

VI. Other Time Frequency Distributions

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 6, Prentice Hall, N.J., 1996.

Requirements for time-frequency analysis:

- (1) higher clarity $\xleftrightarrow{\text{tradeoff}}$ (2) avoid cross-term
- (3) less computation time (4) good mathematical properties

VI-A Cohen's Class Distribution

VI-A-1 Ambiguity Function

$$A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt$$

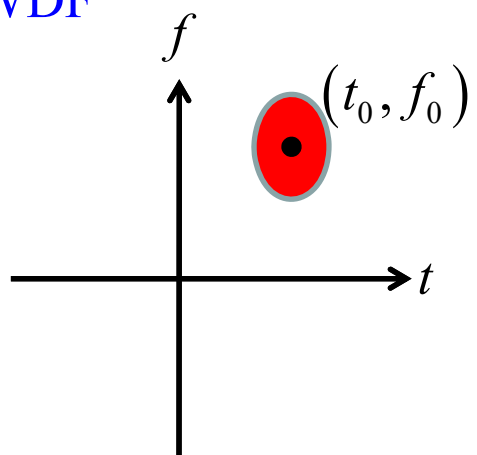
(1) If $x(t) = \exp[-\alpha\pi(t - t_0)^2 + j2\pi f_0 t]$

$$\begin{aligned} A_x(\tau, \eta) &= \int_{-\infty}^{\infty} e^{-\alpha\pi(t+\tau/2-t_0)^2 + j2\pi f_0(t+\tau/2)} e^{-\alpha\pi(t-\tau/2-t_0)^2 - j2\pi f_0(t-\tau/2)} \cdot e^{-j2\pi t\eta} \cdot dt \\ &= \int_{-\infty}^{\infty} e^{-\alpha\pi[2(t-t_0)^2 + \tau^2/2] + j2\pi f_0 \tau} \cdot e^{-j2\pi t\eta} \cdot dt \\ &= \int_{-\infty}^{\infty} e^{-\alpha\pi[2t^2 + \tau^2/2] + j2\pi f_0 \tau} \cdot e^{-j2\pi t\eta} e^{-j2\pi t_0\eta} \cdot dt \end{aligned}$$

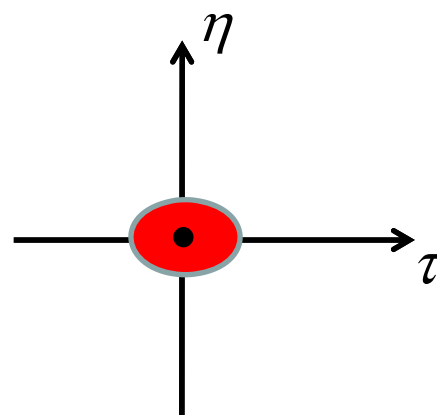
$$A_x(\tau, \eta) = \sqrt{\frac{1}{2\alpha}} \exp\left[-\pi\left(\frac{\alpha\tau^2}{2} + \frac{\eta^2}{2\alpha}\right)\right] \exp[j2\pi(f_0 \tau - t_0\eta)]$$

WDF and AF for the signal with **only 1 term**

WDF



AF



$$(2) \text{ If } x(t) = \underbrace{\exp[-\alpha_1 \pi (t - t_1)^2 + j2\pi f_1 t]}_{x_1(t)} + \underbrace{\exp[-\alpha_2 \pi (t - t_2)^2 + j2\pi f_2 t]}_{x_2(t)}$$

$$\begin{aligned}
 A_x(\tau, \eta) &= \int_{-\infty}^{\infty} x_1(t + \tau/2) \cdot x_1^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \leftarrow A_{x_1}(\tau, \eta) \\
 &\int_{-\infty}^{\infty} x_2(t + \tau/2) \cdot x_2^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \leftarrow A_{x_2}(\tau, \eta) \\
 &\int_{-\infty}^{\infty} x_1(t + \tau/2) \cdot x_2^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \leftarrow A_{x_1 x_2}(\tau, \eta) \\
 &\int_{-\infty}^{\infty} x_2(t + \tau/2) \cdot x_1^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \leftarrow A_{x_2 x_1}(\tau, \eta)
 \end{aligned}$$

$$A_x(\tau, \eta) = A_{x_1}(\tau, \eta) + A_{x_2}(\tau, \eta) + A_{x_1 x_2}(\tau, \eta) + A_{x_2 x_1}(\tau, \eta)$$

$$A_{x_1}(\tau, \eta) = \sqrt{\frac{1}{2\alpha_1}} \exp\left[-\pi\left(\frac{\alpha_1 \tau^2}{2} + \frac{\eta^2}{2\alpha_1}\right)\right] \exp[j2\pi(f_1 \tau - t_1 \eta)]$$

$$A_{x_2}(\tau, \eta) = \sqrt{\frac{1}{2\alpha_2}} \exp\left[-\pi\left(\frac{\alpha_2 \tau^2}{2} + \frac{\eta^2}{2\alpha_2}\right)\right] \exp[j2\pi(f_2 \tau - t_2 \eta)]$$

