XII. Wavelet Transform

Main References

[1] R. C. Gonzalez and R. E. Woods, *Digital Image Processing*, Chap. 7, 4th edition, Prentice Hall, New Jersey, 2017. (適合初學者閱讀)

[2] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 3rd edition, 2009. (適合想深入研究的人閱讀)
(若對時頻分析已經有足夠的概念，可以由這本書 Chapter 4 開始閱讀)

[3] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Comm. Pure Appl. Math.*, vol. 4, pp. 909-996, Nov. 1988.
[4] S. Mallat, "Multiresolution approximations and wavelet orthonormal bases of L2(R)," *Trans. Amer. Math. Soc.*, vol. 315, pp. 69-87, Sept. 1989.
[5] C. Heil and D. Walnut, "Continuous and discrete wavelet transforms," *SIAM Rev.*, vol. 31, pp. 628-666, 1989.
[6] I. Daubechies, "The wavelet transform, time-frequency localization and signal analysis," *IEEE Trans. Information Theory*, pp. 961-1005, Sept. 1990.
[7] R. K. Young, *Wavelet Theory and Its Applications*, Kluwer Academic Pub., Boston, 1995.
[8] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chapter 4, Prentice-Hall, New Jersey, 1996.
[9] L. Debnath, *Wavelet Transforms and Time-Frequency Signal Analysis*, Birkhäuser, Boston, 2001.
[10] B. E. Usevitch, "A Tutorial on Modern Lossy Wavelet Image Compression: Foundations of JPEG 2000," *IEEE Signal Processing Magazine*, vol. 18, pp. 22-35, Sept. 2001.

(1) Conventional method for signal analysis

- Fourier transform :　$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$
- Cosine and Sine transforms: if $x(t)$ is even and odd
- Orthogonal Polynomial Expansion

傳統方法共通的問題：

(2) Time frequency analysis

例如， STFT

$$X(t,f)=\int_{-\infty}^{\infty}w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Time frequency analysis 共通的問題：

12-A Haar Transform

一種最簡單又可以反應 time-variant spectrum 的 signal representation

8-point Haar transform

$$H[m,n]=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

8-point Haar transform

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

y_1: low frequency component $y_2 \sim y_8$: high frequency component

$$y_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

$$y_2 = x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8$$

$$y_3 = x_1 + x_2 - x_3 - x_4$$

$$y_4 = x_5 + x_6 - x_7 - x_8$$

$$y_5 = x_1 - x_2$$

$N = 2$

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$N = 4$

$$\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$N = 8$

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

General way to generate the Haar transform:

$$\mathbf{H}_{2N} = \begin{bmatrix} \mathbf{H}_N \otimes [1,1] \\ \mathbf{I}_N \otimes [1,-1] \end{bmatrix}$$

where $\otimes$ means the Kronecker product

$$\mathbf{I}_N = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$N = 2^k$ 時

$$\mathbf{H} = \begin{bmatrix} \phi \\ h_{0,1} \\ h_{1,1} \\ h_{1,2} \\ \vdots \\ \vdots \\ h_{k-1,1} \\ h_{k-1,2} \\ \vdots \\ h_{k-1,2^{k-1}} \end{bmatrix}$$

H 除了第 1 個row 為 $\underbrace{\phi = [1 \quad 1 \quad 1 \quad \cdots \quad 1]}_{N \text{ 個 } 1}$ 以外

第 $2^p + q$ 個row 為 $h_{p,q}[n]$

$$p = 0, 1, \ldots, k-1, \quad q = 1, 2, \ldots, 2^p$$

$$k = \log_2 N$$

$h_{p,q}[n] = 1$ when $(q\text{-}1)2^{k-p} < n \le (q\text{-}1/2)2^{k-p}$

$h_{p,q}[n] = -1$ when $(q\text{-}1/2)2^{k-p} < n \le q2^{k-p}$

- Inverse 2^k-point Haar Transform

$$\mathbf{H}^{-1} = \mathbf{H}^{\mathrm{T}}\mathbf{D}$$

$D[m, n] = 0$ if $m \neq n$

$D[1, 1] = 2^{-k}$, $D[2, 2] = 2^{-k}$,

$D[n, n] = 2^{-k+p}$ if $2^p < n \leq 2^{p+1}$

When $k = 3$,

$$\mathbf{D} = \begin{bmatrix} 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

12-B Characteristics of Haar Transform

(1) No multiplications

(2) Input 和 Output 點數相同

(3) 頻率只分兩種：低頻 (全為 1) 和高頻 (一半為 1，一半為 −1)

