

## II. Short-time Fourier Transform

### II-A Definition

Short-time Fourier transform (STFT)

$$X(t, f) = \int_{-\infty}^{\infty} w(t - \tau) x(\tau) e^{-j2\pi f \tau} d\tau$$

Alternative definition

$$X(t, \omega) = \int_{-\infty}^{\infty} w(t - \tau) x(\tau) e^{-j\omega \tau} d\tau$$

### 參考資料

- [1] S. Qian and D. Chen, [Section 3-1](#) in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.
- [2] S. H. Nawab and T. F. Quatieri, “Short time Fourier transform,” in *Advanced Topics in Signal Processing*, pp. 289-337, Prentice Hall, 1987.

$$\text{STFT} \quad X(t, f) = \int_{-\infty}^{\infty} w(t - \tau) x(\tau) e^{-j2\pi f \tau} d\tau$$

$$X(t, \omega) = \int_{-\infty}^{\infty} w(t - \tau) x(\tau) e^{-j\omega \tau} d\tau$$

**Inverse of the STFT:** To recover  $x(t)$ ,

$$x(t) = w^{-1}(t_1 - t) \int_{-\infty}^{\infty} X(t_1, f) e^{j2\pi f t} df$$

where  $w(t_1 - t) \neq 0$ .

For the alternative definition, the inverse transform is:

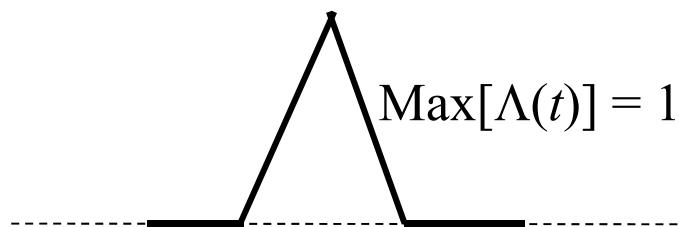
$$x(t) = \frac{1}{2\pi} w^{-1}(t_1 - t) \int_{-\infty}^{\infty} X(t_1, \omega) e^{j\omega t} d\omega$$

The mask function  $w(t)$  always has the property of

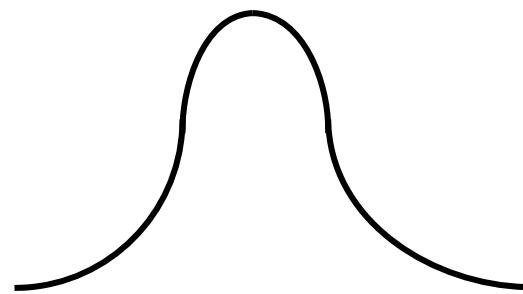
- (a) even:  $w(t) = w(-t)$ , (通常要求這個條件要滿足)
- (b)  $\max(w(t)) = w(0)$ ,  $w(t_1) \geq w(t_2)$  if  $|t_2| > |t_1|$
- (c)  $w(t) \approx 0$  when  $|t|$  is large

$$w(t) = \Lambda(t) \quad (\text{triangular function})$$

$$t = -1 \quad t = 1$$



$$w(t) = \exp(-a|t|^b) \quad (\text{hyper-Laplacian function})$$



## II-B Rec-STFT

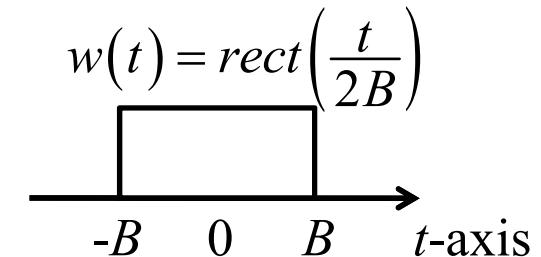
Rectangular mask STFT (rec-STFT)

$$X(t, f) = \int_{t-B}^{t+B} x(\tau) e^{-j2\pi f\tau} d\tau$$

Inverse of the rec-STFT

$$x(t) = \int_{-\infty}^{\infty} X(t_1, f) e^{j2\pi f t} df$$

$$\text{where } t - B < t_1 < t + B$$



The simplest form of the STFT

Other types of the STFT may require more computation time than the rec-STFT.

