

## V. Wigner Distribution Function

### V-A Wigner Distribution Function (WDF)

Definition 1:  $W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} d\tau$

Definition 2:  $W_x(t, \omega) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j\omega\tau} d\tau$

Another way for computation from the frequency domain

Definition 1: 
$$W_x(t, f) = \int_{-\infty}^{\infty} X(f + \eta / 2) \cdot X^*(f - \eta / 2) e^{j2\pi\eta t} d\eta$$

where  $X(f)$  is the Fourier transform of  $x(t)$

Definition 2: 
$$W_x(t, \omega) = \int_{-\infty}^{\infty} X(\omega + \eta / 2) \cdot X^*(\omega - \eta / 2) e^{j\eta t} d\eta$$

## Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 5, Prentice Hall, N.J., 1996.

## Other References

[Ref] E. P. Wigner, “On the quantum correlation for thermodynamic equilibrium,” *Phys. Rev.*, vol. 40, pp. 749-759, 1932.

[Ref] T. A. C. M. Classen and W. F. G. Mecklenbrauker, “The Wigner distribution—A tool for time-frequency signal analysis; Part I,” *Philips J. Res.*, vol. 35, pp. 217-250, 1980.

[Ref] F. Hlawatsch and G. F. Boudreaux–Bartels, “Linear and quadratic time-frequency signal representation,” *IEEE Signal Processing Magazine*, pp. 21-67, Apr. 1992.

[Ref] R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley-Interscience, NJ, 2004.

## The operators that are related to the WDF:

(a) Signal auto-correlation function:

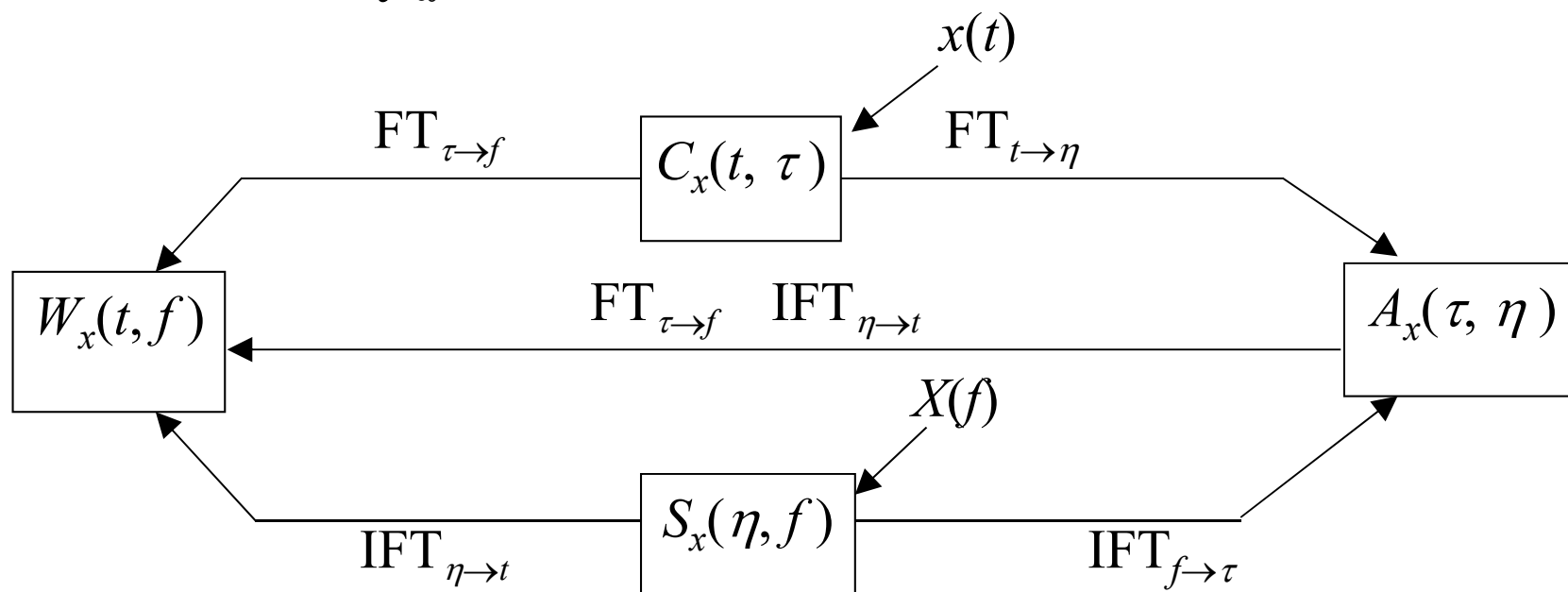
$$C_x(t, \tau) = x(t + \tau/2) \cdot x^*(t - \tau/2)$$

(b) Spectrum auto-correlation function:

$$S_x(\eta, f) = X(f + \eta/2) \cdot X^*(f - \eta/2)$$

(c) Ambiguity function (AF):

$$A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi t\eta} dt$$



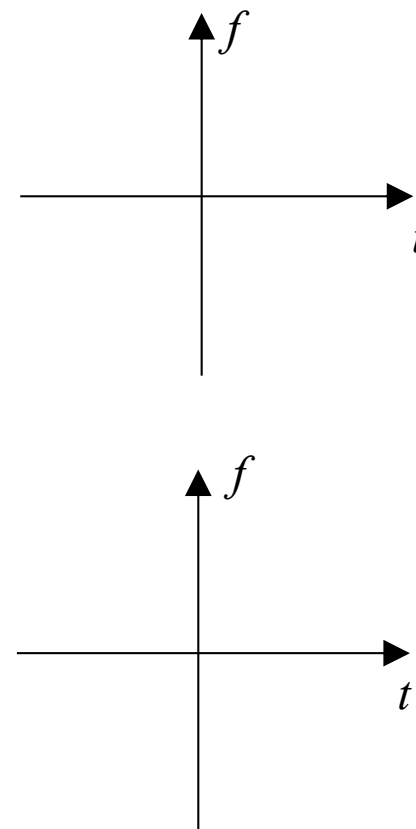
## V-B Why the WDF Has Higher Clarity?