When $\alpha_1 = \alpha_2$

$$A_{x_1x_2}(\tau, \eta) = \sqrt{\frac{1}{2\alpha_\mu}} \exp\left[-\pi\left(\alpha_\mu \frac{(\tau - t_d)^2}{2} + \frac{(\eta - f_d)^2}{2\alpha_\mu}\right)\right] \\ \times \exp\left[j2\pi(f_\mu\tau - t_\mu\eta + f_d t_\mu)\right]$$

$$t_\mu = (t_1 + t_2)/2, \quad f_\mu = (f_1 + f_2)/2, \quad \alpha_\mu = (\alpha_1 + \alpha_2)/2,$$

$$t_d = t_1 - t_2, \quad f_d = f_1 - f_2, \quad \alpha_d = \alpha_1 - \alpha_2$$

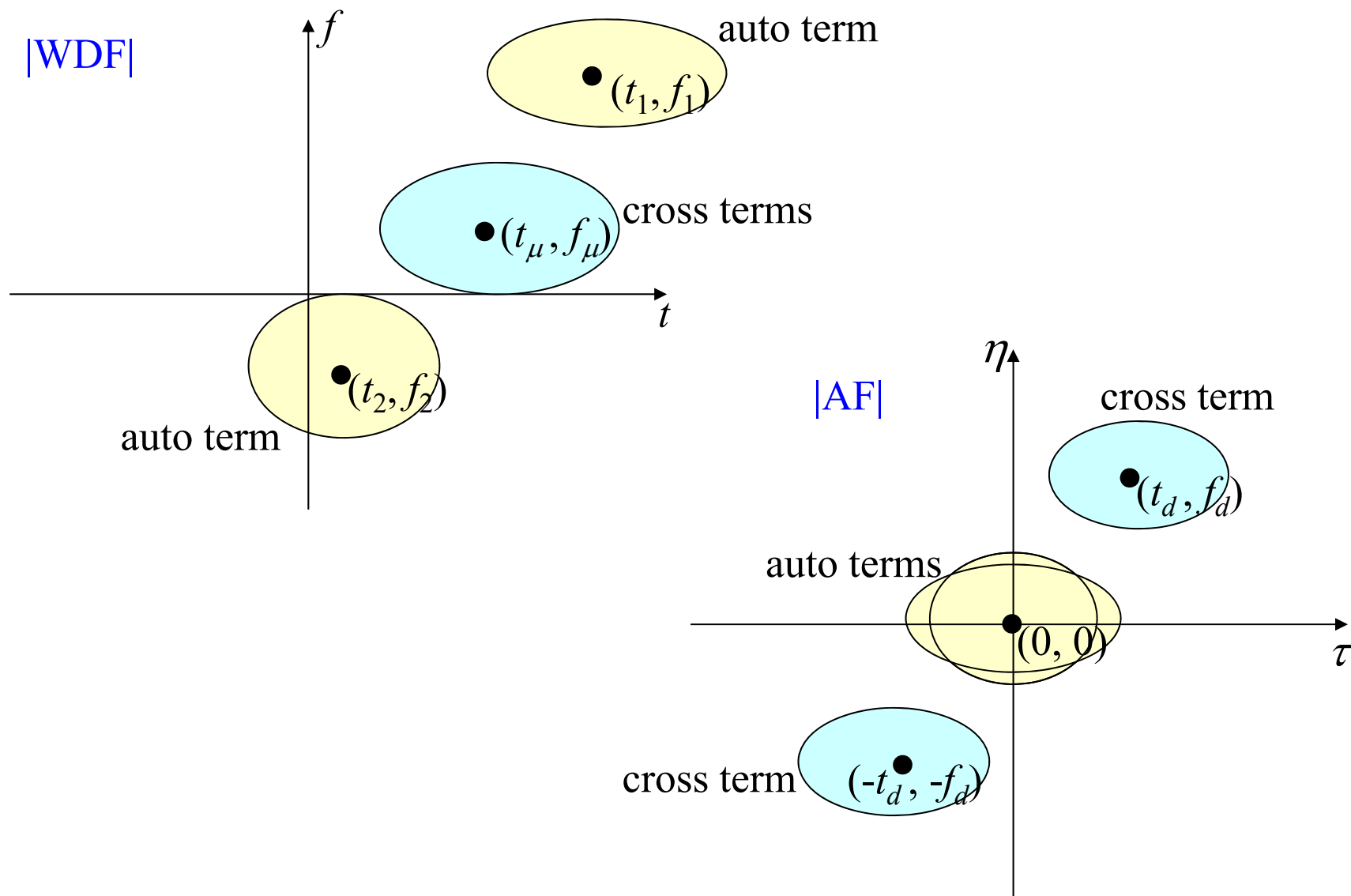
$$A_{x_2x_1}(\tau, \eta) = A_{x_1x_2}^*(-\tau, -\eta)$$

When $\alpha_1 \neq \alpha_2$

$$A_{x_1x_2}(\tau, \eta) = \sqrt{\frac{1}{2\alpha_\mu}} \exp\left[-\pi \frac{[(\eta - f_d) + j(\alpha_1 t_1 + \alpha_2 t_2) - j\alpha_d \tau / 2]^2}{2\alpha_\mu}\right] \\ \exp\left[-\pi\left(\alpha_1\left(t_1 - \frac{\tau}{2}\right)^2 + \alpha_2\left(t_2 + \frac{\tau}{2}\right)^2\right)\right] \exp\left[j2\pi f_\mu \tau\right]$$

$$A_{x_2x_1}(\tau, \eta) = A_{x_1x_2}^*(-\tau, -\eta)$$

WDF and AF for the signal with 2 terms



For the ambiguity function

The **auto term** is always near to the origin

The **cross-term** is always far from the origin

VI-A-2 Definition of Cohen's Class Distribution

The Cohen's Class distribution is a further generalization of the Wigner distribution function

$$C_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_x(\tau, \eta) \Phi(\tau, \eta) \exp(j2\pi(\eta t - \tau f)) d\eta d\tau$$

$$\text{where } A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt$$

is the ambiguity function (AF).

$\Phi(\eta, \tau) = 1 \rightarrow$ WDF

$$C_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(u + \tau/2) x^*(u - \tau/2) \phi(t - u, \tau) du e^{-j2\pi f\tau} d\tau$$

where $\phi(t, \tau) = \int_{-\infty}^{\infty} \Phi(\tau, \eta) \exp(j2\pi\eta t) d\eta$

How does the Cohen's class distribution avoid the cross term?