## II-C Properties of the Rec-STFT

(1) Integration (recovery):

$$(a) \quad \int_{-\infty}^{\infty} X(t, f) e^{j2\pi f v} df = x(v) \quad \text{when } v - B < t < v + B,$$

$$= 0 \quad \text{otherwise}$$

$$(b) \quad \int_{-\infty}^{\infty} X(t, f) df = \int_{t-B}^{t+B} x(\tau) \int_{-\infty}^{\infty} e^{-j2\pi f \tau} df d\tau$$

$$= \int_{t-B}^{t+B} x(\tau) \delta(\tau) d\tau$$

$$= \begin{cases} x(0) & \text{when } t - B < 0 < t + B, \quad -B < t < B \\ 0 & \text{otherwise} \end{cases}$$

**(2) Shifting property (橫的方向移動)**

$$\int_{t-B}^{t+B} x(\tau + \tau_0) e^{-j2\pi f \tau} d\tau = X(t + \tau_0, f) e^{j2\pi f \tau_0}$$

**(3) Modulation property (縱的方向移動)**

$$\int_{t-B}^{t+B} [x(\tau) e^{j2\pi f_0 \tau}] e^{-j2\pi f \tau} d\tau = X(t, f - f_0)$$

**(4) Special inputs:**

(1) When  $x(t) = \delta(t)$ ,

$$X(t, f) = 1 \text{ when } -B < t < B, \quad X(t, f) = 0 \text{ otherwise}$$

(2) When  $x(t) = 1$

$$X(t, f) = 2B \operatorname{sinc}(2Bf) e^{-j2\pi ft}$$

思考：  $B$  值的大小，對解析度的影響是什麼？

## (5) Linearity property

If  $h(t) = \alpha x(t) + \beta y(t)$  and  $H(t, f)$ ,  $X(t, f)$  and  $Y(t, f)$  are their rec-STFTs, then

$$H(t, f) = \alpha X(t, f) + \beta Y(t, f).$$

## (6) Power integration property

$$\int_{-\infty}^{\infty} |X(t, f)|^2 df = \int_{t-B}^{t+B} |x(\tau)|^2 d\tau$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |X(t, f)|^2 df dt = 2B \int_{-\infty}^{\infty} |x(\tau)|^2 d\tau$$

## (7) Energy sum property (Parseval's theorem)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(t, f) Y^*(t, f) df dt = 2B \int_{-\infty}^{\infty} x(\tau) y^*(\tau) d\tau$$

$$\int_{-\infty}^{\infty} X(t, f) Y^*(t, f) df = \int_{t-B}^{t+B} x(\tau) y^*(\tau) d\tau$$

思考：

(1) 哪些性質 Fourier transform 也有？

(2) 其他型態的 STFT 是否有類似的性質？

$$\begin{aligned} \text{Shifting} \quad & \int_{-\infty}^{\infty} w(t-\tau)x(\tau-\tau_0)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} w(t-\tau-\tau_0)x(\tau)e^{-j2\pi f\tau}e^{-j2\pi f\tau_0}d\tau \\ &= X(t-\tau_0, f)e^{-j2\pi f\tau_0} \end{aligned}$$