(4) 可以分析一個信號的 localized feature

(5) Very fast, but not accurate

Example:

$$\mathbf{H}\begin{bmatrix}1.2\\1.2\\1.8\\0.8\\2\\2\\1.9\\2.1\end{bmatrix}=\begin{bmatrix}13\\-3\\-0.2\\0\\0\\1\\0\\-0.2\end{bmatrix}$$

Transforms	Running Time	terms required for NRMSE < 10^{-5}
DFT	9.5 sec	43
Haar Transform	0.3 sec	128

References

- A. Haar, “Zur theorie der orthogonalen funktionensysteme ,” *Math. Annal.*, vol. 69, pp. 331-371, 1910.
- H. F. Harmuth, *Transmission of Information by Orthogonal Functions*, Springer-Verlag, New York, 1972.

The Haar Transform is closely related to the Wavelet transform (especially the discrete wavelet transform).

12-C History of the Wavelet Transform

- 1910, Haar families.
- 1981, Morlet, wavelet concept.
- 1984, Morlet and Grossman, "wavelet".
- 1985, Meyer, "orthogonal wavelet".
- 1987, International conference in France.
- 1988, Mallat and Meyer, multiresolution.
- 1988, Daubechies, compact support orthogonal wavelet.
- 1989, Mallat, fast wavelet transform.
- 1990s, Discrete wavelet transforms
- 1999, Directional wavelet transform
- 2000, JPEG 2000

12-D Three Types of Wavelets

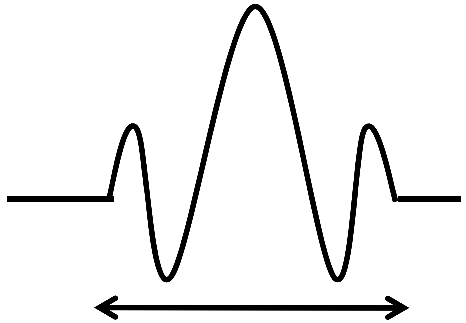
Wavelet 以 continuous / discrete 來分，有 3 種

	Input	Output	Name
Type 1	Continuous	Continuous	Continuous Wavelet Transform
Type 2	Continuous	Discrete	有時被稱為 discrete wavelet transform，但其實是continuous wavelet transform with discrete coefficients
Type 3	Discrete	Discrete	Discrete Wavelet Transform

比較：Fourier transform 有四種

12-E Continuous Wavelet Transform (WT)

Definition: $$X_w(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\psi\left(\frac{t-a}{b}\right)dt$$



$x(t)$: input, $\psi(t)$: mother wavelet

a: location, b: scaling

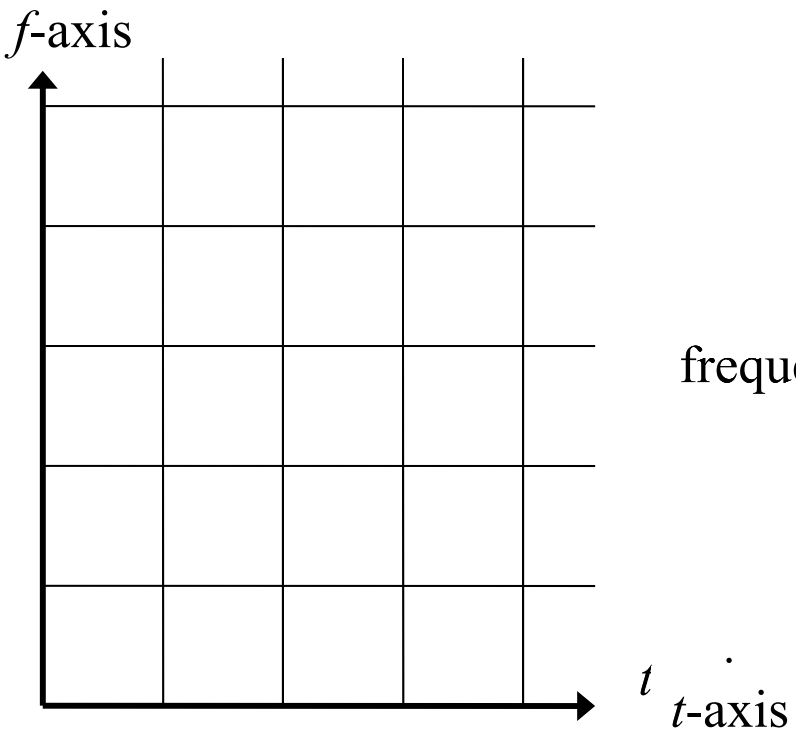
a is any real number, b is any positive real number

Compare with time-frequency analysis:

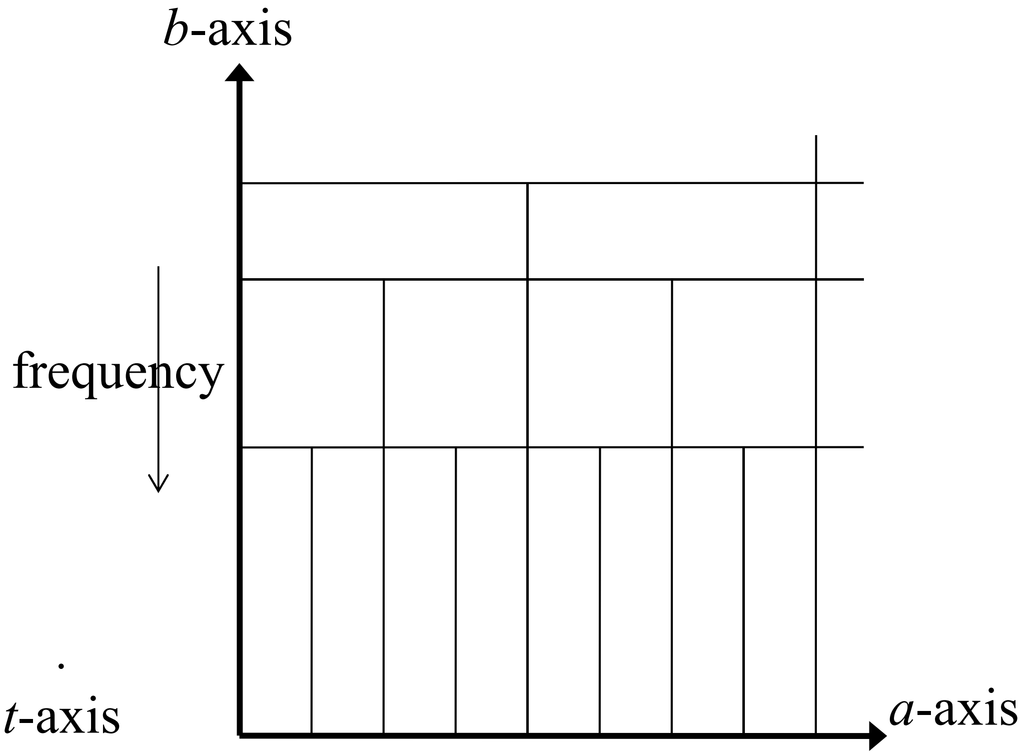
location + modulation

Gabor Transform $$G_x(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)d\tau$$

