = \mathcal{L}_{t\to s}\{u(0,t)\} = \frac{1}{s} + \frac{1}{s^2}$

we have $A = \frac{1}{s^2}$

$\therefore \quad U = \frac{1}{s^2} e^{sx} + \frac{1}{s}$

$$u(x,t) = (t+x)\underbrace{u}_{\text{unit step}}(t+x) + 1$$

4. $p_0 = 1$, $p_1 = \cos x$

$$\phi_0 = \frac{p_0}{\|p_0\|} = \frac{1}{\sqrt{\int_0^{\pi/2} 1^2\,dx}} = \sqrt{\frac{2}{\pi}}$$

$$g_1 = p_1 - \langle p_1, \phi_0 \rangle \phi_0 = \cos x - \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} = \cos x - \frac{2}{\pi}$$

$$\|g_1\| = \int_0^{\pi/2} \left(\cos x - \frac{2}{\pi}\right)^2 dx = \frac{\pi}{4} - \frac{2}{\pi}$$

$$\phi_1 = \frac{g_1}{\|g_1\|} = \frac{\cos x - \frac{2}{\pi}}{\sqrt{\frac{\pi}{4} - \frac{2}{\pi}}}$$

$$\{\phi_0, \phi_1\} = \left\{ \sqrt{\frac{2}{\pi}}, \frac{\cos x - \frac{2}{\pi}}{\sqrt{\frac{\pi}{4} - \frac{2}{\pi}}} \right\}$$

(b) $g(x) = a_0\phi_0 + a_1\phi_1$

$$a_0 = \int_0^{\frac{\pi}{2}} g(x)\cdot\phi_0 = \int_0^{\frac{\pi}{2}} \sin x\sqrt{\frac{2}{\pi}}\,dx = \frac{\sqrt{2\pi}}{\pi}$$

$$a_1 = \int_0^{\frac{\pi}{2}} g(x)\cdot\phi_1\,dx = \int_0^{\frac{\pi}{2}} \sin x\,\frac{\cos x - \frac{2}{\pi}}{\sqrt{\frac{\pi}{4}-\frac{2}{\pi}}}\,dx$$

$$= \sqrt{\frac{\pi}{\pi^2-8}}\cdot\frac{\pi-4}{\pi}$$

$$g = a_0\sqrt{\frac{2}{\pi}} + a_1\sqrt{\frac{4\pi}{\pi^2-8}}\left(\cos x - \frac{2}{\pi}\right) = c_0 + c_1\cos x$$

$$\Rightarrow c_0 = a_0\sqrt{\frac{2}{\pi}} - a_1\sqrt{\frac{4\pi}{\pi^2-8}}\cdot\frac{2}{\pi}$$

$$c_1 = a_1\sqrt{\frac{4\pi}{\pi^2-8}}$$

simplify $c_0 = \frac{2}{\pi} - \frac{\pi}{\pi^2-8}\cdot\frac{\pi-4}{\pi}\cdot 2\,\frac{2}{\pi}$

$$= \frac{2}{\pi}\left(\frac{\pi^2-8-2\pi+8}{\pi^2-8}\right) = \frac{2\pi-4}{\pi^2-8}$$

$$c_1 = \sqrt{\frac{\pi}{\pi^2-8}}\cdot\frac{\pi-4}{\pi}\cdot\sqrt{\frac{4\pi}{\pi^2-8}} = \frac{\pi}{\pi^2-8}\cdot\frac{\pi-4}{\pi}\cdot 2$$

$$= \frac{2\pi-8}{\pi^2-8}$$