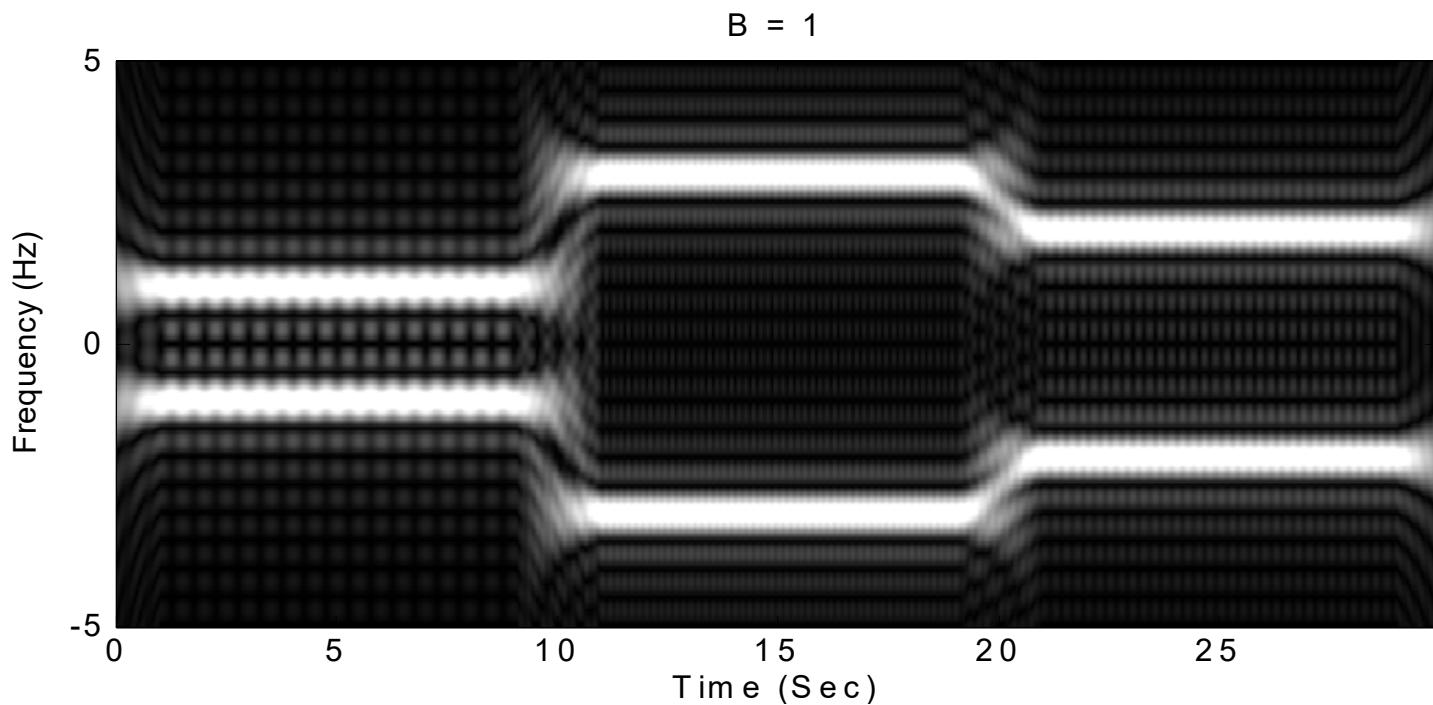
Modulation

$$\int_{-\infty}^{\infty} w(t-\tau)[x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t, f - f_0)$$

