

XIII. Continuous WT with Discrete Coefficients

13-A Definition

The parameters a and b are not chosen arbitrarily.

For example,

$$a = n2^{-m} \quad \text{and} \quad b = 2^{-m}.$$

$$X_w(n, m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

$$\begin{array}{l} n \in \mathbb{Z}, \quad n \in (-\infty, \infty) \\ m \in \mathbb{Z}, \quad m \in (-\infty, \infty) \end{array}$$

註：某些文獻把這個式子稱作是 discrete wavelet transform，實際上仍然是 continuous wavelet transform 的特例

- Main reason for constrain a and b to be $n2^{-m}$ and 2^{-m} :

Easy to implementation

$X_w(n, m)$ can be computed from $X_w(n, m-1)$ by digital convolution.

13-B Inverse Wavelet Transform

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$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi(2^m t - n) X_w(n, m)$$

$$\begin{aligned} x(t) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi(2^m t - n) 2^{m/2} \int_{-\infty}^{\infty} x(t_1) \psi(2^m t_1 - n) dt_1 \\ &= \int_{-\infty}^{\infty} \left\{ \sum_{m_1=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^m \psi(2^m t - n) \psi(2^{m_1} t_1 - n) \right\} x(t_1) dt_1 \end{aligned}$$

since $x(t) = \int_{-\infty}^{\infty} \delta(t - t_1) x(t_1) dt_1$

Constraint: $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^m \psi(2^m t - n) \psi(2^{m_1} t_1 - n) = \delta(t - t_1)$

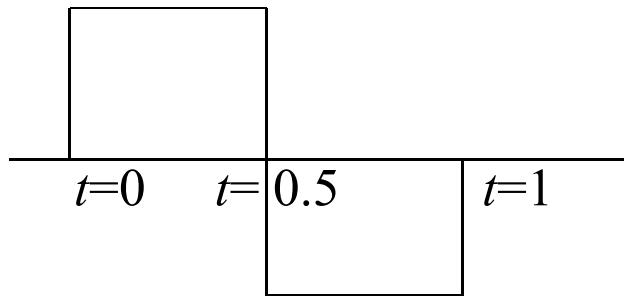
i.e., $\int_{-\infty}^{\infty} 2^m \psi(2^{m_1} t - n_1) \psi(2^m t - n) dt = \delta(m - m_1) \delta(n - n_1)$

should be satisfied.

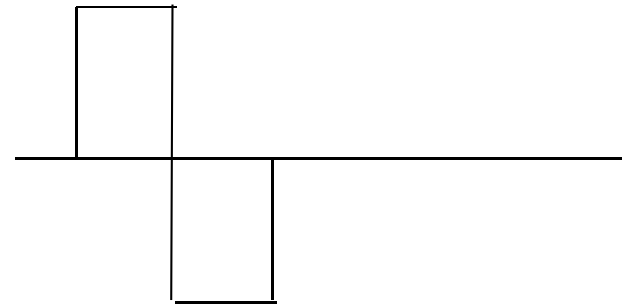
13-C Haar Wavelet

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$\psi(t)$ mother wavelet
(wavelet function)



$\psi(2t)$

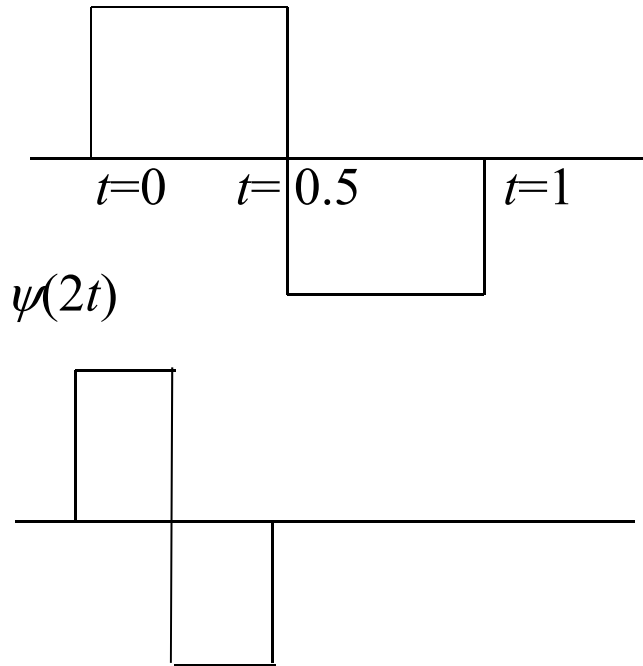


The Haar wavelet satisfies

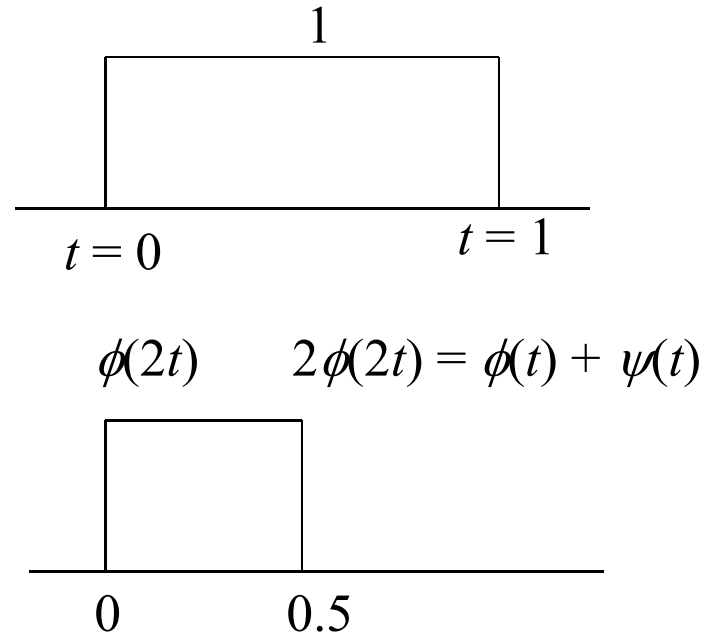
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$$2^m \int_{-\infty}^{\infty} \psi(2^{m_1} t - n_1) \psi(2^m t - n) dt = \delta(m - m_1) \delta(n - n_1)$$

