5. Sampling and Discrete Fourier Transform

Section 5.1 Sampling and Reconstruction

Section 5.2 Discrete Fourier Transform

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

Sampling and Discrete Fourier Transform

- Sampling and Discrete Fourier Transform
  - Sampling (Sec. 5-1-2)
    - Varying the Sampling Rate (Sec. 5-1-4)
  - Reconstruction (Sec. 5-1-3)
  - Discrete Fourier Transform (Sec. 5-2-1)
    - Implementing the Continuous FT (Sec. 5-2-2)
    - Properties (Sec. 5-2-3)
    - Complexity (Sec. 5-2-4)
    - 2D Version (Sec. 5-2-5)

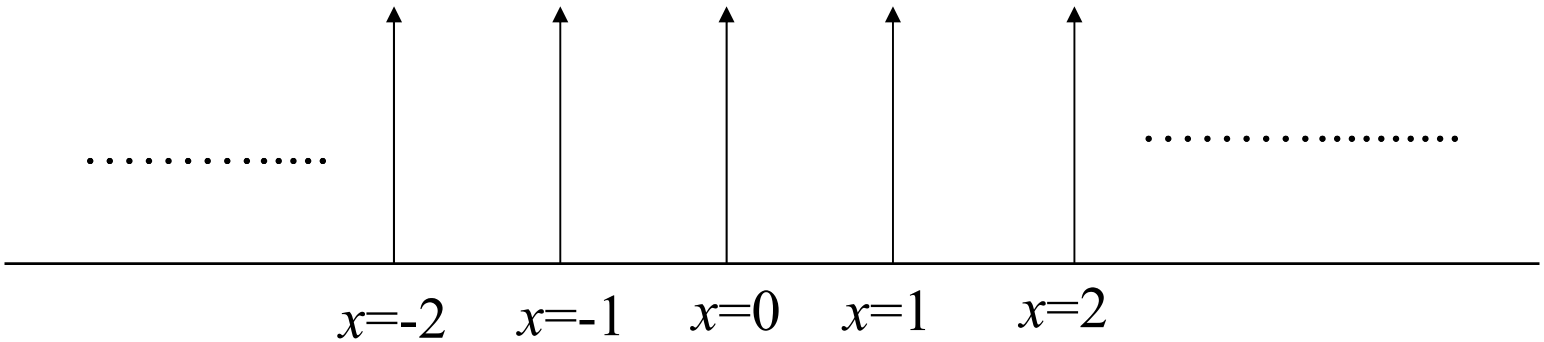
5.1 Sampling

5.1.1 Impulse Train

Impulse Train

$$p(x) = \sum_{n} \delta(x-n) = \cdots + \delta(x+1) + \delta(x) + \delta(x-1) + \delta(x-2) + \cdots\cdots$$

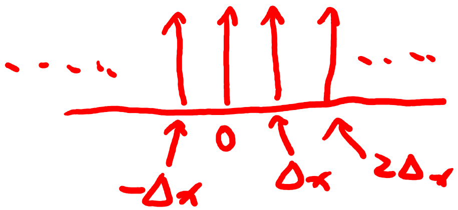
It is also called the comb function.



Signal sampling can be express in terms of the impulse train

$$g(x) \xrightarrow{sampling} g_s(x) = g(x) \sum_n \delta(x - n\Delta_x)$$

$$= \sum_n g_n \delta(x - n\Delta_x)$$



where $g_n = g(n\Delta_x)$

Since

page 349 (4)

$$\sum_n \delta(x - n\Delta_x) = \frac{1}{\Delta_x} \sum_n \delta\left(\frac{x}{\Delta_x} - n\right) = \frac{1}{\Delta_x} p\left(\frac{x}{\Delta_x}\right)$$

the sampled signal $g_s(x)$ can be expressed in terms of

$$g_s(x) = \frac{1}{\Delta_x} g(x) p\left(\frac{x}{\Delta_x}\right)$$

[**Theorem 5.1.1**] The impulse train is also an eigenfunction of the Fourier transform, i.e.,

$$P(f) = \Im\{p(x)\} = \sum_n \delta(f-n)$$

(Proof): Note that the impulse train is a periodic function

$$p(x) = p(x+1)$$

Therefore, it can be expanded by the Fourier series (page 327) of the complex form with $T = 1$

$$p(x) = \sum_n c_n \exp(j2\pi nx)$$

where

$$c_n = \frac{1}{1}\int_{-1/2}^{1/2} p(x)\exp(-j2\pi nx)\,dx = \int_{-1/2}^{1/2} \delta(x)\exp(-j2\pi nx)\,dx = 1$$

conjugate

(page 348(2))

k=0

Therefore,

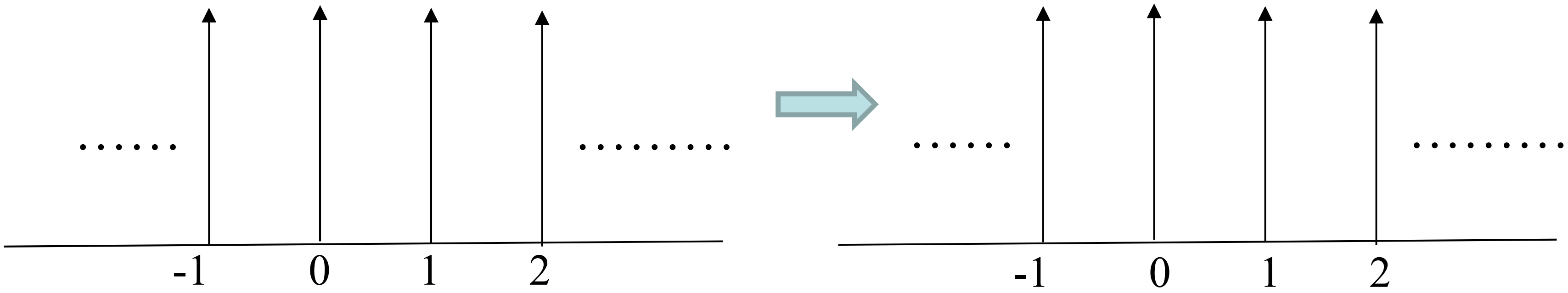
$$p(x) = \sum_n \exp(j2\pi nx)$$

$$\mathscr{F}[p(x)] = \sum_n \mathscr{F}[\exp(j2\pi nx)]$$

$$= \sum_n \delta(f-n) = p(f)$$

page 340

$$\Im\left\{\sum_n \delta(x-n)\right\} = \sum_n \delta(f-n)$$



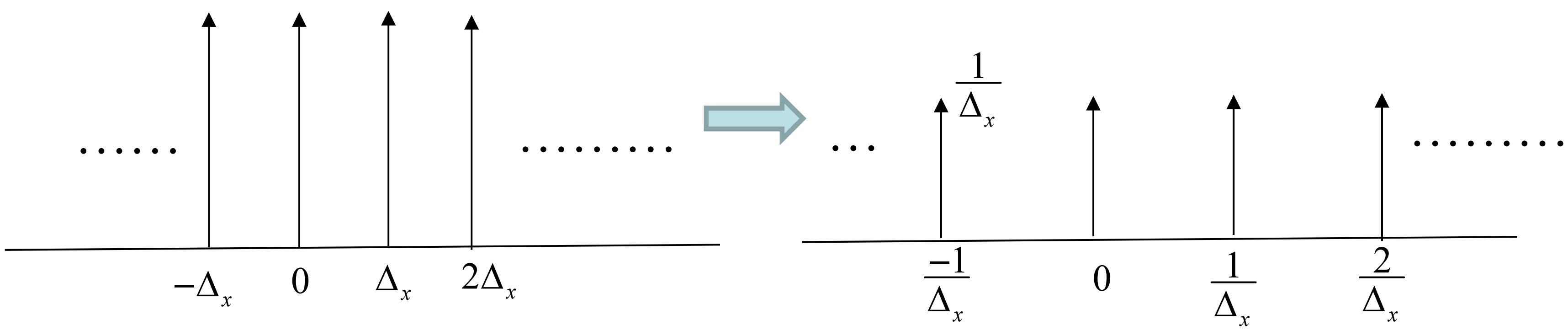
Varying the interval of the impulse train

$$\sum_n \delta(x-n\Delta_x)=\frac{1}{\Delta_x}\sum_n \delta\left(\frac{x}{\Delta_x}-n\right)=\underline{\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)}$$ (page 349(4))

$\downarrow$FT

$$\mathcal{F}\left[\sum_n \delta(x-n\Delta_x)\right]=\mathcal{F}\left[\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)\right]=\underline{p(\Delta_x f)}$$ (page 355(5))

$$=\sum_n \delta(\Delta_x f-n)=\frac{1}{\Delta_x}\sum_n \delta\left(f-\frac{n}{\Delta_x}\right)$$



5.1.2 Sampling Theory

[**Theorem 5.1.2**] Suppose that we perform sampling for a continuous signal with sampling interval Δ_x

$$g(x) \xrightarrow{sampling} g_s(x) = g(x)\sum_n \delta(x - n\Delta_x)$$

$$= \sum_n g_n \delta(x - n\Delta_x)$$

where

$$g_n = g(n\Delta_x)$$

then

$$\boxed{G_s(f) = \frac{1}{\Delta_x}\sum_n G\left(f - \frac{n}{\Delta_x}\right)}$$

(Proof): Since $\quad g_s(x) = g(x)\dfrac{1}{\Delta_x}p\left(\dfrac{x}{\Delta_x}\right)$

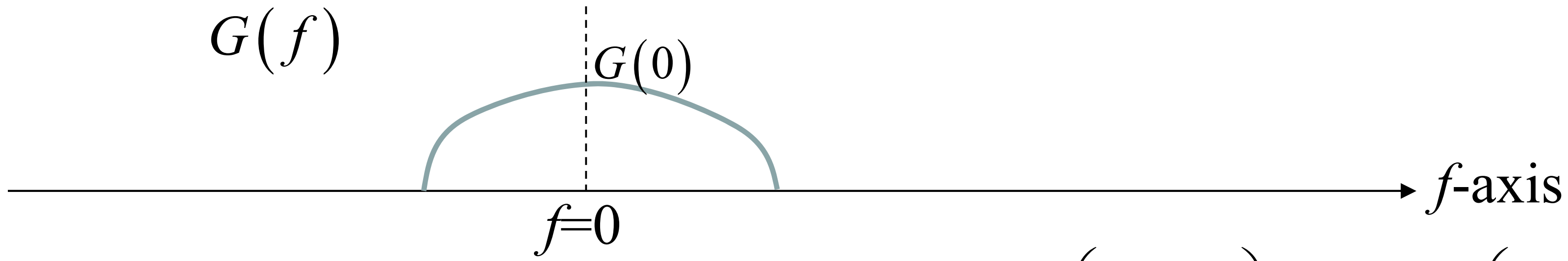
$$\mathcal{F}\left[g_s(x)\right] = \mathcal{F}\left[g(x)\right] * \mathcal{F}\left[\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)\right]$$

$$G_s(f) = G(f) * p(\Delta_x f) = \frac{1}{\Delta_x}\sum_n G(f) * \delta\left(f - \frac{n}{\Delta_x}\right) = \frac{1}{\Delta_x}\sum_n G\left(f - \frac{n}{\Delta_x}\right)$$

(page 393)

