VII. Other Time Frequency Distributions (II)

The trend of time-frequency analysis in recent years:

(1) S transform and its generalization

(2) Time-variant signal expansion (Compressive sensing)

(3) Improvement for the Hilbert-Huang transform

VII-A S Transform

(Modification from the Gabor transform)

$$S_x(t,f) = |f| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 f^2\right] \exp(-j2\pi f \tau)\, d\tau$$

closely related to the wavelet transform

advantages and disadvantages

[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

S transform 和 Gabor transform 相似。

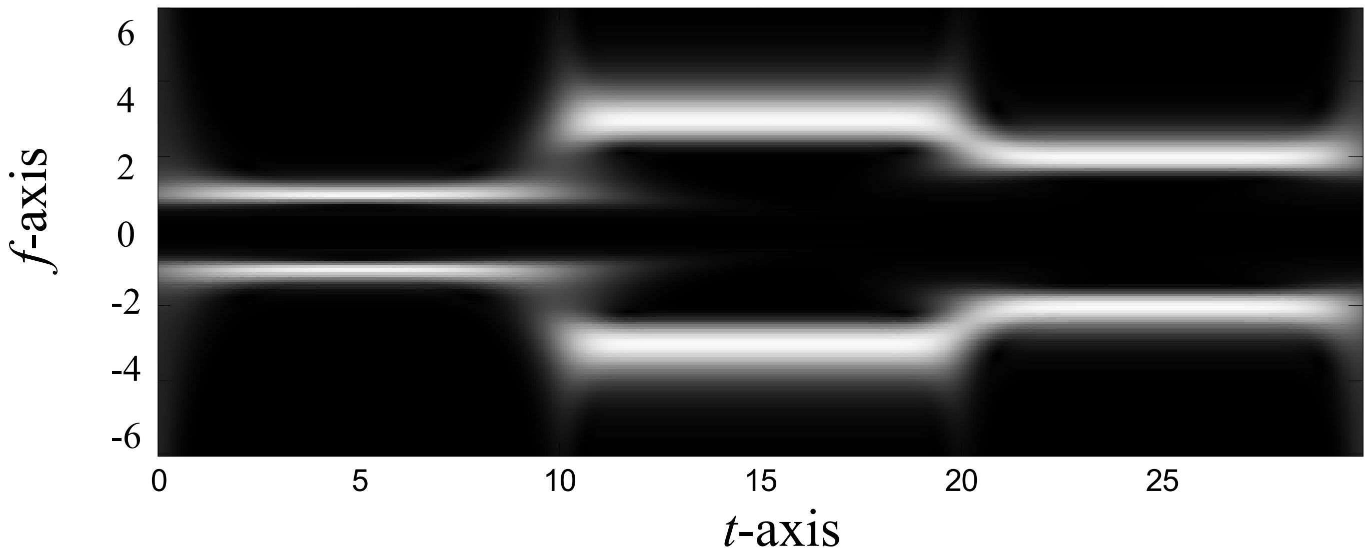
但是 Gaussian window 的寬度會隨著 f 而改變

$$w(t) = \exp\left[-\pi t^2\right] \qquad w(t) = |f| \exp\left[-\pi t^2 f^2\right]$$

低頻： worse time resolution, better frequency resolution

高頻： better time resolution, worse frequency resolution

The result of the S transform (compared with page 91)



- General form

$$S_x(t,f)=\left|s(f)\right|\int_{-\infty}^{\infty}x(\tau)\exp\left[-\pi(t-\tau)^2 s^2(f)\right]\exp(-j2\pi f\tau)\,d\tau$$

s(*f*) increases with *f*

C. R. Pinnegar and L. Mansinha, "The S-transform with windows of arbitrary and varying shape," *Geophysics*, vol. 68, pp. 381-385, 2003.

Fast algorithm of the S transform

When f is fixed, the S transform can be expressed as a convolution form:

$$S_x(t,f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 s^2(f)\right] \exp(-j2\pi f\tau)\, d\tau$$

$$\Downarrow$$

$$S_x(t,f) = |s(f)| \left(x(t)\exp(-j2\pi ft) \underset{\substack{\text{convolution}\\ \text{along } t\text{-axis}}}{*} \exp\left[-\pi t^2 s^2(f)\right] \right)$$

(for every fixed f)

Remember: $g(t) * h(t) = \int g(\tau) h(t-\tau)\, d\tau$

Q: Can we use the FFT-based method on page 115 to implement the S transform?