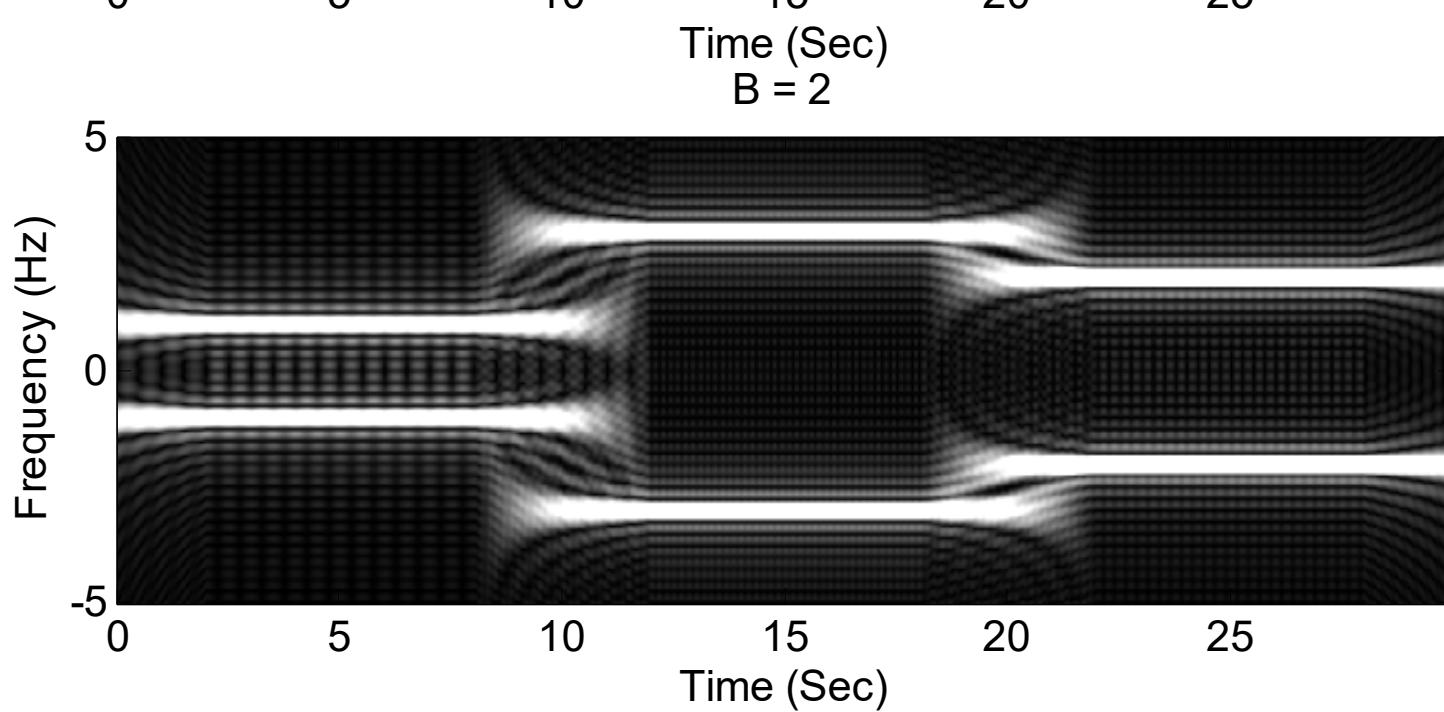
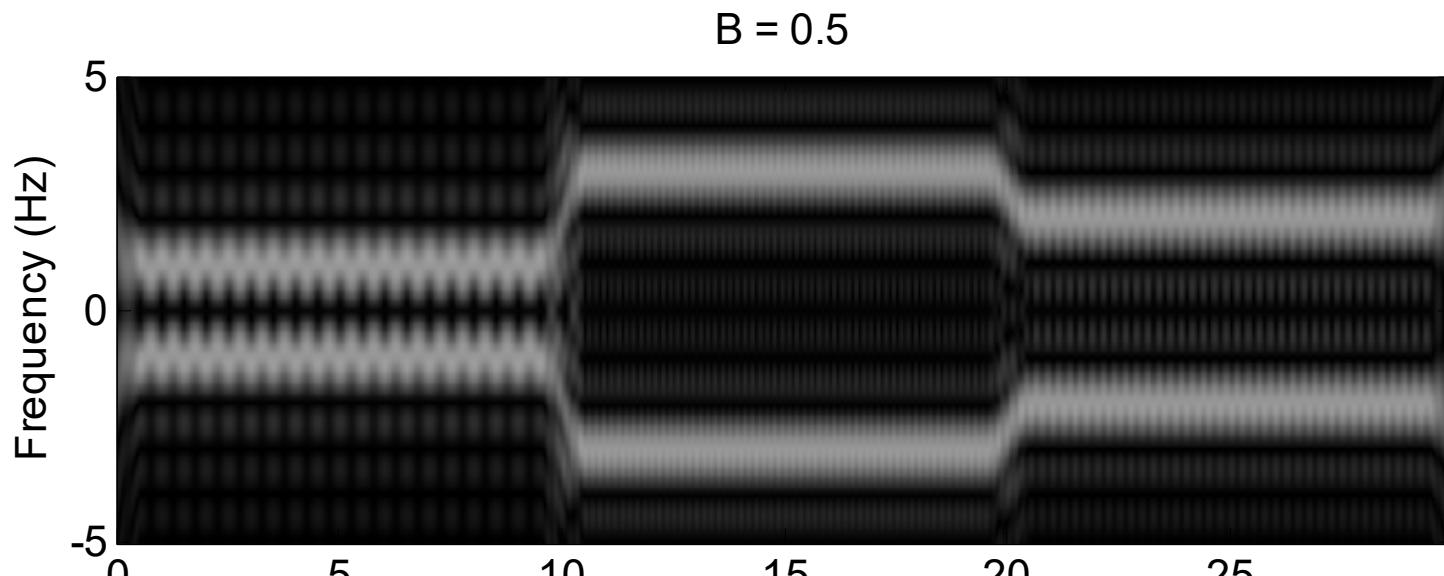
Example:  $x(t) = \cos(2\pi t)$  when  $t < 10$ ,