If we set $f_s = \dfrac{1}{\Delta_x}$ (f_s is call the sampling frequency)

then $$\boxed{G_s(f) = f_s \sum_n G(f - nf_s)}$$

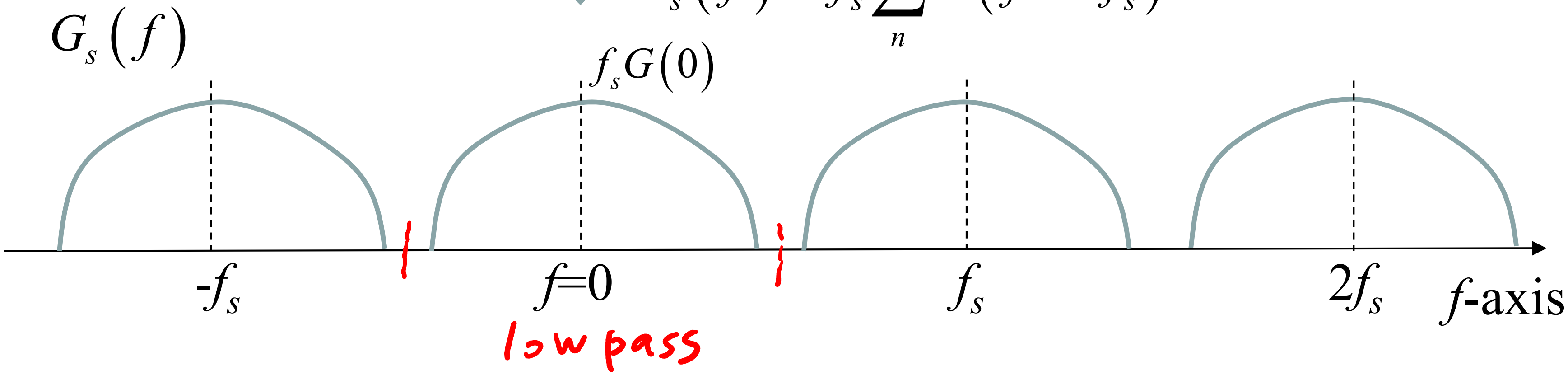


$g_n = g(n\Delta_x)$

sampling

$$g_s(x) = g(x)\sum_n \delta\left(x - \frac{n}{f_s}\right) = \sum_n g_n \delta\left(x - \frac{n}{f_s}\right)$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$



[Sampling Theory]

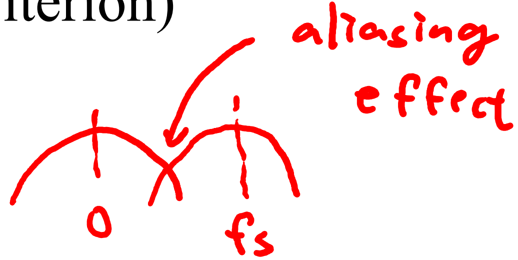
The sampling frequency $f_s = \dfrac{1}{\Delta_x}$ should be larger than twice of the bandwidth of the original continuous function:

$$f_s - B > B$$

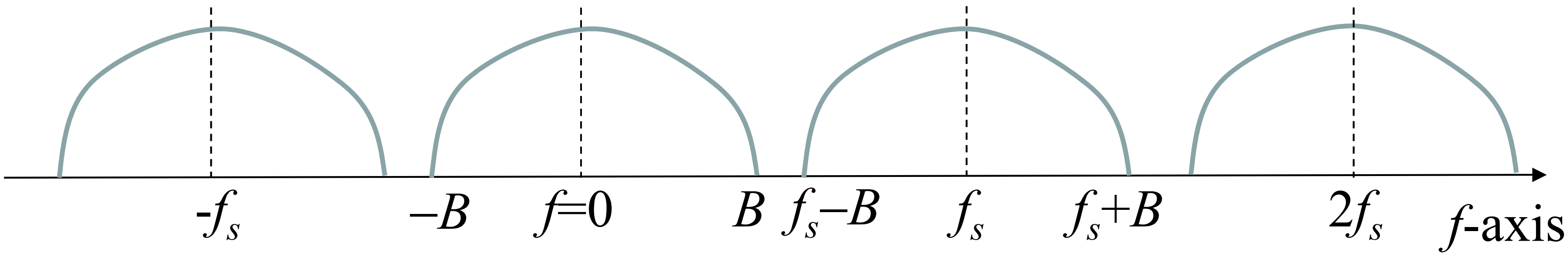
$$\underline{f_s} > \underline{2B} \qquad \text{(Nyquist criterion)}$$

where

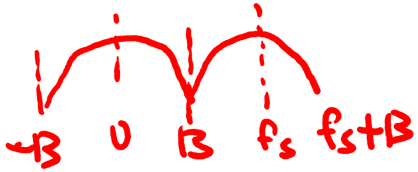
$$G(f) = 0 \qquad \text{when } f > B.$$



Otherwise, the original function cannot be reconstructed and the aliasing effect is led.

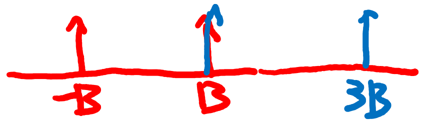


Q: What will happen if $f_s = 2B$?



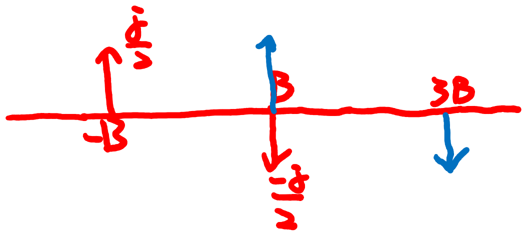
Even component with frequency $f = \pm B \rightarrow$ preserved

$\cos(2\pi Bx)$



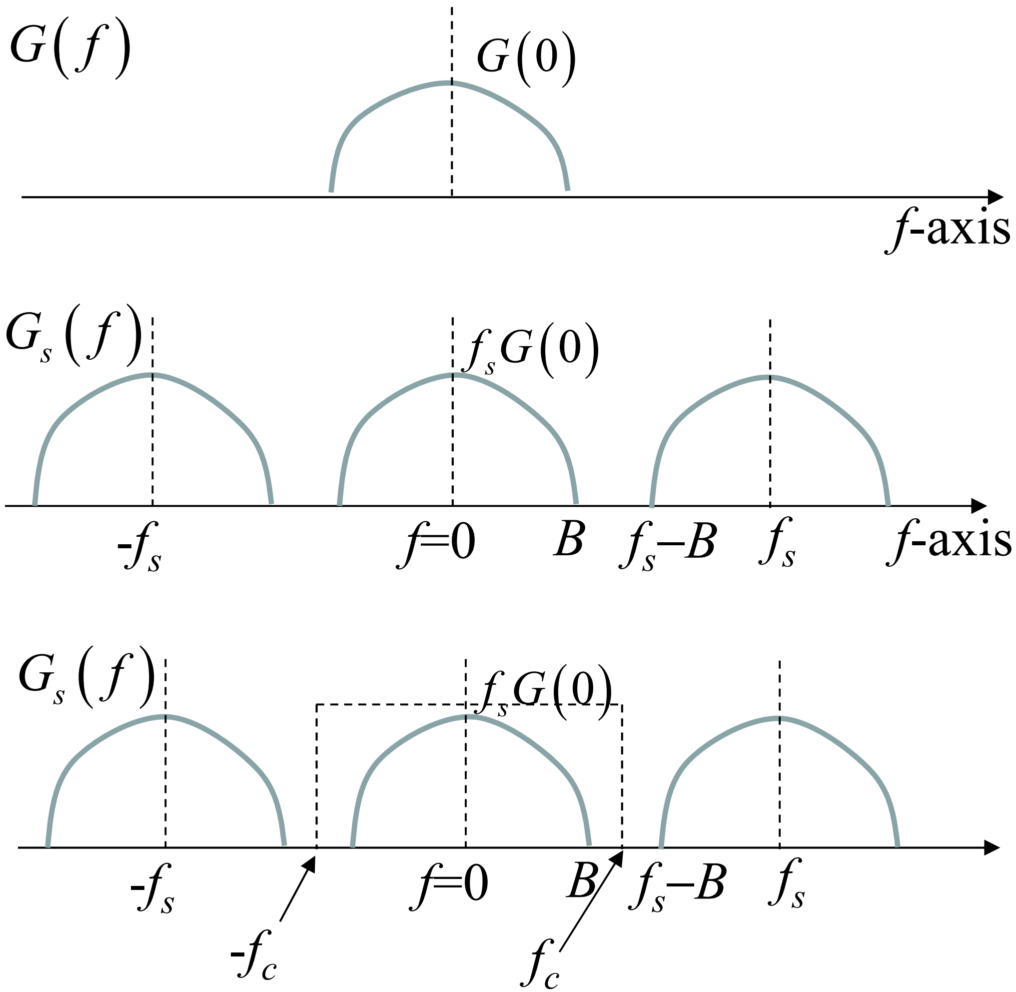
Odd component with frequency $f = \pm B \rightarrow$ destroyed

$\sin(2\pi Bx)$



5.1.3 Reconstruction (Digital to Analogous)

When the Nyquist criterion is satisfied, one can apply the lowpass filter to reconstruct the original signal.



Frequency Domain

$$G(f)$$

$$\downarrow$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$

$$\downarrow$$

$$G(f) = \frac{1}{f_s} \Pi\left(\frac{f}{2f_c}\right) G_s(f)$$

where $B < f_c < f_s - B$

Time Domain

Frequency Domain

$$g(x)$$

$$\Big\downarrow \text{sampling}$$

$$g_n = g(n\Delta_x)$$

$$g_s(x) = \sum_n g_n \delta\left(x - \frac{n}{f_s}\right)$$

D/A conversion

$$G(f)$$

$$\Big\downarrow$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$

$$\Big\downarrow \text{reconstruction}$$

A/D conversion

$$g(x) = g_s(x) * \frac{2f_c}{f_s} \operatorname{sinc}(2f_c x)$$

page 355 (5)

$$\Big\downarrow$$

$$G(f) = \frac{1}{f_s} \Pi\left(\frac{f}{2f_c}\right) G_s(f)$$

where $B < f_c < f_s - B$

$$g(x)=\frac{2f_c}{f_s}\int g_s(\tau)\operatorname{sinc}(2f_c(x-\tau))\,d\tau$$

$$=\frac{2f_c}{f_s}\int\sum_n g_n\delta\left(\tau-\frac{n}{f_s}\right)\operatorname{sinc}(2f_c(x-\tau))\,d\tau$$

$$g(x)=\frac{2f_c}{f_s}\sum_n g_n\operatorname{sinc}\left(2f_c(x-\frac{n}{f_s})\right)$$

Specially, when $f_c = f_s/2$

If $f_s > 2B$, then

$B < \frac{f_s}{2} < f_s - B$

$$g(x)=\sum_n g_n\operatorname{sinc}(f_s x-n)$$

Signal Reconstruction Formula:

Shannon's theorem

$$\boxed{g(x)=\sum_n g_n\operatorname{sinc}\left(\frac{x}{\Delta_x}-n\right)}$$

where $g_n = g(n\Delta_x)$

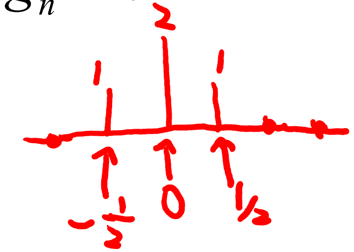
constraint: $\frac{1}{\Delta_x} > 2B$

[Example 1] Suppose that

$$g_n = g\left(\frac{n}{2}\right)$$

$\Delta_x = \frac{1}{2}$, $f_s = 2$

$g_{-1} = g_1 = 1, \quad g_0 = 2, \qquad g_n = 0$ otherwise



$G(f) = 0 \qquad \text{for } f \geq 1$

Try to reconstruct $g(x)$.

