Time Frequency Analysis and Wavelet Transforms

時頻分析與小波轉換

授課者： 丁 建 均

Office：明達館723室， TEL： 33669652

E-mail: jjding@ntu.edu.tw

課程網頁：http://djj.ee.ntu.edu.tw/TFW.htm

歡迎大家來修課，也歡迎有問題時隨時聯絡！

- **評分方式：**

平時分數: 15 scores

基本分12分，各位同學皆可拿到

另外，每週都將會在課堂上問一個問題，請特定學號尾數的同學，在作業上回答 (每位同學每學期會輪到四次左右)，每回答一次 (無論答對否)，只要不離題，都加 0.8分

將來再視疫情決定是否要在上課時問答

Homework: 60 scores

5 times, 每 3 週一次

請自己寫，和同學內容相同 ，將扣 60% 的分數，就算寫錯但好好寫也會給 40~95% 的分數，遲交分數打 8 折，不交不給分。

不知道如何寫，可用 E-mail 和我聯絡，或於下課時和老師討論

Term paper 25 scores

Term paper 25 scores

方式有四種，可任選其中一種

(1) 書面報告

(10頁以上(不含封面)，中英文皆可，11或12的字體，題目可選擇和課程有關的任何一個主題。

格式不限，但儘量和一般寫期刊論文或碩博士論文相同，包括 abstract, conclusion, 及 references，並且要分 sections，必要時有subsections。鼓勵多做實驗及模擬。

嚴禁剪刀漿糊 (Ctrl-C , Ctrl-V) 的情形，否則扣 60% 的分數

(2) Tutorial

限十二個名額，和書面報告格式相同，但 17頁以上(若為加強前人的 tutorial，則頁數為 $(2/3)N + 12$ 以上，N 為前人 tutorial 之頁數)，題目由老師指定，以**清楚且有系統**的介紹一個主題的基本概念和應用為要求，為上課內容的進一步探討和補充，**交Word 檔**。

選擇這個項目的同學，學期成績加 3分

(3) 口頭報告

限四個名額，每個人 40分鐘，題目可選擇和課程有關的任何一個主題。口頭報告將於最後一週 (1月13日)進行。有意願的同學，請儘早告知，以先登記的同學為優先。

口頭報告時，鼓勵大家提問（包括口頭報告的同學，也可針對其他同學的報告內容提問）。曾經提問的同學，期末報告皆加 2 分。

選擇這個項目的同學，學期成績加 2分

(4) 編輯 Wikipedia

中文或英文網頁皆可，至少 2 個條目，但不可同一個條目翻成中文和英文。總計80行以上。限和課程相關者，自由發揮，越有條理、有系統的越好

選擇編輯 Wikipedia 的同學，請於明年1月13日(本學期最後一次上課)前，向我登記並告知我要編緝的條目(2 個以上)，若有和其他同學選擇相同條目的情形，則較晚向我登記的同學將更換要編緝的條目

編輯完成之後，要將連結寄給老師

書面報告、Tutorial、和編輯 Wikipedia，期限是 1月20日

以上若有做實驗模擬，請附上程式碼，會有額外的加分 (鼓勵不強制)

上課方式

(1) 錄影 ，影片將藉由 NTU Cool 下載　http://cool.ntu.edu.tw

(2) 現場 (明達館231室)

作業和報告繳交方式

用 NTU Cool 來繳交作業與報告的電子檔　http://cool.ntu.edu.tw

注意，Tutorial 一定要交Word 或 Latex 原始碼

Wiki 要寄編輯條目的連結給老師

上課時間：15 週

9/23,

9/30,

10/7, 出 HW1

10/14,

10/21, 交 HW1

10/28, 出 HW2

11/4,

11/11, 交 HW2

11/18, 出 HW3

11/25,

12/2, 交 HW3

12/9, 出 HW4

12/16,

12/23, 交 HW4,

12/30, 出 HW5

1/13, Oral

1/20, 交 HW5 及 term paper

課程大綱：

(1) Introduction

(2) Short-Time Fourier Transform

(3) Gabor Transform

(4) Implementation of Time-Frequency Analysis

(5) Wigner Distribution Function

(6) Cohen's Class Time-Frequency Distribution

(7) S Transforms, Gabor-Wigner Transforms, Matching Pursuit, and Other Time Frequency Analysis Methods

(8) Movement in the Time-Frequency Plane and Fractional Fourier Transforms

(9) Filter Design by Time-Frequency Analysis

(10) Modulation, Multiplexing, Sampling, and Other Applications

(續)

課程大綱：

(11) Hilbert Huang Transform

(12) From Haar Transforms to Wavelet Transforms

(13) Continuous Wavelet Transforms

(14) Continuous Wavelet Transforms with Discrete Coefficients

(15) Discrete Wavelet Transform

(16) Applications of the Wavelet Transform

- 上課資料：

(1) 講義 (將放在網頁上，請大家每次上課前先印好)

(2) S. Mallat, *A Wavelet Tour of Signal Processing: The Sparse Way*, Academic Press, 3rd ed., 2009.

(3) S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

(4) P. Flandrin, *Time-frequency / Time Scale Analysis*, translated by J. Stöckler, Academic Press, San Diego, 1999.

(5) K. Grochenig, *Foundations of Time-Frequency Analysis*, Birkhauser, Boston, 2001.

(6) L. Debnath, *Wavelet Transforms and Time-Frequency Signal Analysis*, Birkhäuser, Boston, 2001.

(7) F. Hlawatsch and F. Auger, *Time-frequency Analysis Concepts and Methods*, Wiley, London, 2008.

Matlab Program

Download: 請洽台大各系所

參考書目

洪維恩，Matlab 程式設計，旗標，台北市，2013 . (合適的入門書)

張智星，Matlab 程式設計入門篇，第四版，碁峰，2016.

預計看書學習所花時間： 3~5 天

Python Program

Download: https://www.python.org/

參考書目

葉難， Python程式設計入門，博碩，2015

黃健庭， Python程式設計：從入門到進階應用 ，全華，2020

The Python Tutorial   https://docs.python.org/3/tutorial/index.html

Tutorial 可供選擇的題目(可以略做修改)

(1) Cochleagram (for Acoustic Feature Extraction)

(2) Seismic Wave Analysis Using Time-frequency Analysis

(3) Time-Frequency Analysis for Tremor Analysis in Parkinson's Disease

(4) Time-frequency Analysis of Musical Instruments

(5) Learning-Based Speech Analysis in the Time-Frequency Domain

(6) Learning-Based Bearing Fault Diagnosis in the Time-Frequency Domain

(7) Computational Perception Auditory Structure

(8) Dual Tree Complex Wavelet Transforms

(9) Wavelet Pooling in Convolutional Neural Networks

(10) Contourlet Convolutional Neural Networks

(11) Fresnelet

(12) Prediction Models Using Wavelet Transforms

I. Introduction

Fourier transform (FT)

$X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$ Time-Domain $\rightarrow$ Frequency Domain

↑ *t* varies from $-\infty\sim\infty$

Laplace Transform $X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$

Cosine Transform, Sine Transform, Z Transform.

Some things make the FT not practical:

(1) It less happens that a signal has the interval of $(-\infty\sim\infty)$

Even for a signal with infinite length, we are only interested in the characteristics in a finite interval.

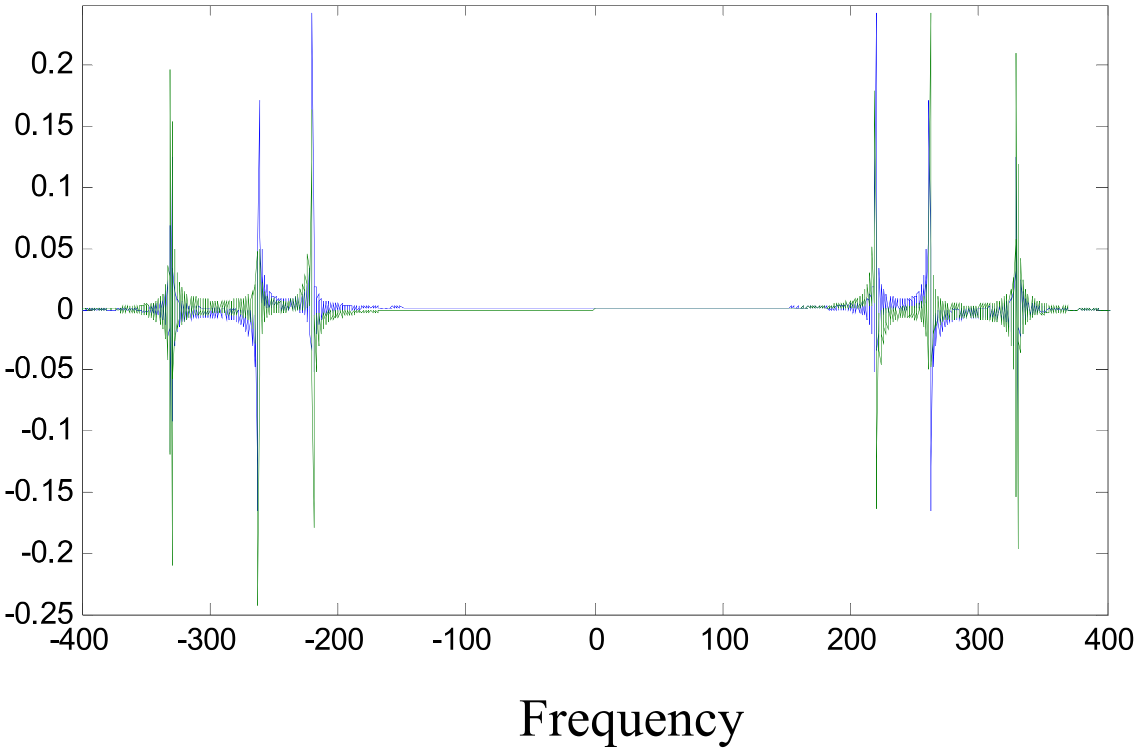
(2) It is hard to observe the variation of spectrum with time by the FT.