Gabor



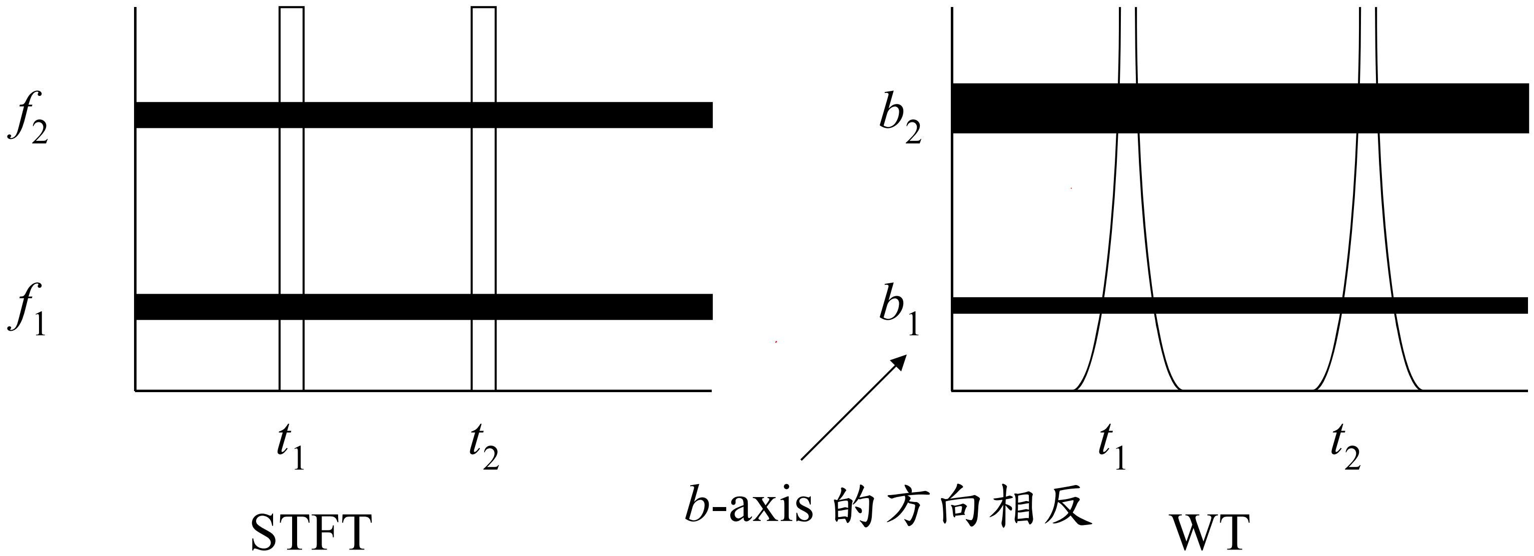
Wavelet transform



$$X_w(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\,\psi\!\left(\frac{t-a}{b}\right) dt \qquad a\text{: location,}\ \ b\text{: scaling}$$

- The resolution of the wavelet transform is invariant along a (location-axis) but variant along b (scaling axis).

If $x(t) = \delta(t - t_1) + \delta(t - t_2) + \exp(j2\pi f_1 t) + \exp(j2\pi f_2 t)$,



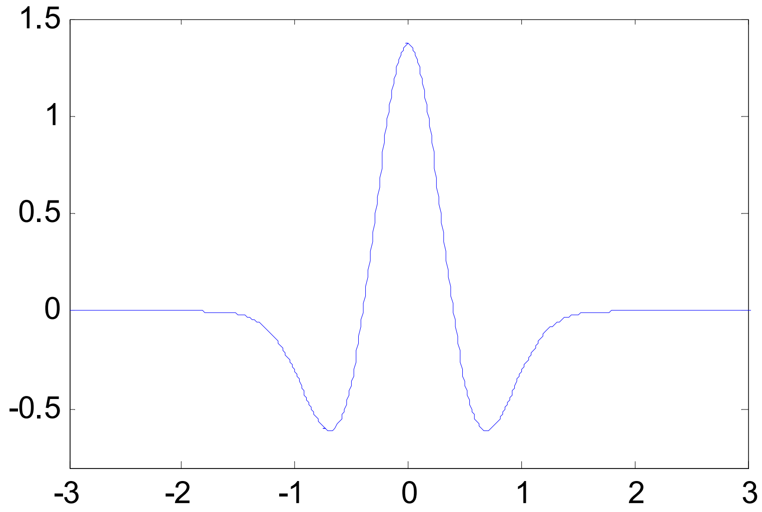
12-F Mother Wavelet

There are many ways to choose the mother wavelet. For example,

- Haar basis

- Mexican hat function $\psi(t) = \frac{2^{5/4}}{\sqrt{3}}\left(1 - 2\pi t^2\right)e^{-\pi t^2}$

In fact, the Mexican hat function is the 2nd order derivation of the Gaussian function.

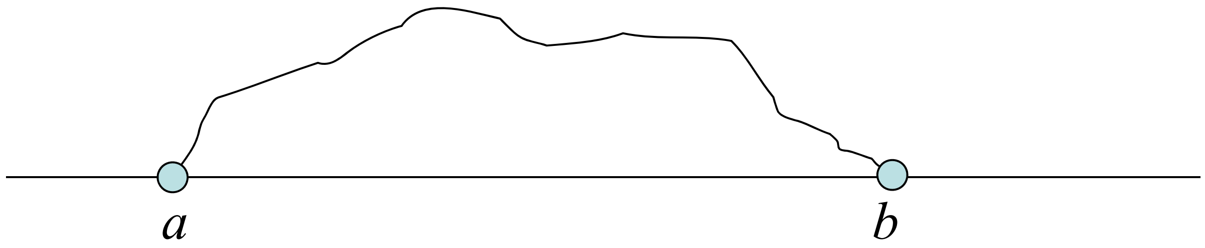


Constraints for the mother wavelet:

(1) Compact Support

support: the region where a function is not equal to zero

compact support: the width of the support is not infinite



(2) Real

(3) Even Symmetric or Odd Symmetric

(4) Vanishing Moments

k^{th} moment: $m_k = \int_{-\infty}^{\infty} t^k \psi(t)\,dt$

If $m_0 = m_1 = m_2 = \ldots.. = m_{p-1} = 0$, we say $\psi(t)$ has p vanishing moments.