Chose $\Phi(\tau, \eta)$ low pass function.

$$\Phi(\tau, \eta) \approx 1 \quad \text{for small } |\eta|, |\tau|$$

$$\Phi(\tau, \eta) \approx 0 \quad \text{for large } |\eta|, |\tau|$$

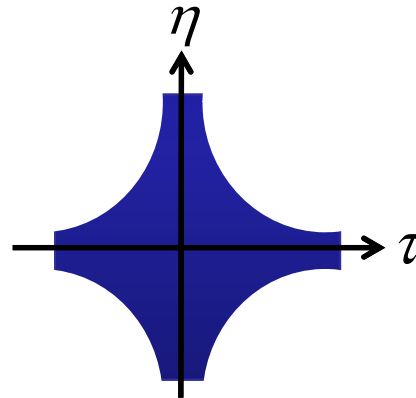
[Ref] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781-806, 1966.

[Ref] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, New York, 1995.

VI-A-3 Several Types of Cohen's Class Distribution

Choi-Williams Distribution (One of the Cohen's class distribution)

$$\Phi(\tau, \eta) = \exp\left[-\alpha(\eta\tau)^2\right]$$

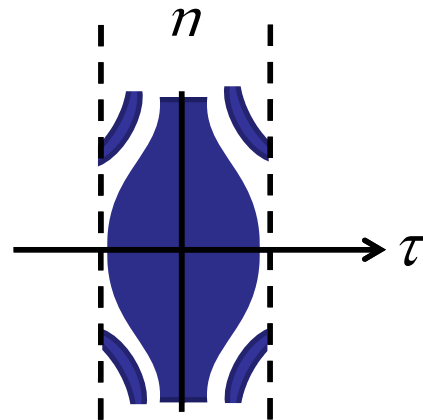


[Ref] H. Choi and W. J. Williams, "Improved time-frequency representation of multicomponent signals using exponential kernels," *IEEE. Trans. Acoustics, Speech, Signal Processing*, vol. 37, no. 6, pp. 862-871, June 1989.

Cone-Shape Distribution (One of the Cohen's class distribution)

$$\phi(t, \tau) = \frac{1}{|\tau|} \exp(-2\pi\alpha\tau^2) \Pi\left(\frac{t}{\tau}\right)$$

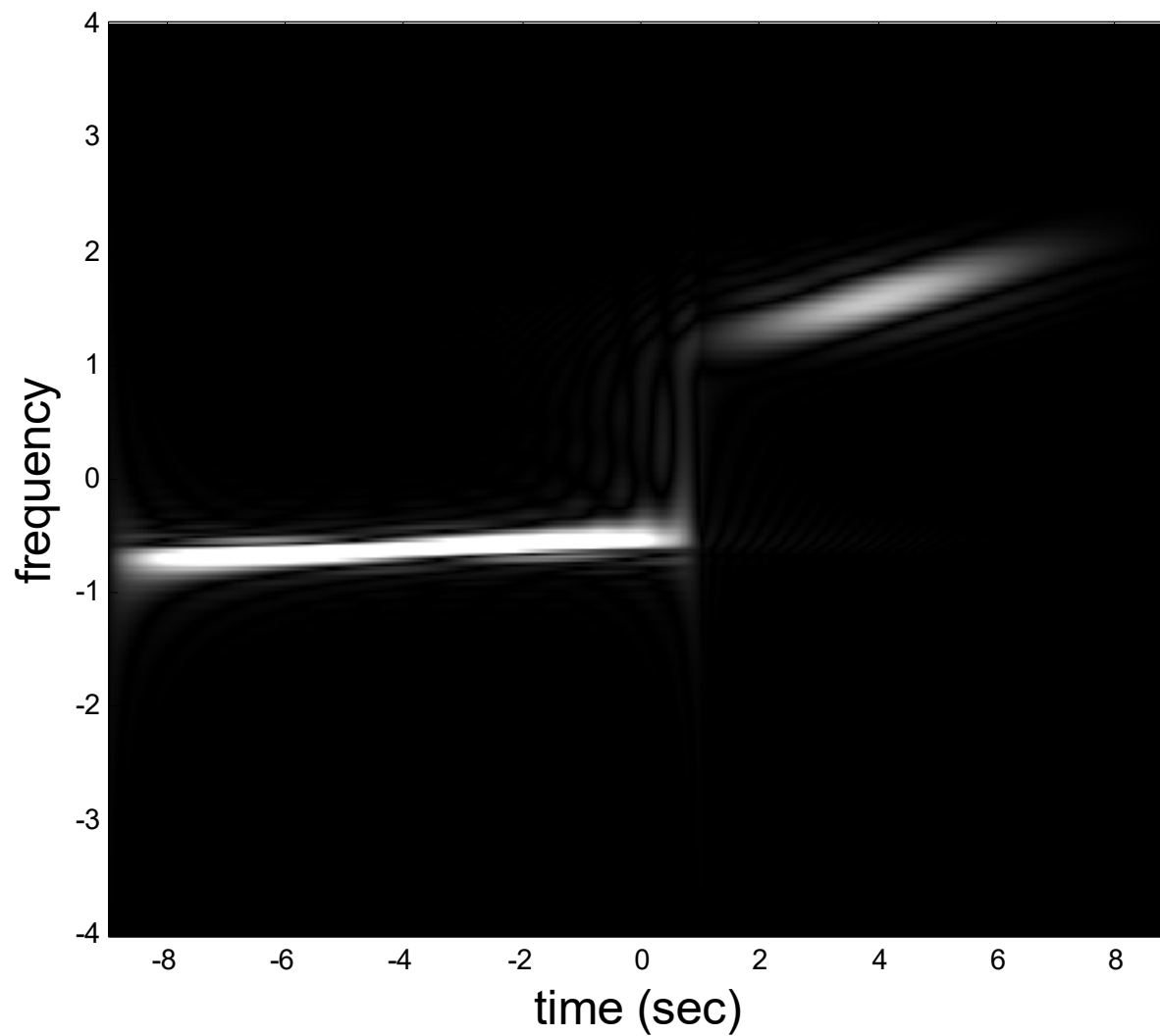
$$\Phi(\tau, \eta) = \text{sinc}(\eta\tau) \exp(-2\pi\alpha\tau^2)$$



[Ref] Y. Zhao, L. E. Atlas, and R. J. Marks, "The use of cone-shape kernels for generalized time-frequency representations of nonstationary signals," *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. 38, no. 7, pp. 1084-1091, July 1990.