Example 1: $x(t) = \cos(440\pi t)$ when $t < 0.5$,

$x(t) = \cos(660\pi t)$ when $0.5 \leq t < 1$,

$x(t) = \cos(524\pi t)$ when $t \geq 1$

The Fourier transform of $x(t)$



(A) Finite-Supporting Fourier Transform

$$X(f)=\int_{t_0-B}^{t_0+B} x(t)e^{-j2\pi f t}dt$$

(B) Short-Time Fourier Transform (STFT)

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f \tau}d\tau$$

$w(t)$: window function 或 mask function

STFT 也稱作 windowed Fourier transform 或

time-dependent Fourier transform

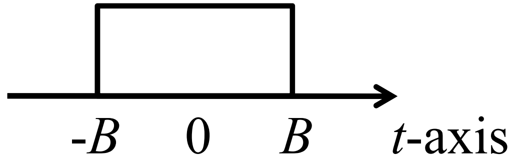
[Ref] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, New York, 1995.

[Ref] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

最簡單的例子： $w(t) = 1$ for $|t| \le B$,

$w(t) = 0$ otherwise

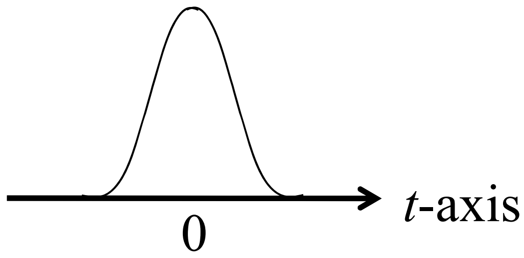


此時 Short-time Fourier transform 可以改寫

$$X(t,f)=\int_{t-B}^{t+B} x(\tau)e^{-j2\pi f\tau}d\tau$$

其他的例子：

$$w(t)=\exp\left(-\sigma t^2\right)$$



一般我們把 $\exp(-\sigma t^2)$ 稱作為 Gaussian function 或 Gabor function

此時的 Short-Time Fourier Transform 亦稱作 Gabor Transform

(C) Gabor Transform

$$G_x(t,f)=\int_{-\infty}^{\infty} e^{-\pi\sigma(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau$$

$$G_x(t,\omega)=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\sigma(\tau-t)^2}{2}} e^{-j\omega\tau} x(\tau)\,d\tau$$

• S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice Hall, N.J., 1996.

Without cross term, poor clarity

(D) Wigner Distribution Function

$$G_x(t,f)=\int_{-\infty}^{\infty} e^{-j2\pi f\tau} x(t+\tau/2)\,x^*(t-\tau/2)\,d\tau$$

$$G_x(t,\omega)=\int_{-\infty}^{\infty} e^{-j\omega\tau} x(t+\tau/2)\,x^*(t-\tau/2)\,d\tau$$

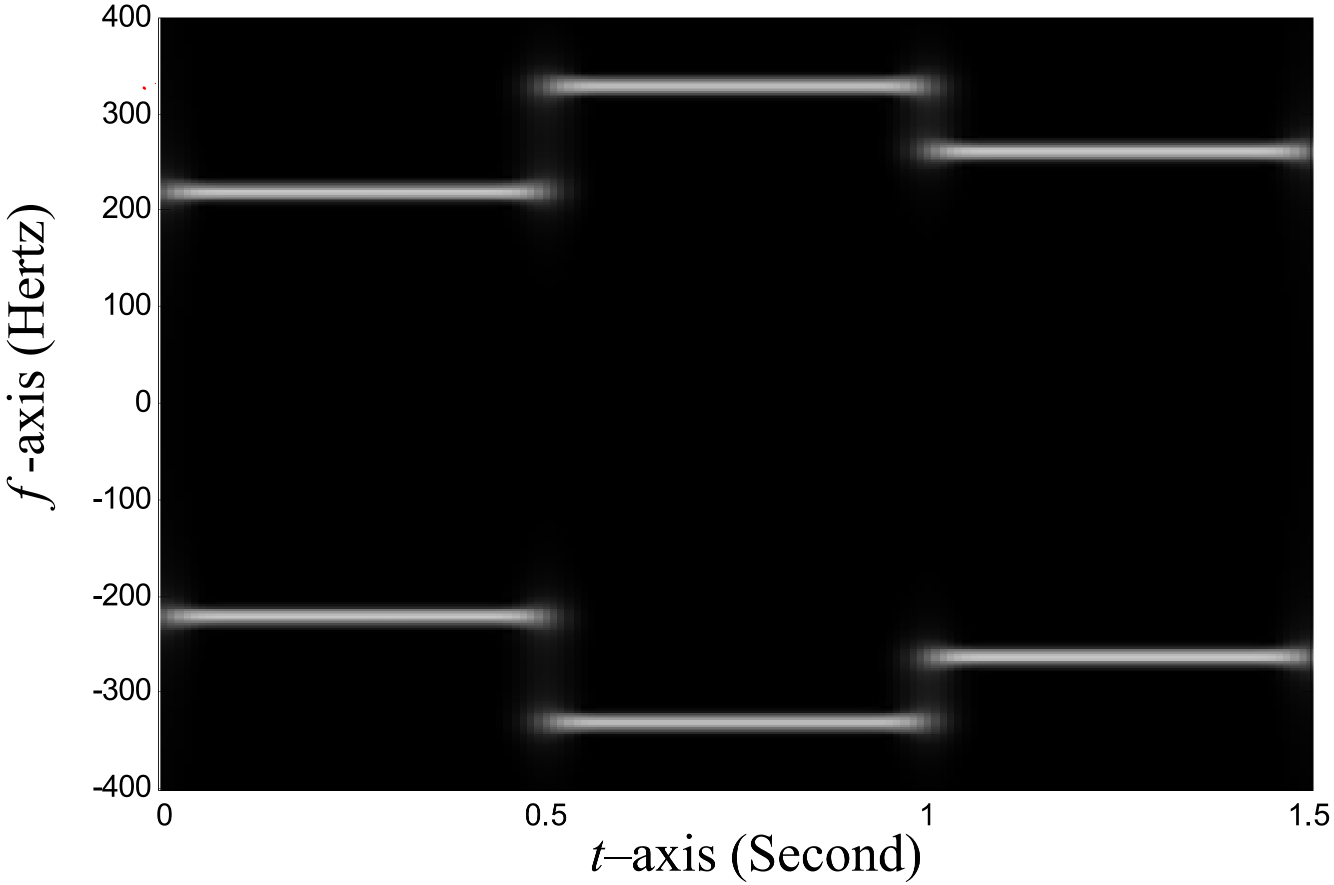
With cross term, high clarity

Example: $x(t) = \cos(440\pi t)$ when $t < 0.5$,

$x(t) = \cos(660\pi t)$ when $0.5 \le t < 1$,

$x(t) = \cos(524\pi t)$ when $t \ge 1$

The Gabor transform of $x(t)$ ($\sigma = 200$)



用 Gray level 來表示 $X(t, f)$ 的 amplitude

Instantaneous Frequency 瞬時頻率

If $x(t) = \sum_{k=1}^{N} a_k \cdot \exp(j \cdot \phi_k(t))$ around t_0

then the instantaneous frequency of $x(t)$ at t_0 are

$$\frac{\phi_1'(t_0)}{2\pi}, \frac{\phi_2'(t_0)}{2\pi}, \frac{\phi_3'(t_0)}{2\pi}, \cdots, \frac{\phi_N'(t_0)}{2\pi}$$ (以頻率 frequency 表示)

$$\phi_1'(t_0), \phi_2'(t_0), \phi_3'(t_0), \cdots, \phi_N'(t_0)$$ (以角頻率 angular frequency 表示)：

If the order of $\phi_k(t)$ > 1, then instantaneous frequency varies with time

自然界中，頻率會隨著時間而改變的例子

Frequency Modulation

Music

Speech

Others (Animal voice, Doppler effect, seismic waves, radar system, optics, rectangular function)

In fact, in addition to **sinusoid-like functions**, the instantaneous frequencies of other functions will inevitably vary with time.

- Sinusoid Function

- Chirp function

$$\exp\left[\, j(\alpha_2 t^2 + \alpha_1 t + \alpha_0)\right] \qquad \text{Instantaneous frequency} = \frac{\alpha_2}{\pi} t + \frac{\alpha_1}{2\pi}$$

acoustics, wireless communication, radar system, optics

例： ㄚ (F_1 = 900Hz, F_2 = 1200Hz) , 一 (F_1 = 300Hz, F_2 = 2300Hz)