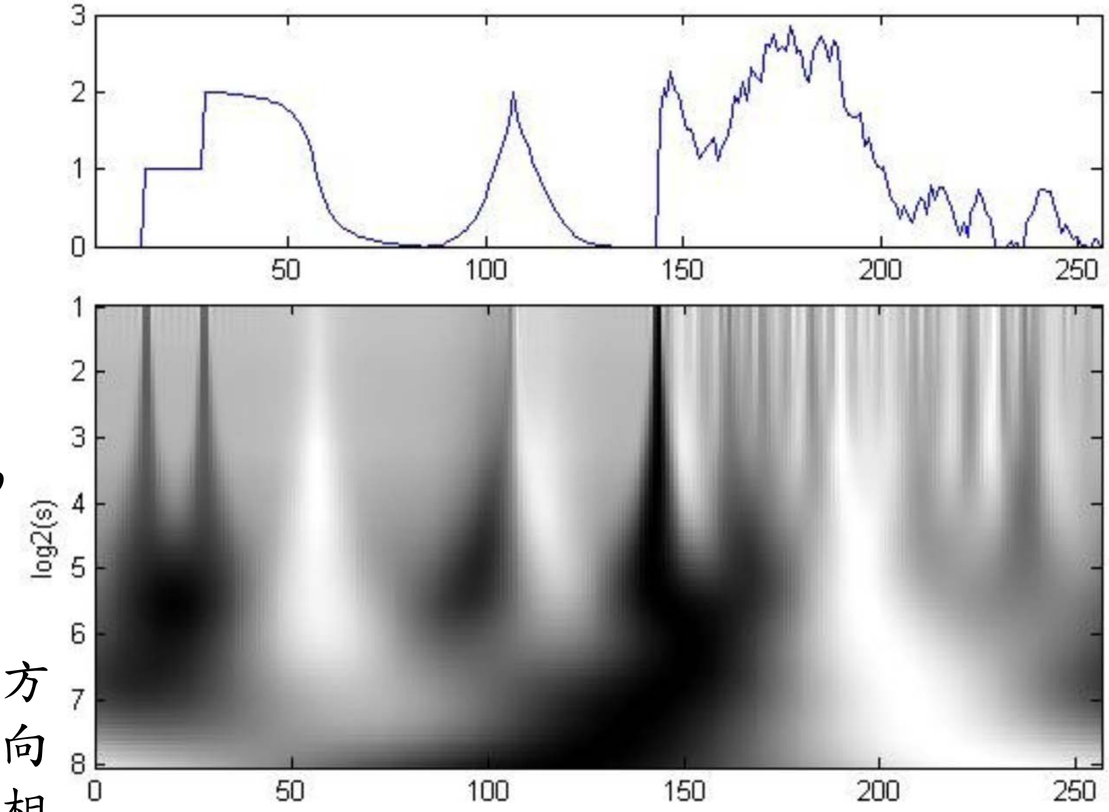
Vanish moment 越高，經過內積後被濾掉的低頻成分越多

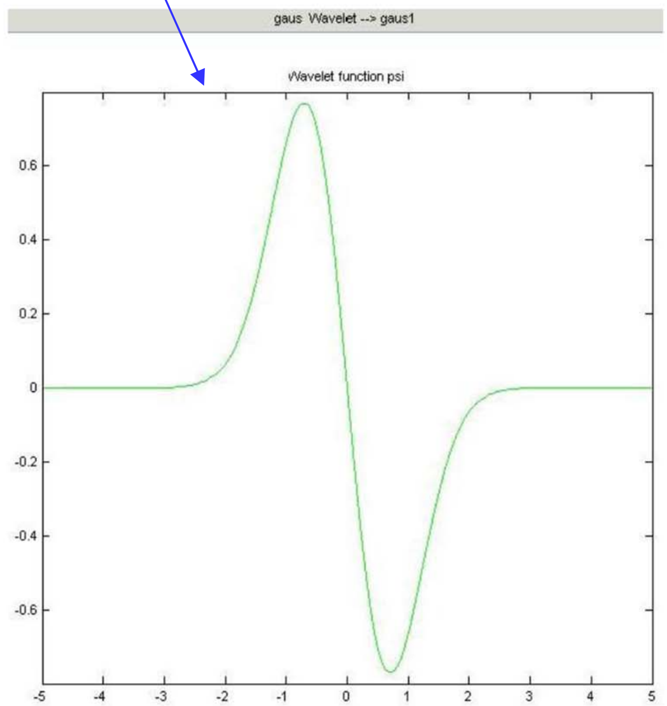
Question：為什麼要求 $\int_{-\infty}^{\infty} \psi(t)\,dt = 0$?