$\psi(t)$ mother wavelet
(wavelet function)

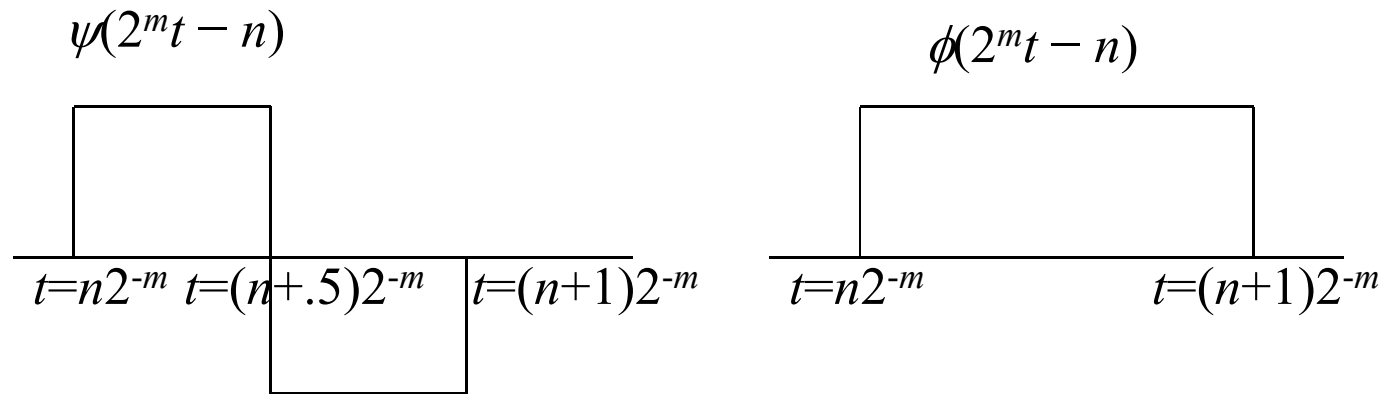


$\phi(t)$ scaling function



$$\phi(t) = \phi(2t) + \phi(2t-1)$$

$$\psi(t) = \phi(2t) - \phi(2t-1)$$



- Advantages of Haar wavelet
 - (1) Simple
 - (2) Fast algorithm
 - (3) Orthogonal \rightarrow reversible
 - (4) Compact, real, odd
- Disadvantages of Haar wavelet

Vanish moment =

Properties

(1) 任何 function 都可以由 $\phi(t), \phi(2t), \phi(4t), \phi(8t), \phi(16t), \dots$ 以及它們的位移所組成



(2) 任何平均為 0 的 function 都可以由 $\psi(t), \psi(2t), \psi(4t), \psi(8t), \psi(16t), \dots$ 所組成

換句話說..... 任何 function 都可以由 constant, $\psi(t), \psi(2t), \psi(4t), \psi(8t), \psi(16t), \dots$ 所組成