F_1 由嘴唇的大小決定, F_2 - F_1 由如面的高低決定

- Higher order exponential function

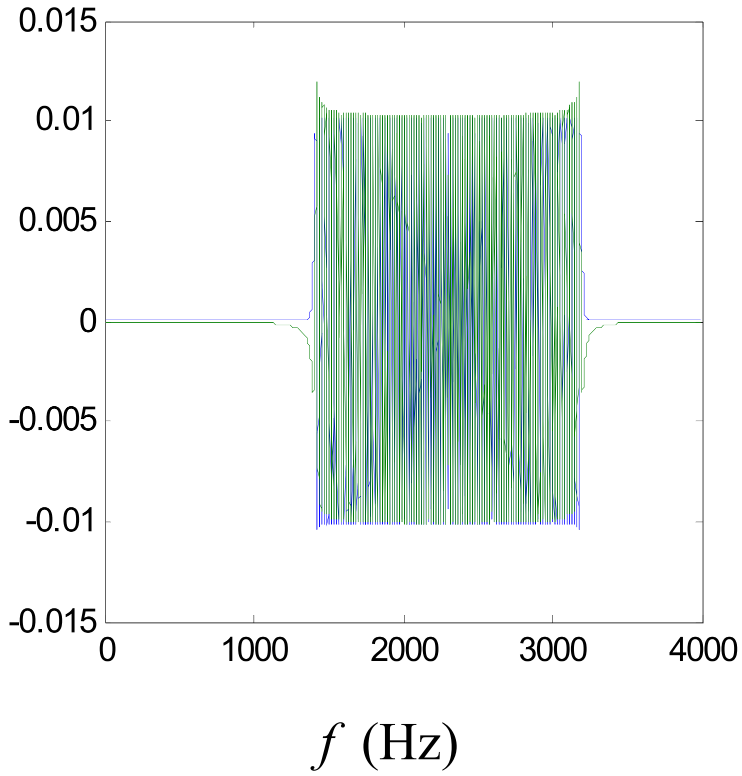
Example 2

(1) $$x(t) = 0.5\cos\left(6400\pi t - 600\pi t^2\right) \qquad t \in [0, 3]$$

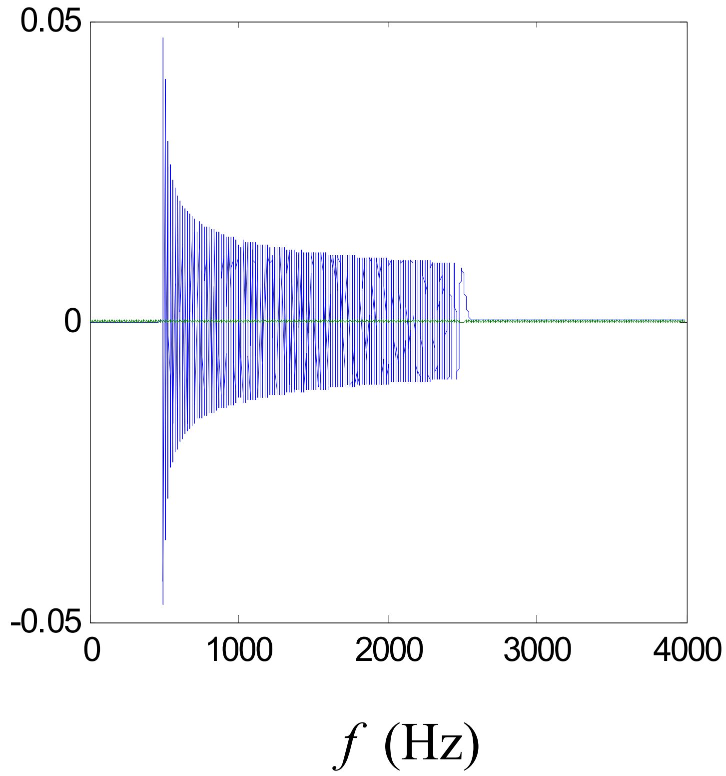
(2) $$x(t) = 0.5\cos\left(600\pi t^3 - 2700\pi t^2 + 5050\pi t\right) \quad t \in [0, 3]$$

Fourier transform

$$x(t) = 0.5\cos\left(6400\pi t - 600\pi t^2\right)$$

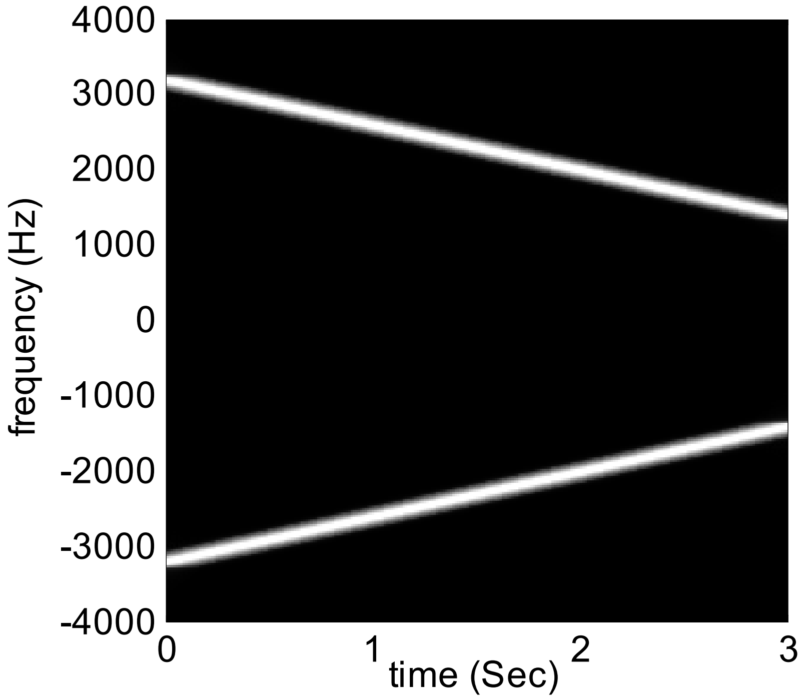


$$x(t) = 0.5\cos\left(600\pi t^3 - 2700\pi t^2 + 5050\pi t\right)$$



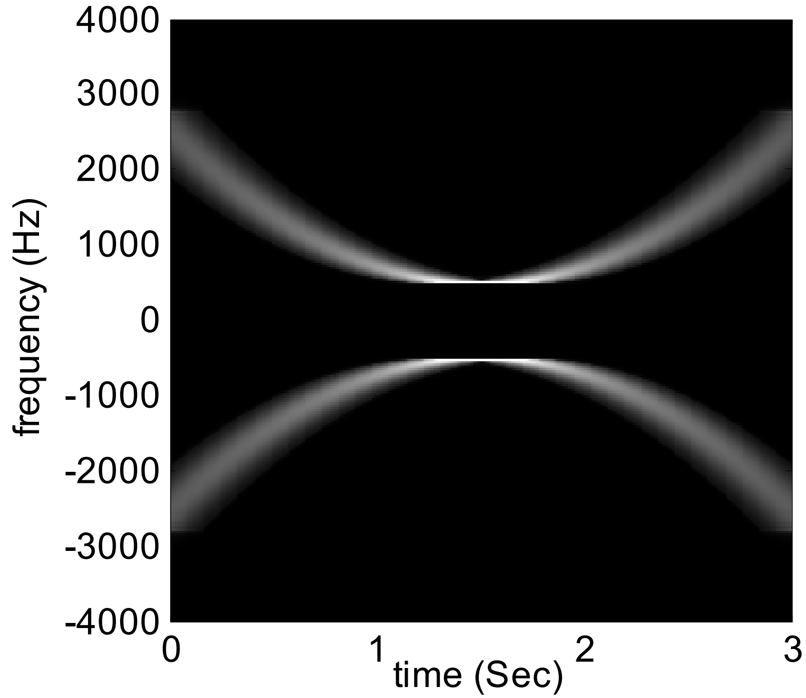
(1)

$$x(t) = 0.5\cos\left(6400\pi t - 600\pi t^2\right)$$



(2)

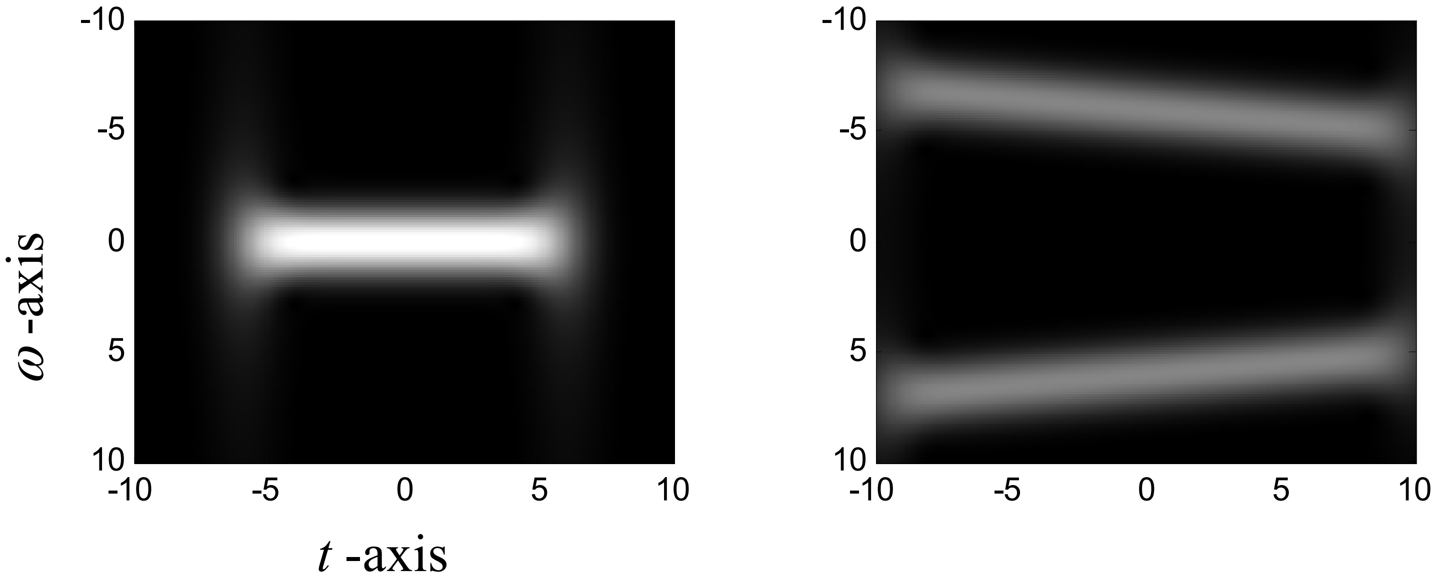
$$x(t) = 0.5\cos\left(600\pi t^3 - 2700\pi t^2 + 5050\pi t\right)$$