註：感謝 2006年修課的張育思同學

Vanish moment = 1

the 1st order derivation of the Gaussian function





b

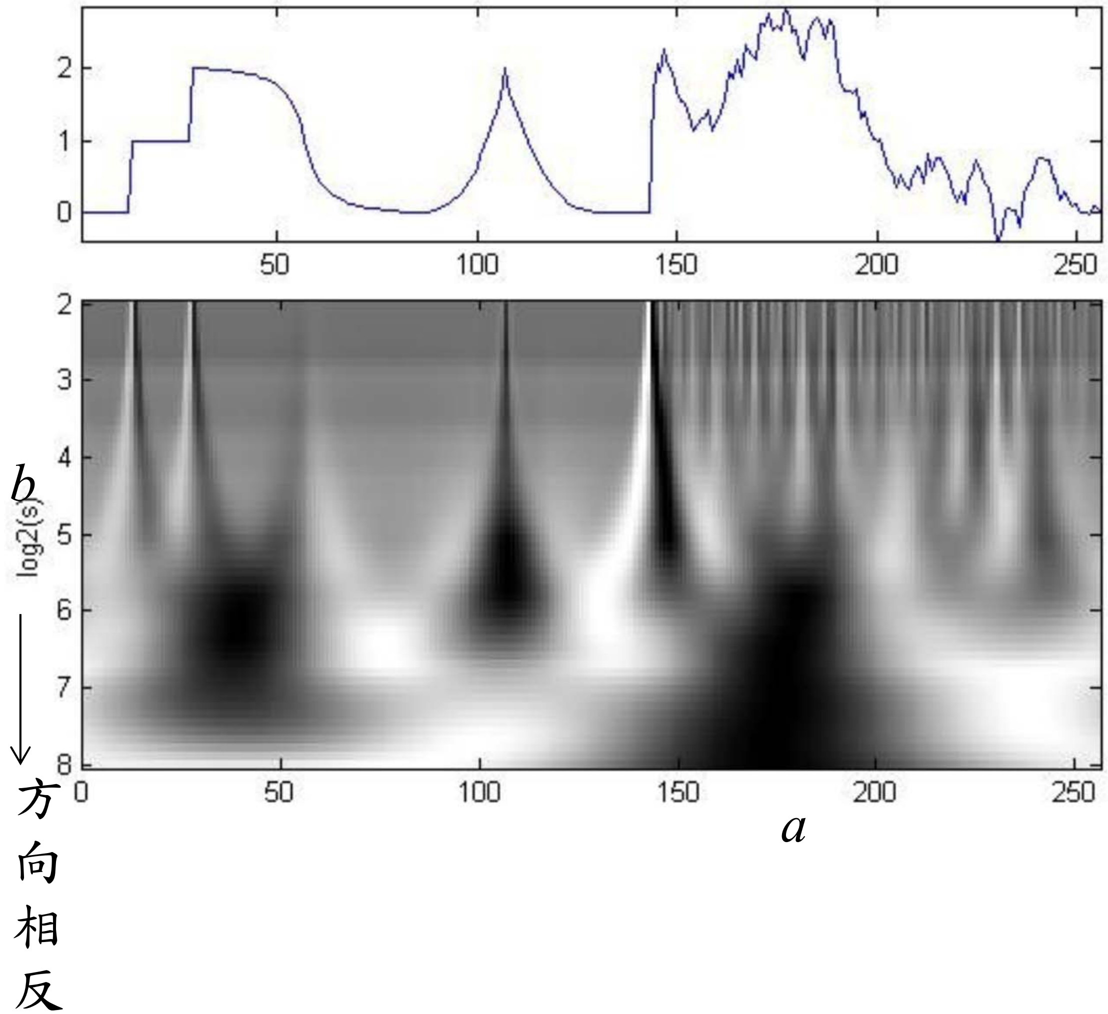
↓方向相反

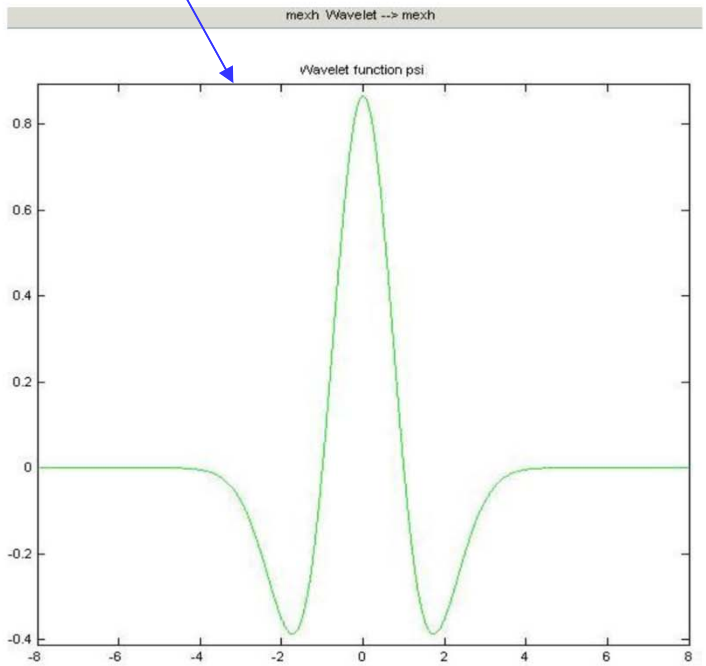
a

[Ref] S. Mallat, *A Wavelet Tour of Signal Processing*, 2nd Ed., Academic Press, San Diego, 1999.