VII-B Generalized Spectrogram

[Ref] P. Boggiatto, G. De Donno, and A. Oliaro, "Two window spectrogram and their integrals," *Advances and Applications*, vol. 205, pp. 251-268, 2009.

Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^{*}_{x,w_2}(t,f)$

$$G_{x,w_1}(t,f) = \int_{-\infty}^{\infty} w_1(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$G_{x,w_2}(t,f) = \int_{-\infty}^{\infty} w_2(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

Original spectrogram: $w_1(t) = w_2(t)$

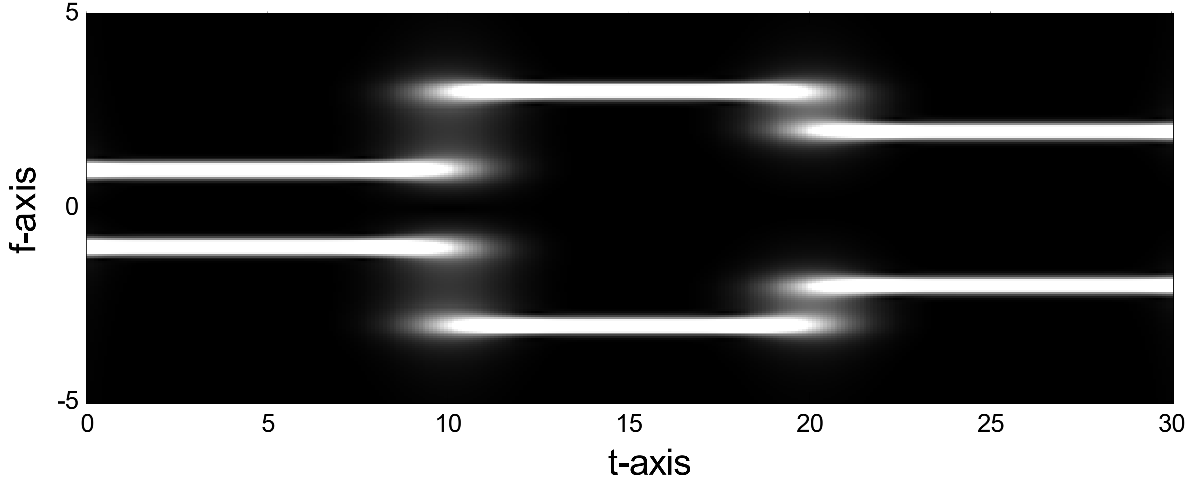
To achieve better clarity, $w_1(t)$ can be chosen as a wider window,
$w_2(t)$ can be chosen as a narrower window.

$x(t) = \cos(2\pi t)$ when $t < 10$,

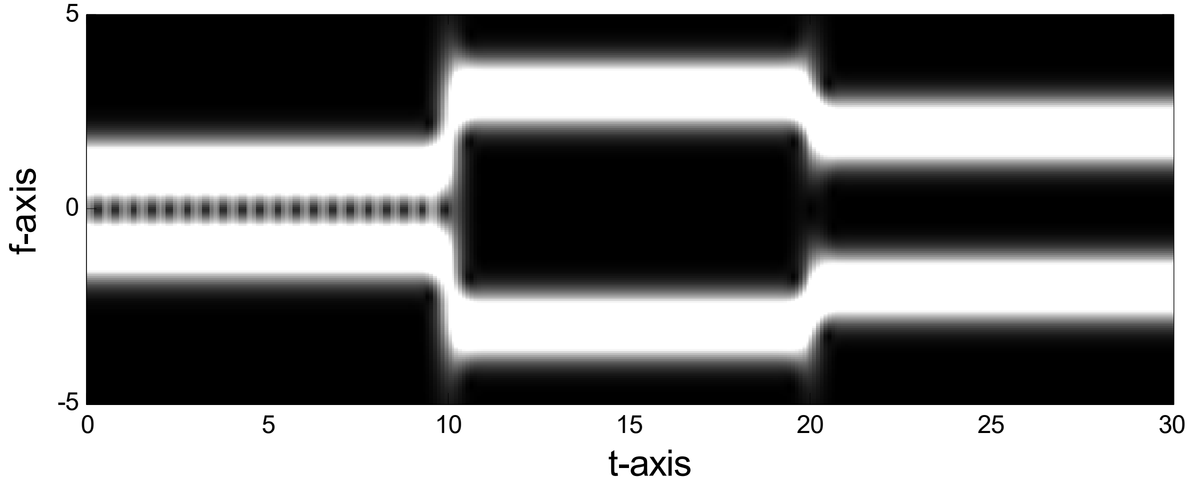
$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$