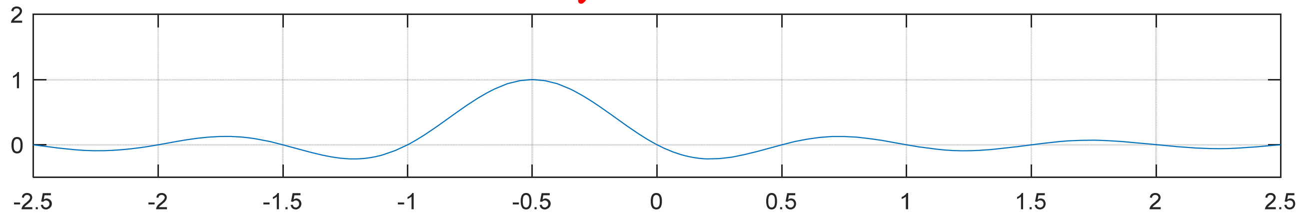
(Solution): $\Delta_x = 1/2$

$$g(x) = \text{sinc}(2x+1) + 2\,\text{sinc}\,2x + \text{sinc}(2x-1)$$

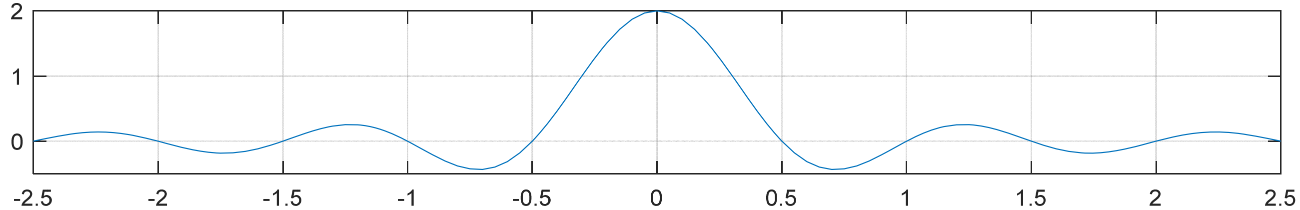
$$g(x) = \mathrm{sinc}(2x+1) + 2\,\mathrm{sinc}\,2x + \mathrm{sinc}(2x-1)$$

$\mathrm{sinc}(2x+1)=0$ for x: $\cdots -2, -1.5, -1, 0, 0.5, 1, \cdots$

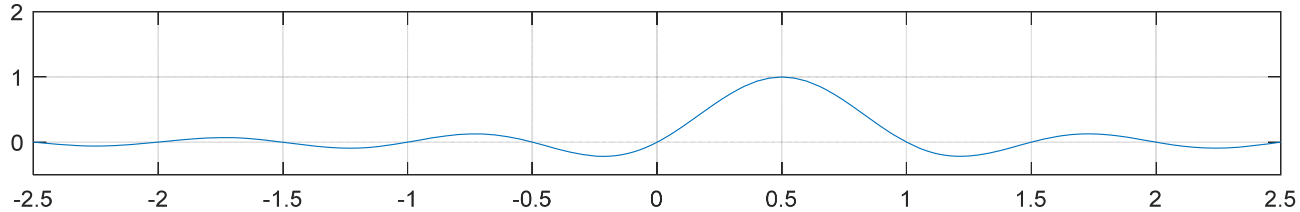
$\mathrm{sinc}(2x+1)$



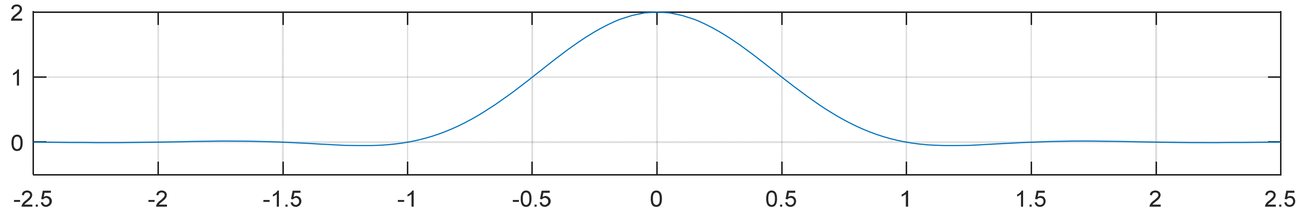
$2\,\mathrm{sinc}(2x)$



$\mathrm{sinc}(2x-1)$



$g(x)$



(Note): $\mathrm{sinc}(2x+1)$, $2\,\mathrm{sinc}(2x)$, $\mathrm{sinc}(2x-1)$

do not interfere with one another at $x = n/2$.

5.1.4 Varying the Sampling Rate

(i) D/A conversion

$$g(x) = \sum_n g_n \operatorname{sinc}\left(\frac{x}{\Delta_x} - n\right)$$

(ii) Re-sampling

$$\hat{g}_n = g(n\Delta_{new}) = \sum_m g_m \operatorname{sinc}\left(\frac{n\Delta_{new}}{\Delta_x} - m\right)$$

Note: When $\Delta_{new} = k\Delta_x$ and k is an integer

$$\hat{g}_n = g_{kn}$$

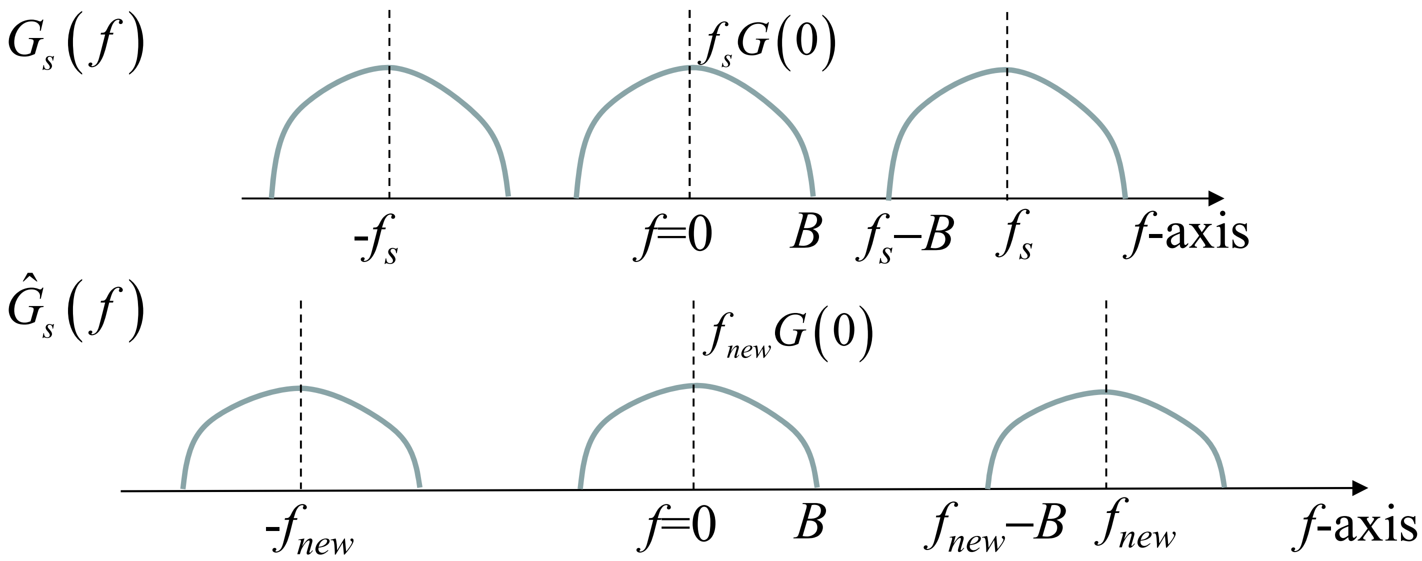
From the view point of the spectrum, if

$$g_s(x) = \sum_n g_n \delta(x - n\Delta_x) \qquad \hat{g}_s(x) = \sum_n \hat{g}_n \delta(x - n\Delta_{new})$$

then

$$G_s(f) = f_s \sum_n G(f - nf_s) \qquad \hat{G}_s(f) = f_{new} \sum_n G(f - nf_{new})$$

where $\quad f_s = 1/\Delta_x, \quad f_{new} = 1/\Delta_{new}$



5.2 Discrete Fourier Transform

5.2.1 Derivation and Definitions of the Discrete Fourier Transform

To process discrete functions, the continuous Fourier transform should be converted into the discrete version.

Continuous Fourier transform:

$$G(f) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx$$

If we set

$$f = m\Delta_f, \quad x = n\Delta_x$$

then

$$G(m\Delta_f) = \sum_n g(n\Delta_x) e^{-j2\pi mn\Delta_f\Delta_x} \underline{\Delta_x}$$

$dx \to \Delta x$

$$G\left(m\Delta_f\right) = \sum_n g\left(n\Delta_x\right) e^{-j2\pi mn\Delta_f\Delta_x}\Delta_x$$

Specially, if

$$\Delta_f\Delta_x = 1/N$$

then

$$G\left(m\Delta_f\right) = \sum_n g\left(n\Delta_x\right) e^{-j\frac{2\pi mn}{N}}\underline{\underline{\Delta_x}}$$

Similarly, for the continuous inverse Fourier transform:

$$g(x) = \int_{-\infty}^{\infty} G(f)e^{j2\pi fx}df$$

$$g\left(n\Delta_x\right) = \sum_n G\left(m\Delta_f\right) e^{j2\pi mn\Delta_f\Delta_x}\Delta_f$$

修正

$$g\left(n\Delta_x\right) = \sum_m G\left(m\Delta_f\right) e^{j\frac{2\pi mn}{N}}\Delta_f$$

Discrete Fourier Transform (DFT)

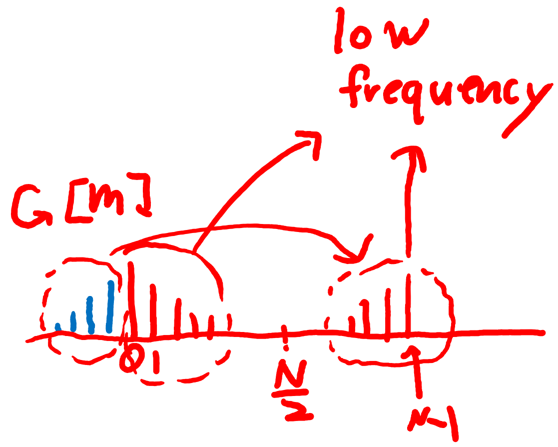
$$G[m] = DFT\{g[n]\} = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

$m, n = 0, 1, 2, \ldots, N\text{-}1$

$f = m\Delta_f$

Inverse Discrete Fourier Transform (IDFT)

$$g[n] = IDFT\{G[m]\} = \frac{1}{N}\sum_{m=0}^{N-1} G[m] e^{j\frac{2\pi mn}{N}}$$