Vanish moment = 2

the 2^{nd} order derivation of the Gaussian function





Similarly, when

$$\psi(t) = \frac{d^p}{dt^p} e^{-\pi t^2}$$

the vanish moment is p

(5) Admissibility Criterion

$$C_\psi = \int_0^\infty \frac{|\Psi(f)|^2}{|f|} df < \infty$$,where $\Psi(f)$ is the Fourier transform of $\psi(t)$

For reversible

[Ref] A. Grossman and J. Morlet, “Decomposition of hardy functions into square integrable wavelets of constant shape,” *SIAM J. Appl. Math.*, vol. 15, pp. 723-736, 1984.

12-G Inverse Wavelet Transform

$$x(t) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty \frac{1}{b^{5/2}} X_w(a,b)\psi\left(\frac{t-a}{b}\right) da\, db$$

where $C_\psi = \int_0^\infty \frac{|\Psi(f)|^2}{|f|} df < \infty$

simplified $x(t) \simeq \frac{1}{C_\psi} \int_0^\infty \int_{t-bt_0}^{t+bt_0} \frac{1}{b^{5/2}} X_w(a,b)\psi\left(\frac{t-a}{b}\right) da\, db$ if $\psi(t) \cong 0$ for $|t| > t_0$

(Proof): Since $X_w(a,b) = x(a) * \frac{1}{\sqrt{b}}\psi\left(\frac{-a}{b}\right)$

if $$y(t) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty \frac{1}{b^{5/2}} X_w(a,b)\psi\left(\frac{t-a}{b}\right) da\, db$$

then $$y(t) = \frac{1}{C_\psi} \int_0^\infty x(t) * \psi\left(\frac{-t}{b}\right) * \psi\left(\frac{t}{b}\right) \frac{db}{b^3}$$

$$y(t) = \frac{1}{C_\psi}\int_0^\infty x(t) * \psi\left(\frac{-t}{b}\right) * \psi\left(\frac{t}{b}\right)\frac{db}{b^3}$$

$$Y(f) = \frac{1}{C_\psi}\int_0^\infty X(f)\Psi(-bf)\Psi(bf)\frac{db}{b}$$

where

$$Y(f) = FT[y(t)]$$
$$X(f) = FT[x(t)]$$
$$\Psi(f) = FT[\psi(t)]$$

If $\psi(t)$ is real, $\Psi(-f) = \Psi^*(f)$, $\Psi(-bf)\,\Psi(bf) = \Psi^*(bf)\,\Psi(bf) = |\Psi(bf)|^2$

$$Y(f) = X(f)\frac{1}{C_\psi}\int_0^\infty \left|\Psi(bf)\right|^2 \frac{db}{b}$$

$$= X(f)\frac{1}{C_\psi}\int_0^\infty \left|\Psi(f_1)\right|^2 \frac{df_1}{bf} \qquad (f_1 = bf,\ df_1 = fdb)$$

$$= X(f)\frac{1}{C_\psi}\int_0^\infty \left|\Psi(f_1)\right|^2 \frac{df_1}{f_1}$$

$$= X(f)$$

Therefore, $y(t) = x(t)$.

12-H Scaling Function

定義 scaling function 為

$$\phi(t) = \int_{-\infty}^{\infty} \Phi(f) e^{j2\pi f t} df$$

where $$\left|\Phi(f)\right|^2 = \int_{f}^{\infty} \frac{|\Psi(f_1)|^2}{|f_1|} df_1$$ for $f > 0$, $\Phi(-f) = \Phi^*(f)$

$\phi(t)$ is usually a lowpass filter (Why?)