$x(t) = \cos(6\pi t)$  when  $10 \leq t < 20$ ,

$x(t) = \cos(4\pi t)$  when  $t \geq 20$



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## II-D Advantage and Disadvantage

- Compared with the Fourier transform:

All the time-frequency analysis methods has the advantage of:

The instantaneous frequency can be observed.

All the time-frequency analysis methods has the disadvantage of:

Higher complexity for computation

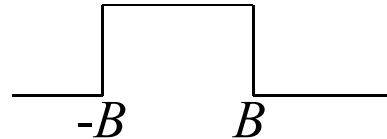
- Compared with other types of time-frequency analysis:

The rec-STFT has an advantage of the least computation time for digital implementation

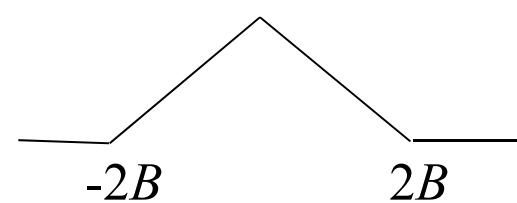
but its performance is worse than other types of time-frequency analysis.

## II-E STFT with Other Windows

(1) Rectangle



(2) Triangle



(3) Hanning

$$w(t) = \begin{cases} 0.5 + 0.5\cos(\pi t / B) & \text{when } |t| \leq B \\ 0 & \text{otherwise} \end{cases}$$

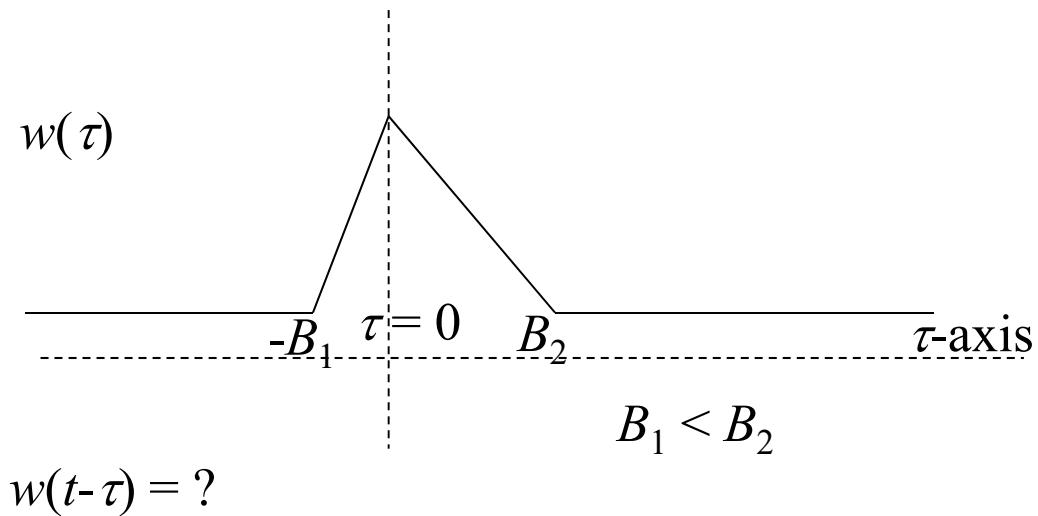
(4) Hamming

$$w(t) = \begin{cases} 0.54 + 0.46\cos(\pi t / B) & \text{when } |t| \leq B \\ 0 & \text{otherwise} \end{cases}$$