(4) 不同寬度 (也就是不同 m) 的 wavelet / scaling functions 之間會有一個關係

$$\phi(t) = \phi(2t) + \phi(2t - 1)$$

$$\phi(t - n) = \phi(2t - 2n) + \phi(2t - 2n - 1)$$

$$\phi(2^m t - n) = \phi(2^{m+1} t - 2n) + \phi(2^{m+1} t - 2n - 1)$$

$$\psi(t) = \phi(2t) - \phi(2t - 1)$$

$$\psi(t - n) = \phi(2t - 2n) - \phi(2t - 2n - 1)$$

$$\psi(2^m t - n) = \phi(2^{m+1} t - 2n) - \phi(2^{m+1} t - 2n - 1)$$

(5) 可以用 $m+1$ 的 coefficients 來算 m 的 coefficients

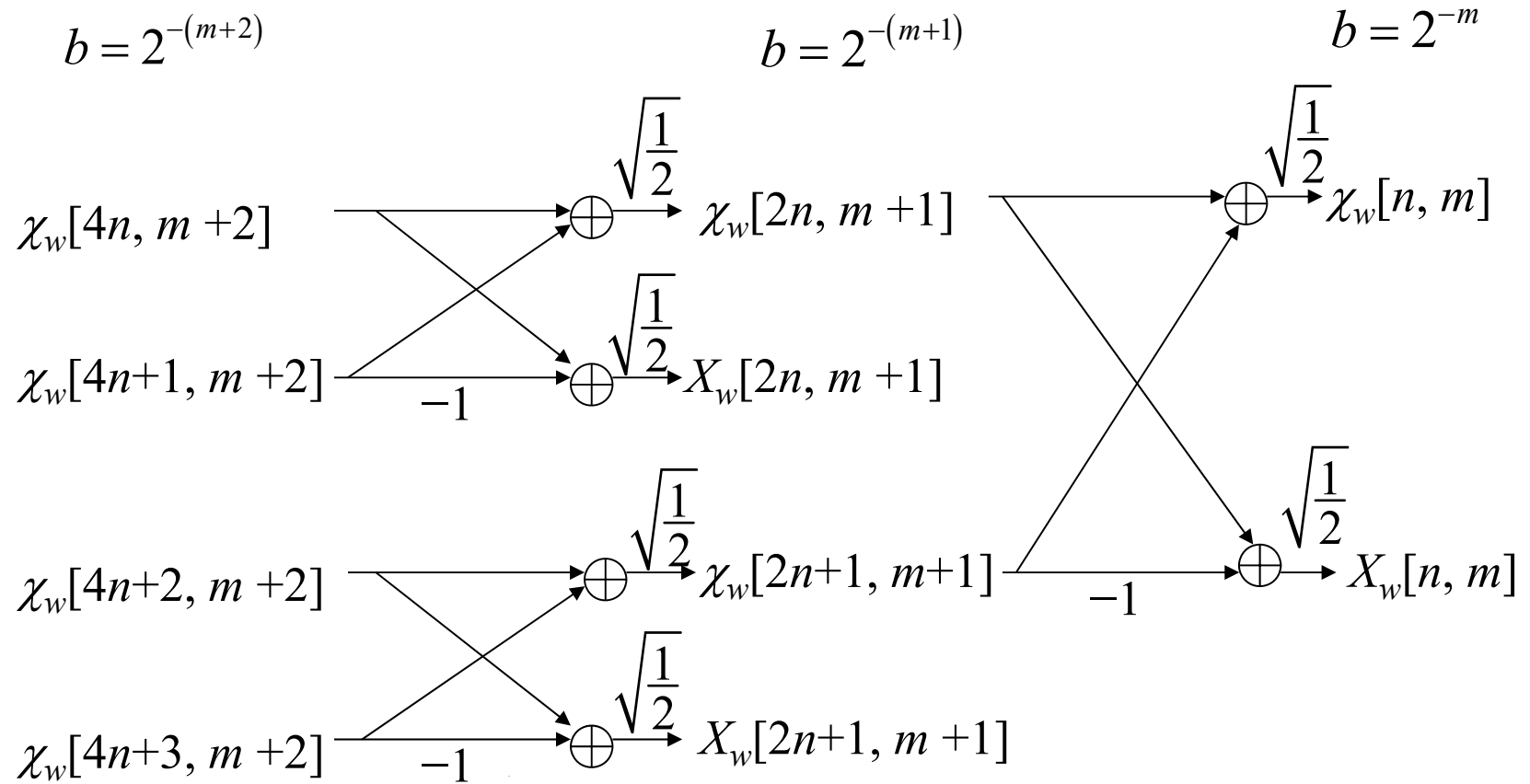
$$\text{若 } \chi_w(n, m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \phi(2^m t - n) dt$$

$$\begin{aligned} \chi_w(n, m) &= 2^{m/2} \int_{-\infty}^{\infty} x(t) \phi(2^{m+1} t - 2n) dt + 2^{m/2} \int_{-\infty}^{\infty} x(t) \phi(2^{m+1} t - 2n - 1) dt \\ &= \sqrt{\frac{1}{2}} (\chi_w(2n, m+1) + \chi_w(2n+1, m+1)) \end{aligned}$$

$$X_w(n, m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

$$\begin{aligned} X_w(n, m) &= 2^{m/2} \int_{-\infty}^{\infty} x(t) \phi(2^{m+1} t - 2n) dt - 2^{m/2} \int_{-\infty}^{\infty} x(t) \phi(2^{m+1} t - 2n - 1) dt \\ &= \sqrt{\frac{1}{2}} (\chi_w(2n, m+1) - \chi_w(2n+1, m+1)) \end{aligned}$$

layer:



structure of [multiresolution analysis \(MRA\)](#)

13-D General Methods to Define the Mother Wavelet and the Scaling Function

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- Constraints:
- (a) nearly compact support
 - (b) fast algorithm
 - (c) real
 - (d) vanish moment
 - (e) orthogonal

和 continuous wavelet transform 比較：

- (1) compact support 放寬為 “near compact support”
- (2) 沒有 even, odd symmetric 的限制
- (3) 由於只要是 complete and orthogonal, 必定可以 reconstruction
所以不需要 admissibility criterion 的限制
- (4) 多了對 fast algorithm 的要求

Higher and lower resolutions 的 recursive relation 的一般化

$$\phi(t) = 2 \sum_k g_k \phi(2t - k) \quad \text{稱作 dilation equation}$$

$$\psi(t) = 2 \sum_k h_k \phi(2t - k)$$

$\psi(t)$: mother wavelet, $\phi(t)$: scaling function

這些關係式成立，才有 fast algorithms

$$\phi(t) = 2 \sum_k g_k \phi(2t - k)$$

$$\psi(t) = 2 \sum_k h_k \phi(2t - k)$$

If $\chi_w(n, m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \phi(2^m t - n) dt$

then
$$\begin{aligned} \chi_w(n, m) &= \sum_k 2^{\frac{m}{2}+1} \int_{-\infty}^{\infty} x(t) g_k \phi(2^{m+1} t - 2n - k) dt \\ &= 2^{\frac{1}{2}} \sum_k g_k \chi_w(2n + k, m + 1) \end{aligned}$$

If $X_w(n, m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$

then
$$\begin{aligned} X_w(n, m) &= \sum_k 2^{\frac{m}{2}+1} \int_{-\infty}^{\infty} x(t) h_k \phi(2^{m+1} t - 2n - k) dt \\ &= 2^{\frac{1}{2}} \sum_k h_k \chi_w(2n + k, m + 1) \end{aligned}$$