Cone-Shape distribution for the example on pages 93, 144

($\alpha = 1$)



Distributions	$\Phi(\tau, \eta)$
Wigner	1
Cohen (circular)	$\Phi(\tau, \eta) = 1$ for $\sqrt{\eta^2 + \tau^2} < r$ $\Phi(\tau, \eta) = 0$ otherwise
Cohen (rectangular)	$\Phi(\tau, \eta) = 1$ for $Max(\eta , \tau) < T$ $\Phi(\tau, \eta) = 0$ otherwise
Choi-Williams	$\exp[-\alpha(\eta\tau)^2]$
Cone-Shape	$\text{sinc}(\eta\tau)\exp(-2\pi\alpha\tau^2)$
Page	$\exp(j\pi\eta \tau)$
Levin (Margenau-Hill)	$\cos(\pi\eta\tau)$
Born-Jordan	$\text{sinc}(\eta\tau)$

註：感謝 2007 年修課的王文阜同學

VI-A-4 Advantages and Disadvantages of Cohen's Class Distributions

The Cohen's class distribution may [avoid the cross term](#) and has [higher clarity](#).

However, it requires more computation time and lacks of well mathematical properties.

Moreover, there is a tradeoff between [the quality of the auto term](#) and [the ability of removing the cross terms](#).

VI-A-5 Implementation for the Cohen's Class Distribution

$$\begin{aligned}
 C_x(t, f) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_x(\tau, \eta) \Phi(\tau, \eta) \exp(j2\pi(\eta t - \tau f)) d\eta d\tau \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \Phi(\tau, \eta) e^{-j2\pi u\eta + j2\pi(\eta t - \tau f)} du d\eta d\tau
 \end{aligned}$$

簡化法 1：不是所有的 $A_x(\eta, \tau)$ 的值都需要算出

If $\Phi(\tau, \eta) = 0$ for $|\eta| > B$ or $|\tau| > C$

$$C_x(t, f) = \int_{-C}^C \int_{-B}^B \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \Phi(\tau, \eta) e^{-j2\pi u\eta + j2\pi(\eta t - \tau f)} du d\eta d\tau$$

簡化法 2：注意， η 這個參數和 input 及 output 都無關

$$\begin{aligned} C_x(t, f) &= \int_{-C}^C \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \left[\int_{-B}^B \Phi(\tau, \eta) e^{j2\pi\eta(t-u)} d\eta \right] e^{-j2\pi\tau f} dud\tau \\ &= \int_{-C}^C \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \Psi(\tau, t-u) e^{-j2\pi\tau f} dud\tau \end{aligned}$$

$$\Psi(\tau, t) = \int_{-B}^B \Phi(\tau, \eta) e^{j2\pi\eta t} d\eta$$

由於 $\Psi(\tau, t)$ 和 input 無關，可事先算出，所以只剩 2 個積分式

VI-B Modified Wigner Distribution Function

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} x(t + \tau / 2) \cdot x^*(t - \tau / 2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} X(f + \eta / 2) \cdot X^*(f - \eta / 2) e^{j2\pi t\eta} \cdot d\eta \\
 &\quad \text{where } X(f) = FT[x(t)]
 \end{aligned}$$

Modified Form I

$$W_x(t, f) = \int_{-B}^B w(\tau) x(t + \tau / 2) \cdot x^*(t - \tau / 2) e^{-j2\pi\tau f} \cdot d\tau$$

Modified Form II

$$W_x(t, f) = \int_{-B}^B w(\eta) X(f + \eta / 2) \cdot X^*(f - \eta / 2) e^{j2\pi t\eta} \cdot d\eta$$

Modified Form III (Pseudo L -Wigner Distribution)

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x^L\left(t + \frac{\tau}{2L}\right) \cdot \overline{x^L\left(t - \frac{\tau}{2L}\right)} e^{-j2\pi\tau f} \cdot d\tau$$

增加 L 可以減少 cross term 的影響 (但是不會完全消除)

[Ref] L. J. Stankovic, S. Stankovic, and E. Fakultet, “An analysis of instantaneous frequency representation using time frequency distributions-generalized Wigner distribution,” *IEEE Trans. on Signal Processing*, pp. 549-552, vol. 43, no. 2, Feb. 1995

P.S.: 感謝2006年修課的林政豪同學

Modified Form IV (Polynomial Wigner Distribution Function)

$$W_x(t, f) = \int_{-\infty}^{\infty} \left[\prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) \right] e^{-j2\pi \tau f} d\tau$$

When $q = 2$ and $d_1 = d_{-1} = 0.5$, it becomes the original Wigner distribution function.

It can avoid the cross term when **the order of phase** of the exponential function is **no larger than $q/2 + 1$** .

However, the cross term between two components cannot be removed.

[Ref] B. Boashash and P. O'Shea, "Polynomial Wigner-Ville distributions & their relationship to time-varying higher order spectra," *IEEE Trans. Signal Processing*, vol. 42, pp. 216–220, Jan. 1994.