Due to signal auto-correlation function

(1) If  $x(t) = 1$

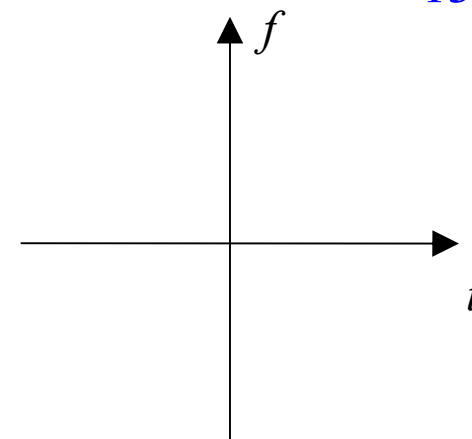
(2) If  $x(t) = \exp(j2\pi h t)$

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} e^{j2\pi h(t+\tau/2)} e^{-j2\pi h(t-\tau/2)} \cdot e^{-j2\pi\tau f} d\tau \\
 &= \int_{-\infty}^{\infty} e^{j2\pi h\tau} \cdot e^{-j2\pi\tau f} d\tau \\
 &= \int_{-\infty}^{\infty} e^{-j2\pi\tau(f-h)} d\tau \\
 &= \delta(f - h)
 \end{aligned}$$



Comparing: for the case of the STFT

(3) If  $x(t) = \exp(j2\pi k t^2)$



(4) If  $x(t) = \delta(t)$

$$W_x(t, f) = \int_{-\infty}^{\infty} \delta(t + \tau/2) \cdot \delta(t - \tau/2) e^{-j2\pi\tau f} d\tau$$

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公式(2)

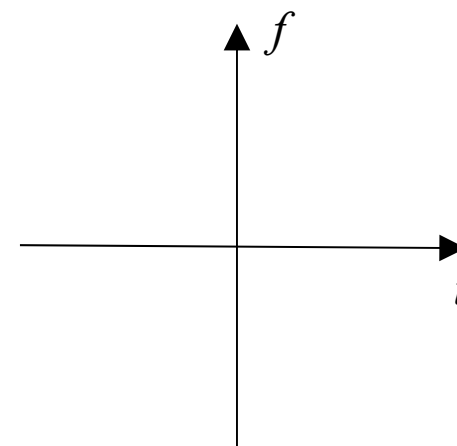
$$= 4 \int_{-\infty}^{\infty} \delta(2t + \tau) \cdot \delta(2t - \tau) e^{-j2\pi\tau f} d\tau$$

$$= 4\delta(4t) e^{j4\pi t f} = \delta(t) e^{j4\pi t f} = \delta(t)$$

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公式(4)

Page 133  
公式(2)

Page 133  
公式(5),  $t_0 = 0$



## V-C The WDF is not a Linear Distribution

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

If  $h(t) = \alpha g(t) + \beta s(t)$

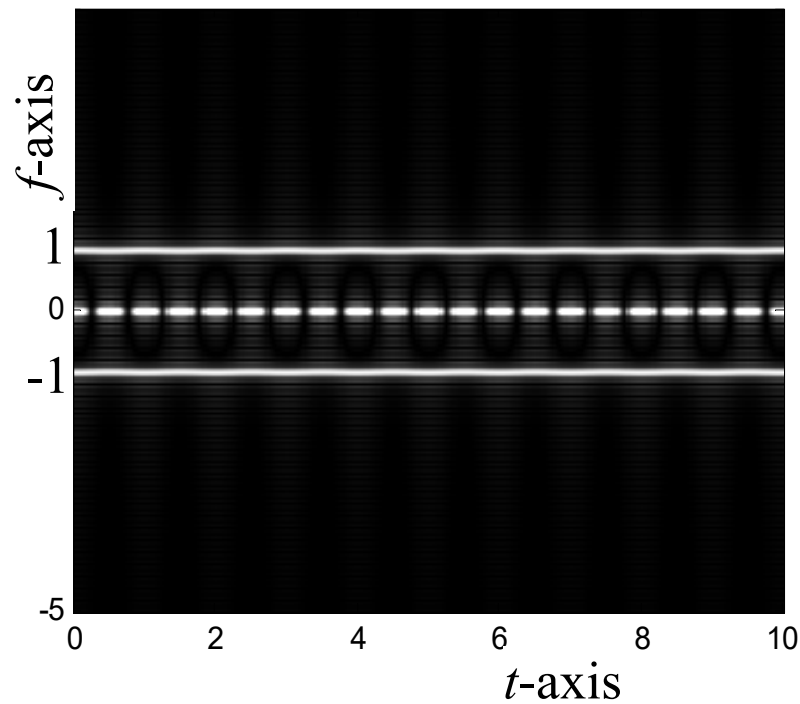
$$\begin{aligned} W_h(t, f) &= \int_{-\infty}^{\infty} h(t + \tau/2) \cdot h^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\ &= \int_{-\infty}^{\infty} [\alpha g(t + \tau/2) + \beta s(t + \tau/2)] [\alpha^* g^*(t - \tau/2) + \beta^* s^*(t - \tau/2)] e^{-j2\pi\tau f} d\tau \\ &= \int_{-\infty}^{\infty} [|\alpha|^2 g(t + \tau/2) g^*(t - \tau/2) + |\beta|^2 s(t + \tau/2) s^*(t - \tau/2) \\ &\quad + \alpha\beta^* g(t + \tau/2) s^*(t - \tau/2) + \alpha^*\beta g^*(t - \tau/2) s(t + \tau/2)] e^{-j2\pi\tau f} d\tau \\ &= |\alpha|^2 W_g(t, f) + |\beta|^2 W_s(t, f) \\ &\quad + \int_{-\infty}^{\infty} [\alpha\beta^* g(t + \tau/2) s^*(t - \tau/2) + \alpha^*\beta g^*(t - \tau/2) s(t + \tau/2)] e^{-j2\pi\tau f} d\tau \end{aligned}$$