(Step 1) convolution

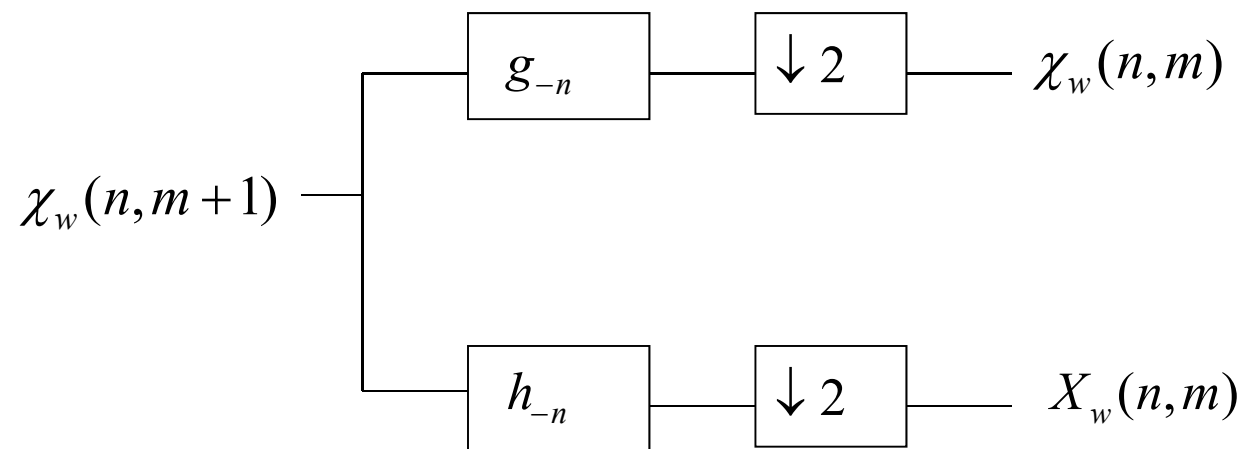
$$\tilde{\chi}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{g}_k \chi_w(n-k, m+1) \quad \tilde{g}_k = g_{-k}$$

$$\tilde{X}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{h}_k \chi_w(n-k, m+1) \quad \tilde{h}_k = h_{-k}$$

(Step 2) down sampling

$$\chi_w(n, m) = \tilde{\chi}_w(2n)$$

$$X_w(n, m) = \tilde{X}_w(2n)$$



m 越大，越屬於細節

- To satisfy $\phi(t) = 2 \sum_k g_k \phi(2t - k)$,

$$\phi(t/2) = 2 \sum_k g_k \phi(t - k)$$

$$\begin{array}{ccc} \text{FT} \downarrow & & \text{FT} \downarrow \\ & & \end{array}$$

$$2\Phi(2f) = 2G(f)\Phi(f)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

where $\Phi(f) = FT[\phi(t)] = \int_{-\infty}^{\infty} \phi(t) e^{-j2\pi f t} dt$

$$G(f) = FT\left[\sum_k g_k \delta(t - k)\right]$$

$$= \sum_k g_k \int_{-\infty}^{\infty} \delta(t - k) e^{-j2\pi f t} dt$$

$$= \sum_k g_k e^{-j2\pi f k}$$

$\Phi(f)$ 是 $\phi(t)$ 的 continuous Fourier transform

$G(f)$ 是 $\{g_k\}$ 的 discrete time Fourier transform

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)G\left(\frac{f}{8}\right)\Phi\left(\frac{f}{8}\right) = \dots$$

$$\Phi(f) = \Phi\left(\frac{f}{2^\infty}\right)\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right) = \Phi(0)\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$$

↑
連乘

$$\Phi(0) = \int_{-\infty}^{\infty} \phi(t) dt \quad (\text{可以藉由 normalization, 讓 } \Phi(0) = 1)$$

$$\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$$

若 $G(f)$ 決定了，則 $\Phi(f)$ 可以被算出來

$G(f)$: 被稱作 generating function

constraint 1

- 同理

$$\psi(t) = 2 \sum_k h_k \phi(2t - k)$$

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi f t} dt$$

$$\psi(t/2) = 2 \sum_k h_k \phi(t - k)$$

$$\Psi(f) = H\left(\frac{f}{2}\right) \Phi\left(\frac{f}{2}\right)$$

$$H(f) = \sum_k h_k e^{-j2\pi f k}$$

$$\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

constraint 2

- 另外，由於

$$\Phi(f) = G\left(\frac{f}{2}\right) \Phi\left(\frac{f}{2}\right)$$

$$\Phi(0) = G(0) \Phi(0) \quad (f=0 \text{ 代入})$$

$$G(0) = 1$$

必需滿足

constraint 3

13-F Real Coefficient Constraints

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Since $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ $\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

If $G(f) = G^*(-f)$ $H(f) = H^*(-f)$ are satisfied,

constraint 4

constraint 5

then $\Phi(f) = \Phi^*(-f)$, $\Psi(f) = \Psi^*(-f)$, and $\phi(t)$, $\psi(t)$ are real.

Note: If these constraints are satisfied, g_k , h_k on page 427 are also real.

13-G Vanish Moment Constraint

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If $\psi(t)$ has p vanishing moments,

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0 \quad \text{for } k = 0, 1, 2, \dots, p-1$$

Since $FT[t^k \psi(t)] = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k} \Psi(f)$

$$\int_{-\infty}^{\infty} x(t) dt = X(0) \quad \text{if } X(f) = FT(x(t))$$

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0 \quad \Longrightarrow \quad \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k} \Psi(f) \Big|_{f=0} = 0$$

Therefore, $\frac{d^k}{df^k} \Psi(f) \Big|_{f=0} = 0$ for $k = 0, 1, 2, \dots, p-1$

Taking the conjugation on both sides, $\frac{d^k}{df^k} \Psi(f) \Big|_{f=0} = 0$ for $k = 0, 1, 2, \dots, p-1$

Since $\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

if $\frac{d^k}{df^k} H(f) \Big|_{f=0} = 0$ for $k = 0, 1, 2, \dots, p-1$ is satisfied,

constraint 6

then $\frac{d^k}{df^k} \Psi(f) \Big|_{f=0} = 0$ for $k = 0, 1, 2, \dots, p-1$ are satisfied

and the wavelet function has p vanishing moments.