(a) Gabor, sigma = 0.1



(b) Gabor, sigmal = 1.6

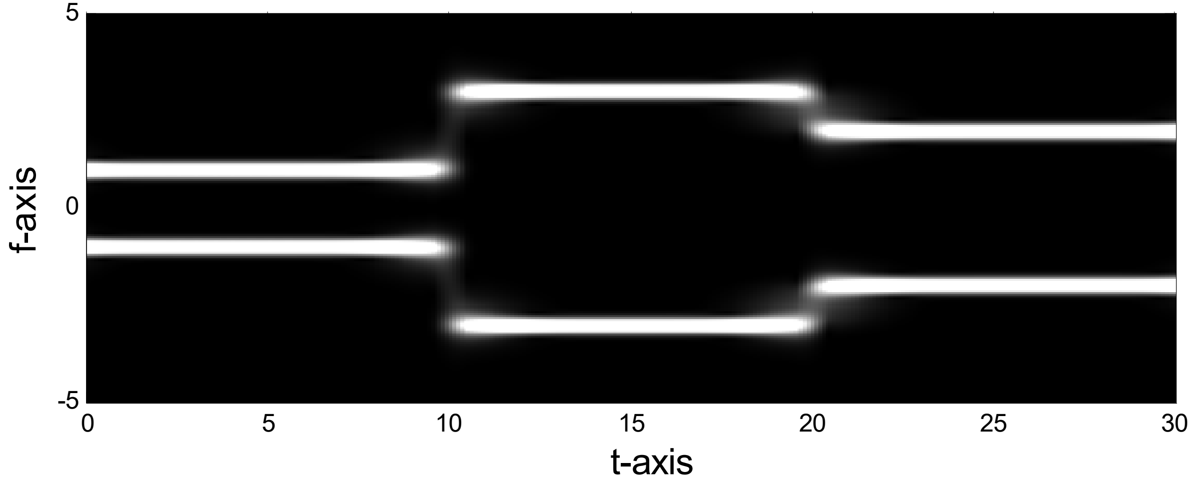


$$x(t) = \cos(2\pi t) \text{ when } t < 10,$$

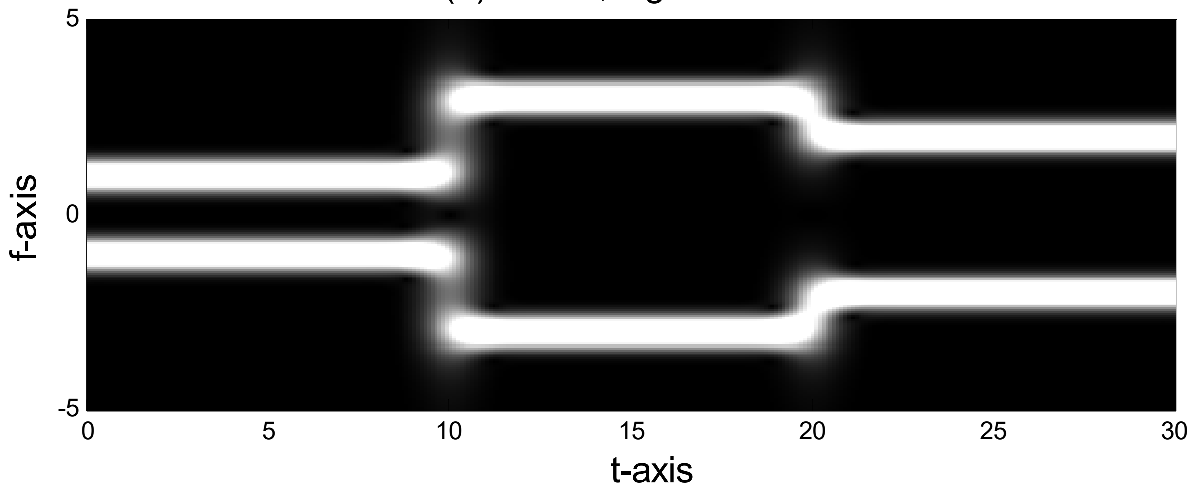
$$x(t) = \cos(6\pi t) \text{ when } 10 \le t < 20,$$

$$x(t) = \cos(4\pi t) \text{ when } t \ge 20$$

(c) Generalized spectrogram



(d) Gabor, sigmal = 0.4



Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^*_{x,w_2}(t,f)$

Further Generalization for the spectrogram:

$$\boxed{SP_{x,w_1,w_2}(t,f) = G^{\alpha}_{x,w_1}(t,f)\,\overline{G^{\beta}_{x,w_2}(t,f)}}$$