cross terms

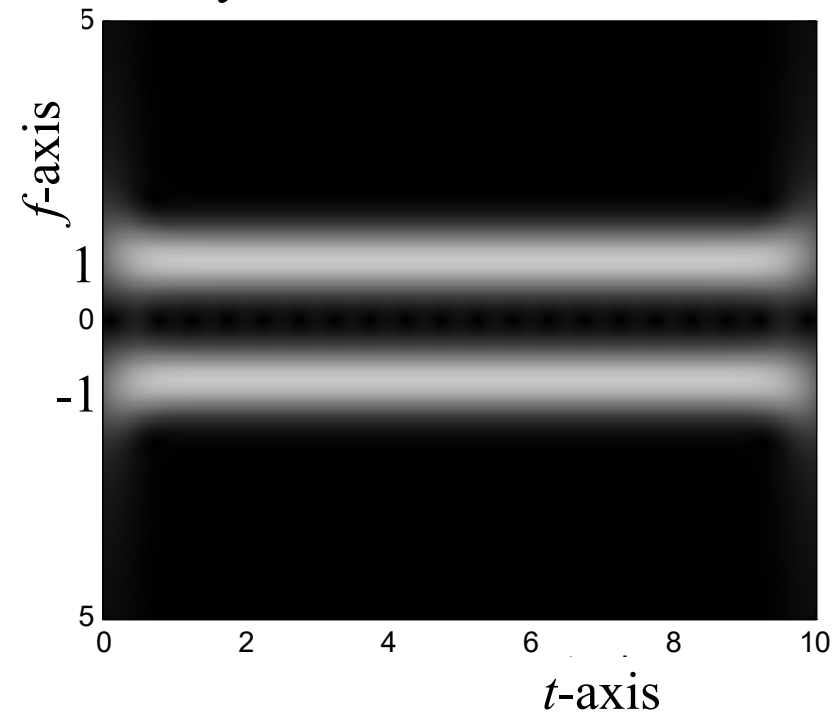
## Simulations

$$x(t) = \cos(2\pi t) = 0.5[\exp(j2\pi t) + \exp(-j2\pi t)]$$

by the WDF



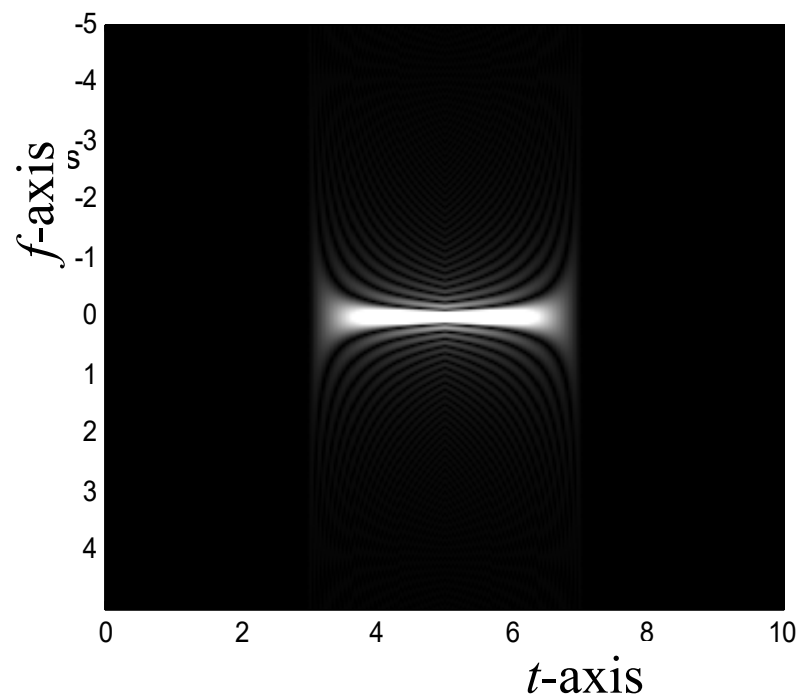
by the Gabor transform



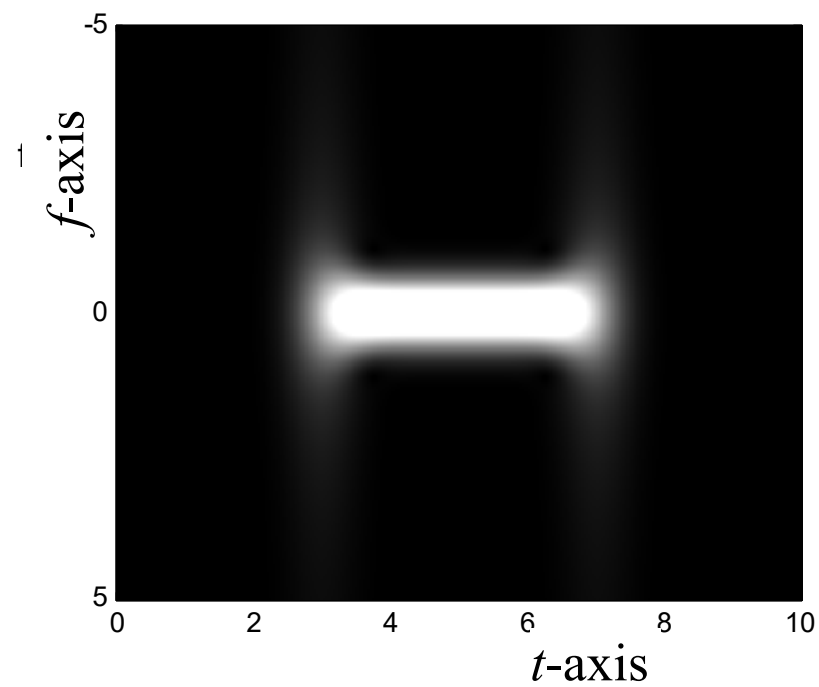


$$x(t) = \Pi((t-5)/4) \quad \Pi: \text{rectangular function}$$

by the WDF

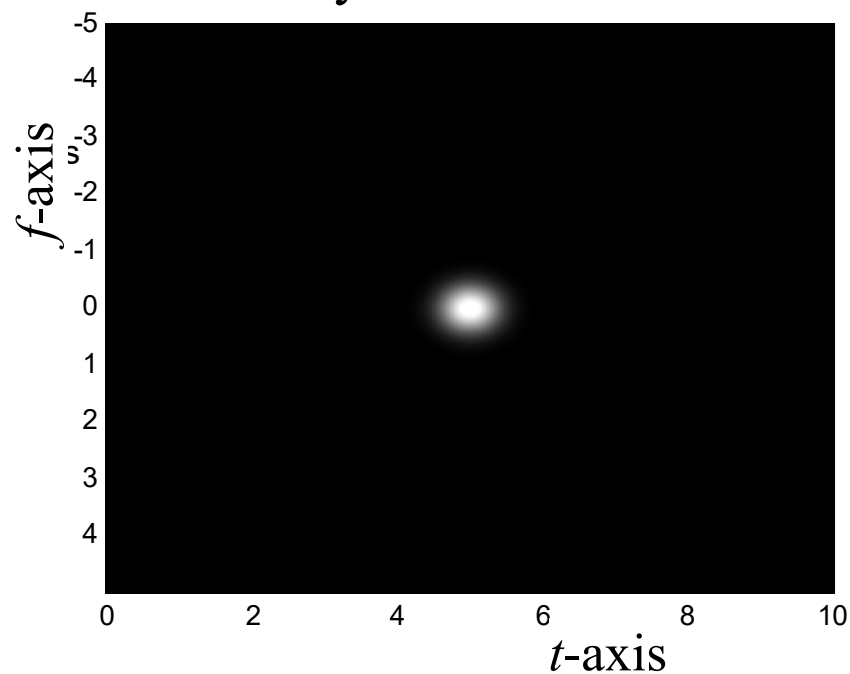


by the Gabor transform

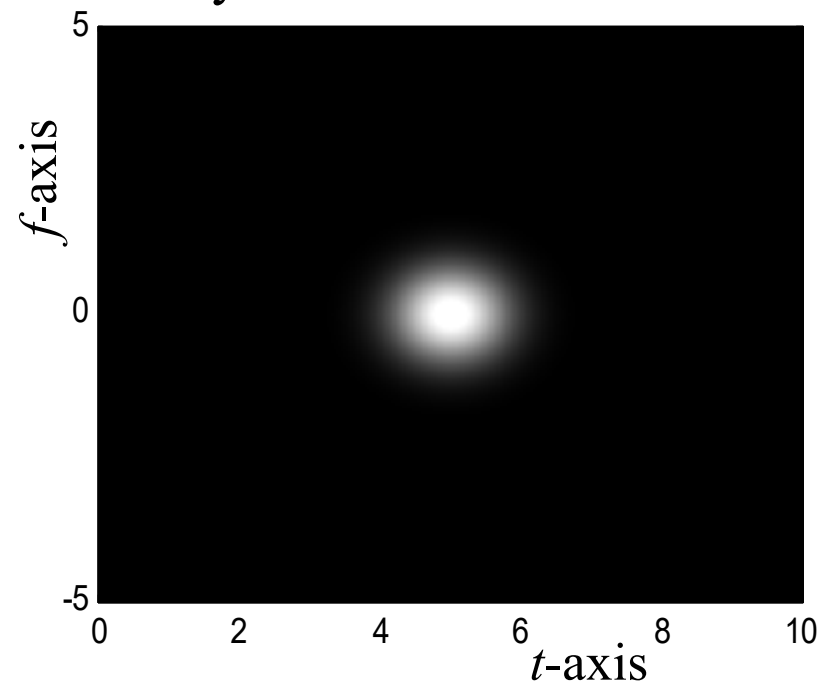