Example 3

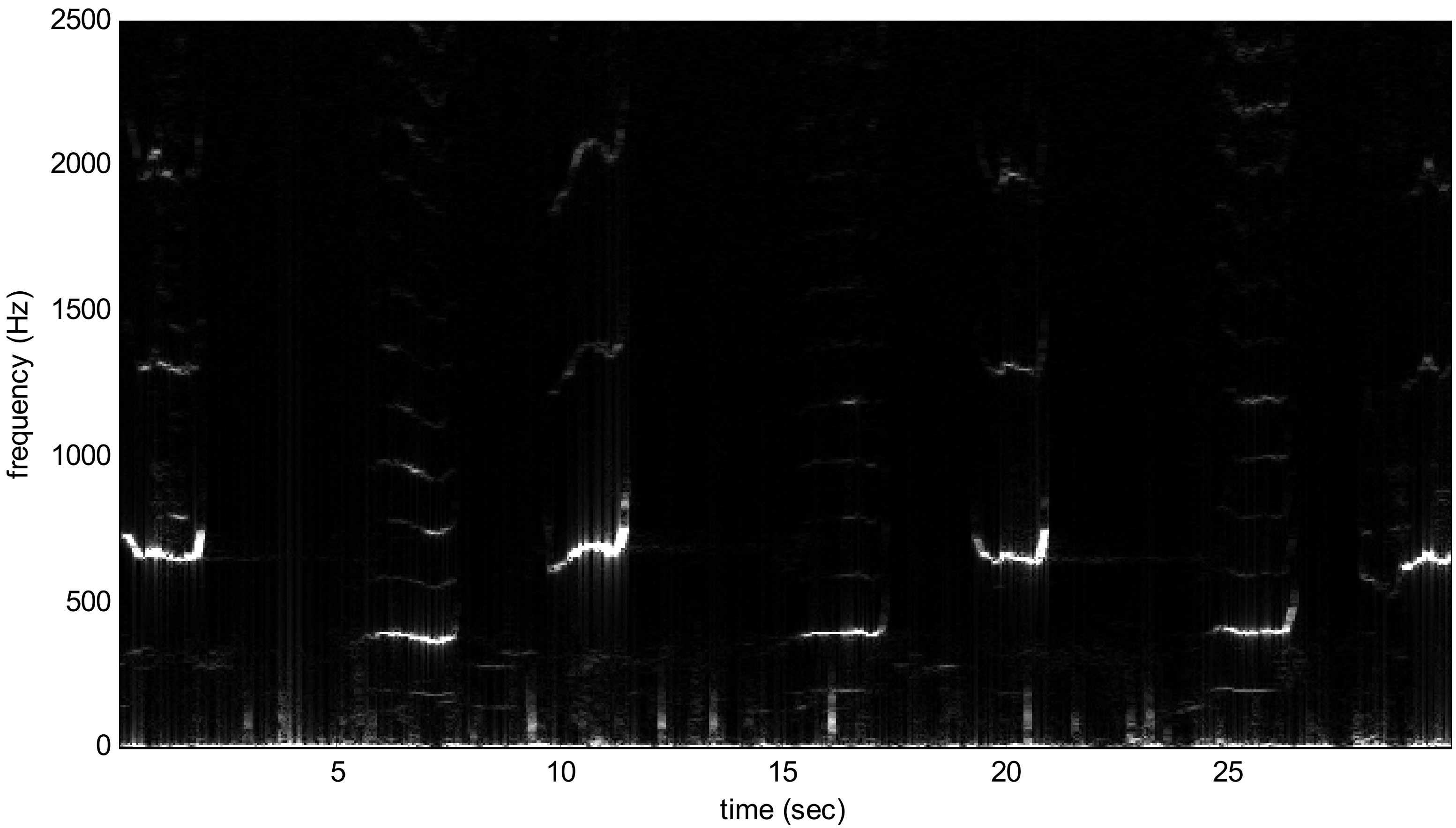
left: $x_1(t) = 1$ for $|t| \leq 6$, $x_1(t) = 0$ otherwise, right: $x_2(t) = \cos(6t - 0.05t^2)$

Gabor transform



Example 4

Data source: http://oalib.hlsresearch.com/Whales/index.html



Why Time-Frequency Analysis is Important?

- Many digital signal processing applications are related to the spectrum or the bandwidth of a signal.

- If the spectrum and the bandwidth can be determined adaptive, the performance can be improved.

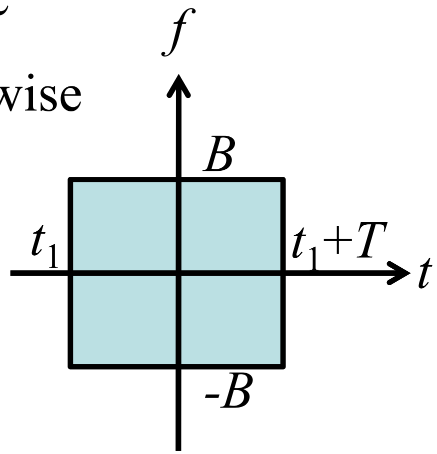
- modulation,
- multiplexing,
- filter design,
- data compression,
- signal analysis,
- signal identification,
- acoustics,
- system modeling,
- radar system analysis
- sampling

Example: Generalization for sampling theory

假設有一個信號，

① The supporting of $x(t)$ is $t_1 \leq t \leq t_1 + T$, $x(t) \approx 0$ otherwise

② The supporting of $X(f) \neq 0$ is $-B \leq f \leq B$, $X(f) \approx 0$ otherwise



根據取樣定理， $\Delta_t \leq 1/F$, $F = 2B$, B:頻寬

所以，取樣點數 N 的範圍是

$$N = T/\Delta_t \geq TF$$

重要定理：一個信號所需要的取樣點數的下限，等於它時頻分佈的面積

Q1：Scaling 對於一個信號的取樣點數有沒有影響？

Hint:

$$g(\sigma t) \xrightarrow{FT} \frac{1}{|\sigma|} G\left(\frac{f}{\sigma}\right)$$

Q2: How to use time-frequency analysis to reduce the number of sampling points?

Time-frequency analysis is an efficient tool for adaptive signal processing.

時頻分析大家族

- (1) Short-time Fourier transform (STFT) (rec-STFT, Gabor, …)
  - square → spectrogram → generalized spectrogram
  - improve → Asymmetric STFT
  - improve → S transform
  - combine (with WDF) → **Gabor-Wigner Transform**
- (2) Wigner distribution function (WDF)
  - combine (with STFT) → **Gabor-Wigner Transform**
  - improve → windowed WDF
  - improve → Cohen's Class Distribution (Choi-Williams, Cone-Shape, Page, Levin, Kirkwood, Born-Jordan, …)
  - improve → Polynomial Wigner Distribution
  - improve → Pseudo L-Wigner Distribution
- (3) Wavelet transform
  - → Haar and Daubechies
  - → Coiflet, Morlet
  - → Directional Wavelet Transform
- (4) Time-Variant Basis Expansion
  - → Matching Pursuit and Compressive Sensing
  - → Prolate Spheroidal Wave Function
- (5) Hilbert-Huang Transform (唯一跳脫 Fourier transform 的架構)

• Continuous Wavelet Transform

forward wavelet transform:

$$X(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\psi\left(\frac{t-a}{b}\right)dt$$

$\psi(t)$: mother wavelet, a: location, b: scaling,

inverse wavelet transform:

$$x(t) = \sum_{a}\sum_{b} X(a,b)\varphi_{a,b}(t)$$

$\varphi_{a,b}(t)$ is dual orthogonal to $\psi(t)$.

	output
Fourier transform	$X(f)$, f: frequency
time-frequency analysis	$X(t, f)$, t: time, f: frequency
wavelet transform	$X(a, b)$, a: time, b: scaling

限制：

(1) $$\frac{1}{\sqrt{b}}\int_{-\infty}^{\infty}\varphi_{a_1,b_1}(t)\psi\left(\frac{t-a}{b}\right)dt=1$$ when $a_1 = a$ and $b_1 = b$,

$$\frac{1}{\sqrt{b}}\int_{-\infty}^{\infty}\varphi_{a_1,b_1}(t)\psi\left(\frac{t-a}{b}\right)dt=0$$ otherwise