[Ref] J. J. Ding, S. C. Pei, and Y. F. Chang, "Generalized polynomial Wigner spectrogram for high-resolution time-frequency analysis," *APSIPA ASC*, Kaohsiung, Taiwan, Oct. 2013.

d_l should be chosen properly such that

$$\prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} n a_n t^{n-1} \tau\right)$$

$$\text{when } x(t) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n\right)$$

then

$$W_x(t, f) = \int_{-\infty}^{\infty} \exp\left(-j2\pi\left(f - \sum_{n=1}^{q/2+1} n a_n t^{n-1}\right)\tau\right) d\tau \cong \delta\left(f - \sum_{n=1}^{q/2+1} n a_n t^{n-1}\right)$$

$$\prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} n a_n t^{n-1} \tau\right)$$

$$x(t) = \exp\left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n\right)$$

when $q = 2$ $x(t) = \exp(j2\pi(a_1 t + a_2 t^2))$

$$x(t + d_1 \tau) x^*(t - d_{-1} \tau) = \exp(j2\pi(a_1 + 2a_2 t)\tau)$$

$$a_2 (t + d_1 \tau)^2 + a_1 (t + d_1 \tau) - a_2 (t - d_{-1} \tau)^2 - a_1 (t - d_{-1} \tau) = 2a_2 t \tau + a_1 \tau$$

$$2a_2 (d_1 + d_{-1}) t \tau + a_2 (d_1^2 - d_{-1}^2) \tau^2 + a_1 (d_1 + d_{-1}) \tau = 2a_2 t \tau + a_1 \tau$$

$$\Rightarrow d_1 + d_{-1} = 1 \quad d_1 - d_{-1} = 0$$

$$\Rightarrow d_1 = d_{-1} = 1/2$$

When $q = 4$ $x(t) = \exp(j2\pi(a_1t + a_2t^2 + a_3t^3))$

$$\prod_{l=1}^2 x(t + d_l\tau)x^*(t - d_{-l}\tau) = \exp\left(j2\pi \sum_{n=1}^3 na_n t^{n-1}\tau\right)$$

$$x(t + d_1\tau)x^*(t - d_{-1}\tau)x(t + d_2\tau)x^*(t - d_{-2}\tau) = \exp\left(j2\pi \sum_{n=1}^3 na_n t^{n-1}\tau\right)$$

$$a_3(t + d_1\tau)^3 + a_2(t + d_1\tau)^2 + a_1(t + d_1\tau)$$

$$+ a_3(t + d_2\tau)^3 + a_2(t + d_2\tau)^2 + a_1(t + d_2\tau)$$

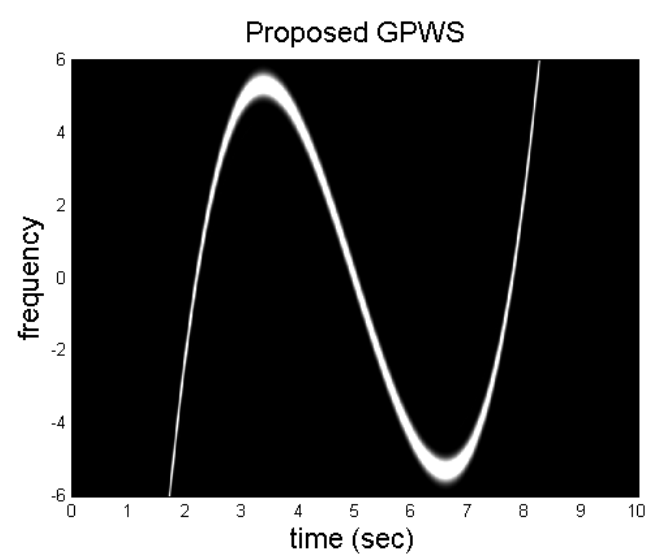
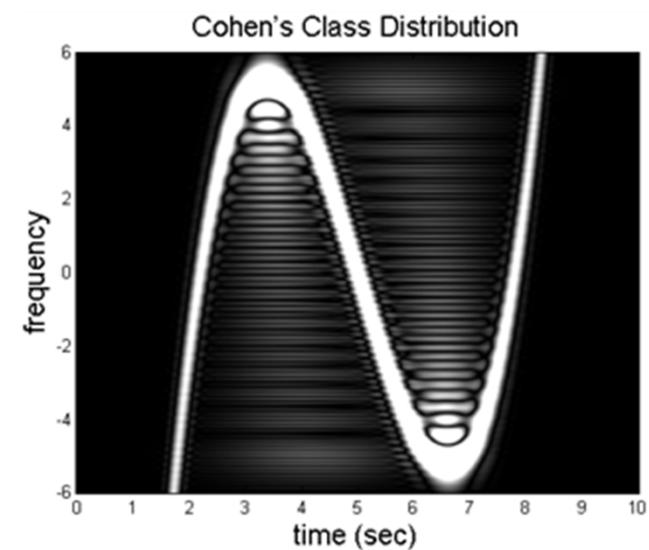
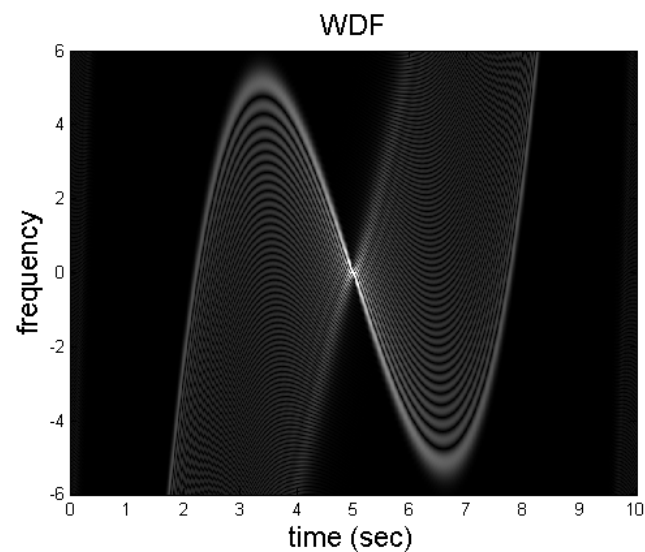
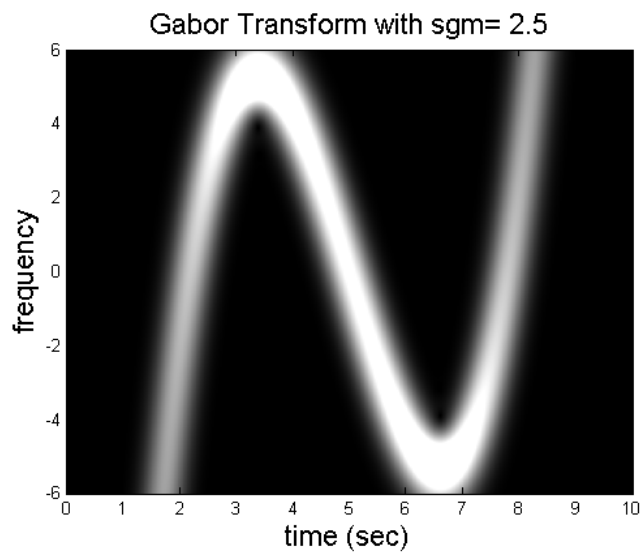
$$- a_3(t - d_{-1}\tau)^3 - a_2(t - d_{-1}\tau)^2 - a_1(t - d_{-1}\tau)$$