Note that the output of $g[n]$ is periodic

$$G[m] = G[m+N]$$

Also note that, on page 445,

$$G(m\Delta_f) = DFT(g(n\Delta_x))\Delta_x$$

The DFT and the IDFT form a transform pair since

$$\frac{1}{N}\sum_{m=0}^{N-1}G[m]e^{j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{m=0}^{N-1}\underbrace{\boxed{\sum_{k=0}^{N-1}g[k]e^{-j\frac{2\pi mk}{N}}}}_{G[m]}e^{j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}g[k]\left(\sum_{m=0}^{N-1}e^{-j\frac{2\pi mk}{N}}e^{j\frac{2\pi mn}{N}}\right)$$

Because $a = n-k$

$a_0 + a_0 r + a_0 r^2 + \cdots a_0 r^{N-1} = \frac{a_0(1-r^N)}{1-r}$

$$\sum_{m=0}^{N-1}e^{j\frac{2\pi a}{N}m} = \frac{1-e^{j\frac{2\pi a}{N}N}}{1-e^{j\frac{2\pi a}{N}}} = \frac{1-e^{j2\pi a}}{1-e^{j\frac{2\pi a}{N}}} = 0 \qquad \text{if } a \neq 0,$$

$$\sum_{m=0}^{N-1}e^{j\frac{2\pi a}{N}m} = N \qquad \text{if } a = 0,$$

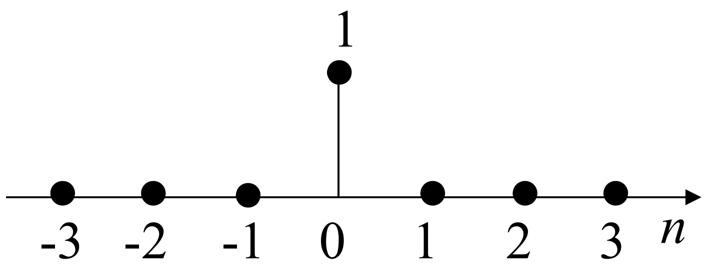
$$\sum_{m=0}^{N-1}e^{j\frac{2\pi m}{N}(n-k)} = N\delta_d[n-k]$$

unit impulse function
(discrete Dirac delta function)

$$\frac{1}{N}\sum_{m=0}^{N-1}G[m]e^{j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}g[k]N\delta_d[n-k] = g[n]$$

Unit Impulse Function (discrete Dirac delta function)

$$\delta_d[n] = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n \neq 0 \end{cases}$$



The unit impulse function has an explicit form. It does not a limitation of a distribution.

Compared to page 347,

$$\sum_n \delta_d[n] = 1$$

$$\delta_d[n] = 0 \quad if\ n \neq 0$$

$$\delta_d[n] = \delta_d[-n]$$

Other possible definitions of the DFT

DFT $$G[m] = \sum_{n=n_0}^{n_0+N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

IDFT $$g[n] = \frac{1}{N} \sum_{m=m_0}^{m_0+N-1} G[m] e^{j\frac{2\pi mn}{N}}$$

DFT $$G[m] = \sqrt{\frac{1}{N}} \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

IDFT $$g[n] = \sqrt{\frac{1}{N}} \sum_{m=0}^{N-1} G[m] e^{j\frac{2\pi mn}{N}}$$

5.2.2 Implementing Continuous FT by the DFT

Suppose that we want to calculate the continuous FT of $g(x)$ digitally and

$$g(x)=0 \qquad \text{for } x \notin [x_1, x_1+T]$$

(i) Shifting

$$g_1(x)=g(x+x_1)$$

$\because$ for DFT $m, n = 0, 1, \cdots N-1$

Note: $g_1(x)=0 \qquad \text{for } x \notin [0, T]$

(ii) Sampling

$$g_d[n]=g_1(n\Delta_x)$$

(iii) DFT

$$G_d[m]=\sum_{n=0}^{N-1} g_d[n]e^{-j\frac{2\pi mn}{N}}$$

(iv) Mapping to the true frequency

$$G_1\left(m\Delta_f\right) = G_d[m]\Delta_x \qquad \text{(from pages 445 and 446)}$$

$f = m\Delta_f = m\frac{f_s}{N}$

Since

$$\Delta_f\Delta_x = 1/N \qquad \Delta_f = \frac{1}{N\Delta_x} = \frac{f_s}{N}$$

when $m \geq \frac{N}{2}$

$f = (m-N)\Delta_f = (m-N)\frac{f_s}{N}$

$= \frac{mf_s}{N} - f_s$

Therefore,

$$G_1\left(m\frac{f_s}{N}\right) = G_d[m]\Delta_x \qquad \text{if } 0 \leq m \leq N/2,$$

$$G_1\left(m\frac{f_s}{N} - f_s\right) = G_d[m]\Delta_x \qquad \text{if } N/2 \leq m \leq N\text{-}1$$

(v) Using the modulation property

$$G(f) = e^{-j2\pi x_1 f}G_1(f)$$ page 453(4)

[Example 1]：Suppose that

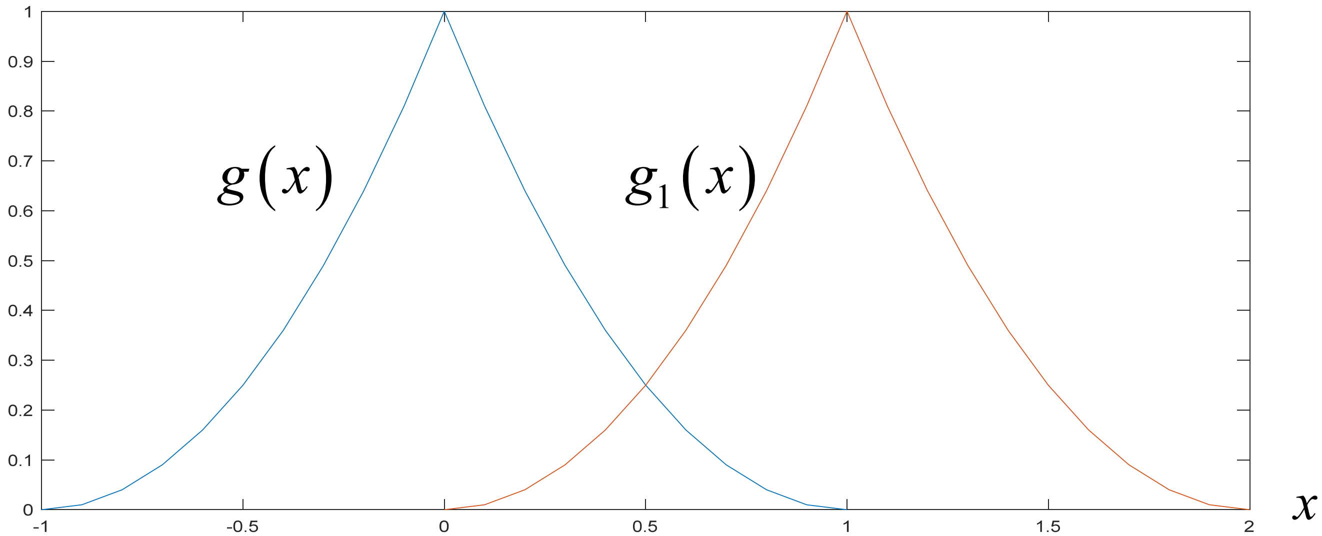
$g(x) = (1-|x|)^2$ for $-1 \le x \le 1$ $g(x) = 0$ otherwise

Sampling interval： $\Delta_x = 0.1$ $f_s = 10$

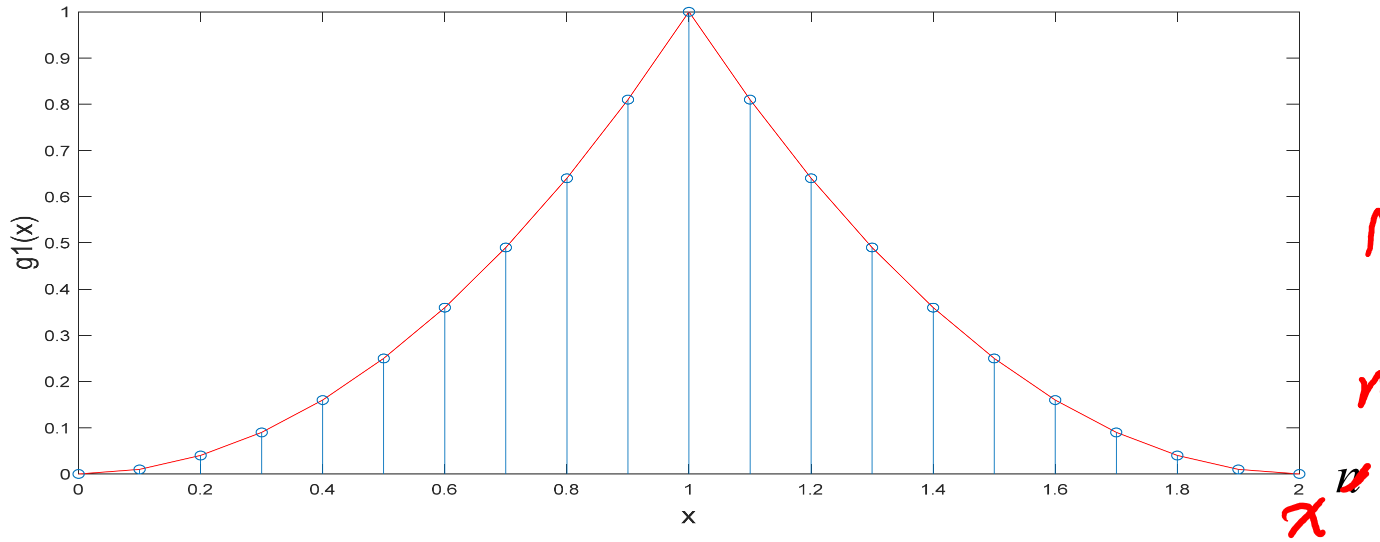
How do we obtain the FT of $g(x)$ by the DFT?