修正型的 Wavelet transform

(1) $$X_w(a,b)=\frac{1}{\sqrt{b}}\int_{-\infty}^{\infty}x(t)\psi\left(\frac{t-a}{b}\right)dt$$

a is any real number, $0<b<b_0$

(2) $$LX_w(a,b_0)=\frac{1}{\sqrt{b_0}}\int_{-\infty}^{\infty}x(t)\phi\left(\frac{t-a}{b_0}\right)dt$$

reconstruction:

$$x(t)=\frac{1}{C_\psi}\left[\int_0^{b_0}\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db+\int_{-\infty}^{\infty}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da\right]$$

由 b_0 至 ∞ 的積分被第二項取代

If $\psi(t)\cong 0$ for $|t|>t_0$, $\phi(t)\cong 0$ for $|t|>t_1$

$$x(t)\cong\frac{1}{C_\psi}\left[\int_0^{b_0}\int_{t-bt_0}^{t+bt_0}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db+\int_{t-b_0t_1}^{t+b_0t_1}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da\right]$$

(Proof): If $y_1(t) = \frac{1}{C_\psi}\int_0^{b_0}\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db$

$$y_2(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da$$

$$Y_1(f) = X(f)\frac{1}{C_\psi}\int_0^{b_0}\left|\Psi(bf)\right|^2\frac{db}{b}$$

(from the similar process on pages 384 and 385)

$$= X(f)\frac{1}{C_\psi}\int_0^{b_0 f}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1}$$

$$y_2(t) = \frac{1}{b_0^2 C_\psi}x(t)*\phi\left(\frac{-t}{b_0}\right)*\phi\left(\frac{t}{b_0}\right)$$

$$Y_2(f) = X(f)\frac{1}{C_\psi}\Phi(-b_0 f)\Phi(b_0 f) = X(f)\frac{1}{C_\psi}\Phi^*(b_0 f)\Phi(b_0 f)$$

$$= X(f)\frac{1}{C_\psi}\left|\Phi(b_0 f)\right|^2$$

關鍵

$$= X(f)\frac{1}{C_\psi}\int_{b_0 f}^{\infty}\frac{|\Psi(f_1)|^2}{|f_1|}df_1$$

Therefore, if $y(t) = y_1(t) + y_2(t)$,

$$\begin{aligned}
Y(f) &= Y_1(f) + Y_2(f) \\
&= X(f)\frac{1}{C_\psi}\int_0^{b_0 f}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1} + X(f)\frac{1}{C_\psi}\int_{b_0 f}^{\infty}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1} \\
&= X(f)\frac{1}{C_\psi}\int_0^{\infty}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1} \\
&= X(f)
\end{aligned}$$

$$y(t) = x(t)$$

12-I Property

(1) real input $\longrightarrow$ real output

(2) If $x(t) \longrightarrow X_w(a, b)$, then $x(t-\tau) \longrightarrow X_w(a-\tau, b)$,

(3) If $x(t) \longrightarrow X_w(a, b)$, then $x(t/\sigma) \longrightarrow \sqrt{\sigma} X_w(a/\sigma, b/\sigma)$

(4) Parseval's Theory:

$$\int |x(t)|^2 \, dt = \frac{1}{C} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{b^2} |X_w(a,b)|^2 \, da \, db$$

12-J Scalogram

Scalogram 即 Wavelet transform 的絕對值平方

$$Sc_x(a,b) = \left|X_w(a,b)\right|^2 = \frac{1}{|b|}\left|\int_{-\infty}^{\infty} x(t)\psi\left(\frac{t-a}{b}\right)dt\right|^2$$

有時，會將 Scalogram 定義成

$$Sc_x(a,\zeta) = \left|X_w\left(a,\frac{\eta}{\zeta}\right)\right|^2$$

$$\eta = \frac{\int_0^{\infty} f\left|\Psi(f)\right|^2 df}{\int_0^{\infty}\left|\Psi(f)\right|^2 df}$$

$$\Psi(f) = \int_{-\infty}^{\infty}\psi(t)e^{-j2\pi f t}dt$$

12-K Problems

Problems of the continuous WT

(1) hard to implement

(2) hard to find $\phi(t)$

Continuous WT is good in mathematics.

In practical, the discrete WT and the continuous WT with discrete coefficients are more useful.

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馬佐平 (微電子，2012年當選院士)
張懋中 (電子元件，2012年當選院士)
林本堅 (積體電路與傅氏光學，2014年當選院士)
陳陽闓 (高速半導體，2016年當選院士)
王康隆 (自旋電子學，2016年當選院士)
李琳山 (語音訊號處理，2016年當選院士)
戴聿昌 (微積電系統與醫工，2016年當選院士)
張世富 (多媒體信號處理，2018年當選院士)
盧志遠 (半導體技術，2018年當選院士)

註：歷年中研院院士當中，屬於電機+資訊相關領域的有37人，佔了全部的 7.7 %

其中和通信、信號處理、影像處理相關的有9位，大多是2004年以後當選院士