(2) $\psi(t)$ has a finite time interval

Two parameters, a: 調整位置, b: 調整寬度

應用： adaptive signal analysis

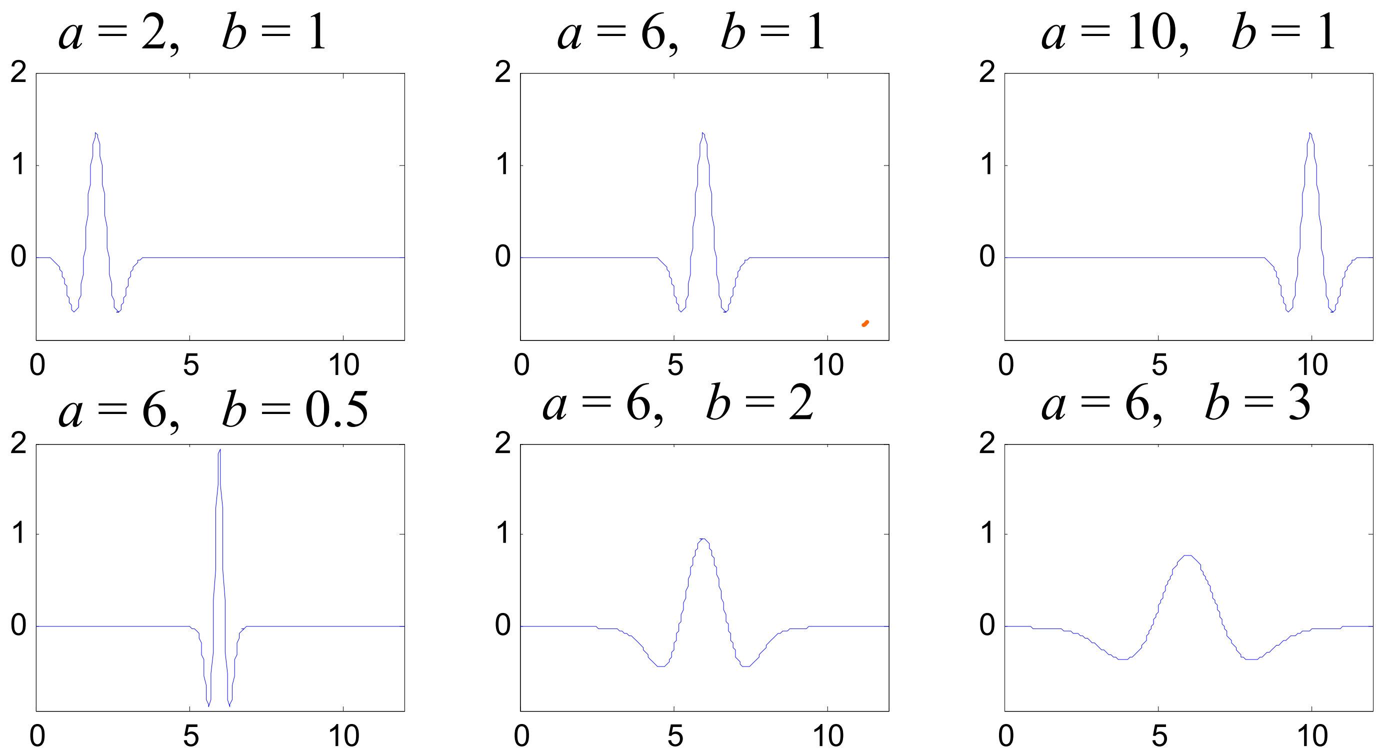
思考：需要較高解析度的地方，b 的值應該如何？

Wavelet 的種類甚多

Mexican hat wavelet, Haar Wavelet, Daubechies wavelet, triangular wavelet, …………

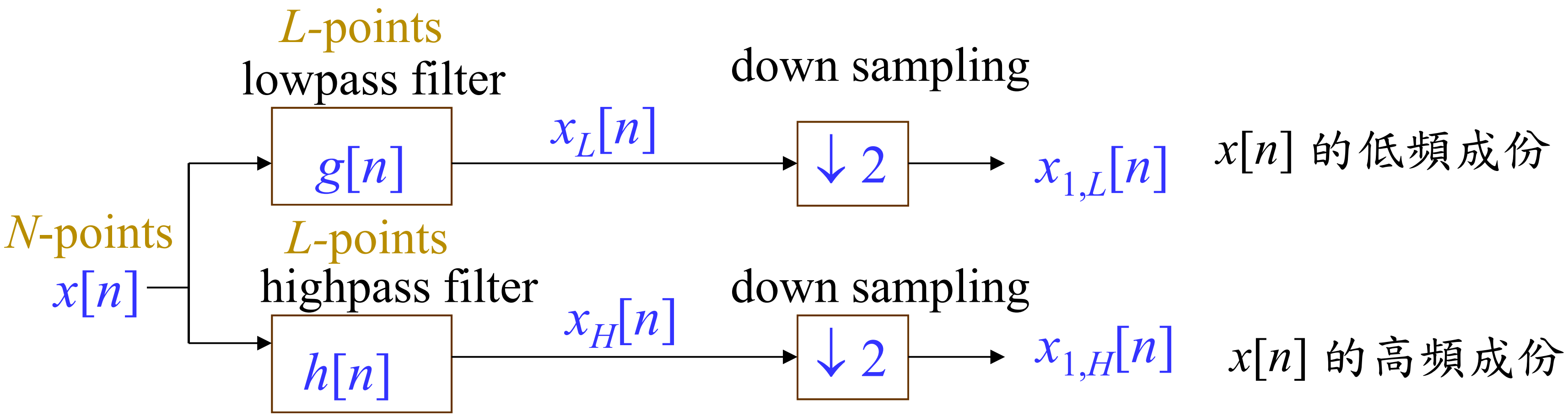
例子： Mexican hat wavelet 隨 a and b 變化之情形

$$\psi(t) = \frac{2^{5/4}}{\sqrt{3}}\left(1-2\pi t^2\right)e^{-\pi t^2} \qquad \frac{1}{\sqrt{b}}\psi\left(\frac{t-a}{b}\right)$$



• Discrete Wavelet Transform (DWT)

The discrete wavelet transform is very different from the continuous wavelet transform. It is simpler and more useful than the continuous one.



$$x_L[n] = \sum_k x[n-k]g[k] \qquad x_{1,L}[n] = \sum_k x[2n-k]g[k]$$

$$x_H[n] = \sum_k x[n-k]h[k] \qquad x_{1,H}[n] = \sum_k x[2n-k]h[k]$$

$$x_{1,L}[n]=\sum_k x[2n-k]\,g[k] \qquad x_{1,H}[n]=\sum_k x[2n-k]\,h[k]$$

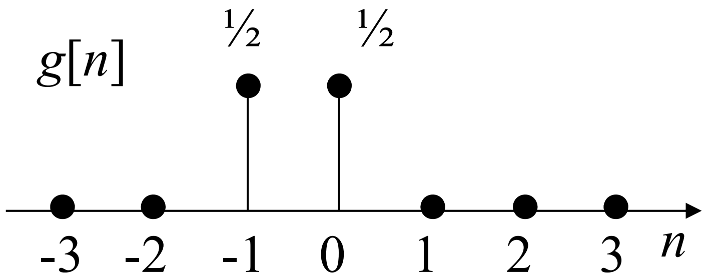
例子：2-point Haar wavelet

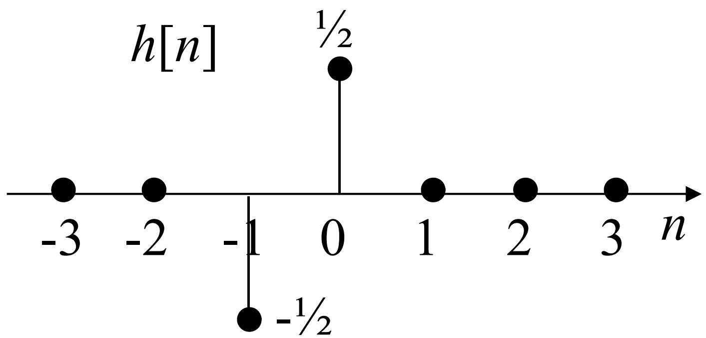
$g[n] = 1/2$ for $n = -1, 0$

$g[n] = 0$ otherwise

$h[0] = 1/2,\quad h[-1] = -1/2,$

$h[n] = 0$ otherwise





then

$$x_{1,L}[n]=\frac{x[2n]+x[2n+1]}{2}$$

（兩點平均）

$$x_{1,H}[n]=\frac{x[2n]-x[2n+1]}{2}$$

（兩點之差）

Discrete wavelet transform 有很多種

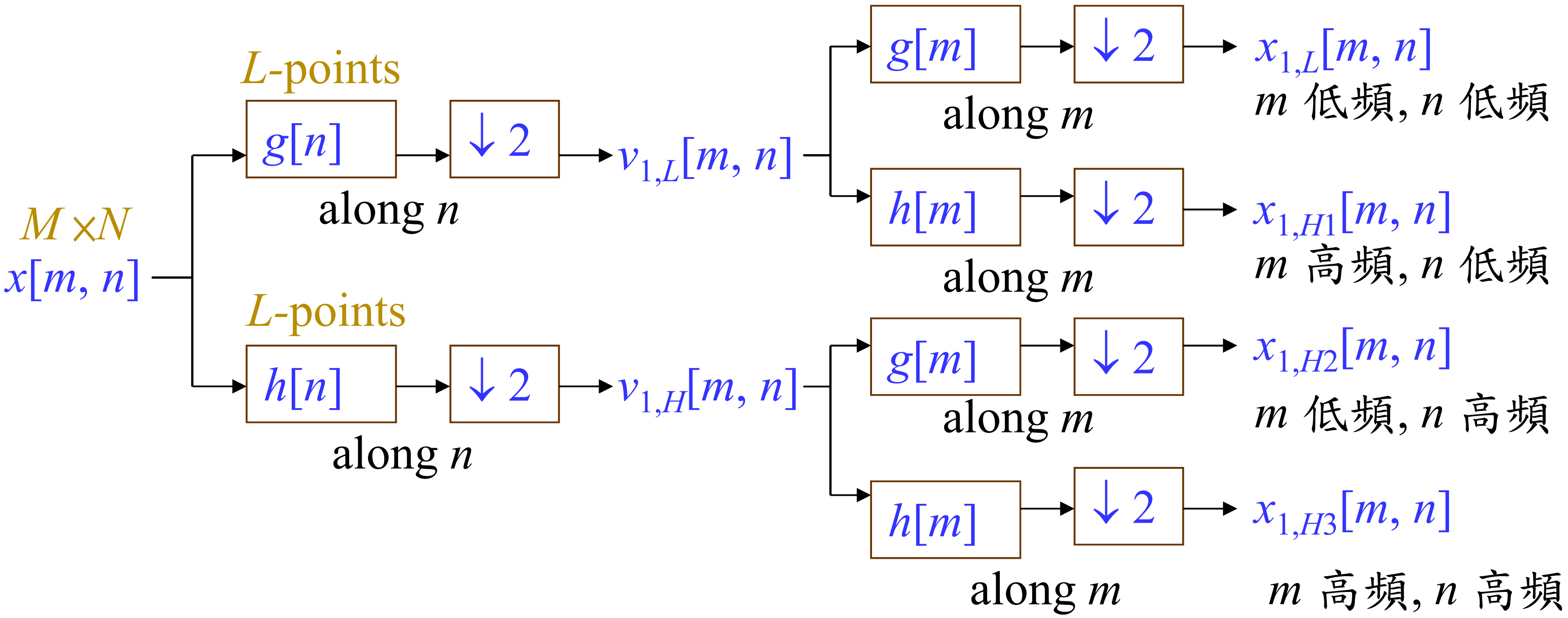
(discrete Haar wavelet, discrete Daubechies wavelet, B-spline DWT, symlet, coilet, ……..)