13-H Orthogonality Constraints

- orthogonality constraint:

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1} t - n_1) \psi(2^m t - n) dt = \delta(m - m_1) \delta(n - n_1)$$

$\psi(t)$: wavelet function

If the above equality is satisfied,

forward wavelet transform:

$$X_w(n, m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

inverse wavelet transform:

$$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi(2^m t - n) X_w(m, n)$$

(much easier for inverse)

$C = \text{mean of } x(t)$

(證明於後頁)

If
$$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi(2^m t - n) X_w(m, n)$$

and
$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1} t - n_1) \psi(2^m t - n) dt = \delta(m - m_1) \delta(n - n_1),$$

then
$$2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} \left[C + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \psi(2^{m_1} t - n_1) X_w(m_1, n_1) \right] \psi(2^m t - n) dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} C \psi(2^m t - n) dt + 2^{m/2} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \int_{-\infty}^{\infty} \psi(2^{m_1} t - n_1) \psi(2^m t - n) dt X_w(m_1, n_1)$$

$$= 0 + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \delta(m_1 - m) \delta(n_1 - n) X_w(m_1, n_1)$$

$$= X_w(m, n)$$

due to $\int_{-\infty}^{\infty} \psi(t) dt = 0$

Therefore, $2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$ is the inverse operation of

$$C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi(2^m t - n) X_w(m, n) \quad \#$$

※ 要滿足

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1} t - n_1) \psi(2^m t - n) dt = \delta(m - m_1) \delta(n - n_1)$$

之前，需要滿足以下三個條件

$$(1) \quad \int_{-\infty}^{\infty} \psi(t - n_1) \psi(t - n) dt = \delta(n_1 - n) \quad \text{for mother wavelet}$$

這個條件若滿足， $\int_{-\infty}^{\infty} 2^m \psi(2^m t - n_1) \psi(2^m t - n) dt = \delta(n - n_1)$

對所有的 m 皆成立

$$(2) \quad \int_{-\infty}^{\infty} \phi(t - n_1) \phi(t - n) dt = \delta(n_1 - n) \quad \text{for scaling function}$$

嚴格來說，這並不是必要條件，但是可以簡化第(3)個條件的計算

$$(3) \int_{-\infty}^{\infty} \psi(t - n_1) \psi(2^{-k}t - n) dt = 0 \quad \text{for any } n, n_1 \quad \text{if } k > 0$$

若 (1) 和 (3) 的條件滿足，則

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t - n_1) \psi(2^m t - n) dt = \delta(m - m_1) \delta(n - n_1)$$

也將滿足

(Proof): Set $t_1 = 2^m t$, $dt_1 = 2^m dt$

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t - n_1) \psi(2^m t - n) dt = \int_{-\infty}^{\infty} \psi(2^{m_1-m}t_1 - n_1) \psi(t_1 - n) dt_1$$

If (3) is satisfied,

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t - n_1) \psi(2^m t - n) dt = 0 \quad \text{when } m \neq m_1$$

In the case where $m = m_1$, if (1) is satisfied, then

$$\int_{-\infty}^{\infty} 2^m \psi(2^m t - n_1) \psi(2^m t - n) dt = \int_{-\infty}^{\infty} \psi(t_1 - n_1) \psi(t_1 - n) dt_1 = \delta(n_1 - n)$$

#

- 由 Page 420 的條件 (1)

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \psi(t - n_1) \psi(t - n) dt \\
 &= \int_{-\infty}^{\infty} e^{-j2\pi n_1 f} \Psi(f) e^{j2\pi n f} \Psi^*(f) df \\
 &= \int_{-\infty}^{\infty} e^{j2\pi(n - n_1)f} \Psi(f) \Psi^*(f) df \\
 &= \sum_{p=-\infty}^{\infty} \int_0^1 e^{j2\pi(n - n_1)(f' + p)} \Psi(f' + p) \Psi^*(f' + p) df' \\
 &= \int_0^1 e^{j2\pi(n - n_1)f'} \sum_{p=-\infty}^{\infty} |\Psi(f' + p)|^2 df' = \delta(n - n_1)
 \end{aligned}$$

Parseval's theorem
 $\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$

$e^{j2\pi(n - n_1)(f' + p)} = e^{j2\pi(n - n_1)f'}$
 if p is an integer

Therefore,

$$\int_0^1 e^{-j2\pi n_2 f'} \sum_{p=-\infty}^{\infty} |\Psi(f' + p)|^2 df' = \delta(-n_2) = \delta(n_2)$$

$$\sum_{p=-\infty}^{\infty} |\Psi(f' + p)|^2 = 1$$

for all f' should be satisfied

- 同理，由 Page 420 的條件 (2)

$$\int_{-\infty}^{\infty} \phi(t - n_1) \phi(t - n) dt = \delta(n_1 - n) \quad \text{for scaling function}$$

↓ 推導過程類似 page 422

$$\sum_{p=-\infty}^{\infty} |\Phi(f + p)|^2 = 1$$

for all f should be satisfied

衍生的條件：將 $\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty} |\Psi(f+p)|^2 = 1$

(page 422)

$$\sum_{p=-\infty}^{\infty} \left| H\left(\frac{f}{2} + \frac{p}{2}\right) \Phi\left(\frac{f}{2} + \frac{p}{2}\right) \right|^2 = 1$$

$$\sum_{q=-\infty}^{\infty} \left| H\left(\frac{f}{2} + q\right) \Phi\left(\frac{f}{2} + q\right) \right|^2 + \sum_{q=-\infty}^{\infty} \left| H\left(\frac{f}{2} + q + \frac{1}{2}\right) \Phi\left(\frac{f}{2} + q + \frac{1}{2}\right) \right|^2 = 1$$

因為 h_k 是 discrete sequence, $H(f)$ 是 h_k 的 discrete-time Fourier transform

$$H(f) = H(f+1) = H(f+2) = \dots$$

$$\left| H\left(\frac{f}{2}\right) \right|^2 \sum_{q=-\infty}^{\infty} \left| \Phi\left(\frac{f}{2} + q\right) \right|^2 + \left| H\left(\frac{f}{2} + \frac{1}{2}\right) \right|^2 \sum_{q=-\infty}^{\infty} \left| \Phi\left(\frac{f}{2} + q + \frac{1}{2}\right) \right|^2 = 1$$

$$\left| H\left(\frac{f}{2}\right) \right|^2 \sum_{q=-\infty}^{\infty} \left| \Phi\left(\frac{f}{2} + q\right) \right|^2 + \left| H\left(\frac{f}{2} + \frac{1}{2}\right) \right|^2 \sum_{q=-\infty}^{\infty} \left| \Phi\left(\frac{f}{2} + q + \frac{1}{2}\right) \right|^2 = 1$$