$$x(t) = \exp[-\pi(t-5)^2]$$

by the WDF



by the Gabor transform



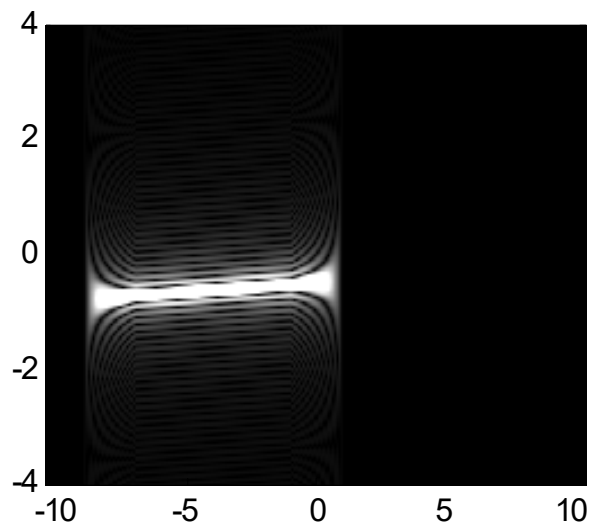
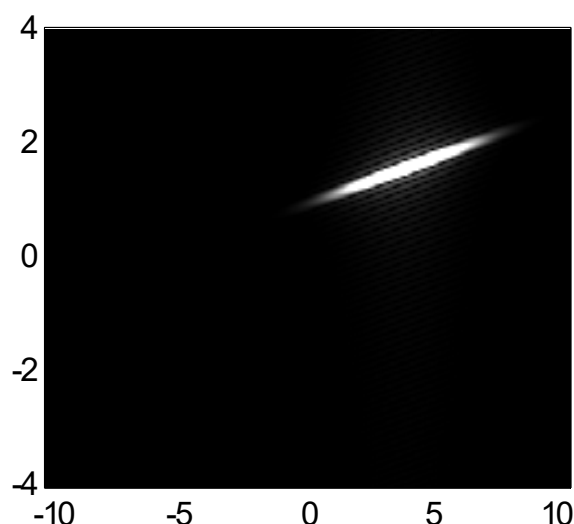
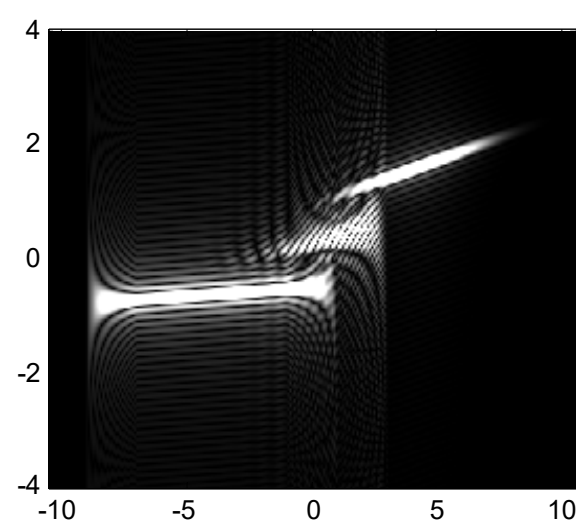
Gaussian function:  $e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

Gaussian function's T-F area is minimal.

$$s(t) = \exp(jt^2 / 10 - j3t) \quad \text{for } -9 \leq t \leq 1, s(t) = 0 \text{ otherwise,}$$