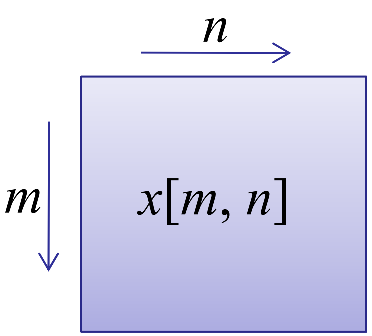
一般的 wavelet, $g[n]$ 和 $h[n]$ 點數會多於 2 點

但是 $g[n]$ 通常都是 lowpass filter 的型態

$h[n]$ 通常都是 highpass filter 的型態

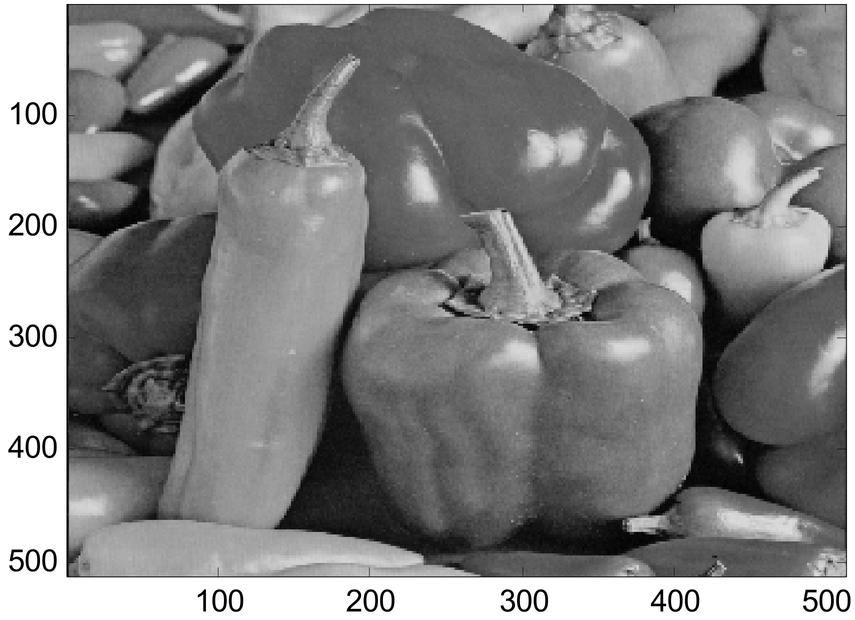
2-D 的情形


L-points
g[n]
↓2
along n
M ×N
x[m, n]
L-points
h[n]
↓2
along n
v1,L[m, n]
v1,H[m, n]
g[m]
↓2
along m
x1,L[m, n]
m 低頻, n 低頻
h[m]
↓2
along m
x1,H1[m, n]
m 高頻, n 低頻
g[m]
↓2
along m
x1,H2[m, n]
m 低頻, n 高頻
h[m]
↓2
along m
x1,H3[m, n]
m 高頻, n 高頻


n
m
x[m, n]

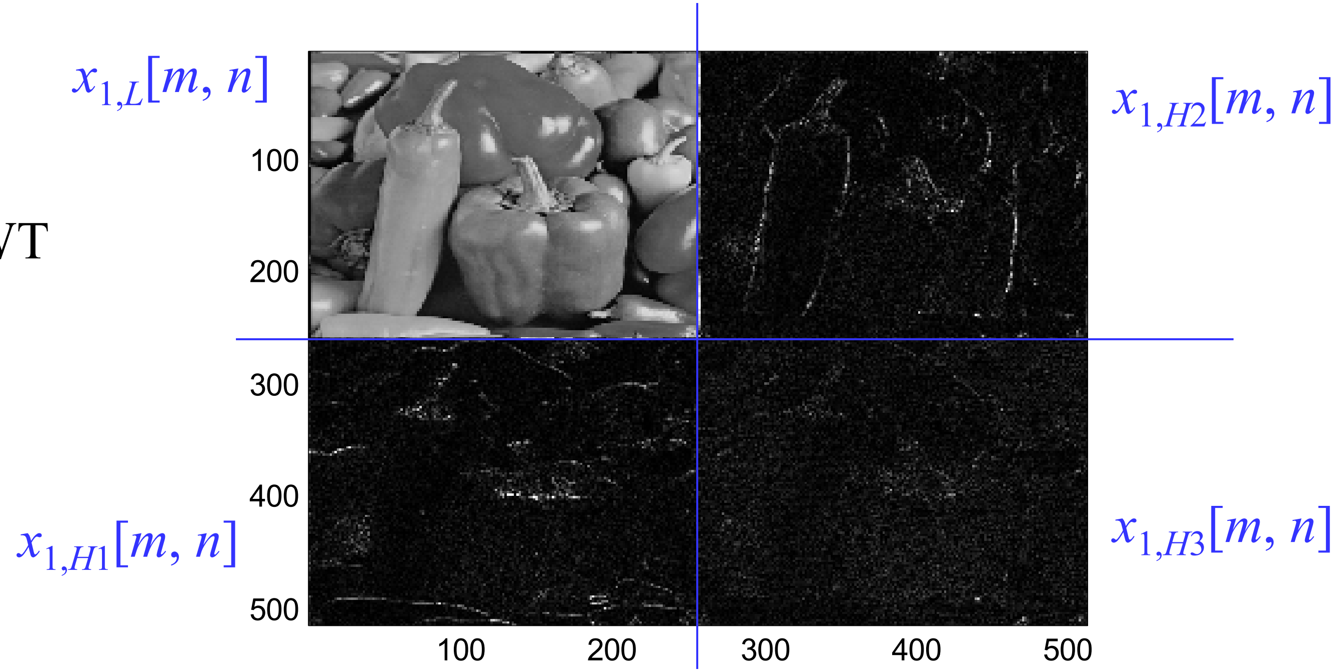
原影像

Pepper.bmp

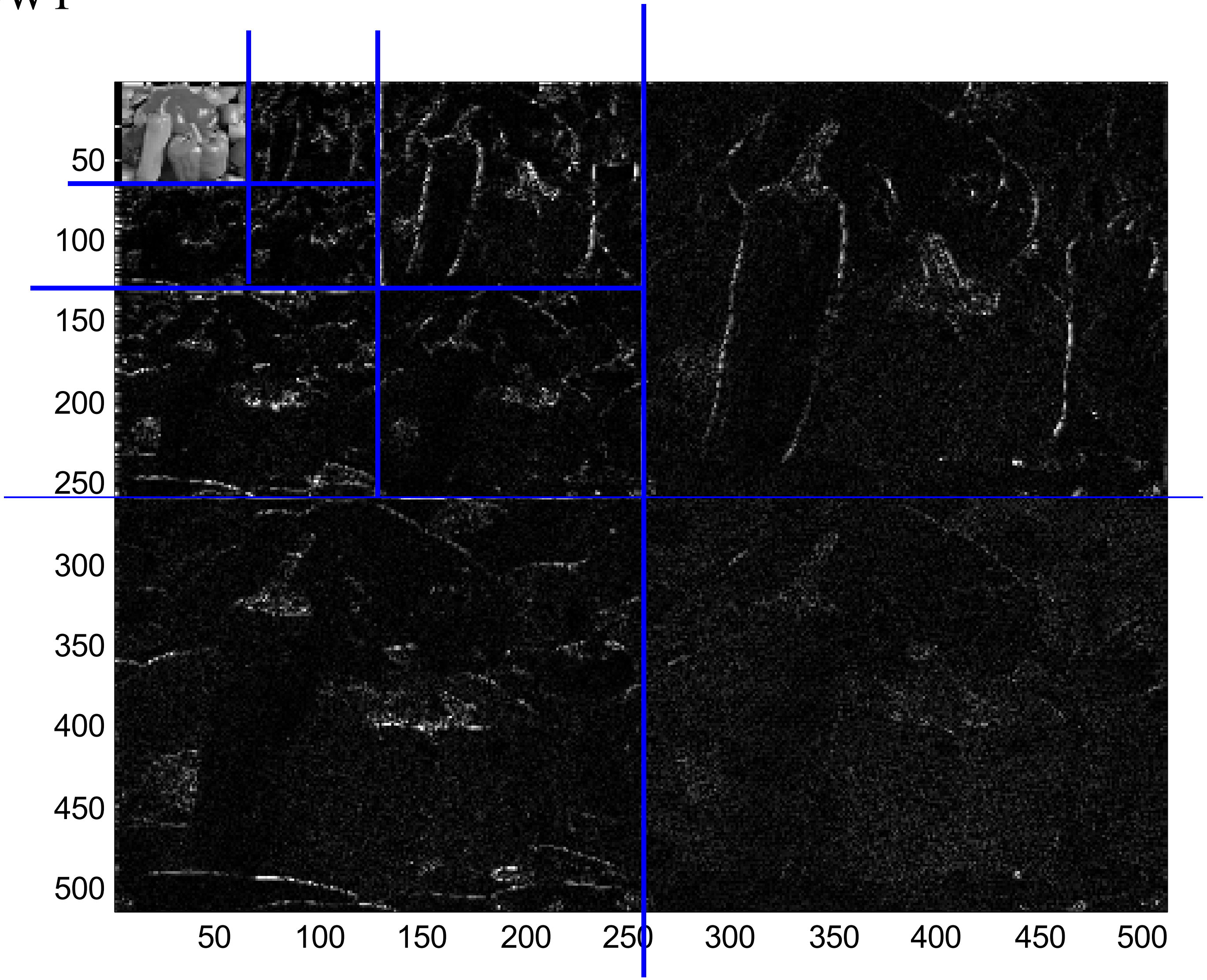


2-D DWT

的結果



3次2-D DWT 的結果


50
100
150
200
250
300
350
400
450
500
50
100
150
200
250
300
350
400
450
500

應用： 影像壓縮 (JPEG 2000)

其他應用：edge detection

corner detection

filter design

pattern recognition

music signal processing

economical data

temperature analysis

feature extraction

biomedical signal processing

附錄一：聲音檔的處理

A. 讀取聲音檔

- 電腦中，沒有經過壓縮的聲音檔都是 *.wav 的型態

- 讀取： **audioread**

- 例： [**x**, **fs**] = audioread('C:\WINDOWS\Media\ringin.wav');
  可以將 ringin.wav 以數字向量 **x** 來呈現。 **fs**: sampling frequency
  這個例子當中 size(**x**) = 9981 1 fs = 11025

- 思考: 所以，取樣間隔多大?