$$- a_3(t - d_{-2}\tau)^3 - a_2(t - d_{-2}\tau)^2 - a_1(t - d_{-2}\tau)$$

$$= 3a_3t^2\tau + 2a_2t\tau + a_1\tau$$

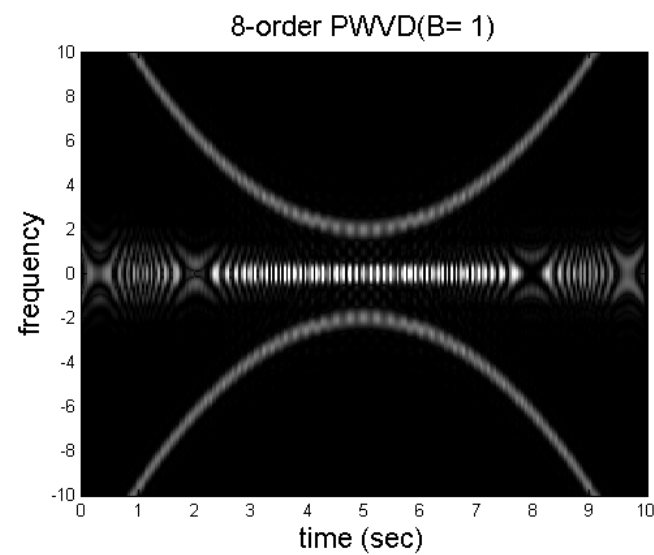
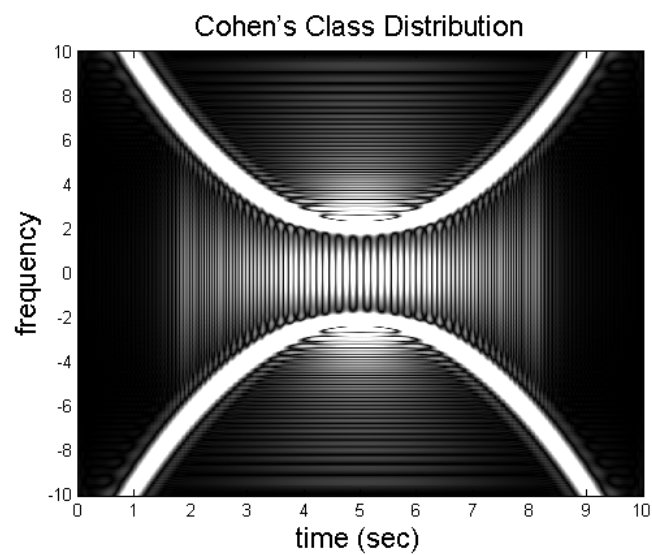
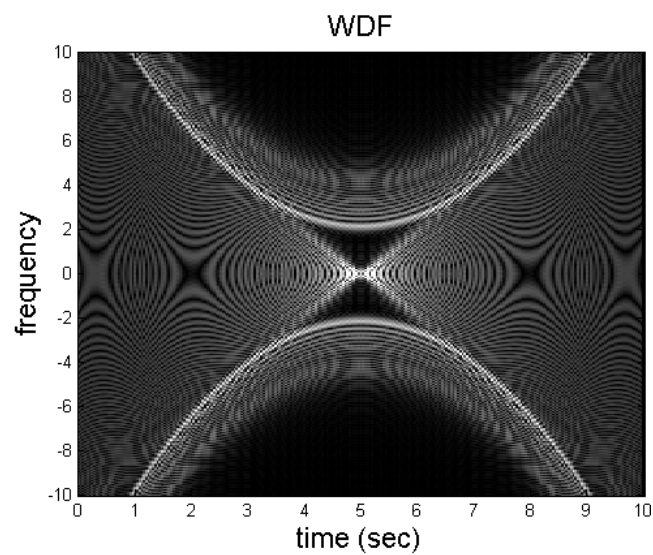
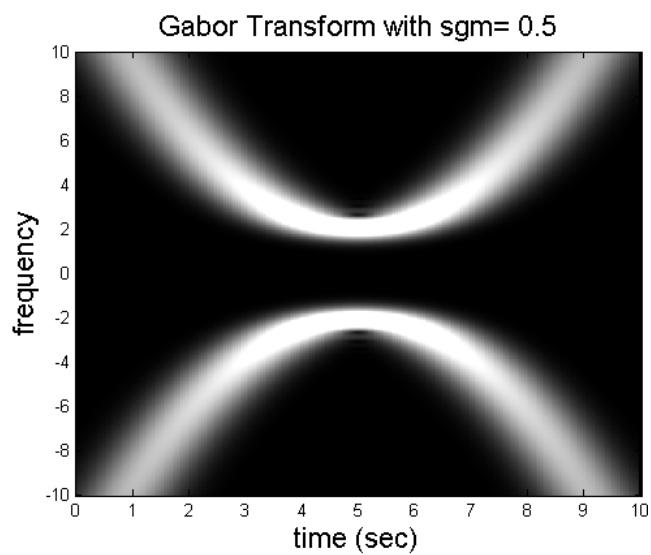
$$\Rightarrow \begin{cases} d_1 + d_2 + d_{-1} + d_{-2} = 1 \\ d_1^2 + d_2^2 - d_{-1}^2 - d_{-2}^2 = 0 \\ d_1^3 + d_2^3 + d_{-1}^3 + d_{-2}^3 = 0 \end{cases}$$

$$x(t) = \exp(j(t-5)^4 - j5\pi(t-5)^2)$$



$q = ?$

$$x(t) = 2 \cos((t - 5)^3 + 4\pi t)$$



VI-C Gabor-Wigner Transform

[Ref] S. C. Pei and J. J. Ding, “Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing,” *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Advantages:

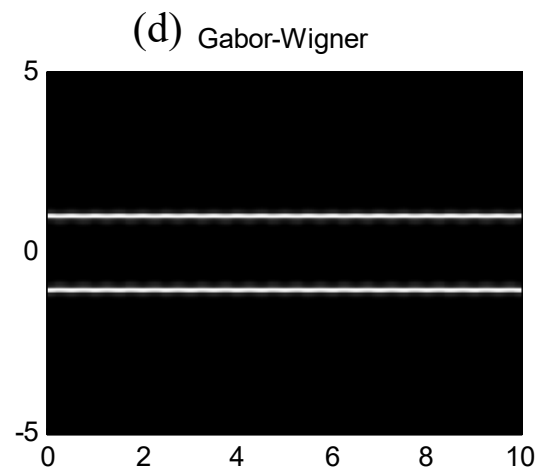
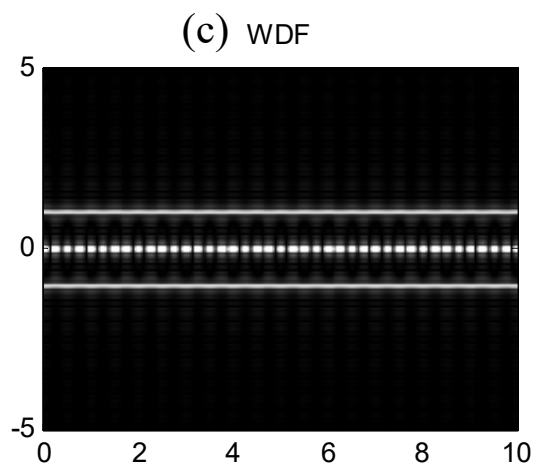
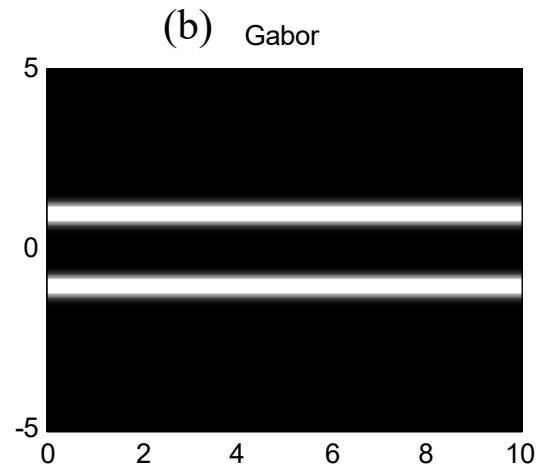
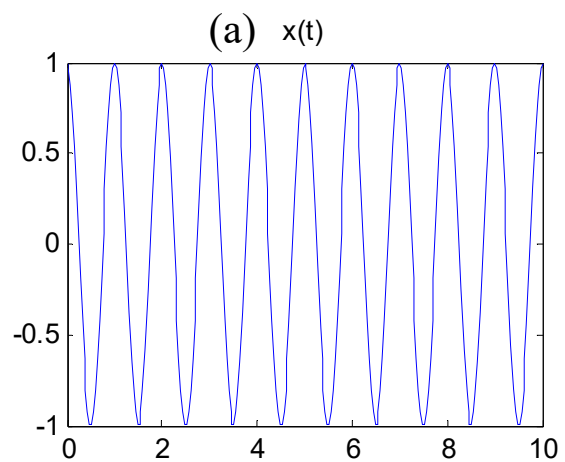
combine the advantage of the WDF and the Gabor transform