$$r(t) = \exp(jt^2 / 2 + j6t) \exp[-(t-4)^2 / 10]$$

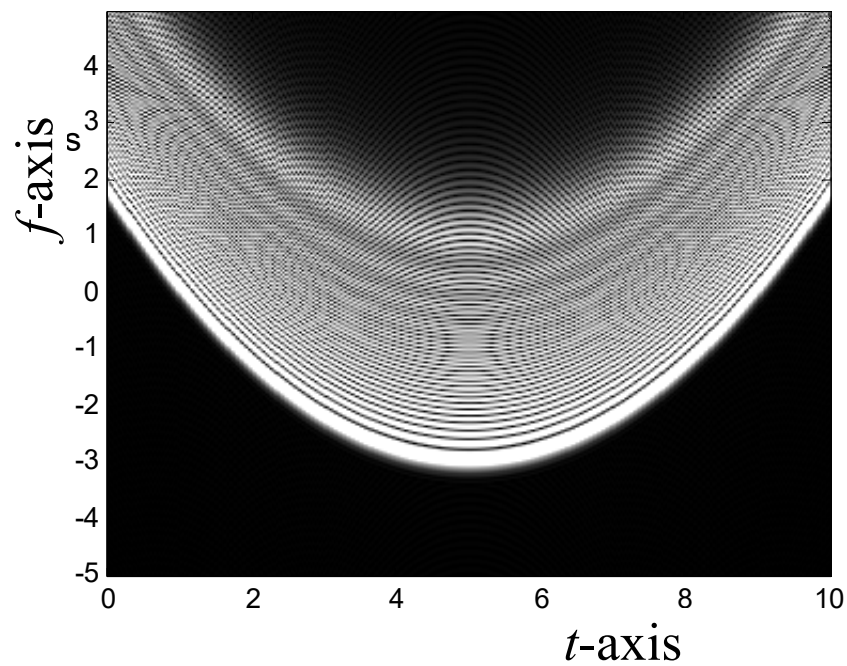
$$f(t) = s(t) + r(t)$$

WDF of  $s(t)$ ,WDF of  $r(t)$ ,WDF of  $s(t) + r(t)$ 

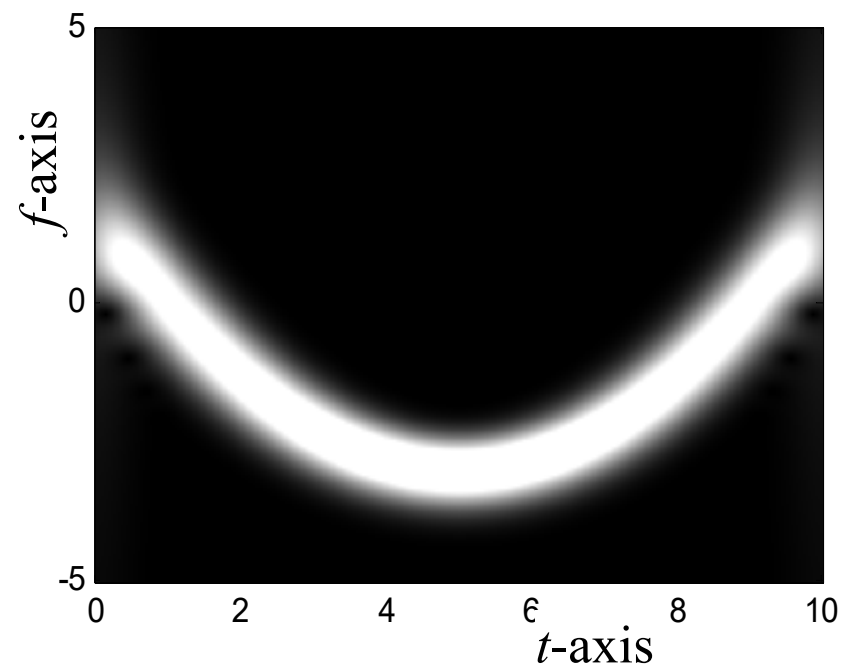
横軸:  $t$ -axis, 縦軸:  $f$ -axis

$$x(t) = \exp(j(t-5)^3 - j6\pi t)$$

by the WDF



by the Gabor transform



## V-E Digital Implementation of the WDF

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau ,$$

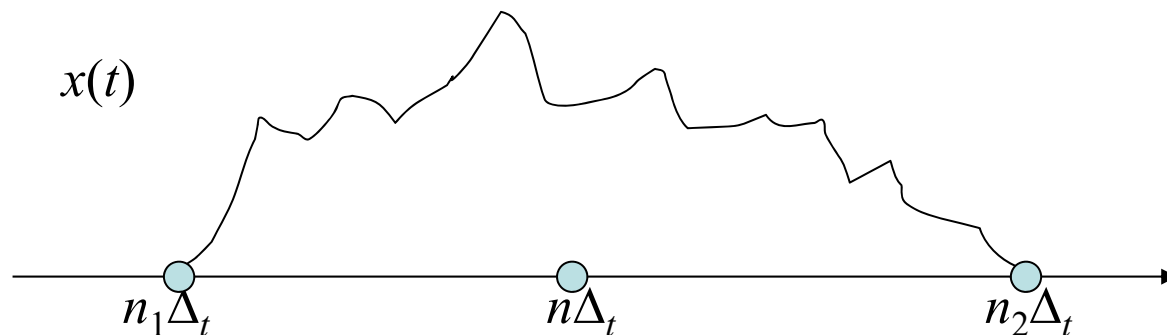
$$W_x(t, f) = 2 \int_{-\infty}^{\infty} x(t + \tau') \cdot x^*(t - \tau') e^{-j4\pi\tau' f} \cdot d\tau' \quad (\text{using } \tau' = \tau/2)$$

Sampling:  $t = n\Delta_t$ ,  $f = m\Delta_f$ ,  $\tau' = p\Delta_t$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t$$

When  $x(t)$  is not a time-limited signal, it is hard to implement.

Suppose that  $x(t) = 0$  for  $t < n_1\Delta_t$  and  $t > n_2\Delta_t$



$$x((n+p)\Delta_t)x^*((n-p)\Delta_t) = 0 \quad \text{if } n+p \notin [n_1, n_2]$$

$$\text{or } n-p \notin [n_1, n_2]$$

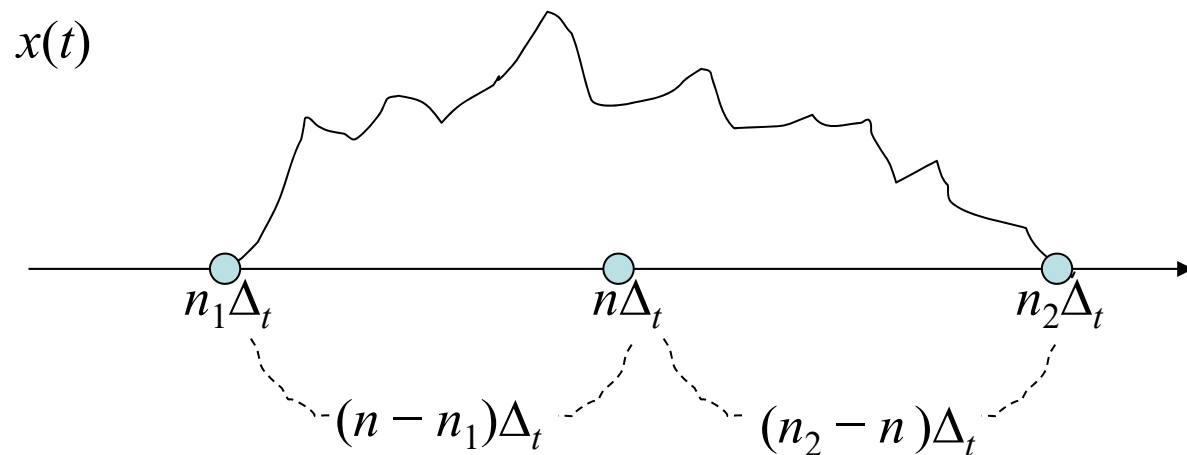
•  $p$  的範圍的問題 (當  $n$  固定時)

$$n_1 \leq n+p \leq n_2 \quad \longrightarrow \quad n_1 - n \leq p \leq n_2 - n$$

$$n_1 \leq n-p \leq n_2 \quad \longrightarrow \quad n_1 - n \leq -p \leq n_2 - n, \quad n - n_2 \leq p \leq n - n_1$$

$$\max(n_1 - n, n - n_2) \leq p \leq \min(n_2 - n, n - n_1)$$

$$-\min(n_2 - n, n - n_1) \leq p \leq \min(n_2 - n, n - n_1)$$



$$-\min(n_2 - n, n - n_1) \leq p \leq \min(n_2 - n, n - n_1)$$

$-Q$

$Q$

$(n_2 - n)\Delta_t, (n - n_1)\Delta_t$ : 離兩個邊界的距離

注意：當  $n > n_2$  或  $n < n_1$  時，

將沒有  $p$  能滿足上面的不等式

If  $x(t) = 0$  for  $t < n_1\Delta_t$  and  $t > n_2\Delta_t$

$$W_x \left( \underset{T\text{點}}{n\Delta_t}, \underset{F\text{點}}{m\Delta_f} \right) = 2 \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t$$

$$Q = \min(n_2 - n, n - n_1). \quad (\text{varies with } n)$$

$$p \in [-Q, Q], \quad n \in [n_1, n_2],$$

possible for implementation

### Method 1: Direct Implementation (brute force method)

唯一的限制條件？



## Method 2: Using the DFT

## 3 大限制條件

When  $\Delta_t \Delta_f = \frac{1}{2N}$  and  $N \geq 2\text{Max}(Q)+1 = 2(n_2-n_1)/2+1 = n_2-n_1+1 = T$

$$W_x \left( \underset{T \text{點}}{n\Delta_t}, \underset{F \text{點}}{m\Delta_f} \right) = 2\Delta_t \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j\frac{2\pi mp}{N}}$$

$$q = p+Q \rightarrow p = q - Q$$

$$W_x \left( n\Delta_t, m\Delta_f \right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{2Q} x((n+q-Q)\Delta_t) x^*((n-q+Q)\Delta_t) e^{-j\frac{2\pi mq}{N}}$$

$$W_x \left( n\Delta_t, m\Delta_f \right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{N-1} c_1(q) e^{-j\frac{2\pi mq}{N}}$$