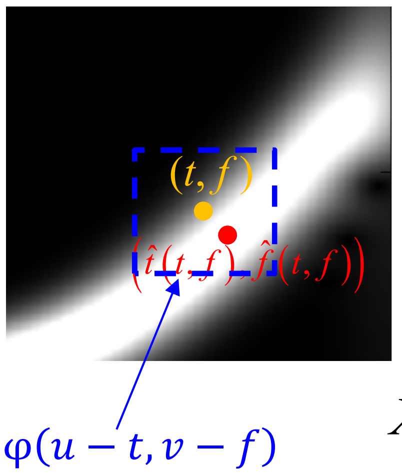
or

$$\boxed{SP_{x,w_1,w_2}(t,f) = \left|G_{x,w_1}(t,f)\right|^{\alpha}\left|G_{x,w_2}(t,f)\right|^{\beta}}$$

VII-C Reassignment Method

After computing the time-frequency distribution, we can use the following way to make the energy even more concentrated.

(1) First, estimate the offset.



$$\hat{t}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}$$

$$\hat{f}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v \cdot \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}$$

$X(t, f)$: time-frequency analysis (STFT, WDF…) of $x(t)$,

$\varphi(u, v) = 1$ when $|u|, |v| < B$

$\varphi(u, v) = 0$ otherwise

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}(t,f), \hat{f}(t,f)\right)$

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}(t,f), \hat{f}(t,f)\right)$

$$\hat{X}(t,f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X(t_1,f_1)\,\delta\left(t-\hat{t}(t_1,f_1)\right)\delta\left(f-\hat{f}(t_1,f_1)\right)dt_1df_1$$

References

[1] F. Auger and P. Flandrin, "Improving the readability of time-frequency and time-scale representations by the reassignment method," *IEEE Trans. Signal Processing*, vol. 43, issue 5, pp. 1068-1089, May 1995.

[2] F. Auger, P. Flandrin, Y.T. Lin, S. McLaughlin, S. Meignen, T. Oberlin, and H.T. Wu, "Time-frequency reassignment and synchrosqueezing: An overview," *IEEE Signal Processing Magazine*, vol. 30, issue 6, pp. 32-41, 2013.

PS: 感謝 2017 年修課的盧德晏同學

VII-D Basis Expansion Time-Frequency Analysis

就如同

- Fourier series: $\varphi_m(t) = \exp(j2\pi f_m t)$, $x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$

$$a_m = \frac{\langle x(t), \varphi_m^*(t) \rangle}{\langle \varphi_m(t), \varphi_m^*(t) \rangle} = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi f_m t)dt \qquad f_m = m/T$$

部分的 Time-Frequency Analysis 也是意圖要將 signal 表示成如下的型態

$$x(t) \approx \sum_{m=1}^{M} a_m \varphi_m(t)$$

並且要求在 M 固定的情形下，

$$\text{approximation error} = \int_{-\infty}^{\infty} \left| x(t) - \sum_{m=1}^{M} a_m \varphi_m(t) \right|^2 dt \qquad \text{為最小}$$

將 $\varphi_m(t)$ 一般化，不同的 basis 之間不只是有 frequency 的差異

(1) Three Parameter Atoms

$$x(t) \approx \sum a_{t_0,f_0,\sigma} \varphi_{t_0,f_0,\sigma}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp(j2\pi f_0 t) \exp(-\frac{\pi(t-t_0)^2}{\sigma^2})$$

3 parameters: t_0 controls the central time
f_0 controls the frequency
σ controls the scaling factor

[Ref] S. G. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397-3415, Dec. 1993.

Since $\varphi_{t_0,f_0,\sigma}(t)$ are not orthogonal, $a_{t_0,f_0,\sigma}$ should be determined by a matching pursuit process.