- 這個聲音檔有多少秒？

一個聲音檔如果太大，我們也可以只讀取它部分的點

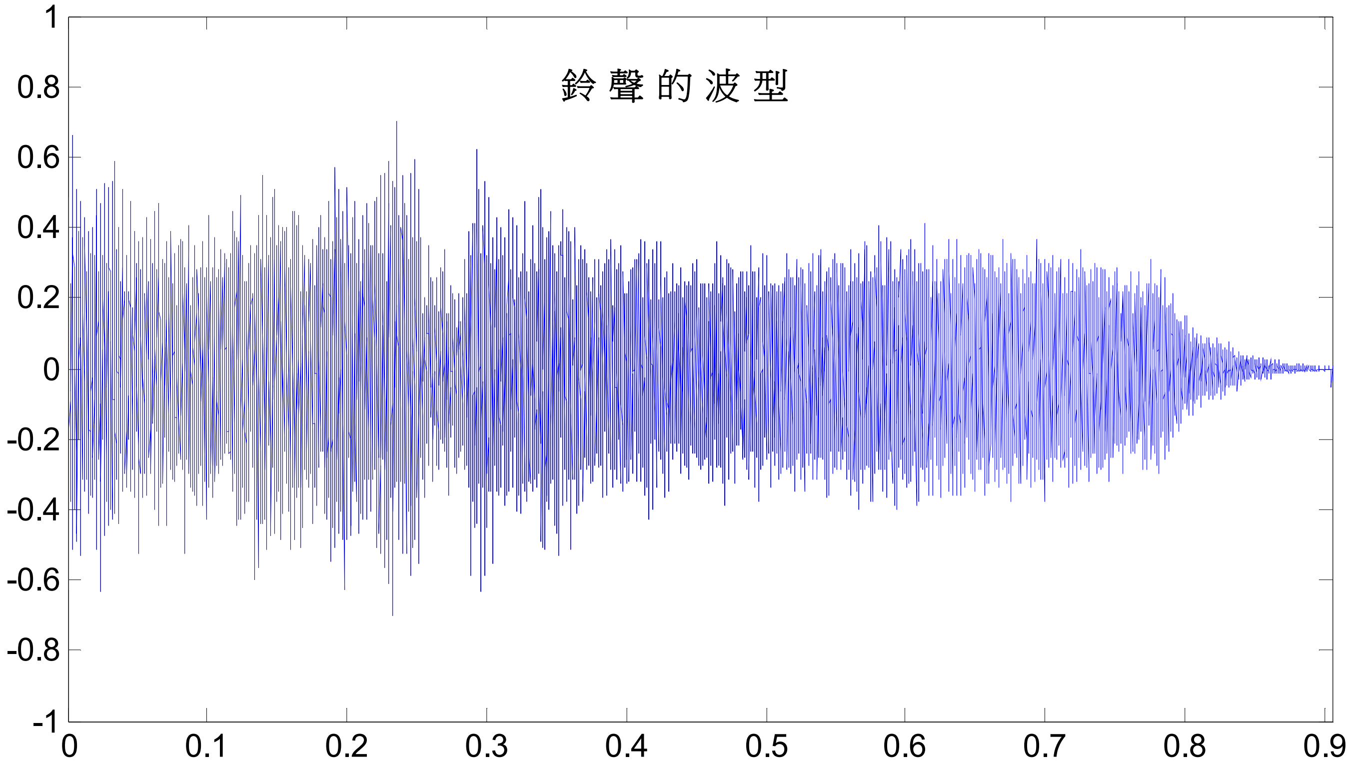
[x, fs]=audioread('C:\WINDOWS\Media\ringin.wav', [4001 5000]);

% 讀取第4001至5000點

畫出聲音的波型

time = [0:length(x)-1]/fs;　　% x 是前頁用 wavread 所讀出的向量

plot(time, x)

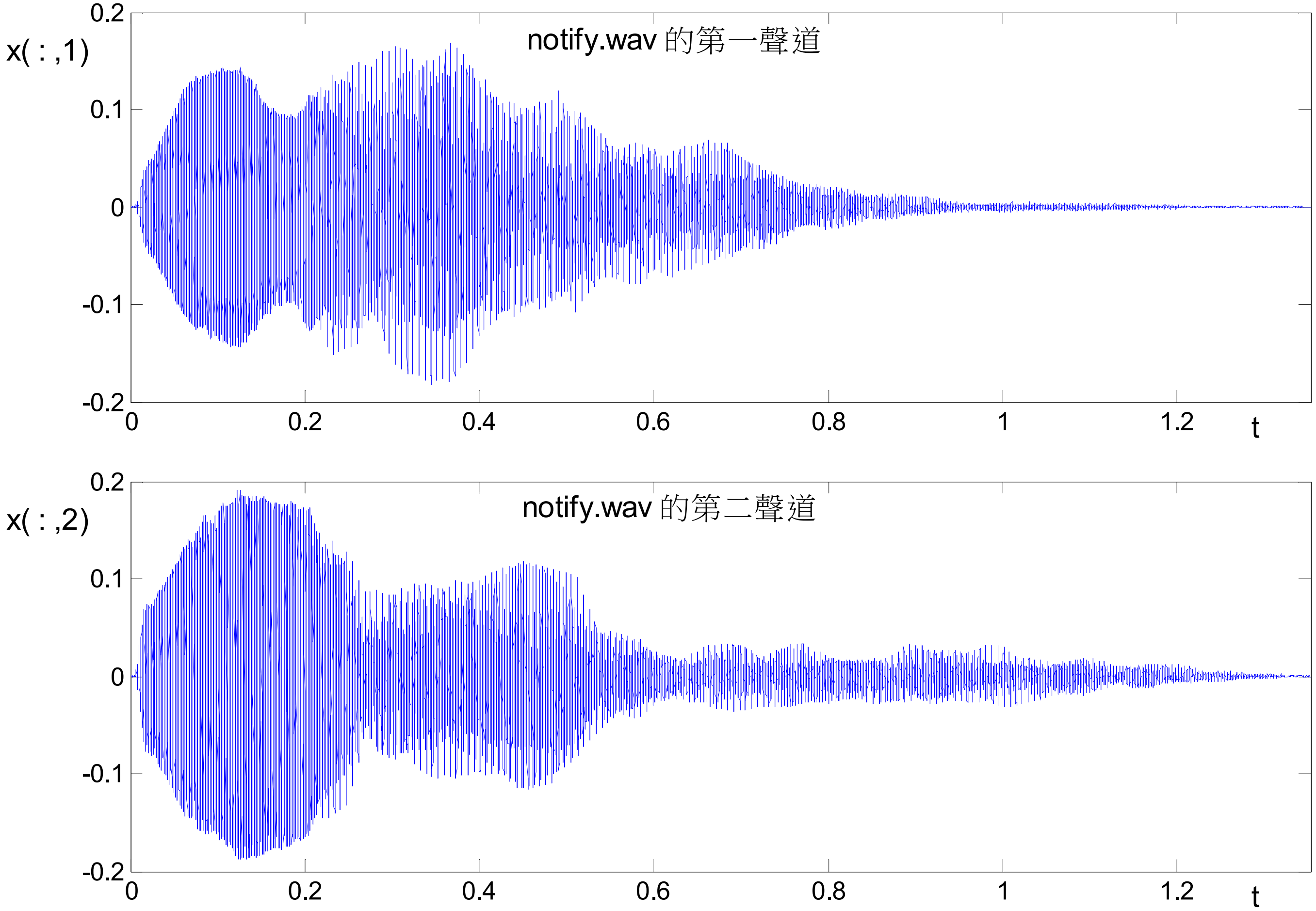


注意：*.wav 檔中所讀取的資料，值都在 –1 和 +1 之間

- 有些聲音檔是 **雙聲道** （**Stereo**）的型態 (俗稱**立體聲**)

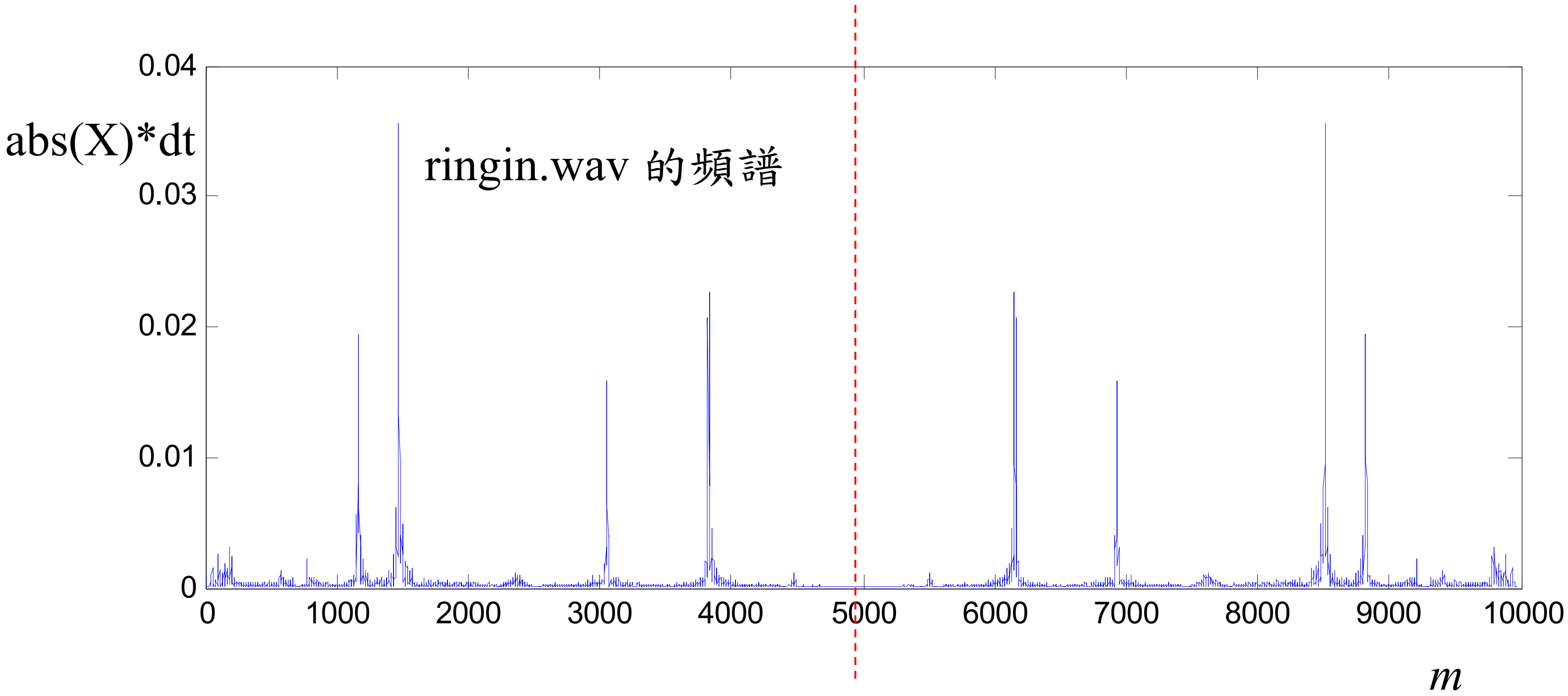
例： [x, fs]=audioread('C:\WINDOWS\Media\notify.wav');