$$Q = \min(n_2-n, n-n_1).$$

$$n \in [n_1, n_2],$$

$$c_1(q) = x((n+q-Q)\Delta_t) x^*((n-q+Q)\Delta_t) \quad \text{for } 0 \leq q \leq 2Q$$

$$\text{i.e., } c_1(Q+k) = x((n+k)\Delta_t) x^*((n-k)\Delta_t) \quad \text{for } -Q \leq k \leq Q \quad (k = q-Q)$$

$$c_1(q) = 0 \quad \text{for } 2Q+1 \leq q \leq N-1$$

假設  $t = n_0\Delta_t, (n_0+1)\Delta_t, (n_0+2)\Delta_t, \dots, n_1\Delta_t$

$f = m_0\Delta_f, (m_0+1)\Delta_f, (m_0+2)\Delta_f, \dots, m_1\Delta_f$

Step 1: Calculate  $n_0, n_1, m_0, m_1, N$

Step 2:  $n = n_0$

Step 3: Determine  $Q$

Step 4: Determine  $c_1(q)$

Step 5:  $C_1(m) = \text{FFT}[c_1(q)]$

Step 6: Convert  $C_1(m)$  into  $C(n\Delta_t, m\Delta_f)$

Step 7: Set  $n = n+1$  and return to Step 3 until  $n = n_1$ .

### Method 3: Using the Chirp Z Transform

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t$$

$$W_x(n\Delta_t, m\Delta_f) = 2\Delta_t e^{-j2\pi m^2\Delta_t\Delta_f} \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j2\pi p^2\Delta_t\Delta_f} e^{j2\pi(p-m)^2\Delta_t\Delta_f}$$

Step 1  $x_1(n, p) = x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j2\pi p^2\Delta_t\Delta_f}$

Step 2  $X_2[n, m] = \sum_{p=-Q}^Q x_1[n, p] c[m-p] \quad c[m] = e^{j2\pi m^2\Delta_t\Delta_f}$

Step 3  $X(n\Delta_t, m\Delta_f) = 2\Delta_t e^{-j2\pi m^2\Delta_t\Delta_f} X_2[n, m]$

思考：Method 1 的複雜度為多少

思考：Method 2 的複雜度為多少

思考：Method 3 的複雜度為多少

The computation time of the WDF is more than those of the rec-STFT and the Gabor transform.

## V-F Properties of the WDF

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|  |  |
|--|--|
| (1) Projection property                              | $ x(t) ^2 = \int_{-\infty}^{\infty} W_x(t, f) df \quad  X(f) ^2 = \int_{-\infty}^{\infty} W_x(t, f) dt$  |
| (2) Energy preservation property                     | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f) dt df = \int_{-\infty}^{\infty}  x(t) ^2 dt = \int_{-\infty}^{\infty}  X(f) ^2 df$  |
| (3) Recovery property                                | $\int_{-\infty}^{\infty} W_x(t/2, f) e^{j2\pi ft} df = x(t) \cdot x^*(0) \quad x^*(0) \text{ 已知}$ $\int_{-\infty}^{\infty} W_x(t, f/2) e^{-j2\pi ft} dt = X(f) \cdot X^*(0)$   |
| (4) Mean condition frequency and mean condition time | <p>If <math>x(t) =  x(t)  \cdot e^{j2\pi\phi(t)}</math>, <math>X(f) =  X(f)  \cdot e^{j2\pi\Psi(f)}</math></p> <p>then <math>\phi'(t) =  x(t) ^{-2} \cdot \int_{-\infty}^{\infty} f \cdot W_x(t, f) \cdot df</math></p> $-\Psi'(f) =  X(f) ^{-2} \int_{-\infty}^{\infty} t \cdot W_x(t, f) \cdot dt$ |
| (5) Moment properties                                | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^n W_x(t, f) dt df = \int_{-\infty}^{\infty} t^n  x(t) ^2 dt ,$ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^n W_x(t, f) dt df = \int_{-\infty}^{\infty} f^n  X(f) ^2 df$  |

|                           |   |
|---------------------------|---|
| (6) $W_x(t, f)$ is real   | $\overline{W_x(t, f)} = W_x(t, f)$  |
| (7) Region properties     | <p>If <math>x(t) = 0</math> for <math>t &gt; t_2</math> then <math>W_x(t, f) = 0</math> for <math>t &gt; t_2</math></p> <p>If <math>x(t) = 0</math> for <math>t &lt; t_1</math> then <math>W_x(t, f) = 0</math> for <math>t &lt; t_1</math></p> |
| (8) Multiplication theory | <p>If <math>y(t) = x(t)h(t)</math>, then</p> $W_y(t, f) = \int_{-\infty}^{\infty} W_x(t, \rho) W_h(t, f - \rho) \cdot d\rho$  |
| (9) Convolution theory    | <p>If <math>y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau) d\tau</math>, then</p> $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(t - \rho, f) \cdot d\rho$  |
| (10) Correlation theory   | <p>If <math>y(t) = \int_{-\infty}^{\infty} x(t + \tau)h^*(\tau) d\tau</math>, then</p> $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(-t + \rho, f) \cdot d\rho$   |

|                             |   |
|-----------------------------|---|
| (11) Time-shifting property | If $y(t) = x(t - t_0)$ , then<br>$W_y(t, f) = W_x(t - t_0, f)$            |
| (12) Modulation property    | If $y(t) = \exp(j2\pi f_0 t)x(t)$ , then<br>$W_y(t, f) = W_x(t, f - f_0)$ |

The STFT (including the rec-STFT, the Gabor transform) does not have real region, multiplication, convolution, and correlation properties.

- Why the **WDF** is always real?

What are the advantages and disadvantages it causes?

- Try to prove of the projection and recovery properties



$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

- Proof of the region properties

If  $x(t) = 0$  for  $t < t_0$ ,

$$x(t + \tau/2) = 0 \quad \text{for } \tau < (t_0 - t)/2 = -(t - t_0)/2,$$

$$x(t - \tau/2) = 0 \quad \text{for } \tau > (t - t_0)/2,$$

Therefore, if  $t - t_0 < 0$ , the nonzero regions of  $x(t + \tau/2)$  and  $x(t - \tau/2)$  does not overlap and  $x(t + \tau/2) x^*(t - \tau/2) = 0$  for all  $\tau$ .