(2) Four Parameter Atoms (Chirplet)

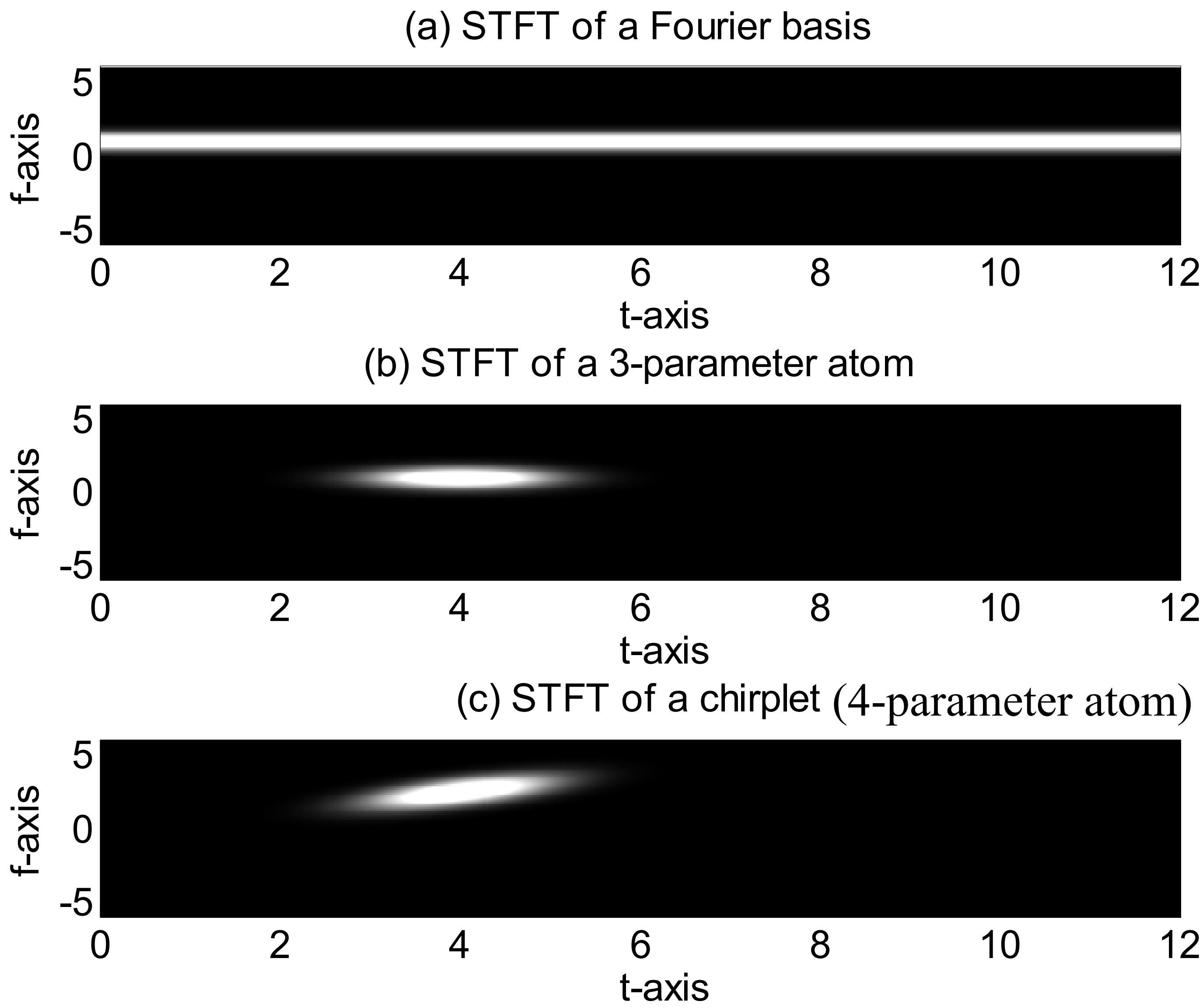
$$x(t) \approx \sum a_{t_0,f_0,\sigma,\eta}\varphi_{t_0,f_0,\sigma,\eta}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}}\exp(j2\pi(f_0 t + \frac{\eta}{2}t^2) - \frac{\pi(t-t_0)^2}{\sigma^2})$$

4 parameters: t_0 controls the central time
f_0 controls the initial frequency
σ controls the scaling factor
η controls the chirp rate

[Ref] A. Bultan, “A four-parameter atomic decomposition of chirplets,” *IEEE Trans. Signal Processing*, vol. 47, no. 3, pp. 731–745, Mar. 1999.