因為 $\sum_{p=-\infty}^{\infty} \left| \Phi(f + p) \right|^2 = 1$ for all f

(page 422 的條件)

$$\left| H\left(\frac{f}{2}\right) \right|^2 + \left| H\left(\frac{f}{2} + \frac{1}{2}\right) \right|^2 = 1$$

$$\boxed{\left| H(f) \right|^2 + \left| H\left(f + \frac{1}{2}\right) \right|^2 = 1}$$

constraint 7

同理，將 $\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$
(page 422)

經過運算可得

$$|G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1$$

constraint 8

• Page 421 條件 (3) 的處理

由於

$\psi(2^{-k}t - n)$ 是 $\phi(2^{-k+1}t - n_1)$ 的 linear combination

$$\psi(t) = 2 \sum_k h_k \phi(2t - k)$$

$\phi(2^{-k+1}t - n_1)$ 是 $\phi(2^{-k+2}t - n_2)$ 的 linear combination

$$\phi(t) = 2 \sum_k g_k \phi(2t - k)$$

$\phi(2^{-k+2}t - n_2)$ 是 $\phi(2^{-k+3}t - n_3)$ 的 linear combination

⋮

⋮

$\phi(2^{-1}t - n_{k-1})$ 是 $\phi(t - n_k)$ 的 linear combination

所以

$\psi(2^{-k}t - n)$ 必定可以表示成 $\phi(t - n_k)$ 的 linear combination

$$\psi(2^{-k}t - n) = \sum_{n_k} b_{n_k} \phi(t - n_k)$$

$$\psi(2^{-k}t - n) = \sum_{n_k} b_{n_k} \phi(t - n_k)$$

所以，若 $\int_{-\infty}^{\infty} \psi(t - n_1) \phi(t - n_k) dt = 0$ for any n_1, n_k 可以滿足

則 $\int_{-\infty}^{\infty} \psi(t - n_1) \psi(2^{-k}t - n) dt = 0$ for any n_1, n_k 必定能夠成立

Page 421 條件 (3) 可改寫成

$$\int_{-\infty}^{\infty} \psi(t - n_1) \phi(t - n_k) dt = 0$$

$$\int_{-\infty}^{\infty} \psi(t) \phi(t - \tau) dt = 0 \quad (\text{將 } t - n_1 \text{ 變成 } t, \quad \tau = n_k - n_1)$$

$$\int_{-\infty}^{\infty} \Psi(f) \Phi^*(f) e^{j2\pi\tau f} df = 0 \quad (\text{from Parseval's theorem})$$

$$\int_{-\infty}^{\infty} \Psi(f) \Phi^*(f) e^{j2\pi\tau f} df = 0$$

$$\text{Since } \Psi(f) = H\left(\frac{f}{2}\right) \Phi\left(\frac{f}{2}\right) \quad \Phi(f) = G\left(\frac{f}{2}\right) \Phi\left(\frac{f}{2}\right)$$

$$\int_{-\infty}^{\infty} H\left(\frac{f}{2}\right) G^*\left(\frac{f}{2}\right) \left|\Phi\left(\frac{f}{2}\right)\right|^2 e^{j2\pi\tau f} df = 0$$

$$\sum_{p=-\infty}^{\infty} \int_0^1 H\left(\frac{f+p}{2}\right) G^*\left(\frac{f+p}{2}\right) \left|\Phi\left(\frac{f+p}{2}\right)\right|^2 e^{j2\pi\tau(f+p)} df = 0$$

$$e^{j2\pi\tau(f+p)} = e^{j2\pi\tau f} \quad (\text{since from page 428, } \tau \text{ is an integer})$$

$$\sum_{q=-\infty}^{\infty} \int_0^1 H\left(\frac{f}{2} + q\right) G^*\left(\frac{f}{2} + q\right) \left|\Phi\left(\frac{f}{2} + q\right)\right|^2 e^{j2\pi\tau f} df$$

$$+ \sum_{q=-\infty}^{\infty} \int_0^1 H\left(\frac{f}{2} + q + \frac{1}{2}\right) G^*\left(\frac{f}{2} + q + \frac{1}{2}\right) \left|\Phi\left(\frac{f}{2} + q + \frac{1}{2}\right)\right|^2 e^{j2\pi\tau f} df = 0$$

Since $H(f) = H(f+1) = H(f+2) = \dots$

$G(f) = G(f+1) = G(f+2) = \dots$

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\int_0^1\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q\right)\right|^2e^{j2\pi\tau f}df$$

$$+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)\int_0^1\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2e^{j2\pi\tau f}df=0$$

Since $\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2=1$ for all f (page 422)

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)=0$$

$$H(f)G^*(f)+H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right)=0$$

constraint 9

整理：設計 mother wavelet 和 scaling function 的九大條件
(皆由 page 406 的 constraints 衍生而來)

- (1) $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm , page 412
- (2) $\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm , page 413
- (3) $G(0) = 1$ for fast algorithm , page 413
- (4) $H(f) = H^*(-f)$ for real , page 414
- (5) $G(f) = G^*(-f)$ for real , page 413
- (6) $\left. \frac{d^k}{df^k} H(f) \right|_{f=0} = 0$ for p vanish moments , page 416
for $k = 0, 1, \dots, p-1$

$$(7) \quad |H(f)|^2 + |H\left(f + \frac{1}{2}\right)|^2 = 1 \quad \text{for orthogonal, page 425}$$

$$(8) \quad |G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1 \quad \text{for orthogonal, page 426}$$

$$(9) \quad H(f)G^*(f) + H\left(f + \frac{1}{2}\right)G^*\left(f + \frac{1}{2}\right) = 0 \quad \text{for orthogonal, page 430}$$

$G(f)$
 $H(f)$ are the discrete-time Fourier transform of $\{g_k\}$
 $\{h_k\}$ on page 407.