The importance of region property

## V-G Advantages and Disadvantages of the WDF

Advantages: clarity

- many good properties

- suitable for analyzing the random process

Disadvantages: cross-term problem

- more time for computation, especial for the signal with long time duration

- not one-to-one

- not suitable for  $\exp(jt^n)$ ,  $n \neq 0,1,2$

## V-H Windowed Wigner Distribution Function

When  $x(t)$  is not time-limited, its WDF is hard for implementation

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

↓ with mask

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$  is real and time-limited

**Advantages:** (1) reduce the computation time

(2) **may** reduce the cross term problem

**Disadvantages:**

$$W_x(t, f) = 2 \int_{-\infty}^{\infty} w(2\tau') x(t + \tau') \cdot x^*(t - \tau') e^{-j4\pi\tau'f} \cdot d\tau'$$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} w(2p\Delta_t) x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t\Delta_f} \Delta_t$$

Suppose that  $w(t) = 0$  for  $|t| > B$

$$w(2p\Delta_t) = 0 \quad \text{for } p < -Q \text{ and } p > Q$$

$$Q = \frac{B}{2\Delta_t}$$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-Q}^Q w(2p\Delta_t) x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t\Delta_f} \Delta_t$$

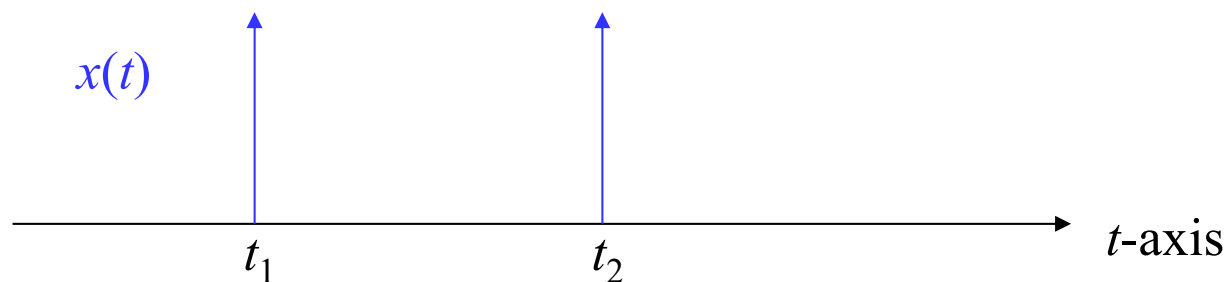
當然，乘上 mask 之後，有一些數學性質將會消失

**(B) Why the cross term problem can be avoided ?**

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$  is real

Viewing the case where  $x(t) = \delta(t - t_1) + \delta(t - t_2)$



理想情形：  $W_x(t, f) = 0$  for  $t \neq t_1, t_2$

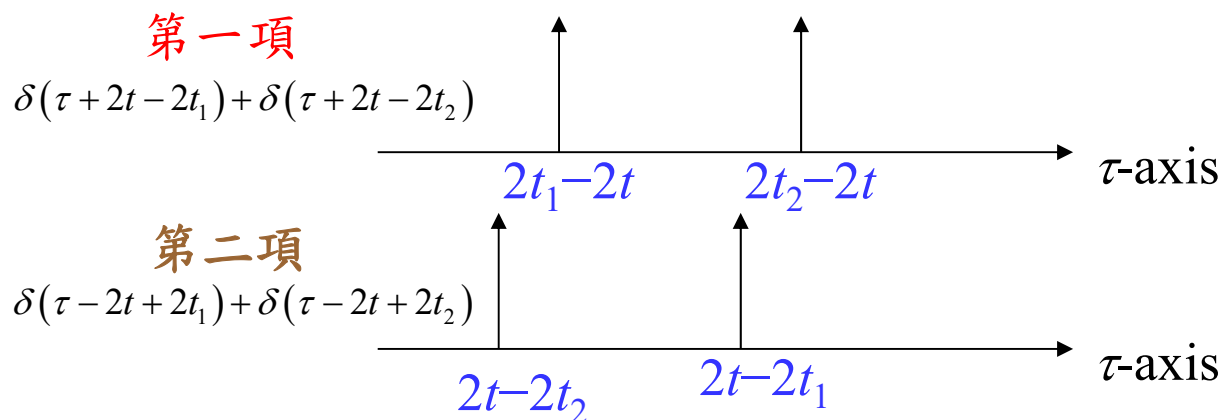
然而，當 mask function  $w(\tau) = 1$  時（也就是沒有使用 mask function）

$$y(t, \tau) = x(t + \tau/2) \quad y^*(t, -\tau) = x^*(t - \tau/2)$$

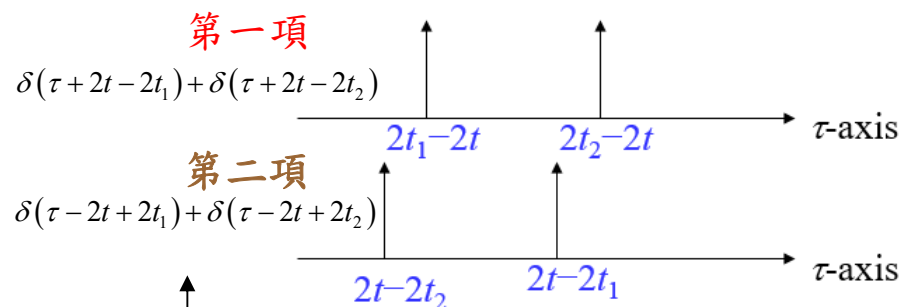
$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} \left[ \delta\left(t + \frac{\tau}{2} - t_1\right) + \delta\left(t + \frac{\tau}{2} - t_2\right) \right] \left[ \delta\left(t - \frac{\tau}{2} - t_1\right) + \delta\left(t - \frac{\tau}{2} - t_2\right) \right] e^{-j2\pi\tau f} \cdot d\tau$$

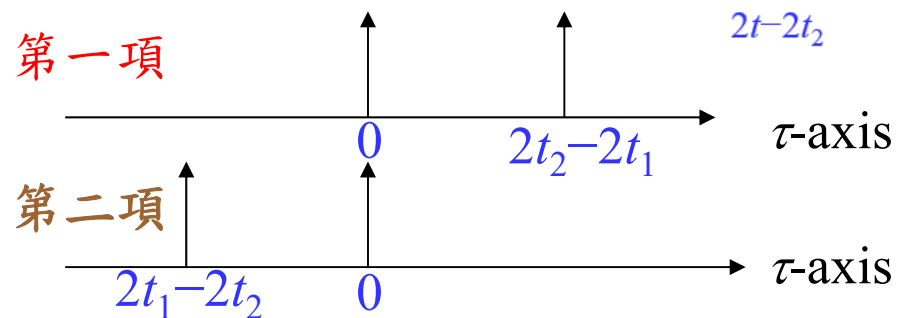
$$= 4 \int_{-\infty}^{\infty} \underbrace{\left[ \delta(\tau + 2t - 2t_1) + \delta(\tau + 2t - 2t_2) \right]}_{\text{第一項}} \underbrace{\left[ \delta(\tau - 2t + 2t_1) + \delta(\tau - 2t + 2t_2) \right]}_{\text{第二項}} e^{-j2\pi\tau f} \cdot d\tau$$