[Ref] C. Capus, and K. Brown. "Short-time fractional Fourier methods for the time-frequency representation of chirp signals," *J. Acoust. Soc. Am*. vol. 113, issue 6, pp. 3253-3263, 2003.


(a) STFT of a Fourier basis
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(b) STFT of a 3-parameter atom
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(c) STFT of a chirplet (4-parameter atom)
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis

(3) Prolate Spheroidal Wave Function (PSWF)

$$x(t) \cong \sum_{n,T,\Omega,t_0,f_0} a_{n,T,\Omega,t_0,f_0} \psi_{n,T,\Omega}(t-t_0)\exp(j2\pi f_0 t)$$

where $\psi_{n,T,\Omega}(t)$ is the prolate spheroidal wave function

[Ref] D. Slepian and H. O. Pollak, “Prolate spheroidal wave functions, Fourier analysis and uncertainty-I,” *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, 1961.

Concept of the prolate spheroidal wave function (PSWF):

- FT: $X(f)=\int_{-\infty}^{\infty}\exp(-j2\pi f\,t)\,x(t)\,dt$, $x, f \in (-\infty, \infty)$.

energy preservation property (Parseval's property)

$$\int_{-\infty}^{\infty}|X(f)|^2\,df = \int_{-\infty}^{\infty}|x(t)|^2\,dt$$

- finite Fourier transform (fi-FT):

$$X_{fi}(f)=\int_{-T}^{T}\exp(-j2\pi f\,t)\,x(t)\,dt$$

space interval: $t \in [-T, T]$, frequency interval: $f \in [-\Omega, \Omega]$

$$0 < \text{energy preservation ratio } = \frac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt} < 1$$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt}$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$

$$\psi_{1,T,\Omega}(t) \text{ can maximize } \frac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt}$$

Among the functions orthogonal to $\psi_{0,T,\Omega}$ and$\psi_{1,T,\Omega}$

$$\psi_{2,T,\Omega}(t) \text{ can maximize } \frac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt}$$

and so on.

- Prolate spheroidal wave functions (PSWFs) are the continuous functions that satisfy: $\int_{-T}^{T} K_{F,\Omega}(t_1,t)\psi_{n,T,\Omega}(t)\,dt = \lambda_{n,T,\Omega}\psi_{n,T,\Omega}(t_1)$,

  where $$K_{F,\Omega}(t_1,t) = \frac{\sin\left[2\pi\Omega(t_1-t)\right]}{\pi(t_1-t)}$$

PSWFs are orthonormal and can be sorted according to the values of $\lambda_{n,T,\Omega}$'s:

$$\int_{-T}^{T} \psi_{m,T,\Omega}(t)\psi_{n,T,\Omega}(t)\,dt = \delta_{m,n}$$

$1 > \lambda_{0,T,\Omega} > \lambda_{1,T,\Omega} > \lambda_{2,T,\Omega} > \ldots\ldots\ldots\ldots > 0.$ (All of $\lambda_{n,T,\Omega}$'s are real)

附錄十： Compressive Sensing and Matching Pursuit 的觀念

Different from orthogonal basis expansion, which applies a complete and orthogonal basis set, compressive sensing is to use an over-complete and non-orthogonal basis set to expand a signal.

Example:

Fourier series expansion is an orthogonal basis expansion method:

$$x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$$

$$\int \exp(j2\pi f_m t)\overline{\exp(j2\pi f_n t)}\, dt = 0 \qquad \text{if } f_m \neq f_n$$

Three-parameter atom expansion, Four-parameter atom (chirplet) expansion, and PSWF expansion are over-complete and non-orthogonal basis expansion methods.

$$x(t) \approx \sum a_{t_0,f_0,\sigma} \varphi_{t_0,f_0,\sigma}(t)$$

$\varphi_{t_0,f_0,\sigma}(t)$ do not form a complete and orthogonal set.

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ ………….. form an over-complete and non-orthogonal basis set.