(Solution):

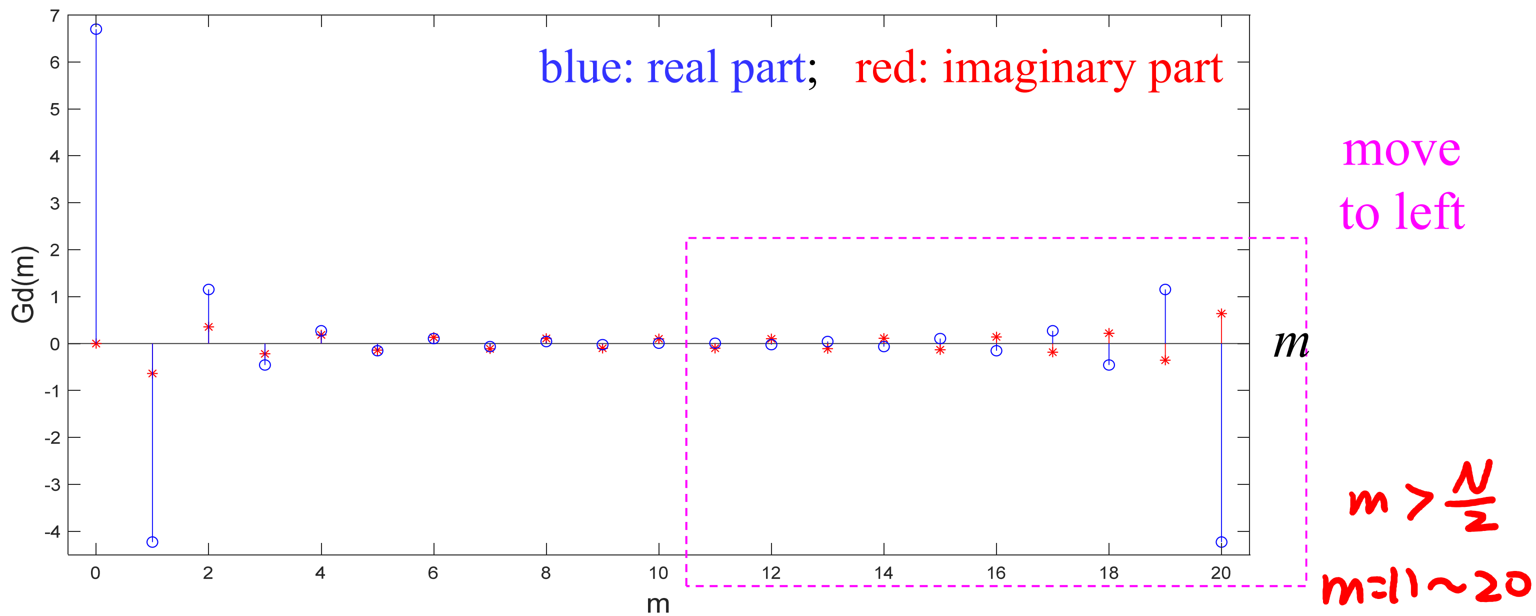
(i) $g_1(x) = g(x-1)$



(ii) $g_d[n] = g_1(n\Delta_x)$

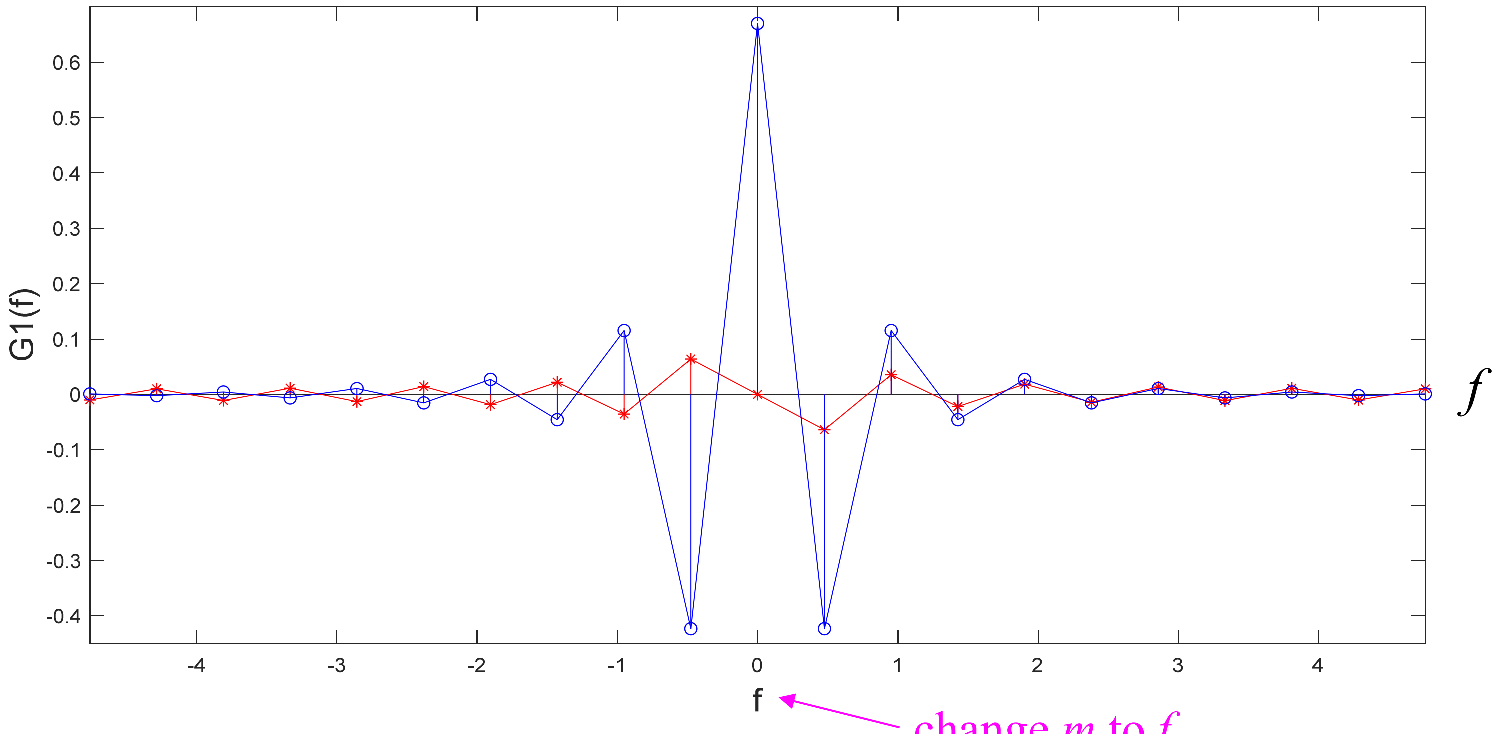


(iii) $G_d[m] = DFT(g_d[n])$



(iv) Mapping

blue: real part; red: imaginary part



$$G_1\left(m\frac{f_s}{N}\right) = G_d[m]\Delta_x \qquad \text{if } 0 \le m \le N/2,$$

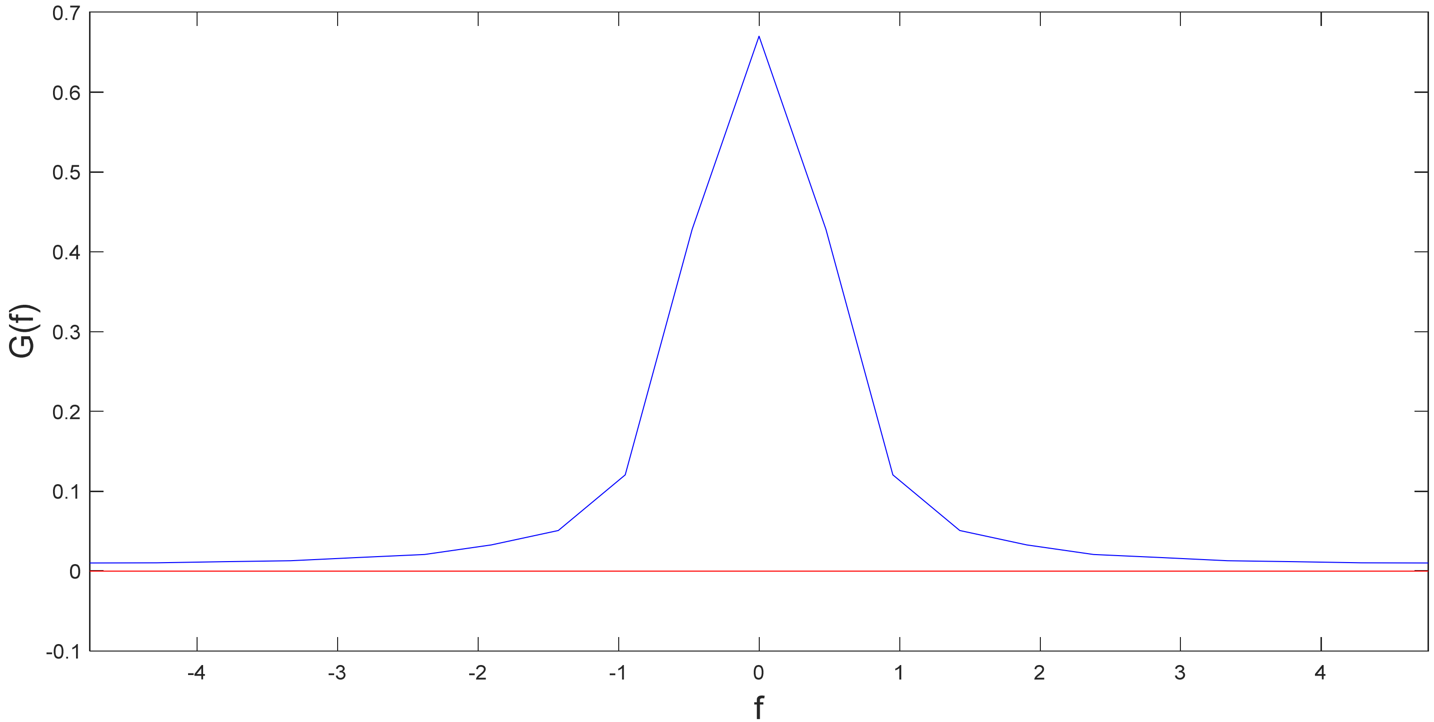
$f_s = 10$, $N = 21$

$\frac{f_s}{N} = 0.4762$

$$G_1\left(m\frac{f_s}{N} - f_s\right) = G_d[m]\Delta_x \qquad \text{if } N/2 \le m \le N\text{-}1$$

(v) $$G(f) = e^{j2\pi f} G_1(f)$$

blue: real part; red: imaginary part



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5.2.3 Transform Pairs and Properties

[Duality Property]

If $G[m] = DFT\{g[n]\}$

then $g[((-m))_N] = \frac{1}{N} DFT\{G[n]\}$ $DFT\{G[n]\} = Ng[((-m))_N]$

$((a))_N$: a modulo N $0 \le ((a))_N \le N-1$

the remainder of a after divided by N

$((12))_{10} = 2$

$((-4))_{10} = 6$

$-4 = 10 \times -1 + 6$

$$g[((-m))_N] = \begin{cases} g[N-m] & \text{if } m = 1, 2, \cdots, N-1 \\ g[0] & \text{if } m = 0 \end{cases}$$

If $G[m] = DFT\{g[n]\}$ then $g[((-m))_N] = \frac{1}{N} DFT\{G[n]\}$

(Proof): $$\frac{1}{N} DFT\{G[n]\} = \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi mn}{N}} G[n]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j\frac{2\pi mn}{N}} \sum_{k=0}^{N-1} e^{-j\frac{2\pi nk}{N}} g[k] = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} e^{-j\frac{2\pi mn}{N}} e^{-j\frac{2\pi nk}{N}} \right) g[k]$$

Since $$\sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}a} = \delta_d[((a))_N]$$

(proved on the next page)

$$\frac{1}{N} DFT\{G[n]\} = \frac{1}{N} \sum_{k=0}^{N-1} N\delta_d[((-m-k))_N] g[k]$$

$$= \sum_{k=0}^{N-1} \delta_d[((m+k))_N] g[k] = g[((-m))_N]$$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = N\delta_d\left[((a))_N\right]$$

$((a))_N$: the remainder of a after divided by N

When $a \neq bN$ where b is some integer

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = \frac{1-e^{j\frac{2\pi a}{N}N}}{1-e^{j\frac{2\pi a}{N}}} = \frac{1-1}{1-e^{j\frac{2\pi a}{N}}} = 0$$

When $a = bN$ where b is some integer (i.e., $((a))_N = 0$)

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = \sum_{n=0}^{N-1} e^{j2\pi bn} = \sum_{n=0}^{N-1} 1 = N$$

[Determine the IDFT by the DFT]

$$g_1[m] = DFT\{G[n]\}$$

$$g[n] = \frac{1}{N} g_1[-n]$$

Note:

(i) Computation loading of the IDFT = Computation loading of the DFT

(ii) In industry, only the chip of the DFT is required.