(5) Gaussian

$$w(t) = \exp(-\pi\sigma t^2)$$

## (6) Asymmetric window



應用 : seismic wave analysis, collision detection

(The applications that require real-time processing)

動腦思考：

- (1) Are there other ways to choose the mask of the STFT?
- (2) Which mask is better?

沒有一定的答案

## II-F Spectrogram

STFT 的絕對值平方，被稱作 Spectrogram

$$SP_x(t, f) = |X(t, f)|^2 = \left| \int_{-\infty}^{\infty} w(t - \tau) e^{-j2\pi f\tau} x(\tau) d\tau \right|^2$$

比較：spectrum 為 Fourier transform 的絕對值平方

文獻上，spectrogram 這個名詞出現的頻率多於 STFT

但實際上，spectrogram 和 STFT 的本質是相同的

## 附錄三：使用 Matlab 將時頻分析結果 Show 出來

可採行兩種方式：

(1) 使用 mesh 指令畫出立體圖

(但結果不一定清楚，且執行時間較久)

(2) 將 amplitude 變為 gray-level，用顯示灰階圖的方法將結果表現出來

假設 y 是時頻分析計算的結果

`image(abs(y)/max(max(abs(y)))*C)` % C 是一個常數，我習慣選 C=400

或 `image(t, f, abs(y)/max(max(abs(y)))*C)`

`colormap(gray(256))` % 變成 gray-level 的圖

`set(gca, 'Ydir', 'normal')` % 若沒這一行，y-axis 的方向是倒過來的

```
set(gca,'Fontsize',12)    % 改變橫縱軸數值的 font sizes  
xlabel('Time (Sec)','Fontsize',12)      % x-axis  
ylabel('Frequency (Hz)','Fontsize',12)    % y-axis  
title('STFT of x(t)','Fontsize',12)      % title
```

### 計算程式執行時間的指令：

**tic** (這指令如同按下碼錶)

**toc** (show 出碼錶按下後已經執行了多少時間)

註：通常程式執行第一次時，由於要做程式的編譯，所得出的執行時間會比較長

程式執行第二次以後所得出的執行時間，是較為正確的結果

## 附錄四：使用 Python 將時頻分析的圖畫出來

事前安裝模組

pip install numpy

pip install matplotlib

假設y為時頻分析結果(應為二維的矩陣數列)，將 y 以灰階方式畫出來

```
import numpy as np
```

```
import matplotlib.pyplot as plt
```

```
C = 400
```

```
y = np.abs(y) / np.max(np.abs(y)) * C
```

```
plt.imshow(y, cmap='gray', origin='lower')
```

```
# 加上 origin='lower' 避免上下相反
```

```
plt.xlabel('Time (Sec)')
```

```
plt.ylabel('Frequency (Hz)')
```

```
plt.show()
```

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若要加上座標軸數值(在plt.show()之前加上以下程式碼)

```
x_label = ['0', '10', '20', '30'] # 橫軸座標值  
y_label = ['-5', '0', '5'] # 縱軸座標值  
plt.xticks(np.arange(0, x_max, step=int(x_max/(len(x_label)-1))), x_label)  
plt.yticks(np.arange(0, y_max, step=int(y_max/(len(y_label)-1))), y_label)
```

Reference :

[https://matplotlib.org/stable/api/\\_as\\_gen/matplotlib.pyplot.xticks.html](https://matplotlib.org/stable/api/_as_gen/matplotlib.pyplot.xticks.html)