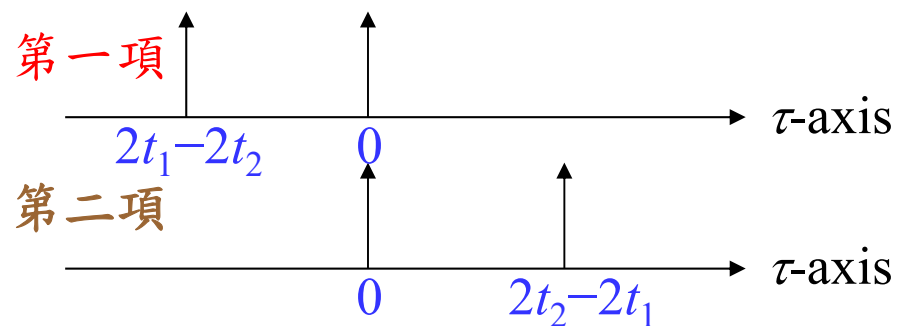
3種情形  $W_x(t, f) \neq 0$



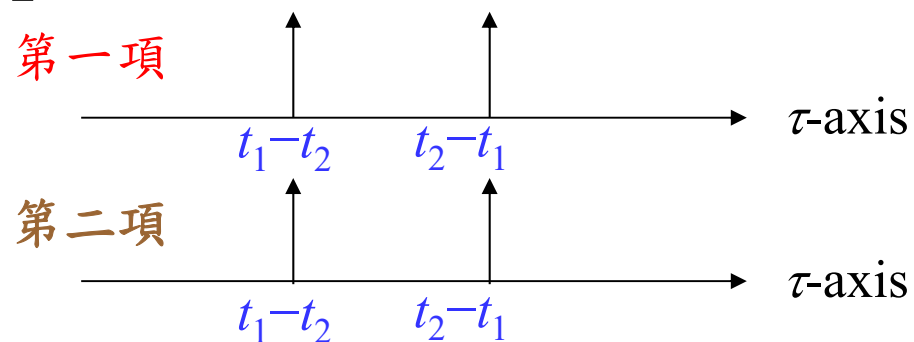
(1) If  $t = t_1$



(2) If  $t = t_2$



(3) If  $t = (t_1 + t_2)/2$



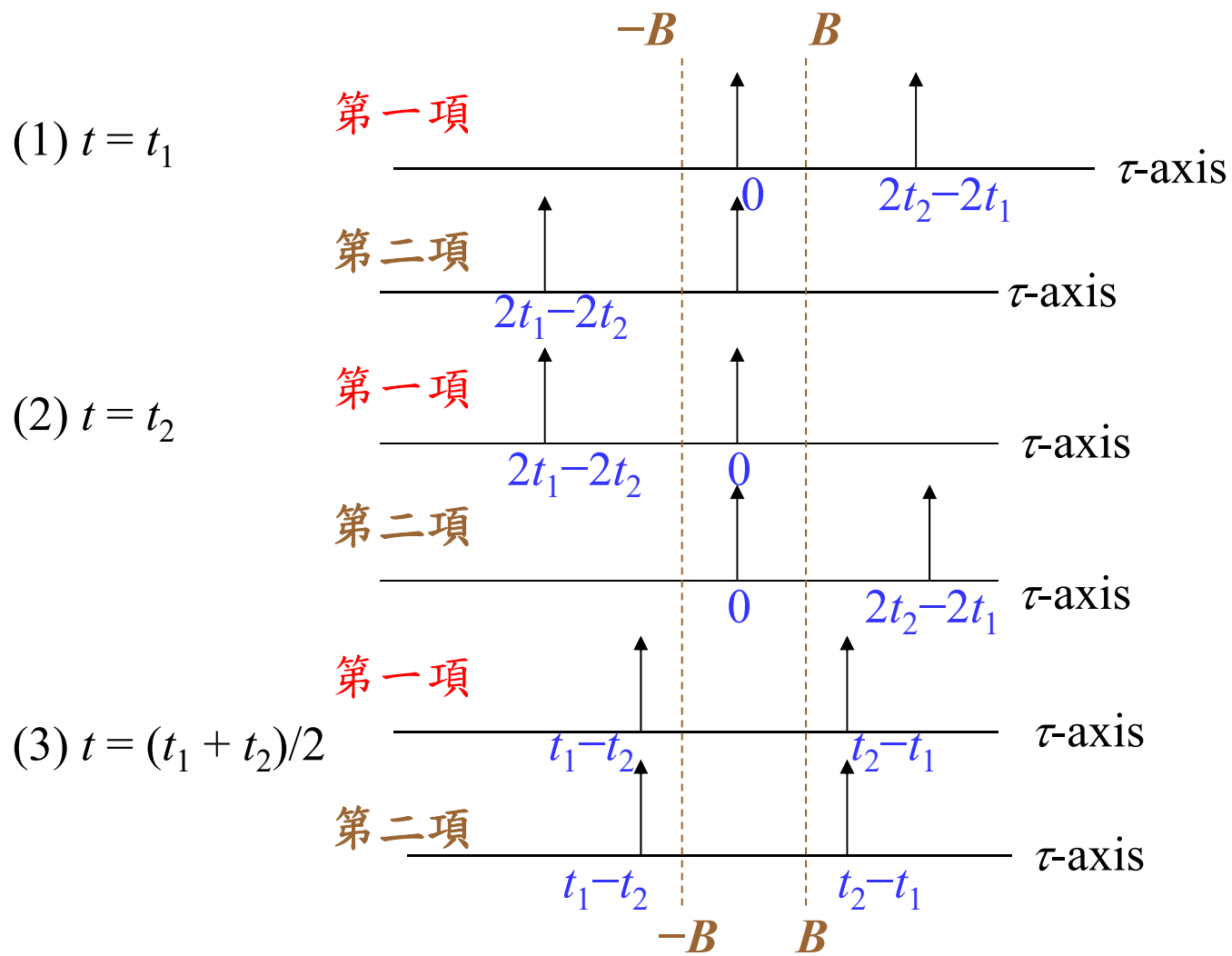
With mask function

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} w(\tau) [\delta(\tau + 2t - 2t_1) + \delta(\tau + 2t - 2t_2)] \\
 &\quad \times [\delta(\tau - 2t + 2t_1) + \delta(\tau - 2t + 2t_2)] e^{-j2\pi\tau f} \cdot d\tau
 \end{aligned}$$

Suppose that  $w(\tau) = 0$  for  $|\tau| > B$ ,  $B$  is positive.

If  $B < t_2 - t_1$





## 附錄八：研究所學習新知識把握的要點

- (1) **Concepts**: 這個方法的核心概念、基本精神是什麼
- (2) **Comparison**: 這方法和其他方法之間，有什麼相同的地方？  
有什麼相異的地方
- (3) **Advantages**: 這方法的優點是什麼  
(3-1) Why? 造成這些優點的原因是什麼
- (4) **Disadvantages**: 這方法的缺點是什麼  
(4-1) Why? 造成這些缺點的原因是什麼
- (5) **Applications**: 這個方法要用來處理什麼問題，有什麼應用
- (6) **Innovations**: 這方法有什麼可以改進的地方  
或是可以推廣到什麼地方

看過一篇論文或一個章節之後，若能夠回答 (1)-(5) 的問題，就代表你已經學通了這個方法

如果你的目標是發明創造出新的方法，可試著回答 (3-1), (4-1), 和 (6) 的問題

每個領域每個月至少都有100篇以上的新論文，閱讀能力再厲害的人，也不可能每一篇都像大學讀書一樣篇篇都逐字讀過，所以，要選讀哪幾篇論文，哪些要詳讀，哪些可以只抓重點，這種擇要的能力，是研究所學生們應該練習的。