advantage of the WDF → higher clarity

advantage of the Gabor transform → no cross-term

$$D_x(t, f) = G_x^2(t, f)W_x(t, f)$$

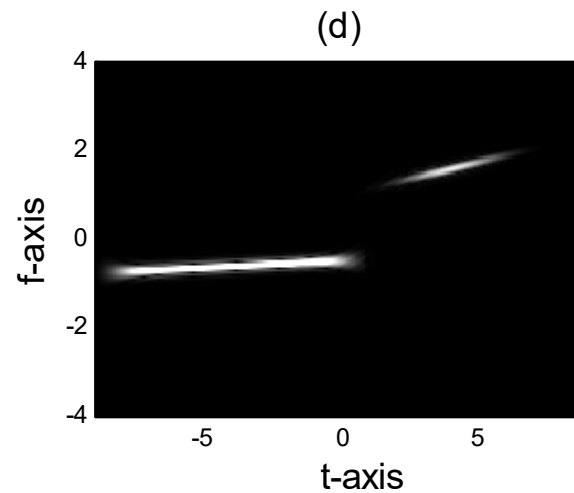
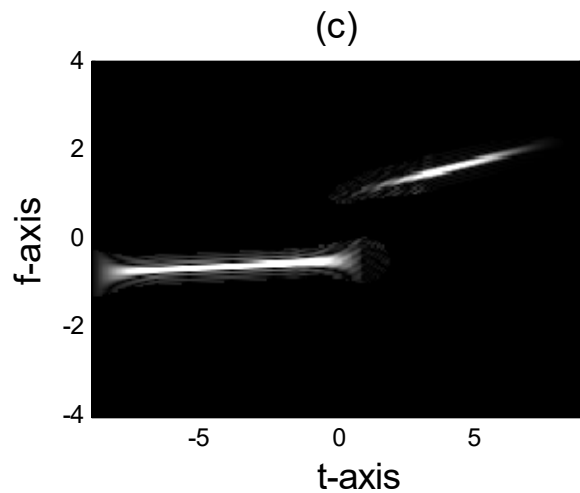
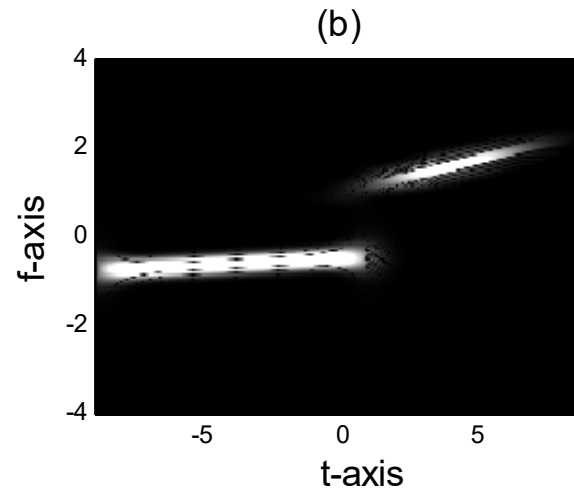
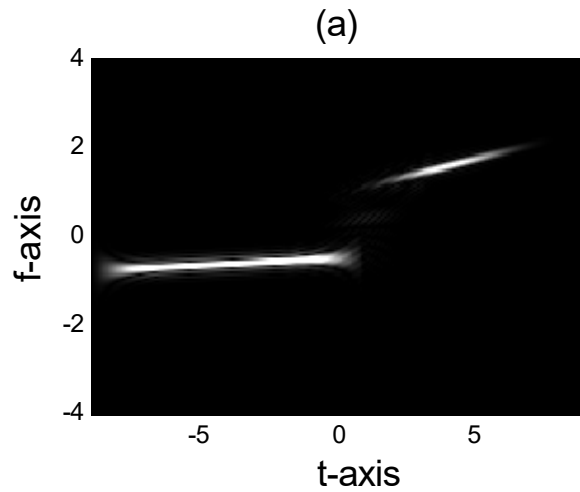
$$x(t) = \cos(2\pi t)$$



(a) $D_x(t, f) = G_x(t, f)W_x(t, f)$ (b) $D_x(t, \omega) = \min(|G_x(t, f)|^2, |W_x(t, f)|)$ 195

(c) $D_x(t, f) = W_x(t, f) \times \{|G_x(t, f)| > 0.25\}$

(d) $D_x(t, f) = G_x^{2.6}(t, f)W_x^{0.7}(t, f)$



(b) 、 (c) are real

思考：

- (1) Which type of the Gabor-Wigner transform is better?
- (2) Can we further generalize the results?

Implementation of the Gabor-Wigner Transform : 簡化技巧

$$(1) \text{ When } G_x(t, f) \approx 0, \quad D_x(t, f) = G_x^\alpha(t, f)W_x^\beta(t, f) \approx 0$$

先算 $G_x(t, f)$

$W_x(t, f)$ 只需算 $G_x(t, f)$ 不近似於 0 的地方

(2) When $x(t)$ is real , 對 Gabor transform 而言

$$X(f) = X^*(-f) \quad \text{if } x(t) \text{ is real, where } X(f) = FT[x(t)]$$

附錄九： Fourier Transform 常用的性質

$$X(f) = FT[x(t)] = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi f t) dt$$

(1) Recovery (inverse Fourier transform)	$x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t) df$
(2) Integration	$x(0) = \int_{-\infty}^{\infty} X(f) df$
(3) Modulation	$FT[x(t) e^{j2\pi f_0 t}] = X(f - f_0)$
(4) Time Shifting	$FT[x(t - t_0)] = X(f) e^{-j2\pi f t_0}$
(5) Scaling	$FT[x(at)] = \frac{1}{ a } X\left(\frac{f}{a}\right)$
(6) Time Reverse	$FT[x(-t)] = X(-f)$

(7) Real / Imaginary Input	<p>If $x(t)$ is real, then $X(f) = X^*(-f)$; If $x(t)$ is pure imaginary, then $X(f) = -X^*(-f)$</p>
(8) Even / Odd Input	<p>If $x(t) = x(-t)$, then $X(f) = X(-f)$; If $x(t) = -x(-t)$, then $X(f) = -X(-f)$;</p>
(9) Conjugation	$FT [x^*(t)] = X^*(-f)$
(10) Differentiation	$FT [x'(t)] = j2\pi f X(f)$
(11) Multiplication by t	$FT [tx(t)] = \frac{j}{2\pi} X'(f)$
(12) Division by t	$FT \left[\frac{x(t)}{t} \right] = -j2\pi \int_{-\infty}^f X(\mu) d\mu$
(13) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} x(t) ^2 dt = \int_{-\infty}^{\infty} X(f) ^2 df$
(14) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$

(15) Linearity	$FT[ax(t) + by(t)] = aX(f) + bY(f)$
(16) Convolution	If $z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$, then $Z(f) = X(f)Y(f)$
(17) Multiplication	If $z(t) = x(t)y(t)$, then $Z(f) = X(f) * Y(f) = \int_{-\infty}^{\infty} X(\mu)Y(f-\mu)d\mu$
(18) Correlation	If $z(t) = \int_{-\infty}^{\infty} x(\tau)y^*(\tau-t)d\tau$, then $Z(f) = X(f)Y^*(f)$
(19) Two Times of Fourier Transforms	$FT\{FT[x(t)]\} = x(-t)$
(20) Four Times of Fourier Transforms	$FT[FT(FT\{FT[x(t)]\})] = x(t)$