- 條件的簡化

Specially, if we set that

$$H(f) = -e^{-j2\pi f} G^*(f + 1/2) \quad h_k = (-1)^k g_{1-k}$$

when the following constraints are satisfied:

$$|G(f)|^2 + |G(f + \frac{1}{2})|^2 = 1$$

$$G(f) = G^*(-f) \quad (\text{條件 (5), (8) 滿足})$$

then $|H(f)|^2 + |H(f + \frac{1}{2})|^2 = |G(f + \frac{1}{2})|^2 + |G(f)|^2 = 1$

$$\begin{aligned} & H(f)G^*(f) + H(f + \frac{1}{2})G^*(f + \frac{1}{2}) \\ &= -e^{-j2\pi f} G^*(f + \frac{1}{2})G^*(f) - e^{-j2\pi(f + \frac{1}{2})} G^*(f)G^*(f + \frac{1}{2}) \\ &= -e^{-j2\pi f} G^*(f + \frac{1}{2})G^*(f) + e^{-j2\pi f} G^*(f)G^*(f + \frac{1}{2}) = 0 \end{aligned}$$

$$H^*(-f) = -e^{-j2\pi f} G(-f + 1/2) = -e^{-j2\pi f} G^*(f - 1/2) = H(f)$$

條件 (4), (7), (9) 也將滿足

整理：設計 mother wavelet 和 scaling function 的幾個要求 (簡化版)

- (1) $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm
- (2) $\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm
- (3) $G(0) = 1$ for fast algorithm
- (4) $G(f) = G^*(-f)$ for real
- (5) $\left. \frac{d^k}{df^k} H(f) \right|_{f=0} = 0$ for p vanish moments
for $k = 0, 1, \dots, p-1$
- (6) $|G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1$ for orthogonal
- (7) $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$

設計時，只要 $G(f)$ ($0 \leq f \leq 1/4$) 決定了，mother wavelet 和 scaling function 皆可決定

$G(f)$: 被稱作 generating function

Design Process (設計流程):

(Step 1): 給定 $G(f)$ ($0 \leq f \leq 1/4$)，滿足以下的條件

(a) $G(0) = 1$

(b) $\left. \frac{d^k}{df^k} G(f) \right|_{f=\frac{1}{2}} = 0$ for $k = 0, 1, 2, \dots, p-1$

(Step 2) 由 $G(f) = G^*(-f)$ 決定 $G(f)$ ($3/4 \leq f < 1$)

(Step 3) 由 $|G(f)|^2 + |G(f + \frac{1}{2})|^2 = 1$ 決定 $G(f)$ ($1/4 < f < 3/4$)

再根據 $G(f) = G(f+1)$ ，決定所有的 $G(f)$ 值

(Step 4) 由 $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$ 決定 $H(f)$

(Step 5) 由 $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$

$\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ 決定 $\Phi(f)$, $\Psi(f)$

註：(1) 當 Step 1 的兩個條件滿足，由於 $|G(f)|^2 + |G(f+1/2)|^2 = 1$

$$\left. \frac{d^k}{df^k} G(f) \right|_{f=1/2} = 0 \quad \text{for } k = 0, 1, 2, \dots, p-1$$

又由於 $H(f) = -e^{-j2\pi f} G^*(f+1/2)$

$$\left. \frac{d^k}{df^k} H(f) \right|_{f=0} = 0 \quad \text{for } k = 0, 1, 2, \dots, p-1$$

$$(2) \quad |G(f)|^2 + |G(f+1/2)|^2 = 1 \quad |G(f)|^2 = |G(-f)|^2$$

所以當 $G(f)$ ($0 \leq f \leq 1/4$) 給定， $|G(f)|$ 有唯一解

(3) 對於離散信號而言， $G(f) = G(f+1)$
有意義的頻率範圍為 $-1/2 < f < 1/2$

$$G(f) = \sum_k g_k e^{-j2\pi f k}$$

13-K Several Continuous Wavelets with Discrete Coefficients

(1) Haar Wavelet

$$g[0] = 1, g[1] = 1$$

$$G(f) = 1 + \exp(-j2\pi f)$$

$$h[0] = 1, h[1] = -1$$

$$H(f) = 1 - \exp(-j2\pi f)$$

或

$$g[0] = 1/2, g[1] = 1/2$$

$$G(f) = [1 + \exp(-j2\pi f)] / 2$$

$$h[0] = 1/2, h[1] = -1/2$$

$$H(f) = [1 - \exp(-j2\pi f)] / 2$$

vanish moment = ?

(2) Sinc Wavelet

$$G(f) = 1 \quad \text{for } |f| \leq 1/4$$

$$G(f) = 0 \quad \text{otherwise}$$

vanish moment = ?