[Transform Pairs] 比較 page 340

$g[n]$	$G[m]$
(1) $\delta_d[n]$ (see page 448)	1
(2) 1	$N\delta_d[m]$
(3) $\delta_d[n-k]$	$\exp[-j2\pi km/N]$
(4) $\exp[j2\pi kn/N]$	$N\delta_d[m-k]$
(5) $\cos[2\pi kn/N]$	$\frac{N}{2}\delta_d[m-k]+\frac{N}{2}\delta_d[m-(N-k)]$
(6) $\sin[2\pi kn/N]$	$-j\frac{N}{2}\delta_d[m-k]+j\frac{N}{2}\delta_d[m-(N-k)]$
(7) $g[n]=1$ for $0\le n\le W$ $g[n]=0$ otherwise	$e^{-j\frac{\pi W}{N}m}\frac{\sin(\pi m(W+1)/N)}{\sin(\pi m/N)}$ $for\ m\neq 0,$ $W+1$ $for\ m=0$
(8) $\exp[-kn]$, $k\neq 0$	$\frac{1-e^{-Nk}}{1-e^{-k-j2\pi m/N}}$

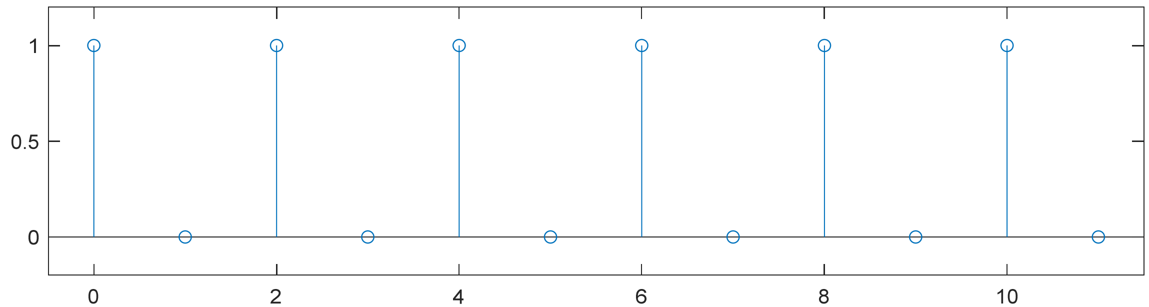
[Discrete Impulse Train]

$$p_c[n] = \begin{cases} 1 & \text{if } n \text{ is a multiple of } c \\ 0 & \text{otherwise} \end{cases}$$

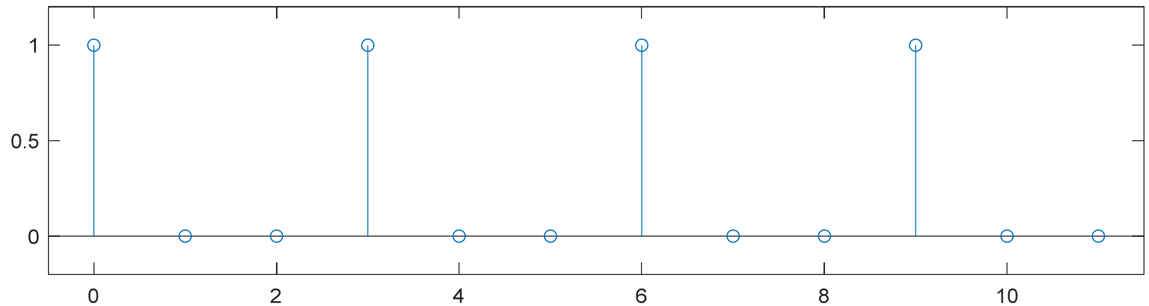
c is a factor of N

$$N = 12$$

$p_2[n]$



$p_3[n]$



[Example 2] Determine the DFT of $p_c[n]$

c is a factor of N

$\underline{n = ck}$

$p_c[n] = 1$ if $n = ck$
$= 0$ otherwise

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi m}{N}n} p_c[n] = \sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N}ck} = \sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k}$$

$$\sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k} = \frac{1-e^{-j\frac{2\pi m}{N/c}\frac{N}{c}}}{1-e^{-j\frac{2\pi m}{N/c}}} = 0$$

if m is not a multiple *of* N/c

$a_0 \frac{1-r^{N/c}}{1-r}$ $a_0 = 1$, $r = e^{-j\frac{2\pi m}{N/c}}$

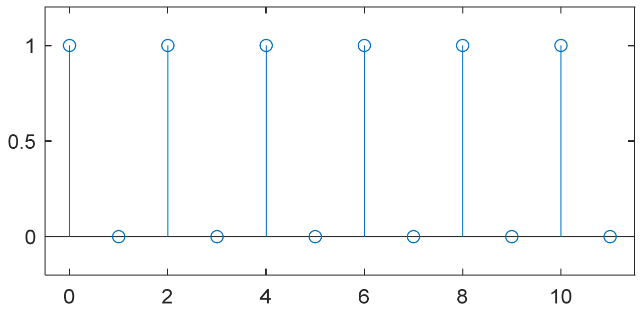
$$\sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k} = \frac{N}{c}$$

if m is a multiple *of* N/c

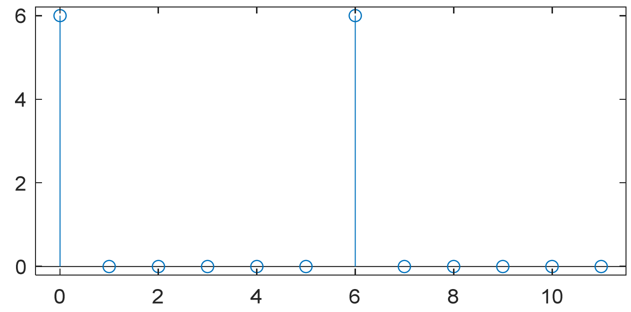
Therefore, $DFT\{p_c[n]\} = \frac{N}{c} p_{N/c}[m]$

$N = 12$

$p_2[n]$

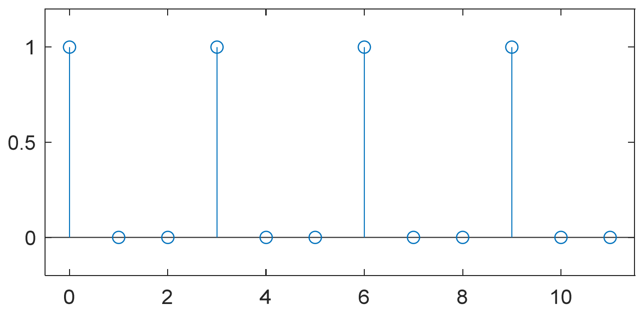


$\xrightarrow{\text{DFT}}$

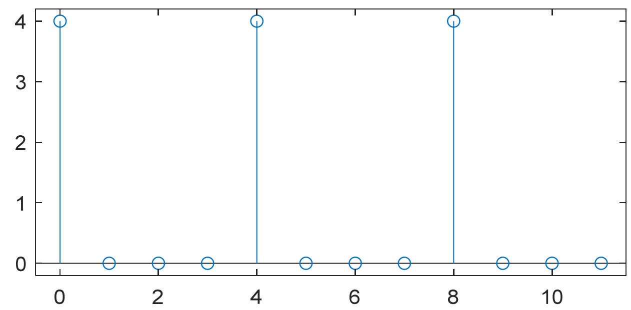


$6p_6[m]$

$p_3[n]$



$\xrightarrow{\text{DFT}}$



$4p_4[m]$

$p_4[n] \xrightarrow{\text{DFT}} 3\,p_3[m]$

Properties 比較 page 355

(1) Linear	$DFT\{ax[n] + by[n]\} = aX[m] + bY[m]$
(2) DC Values	$G[0] = \sum_{n=0}^{N-1} g[n], \quad g[0] = \frac{1}{N} \sum_{n=0}^{N-1} G[m]$
(3) Shifting	$DFT\{g[((n-k))_N]\} = W^{km} G[m]$ where $((n))_N = n$ if $0 \le n \le N$-1 $((n))_N = n+N$ if $-N \le n \le -1$ $((n))_N = n-N$ if $N \le n \le 2N$-1 $W = \exp(-j2\pi / N)$
(4) Modulation	$DFT\{W^{kn} g[n]\} = G[((m+k))_N]$
(5) Time Reverse	$DFT\{g[((-n))_N]\} = G[((-m))_N]$
(6) Even /Odd Input	If $g[n] = g[((-n))_N]$, then $G[m] = G[((-m))_N]$; If $g[n] = -g[((-n))_N]$, then $G[m] = -G[((-m))_N]$;

(7) Conjugate	$DFT\{g^*[n]\} = G^*[((-m))_N]$
(8) Real/Imaginary Input	If $g[n]$ is real, then $G[m] = G^*[((-m))_N]$; If $g[n]$ is pure imaginary, then $G[m] = -G^*[((-m))_N]$;
(9) Circular Convolution ★	If $y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$ Note! then $Y[m] = G[m]H[m]$
(10) Circular Correlation	If $y[n] = g[n] *_c h^*[((-n))_N]$ $= \sum_{k=0}^{N-1} g[((k+n))_N]h^*[k]$ then $Y[m] = G[m]H^*[m]$
(11) Parseval's Theorem (Energy Preservation)	$N\sum_{n=0}^{N-1}\lvert g[n]\rvert^2 = \sum_{m=0}^{N-1}\lvert G[m]\rvert^2$
(12) Generalized Parseval's Theorem	$N\sum_{n=0}^{N-1} g[n]h^*[n] = \sum_{m=0}^{N-1} G[m]H^*[m]$

5.2.4 Discrete Circular Convolution

[Discrete Circular Convolution]

$$g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k] h[((n-k))_N]$$

(Proof of the convolution property)

$$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[m]e^{j\frac{2\pi m}{N}n} = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} g[k]e^{-j\frac{2\pi m}{N}k}\sum_{s=0}^{N-1} h[s]e^{-j\frac{2\pi m}{N}s}e^{j\frac{2\pi m}{N}n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{s=0}^{N-1} g[k]h[s]\sum_{m=0}^{N-1} e^{j\frac{2\pi(n-s-k)}{N}m}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{s=0}^{N-1} g[k]h[s]N\delta_d[((n-s-k))_N]$$

$$= \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$$

Here we apply $\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = N\delta_d[((a))_N]$

$((a))_N$: the remainder of a after divided by N

[Discrete Circular Convolution and Discrete Linear Convolution]

A discrete linear time-invariant (LTI) system can always be expressed a discrete linear convolution:

$$y_1[n] = g[n] * h[n] = \sum_{k=0}^{N-1} g[k]h[n-k]$$

However, the convolution implemented by the DFT is the discrete circular convolution:

If

$$y[n] = IDFT\left(DFT\{g[n]\}\, DFT\{h[n]\}\right) = IDFT\left(G[m]H[m]\right)$$

then

$$y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$$

$((a))_N$: the remainder of a after divided by N

linear convolution: $y_1[n] = g[n] * h[n] = \sum_{k=0}^{N-1} g[k]h[n-k]$

circular convolution: $y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$

For example,

k=3, n=2, n-k=-1

$$y_1[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[-1] + g[4]h[-2] + \cdots\cdots$$

$$y[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + \underline{g[3]h[N-1]} + \underline{g[4]h[N-2]}$$
$$+ \cdots\cdots + \underline{g[N-1]h[3]}$$

k=N-1, n=2, n-k=3-N, $((n-k))_N=3$

The condition where the circular convolution is equal to the linear convolution:

0~M-1 length(g[n])=M

(i) $g[n] = 0$ *for* $n < 0$ *or* $n \geq M$

(ii) $h[n] = 0$ *for* $n < 0$ *or* $n \geq L$ length(h[n])=L

(iii) $\boxed{N \geq M + L - 1}$

The condition where the circular convolution is equal to the linear convolution:

(i) $g[n]=0 \quad for\ n<0 \ \ or \ \ n\geq M$

(ii) $h[n]=0 \quad for\ n<0 \ \ or \ \ n\geq L$

(iii) $N\geq M+L-1$

(Proof): $y[n]=\sum_{k=0}^{N-1}g[k]h[((n-k))_N]$

try to make these terms zero

$$y[n]=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]+g[n+1]h[N-1]+$$
$$g[n+2]h[N-2]+\cdots+g[N+n+1-L]h[L-1]+\cdots+g[N-1]h[n+1]$$
$$=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]$$
$$+g[N+n+1-L]h[L-1]+\cdots+g[N-1]h[n+1]$$
$$=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]=y_1[n]$$

since $h[n]=0$ for $n\geq L$
if $k=N+n+1-L$
$n-k=L-1+N$
$((n-k))_N=L-1$

(Since N+n+1-L ≥ N+1-L ≥ M)

$N+n+1-L\geq M$ should be satisfied
$N\geq M+L-1-n \quad \because n\geq 0$
$N\geq M+L-1$

5.2.5 Complexity

$$G[m]=\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

Direct implementation: Complexity = $O(N^2)$

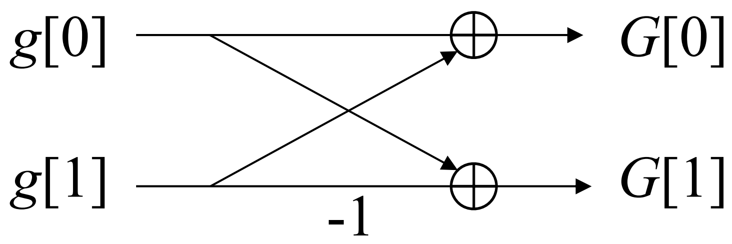
With the fast algorithm: Complexity = $O(N\log_2 N)$

$$G[m]=\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$ $\leftarrow \Theta(N^2) \Rightarrow \Theta(N\log_2 N)$

2-point DFT

$$G[m]=\sum_{n=0}^{1} g[n]\,e^{-j\pi mn}=\sum_{n=0}^{1} g[n]\,(-1)^{mn}$$

$$\begin{bmatrix} G[0] \\ G[1] \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} g[0] \\ g[1] \end{bmatrix}$$



When $N = 2^k$

$$G[m] = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

$$= \sum_{n=0}^{N/2-1} g[2n] e^{-j\frac{2\pi m(2n)}{N}} + \sum_{n=0}^{N/2-1} g[2n+1] e^{-j\frac{2\pi m(2n+1)}{N}}$$

$$= \sum_{n=0}^{N/2-1} g_1[n] e^{-j\frac{2\pi mn}{N/2}} + e^{-j\frac{2\pi m}{N}} \sum_{n=0}^{N/2-1} g_2[n] e^{-j\frac{2\pi mn}{N/2}}$$

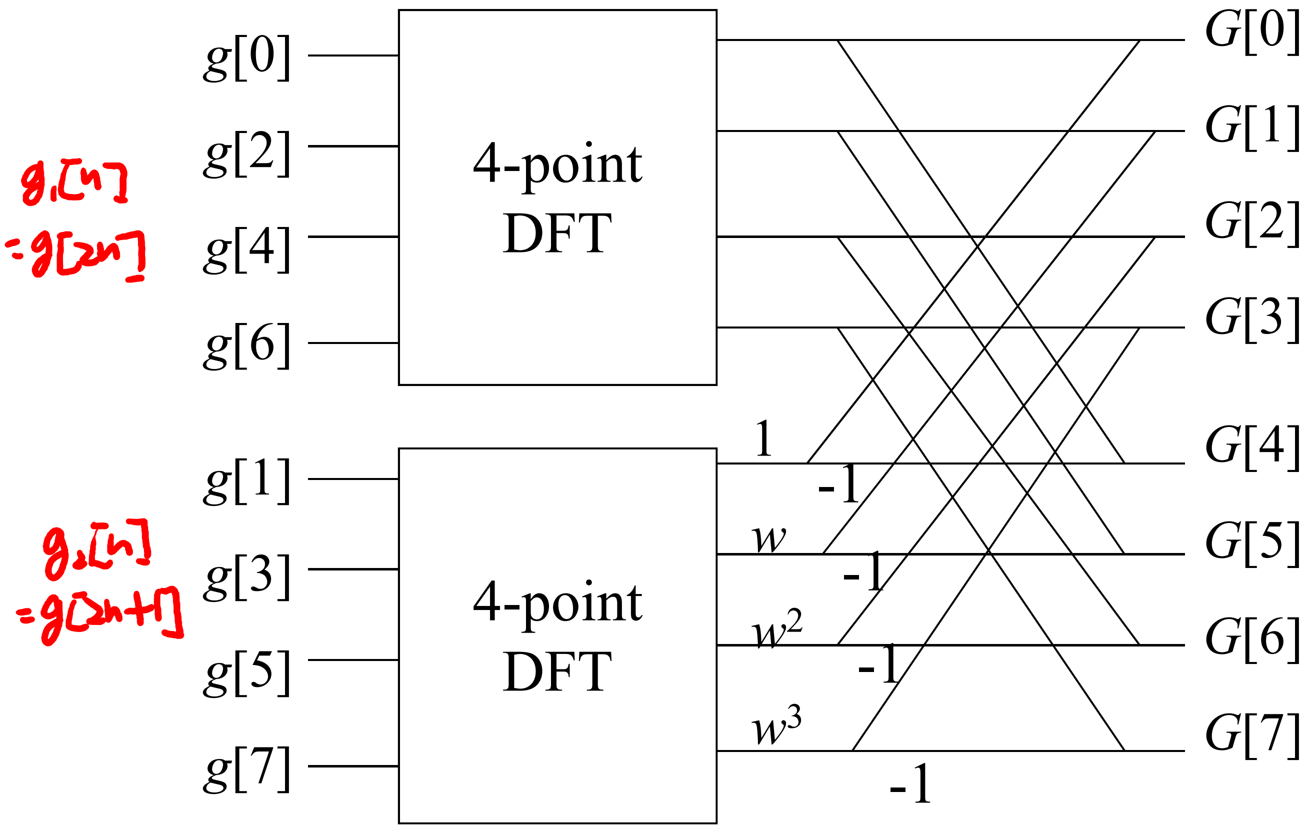
twiddle factors

$$g_1[n] = g[2n], \quad g_2[n] = g[2n+1]$$

Therefore,

one N-point DFT = two $(N/2)$-point DFT + twiddle factors

8-point DFT



When $N = 8$

$$w = e^{-j\frac{2\pi}{N}}$$

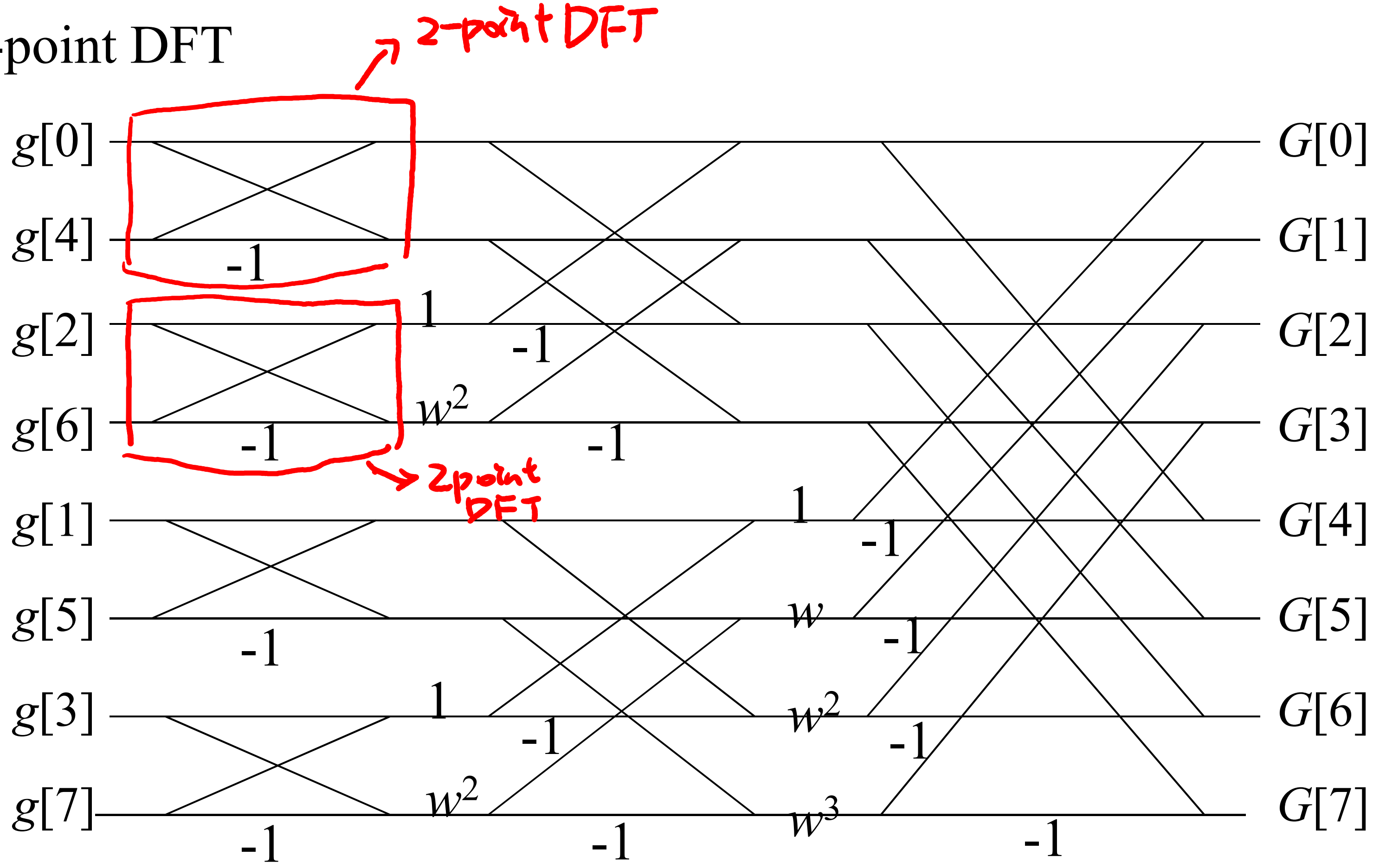
$$w = e^{-j\frac{2\pi}{8}}$$

$$w^4 = -1$$

$$w^8 = 1$$

$$w^{(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}m} e^{-j\frac{2\pi}{N}\frac{N}{2}} = -e^{-j\frac{2\pi}{N}m} = -w^m$$