size(x) = 29823   2   fs = 22050



B. 繪出頻譜

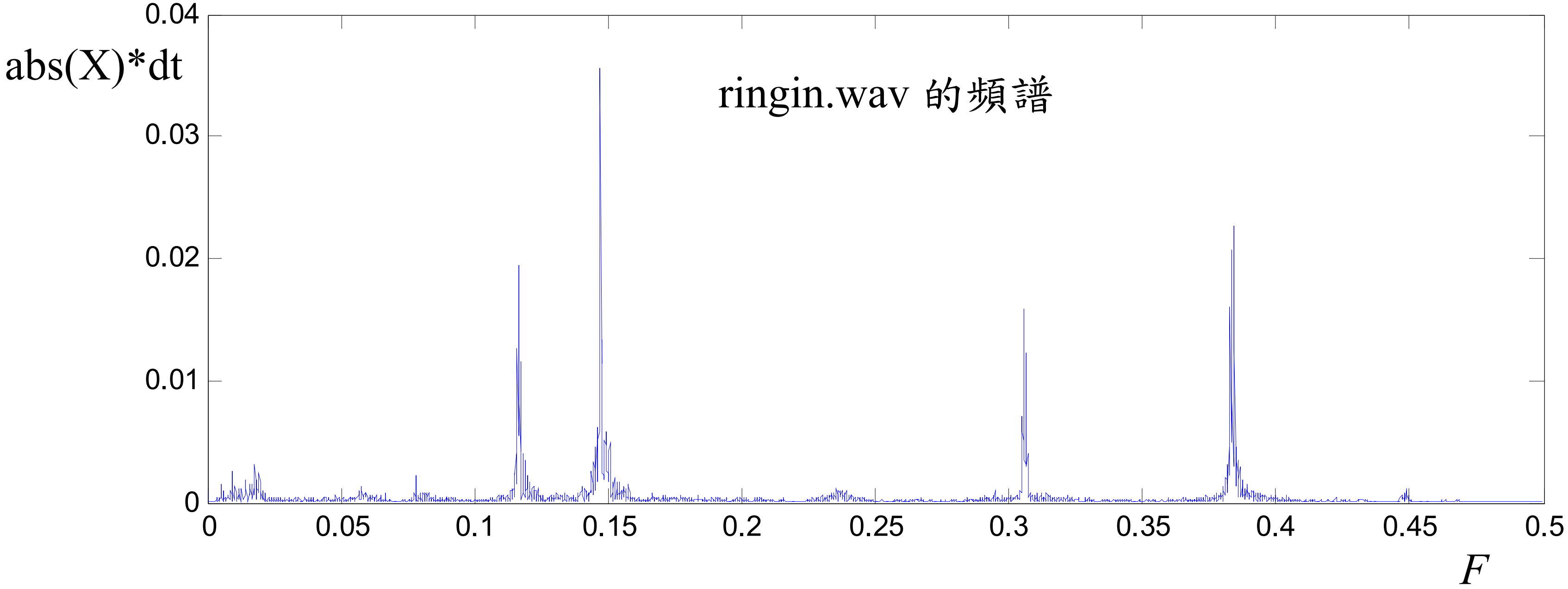
X = fft(x); plot(abs(X)*dt); % dt = 1/fs

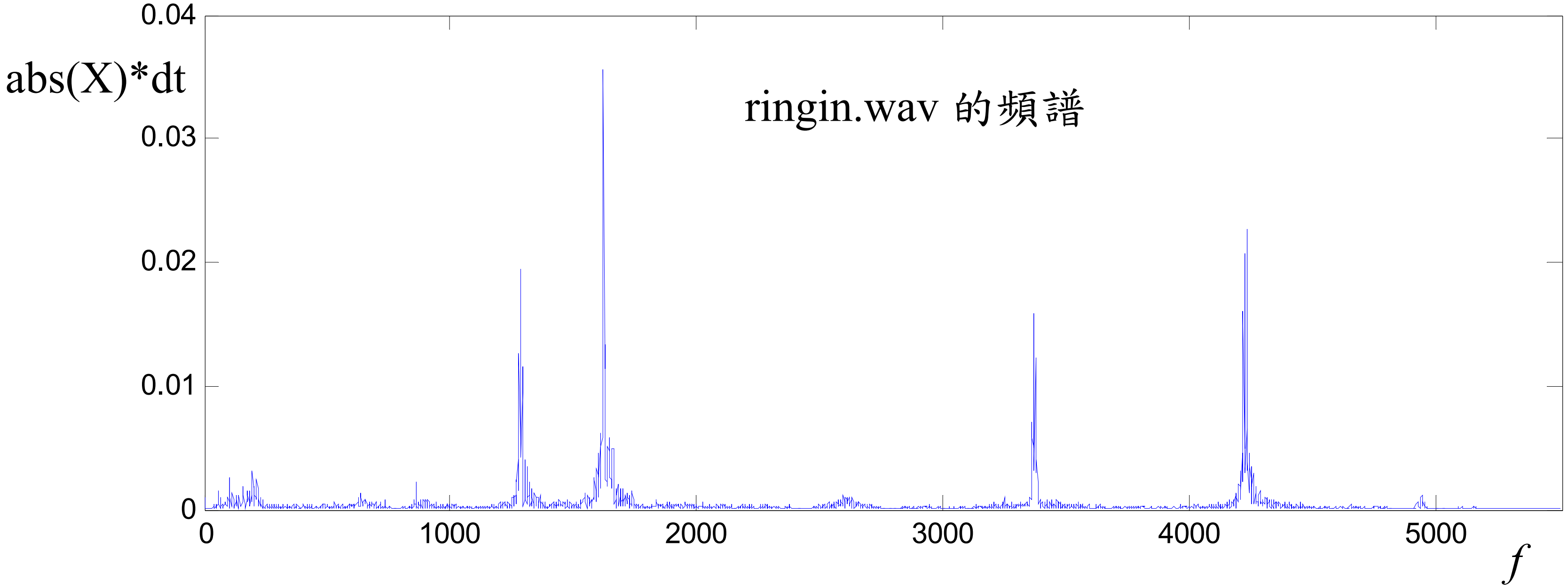


fft 橫軸 轉換的方法

(1) Using normalized frequency F: $F = m / N$.

(2) Using frequency f, $f = F \times f_s = m \times (f_s / N)$.


ringin.wav 的頻譜
abs(X)*dt
0.04
0.03
0.02
0.01
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
F


ringin.wav 的頻譜
abs(X)*dt
0.04
0.03
0.02
0.01
0
0
1000
2000
3000
4000
5000
f

C. 聲音的播放

(1) audioplay(**x**): 將 **x** 以 11025Hz 的頻率播放
(時間間隔 = 1/11025 = 9.07×10^{-5} 秒)

(2) sound(**x**): 將 **x** 以 8192Hz 的頻率播放

(3) sound(**x, fs**) 或 sound(**x, fs**) 或 audioplay(**x , fs**):

將 **x** 以 **fs** Hz 的頻率播放

Note: (1)~(3) 中 **x** 必需是1 個column (或2個 columns)，且 **x** 的值應該介於 –1 和 +1 之間

(4) soundsc(**x, fs**): 自動把 **x** 的值調到 –1 和 +1 之間 再播放

D. 製作音檔： audiowrite

audiowrite(x, fs, waveFile)

將數據 **x** 變成一個 *.wav 檔，取樣速率為 **fs** Hz

① **x** 必需是1 個column (或2個 columns) ② **x** 值應該 介於 –1 和 +1 之間

③ 若沒有設定fs，則預設的fs 為 8000Hz

E. 錄音的方法

錄音之前，要先將電腦接上麥克風，且確定電腦有音效卡

(部分的 notebooks 不需裝麥克風即可錄音)

範例程式：

```
Sec = 3;
Fs = 8000;
recorder = audiorecorder(Fs, 16, 1);
recordblocking(recorder, Sec);
audioarray = getaudiodata(recorder);
```

執行以上的程式，即可錄音。

錄音的時間為三秒，sampling frequency 為 8000 Hz

錄音結果為 audioarray，是一個 column vector (如果是雙聲道，則是兩個 column vectors)

範例程式 (續)：

```
audioplay(audioarray, Fs);                    % 播放錄音的結果
t = [0:length(audioarray)-1]./Fs;
plot (t, audioarray‘);                        % 將錄音的結果用圖畫出來
xlabel('sec','FontSize',16);
audiowrite(audioarray, Fs, ‘test.wav’)    % 將錄音的結果存成 *.wav 檔
```

指令說明：

recorder = audiorecorder(Fs, nb, nch); (提供錄音相關的參數)

Fs: sampling frequency,

nb: using nb bits to record each data

nch: number of channels (1 or 2)

recordblocking(recorder, Sec); (錄音的指令)

recorder: the parameters obtained by the command “audiorecorder”

Sec: the time length for recording

audioarray = getaudiodata(recorder);

(將錄音的結果，變成 audioarray 這個 column vector，如果是雙聲道，則 audioarray 是兩個 column vectors)

以上這三個指令，要並用，才可以錄音