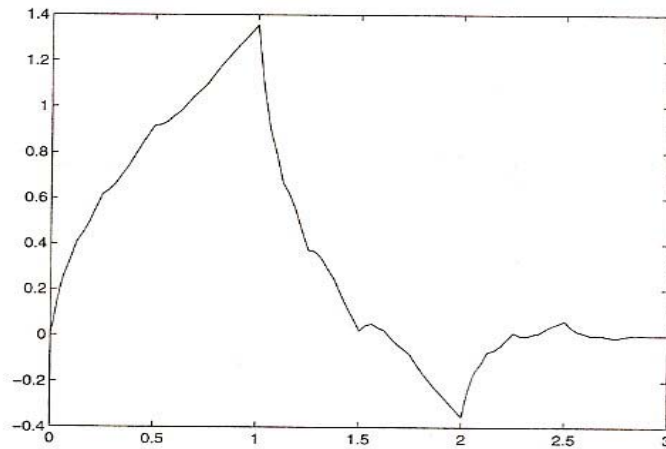
(3) 4-point Daubechies
Wavelet

$$g_k : \left[\frac{1 + \sqrt{3}}{8}, \frac{3 + \sqrt{3}}{8}, \frac{3 - \sqrt{3}}{8}, \frac{1 - \sqrt{3}}{8} \right]$$

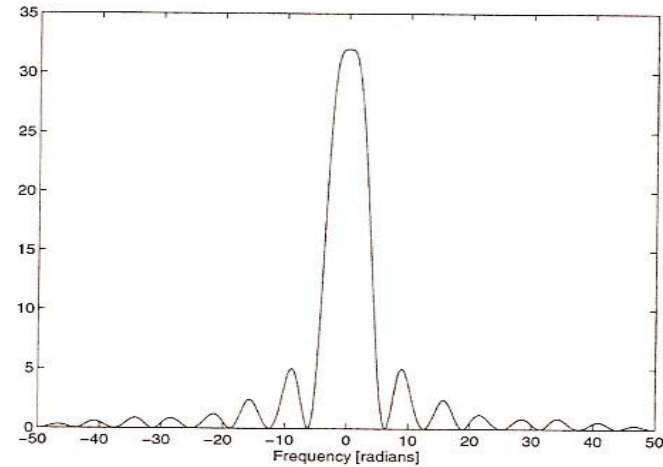
vanish moment = ?

vanish moment VS the number of coefficients

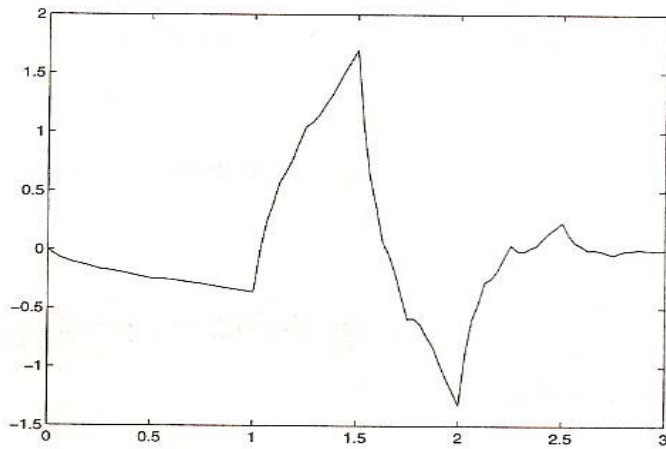
From: S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice Hall, N.J., 1996.



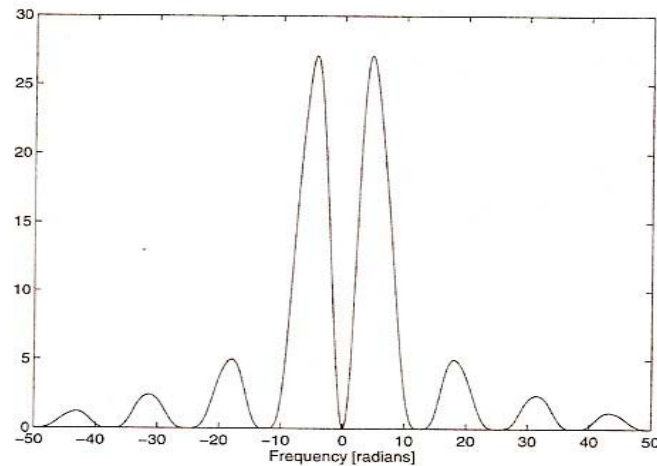
(a) Scaling function $\phi(t)$



(b) $|\Phi(\omega)|$



(c) Daubechies wavelet $\psi(t)$



(d) $|\Psi(\omega)|$

13-L Continuous Wavelet with Discrete Coefficients 優缺點

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- Advantages:

(1) Fast algorithm for MRA

(2) Non-uniform frequency analysis

$$\psi(2^m t - n) \xrightarrow{\text{FT}} 2^{-m} e^{-j2\pi n 2^{-m} f} \Psi(2^{-m} f)$$

(3) Orthogonal

- Disadvantages:

(a) 無限多項連乘

(b) problem of initial

$\chi_w(n, m)$, $X_w(n, m)$ 皆由 $\chi_w(n, m+1)$ 算出

$\chi_w(n, m)|_{m \rightarrow \infty}$ 如何算

(c) 難以保證 compact support

(d) 仍然太複雜

(1) JPEG: 使用 discrete cosine transform (DCT) 和 8×8 blocks 是當前最常用的壓縮格式 (副檔名為 *.jpg 的圖檔都是用 JPEG 來壓縮)

可將圖檔資料量壓縮至原來的 $1/8$ (對灰階影像而言) 或 $1/16$ (對彩色影像而言)

(2) JPEG2000: 使用 discrete wavelet transform (DWT)

壓縮率是 JPEG 的 5 倍左右

(3) JPEG-LS: 是一種 lossless compression

壓縮率較低，但是可以完全重建原來的影像

(4) JPEG2000-LS: 是 JPEF2000 的 lossless compression 版本

(5) JBIG: 針對 bi-level image (非黑即白的影像) 設計的壓縮格式

- (6) GIF: 使用 LZW (Lempel–Ziv–Welch) algorithm (類似字典的建構)
適合卡通圖案和動畫製作，lossless
- (7) PNG: 使用 LZ77 algorithm (類似字典的建構，並使用 sliding window)
lossless
- (8) JPEG XR (又稱 HD Photo): 使用 Integer DCT，lossless
在 lossy compression 的情形下壓縮率可和 JPEG 2000 差不多
- (9) TIFF: 使用標籤，最初是為圖形的印刷和掃描而設計的，lossless