8-point DFT



$$w = e^{-j\frac{2\pi}{8}}$$

number of twiddle factors

$\frac{N}{2}(\log_2 N - 1)$

$\cong \frac{N}{2}\log_2 N$

$O(N\log_2 N)$

complexity: $N\log_2 N$

$N/2$ twiddle factors are required between two stages

Number of stages

- J. W. Cooley and J. W. Tukey, "An algorithm for the machine computation of complex Fourier series," *Mathematics of Computation*, vol. 19, pp. 297-301, Apr. 1965. (Cooley-Tukey)
- C. S. Burrus, "Index Mappings for multidimensional formulation of the DFT and convolution," *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol. 25, pp. 1239-242, June 1977. (Prime factor)

5.2.6 2D DFTs

2-D Discrete Fourier Transform (2-D DFT)

$$G[p,q]=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}g[m,n]e^{-j\frac{2\pi mp}{M}}e^{-j\frac{2\pi nq}{N}}$$

$G[p,q] = G[p+M,q]$
$= G[p,q+N] = G[p+M,q+N]$

2-D Inverse Discrete Fourier Transform (2-D IDFT)

$$g[m,n]=\frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}G[p,q]e^{j\frac{2\pi mp}{M}}e^{j\frac{2\pi nq}{N}}$$

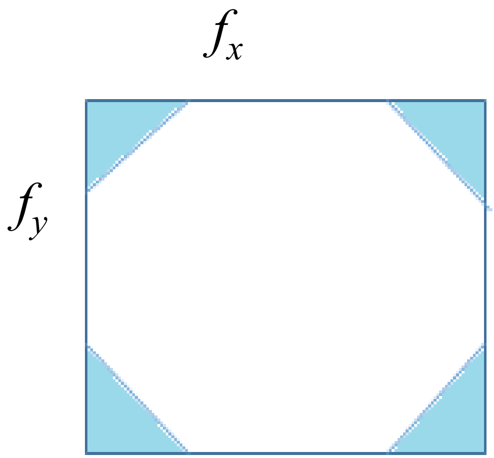
Low Frequency Part $\rightarrow$ Mild Variation $\rightarrow$ Plane

smooth

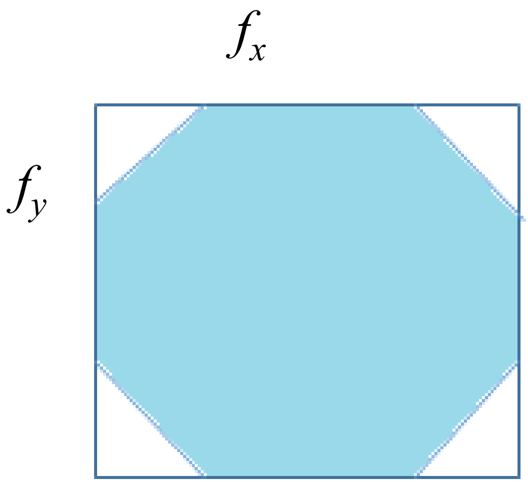
High Frequency Part $\rightarrow$ Large Variation $\rightarrow$ Edge and Noise

large gradient

$G[m,n]$ low frequency part



high frequency part

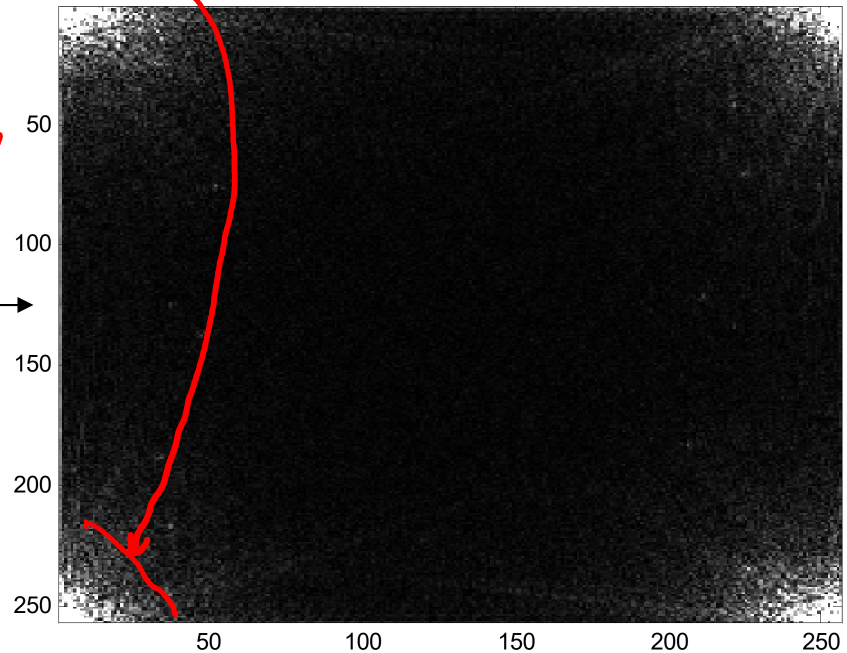


$G[p+M,q] : G[p,q]$

$\xrightarrow[\text{DFT}]{\text{2D}}$

q

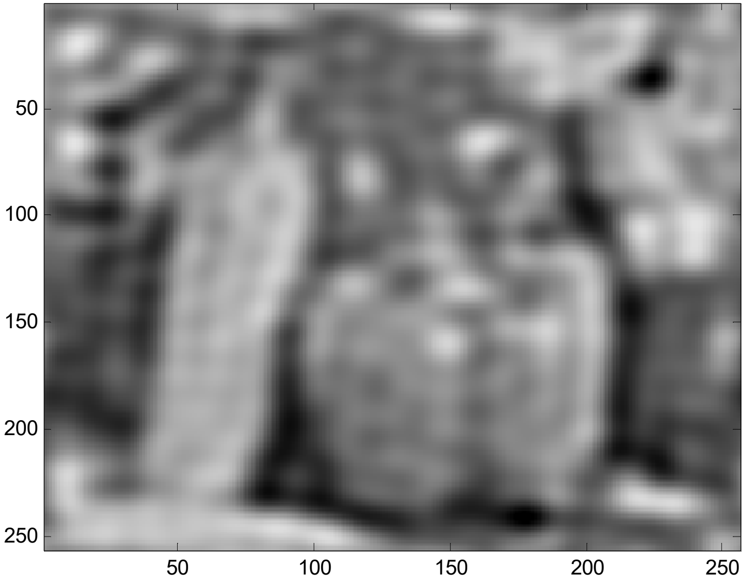
p



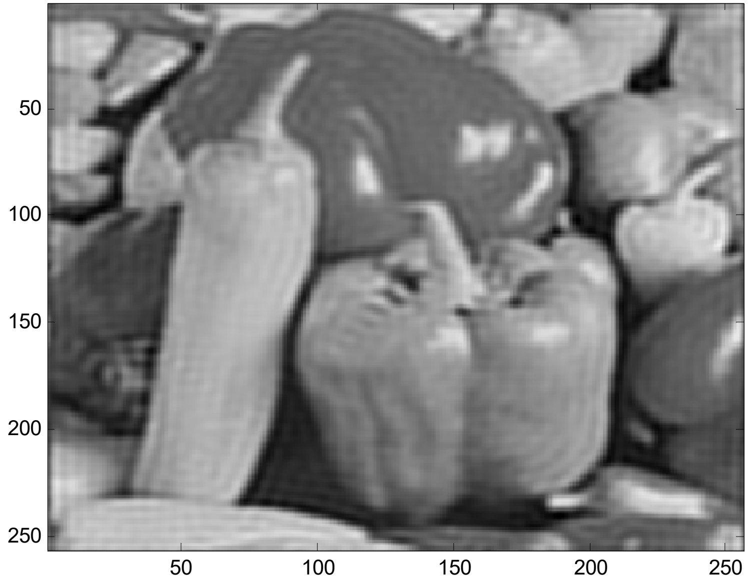
Using the gray level to show the intensity

image(…….)
colormap(gray(256))

Low Frequency Part (similar to the blurred version of the input image)

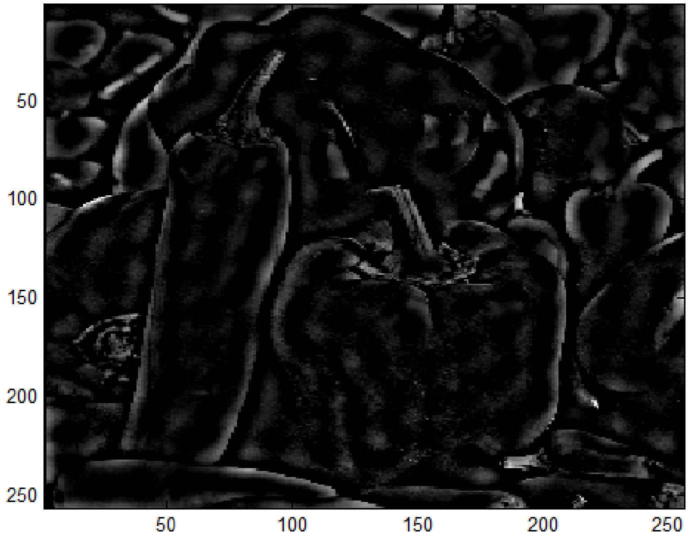


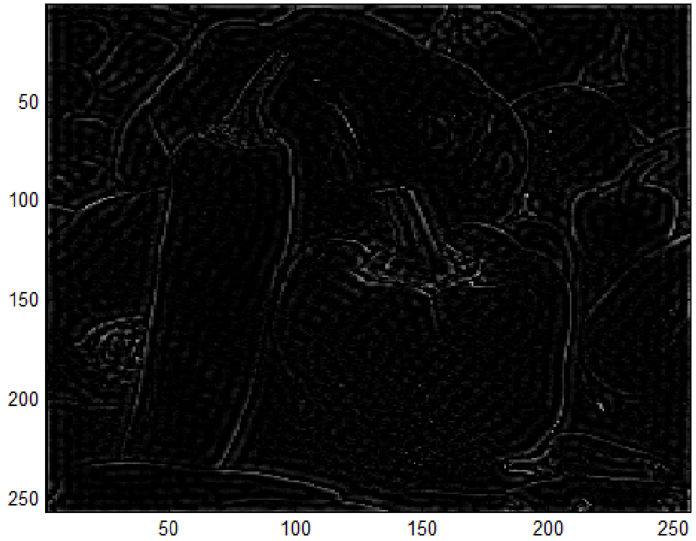
Passband: $|f_x| + |f_y| \leq N/30$



Passband: $|f_x| + |f_y| \leq N/10$

High Frequency Part (similar to the edges)





Passband: $|f_x| + |f_y| > N/30$

Passband: $|f_x| + |f_y| > N/10$