(Problem 1) We want to minimize $||c||_0$ ($||\ ||_0$ 是 L_0 norm，$||c||_0$ 意指 c_m 的值不為 0 的個數） such that

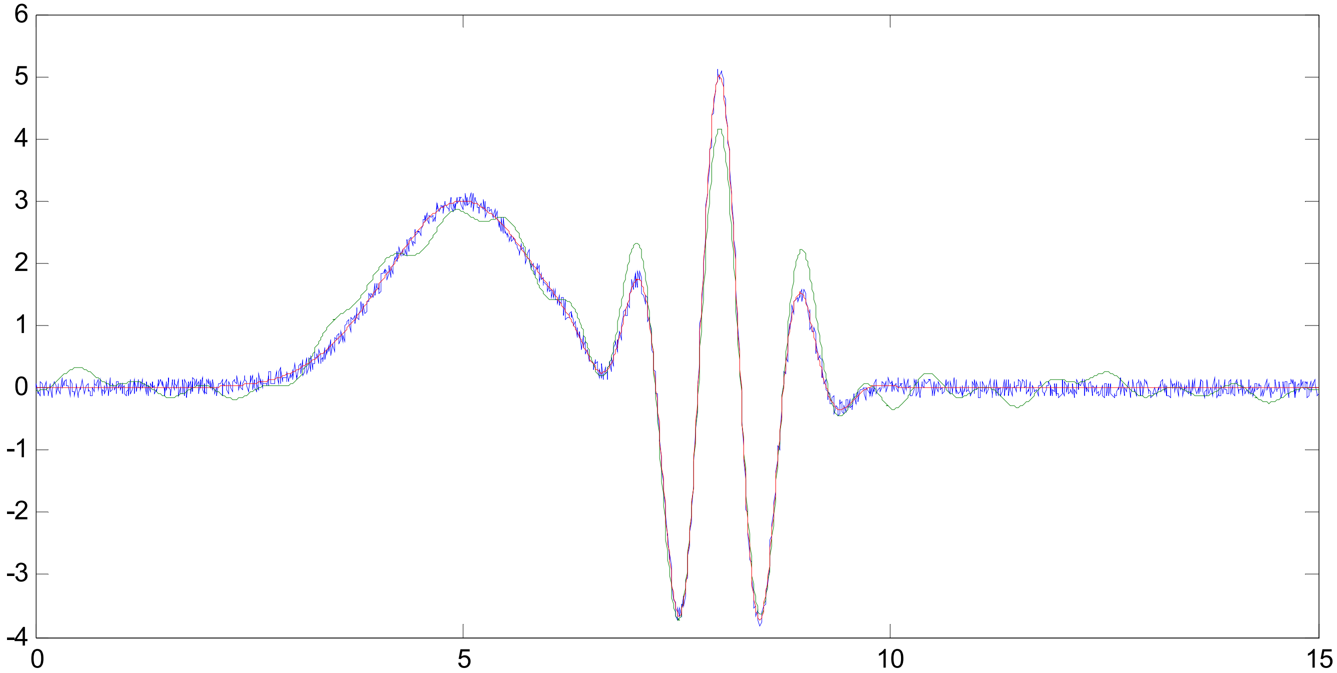
$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_0$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_0$ is limited to M, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$



For example, in the above figure, the blue line is the original signal

- When using three-parameter atoms, the expansion result is the red line

$$x(t) = 3e^{-0.2\pi(t-5)^2} + 2.5e^{-0.4\pi(t-8)^2 + j2\pi t} + 2.5e^{-0.4\pi(t-8)^2 - j2\pi t}$$

Only 3 terms are used and the normalized root square error is 0.39%

- When using Fourier basis, if 31 terms are used, the expansion result is the green line and the normalized root square error is 3.22%

Question: How do we solve the optimization problems on page 221?

Method 1: Matching Pursuit (Greedy Algorithm)

atoms: $b_m(t)$, $m = 1, 2, 3, \ldots.$

$$\left|\int b_m(t) b_m^*(t)\,dt\right| = 1$$

(normalized)

Input: $x(t)$

Initial: $n = 0$, $x_r(t) = x(t)$

↓

Find m such that $\left|\int x_r(t) b_m^*(t)\,dt\right|$ is maximal

↓

Set $\phi_n(t) = b_m(t)$, $\mu_n = \int x(t) b_m^*(t)\,dt$

↓

$x_r(t) = x_r(t) - \mu_n \phi_n(t)$

↓

$n = n+1$

↓

For Problem 1: Whether $x_r(t) = 0$

For Problem 2: Whether $\int x_r^2(t)\,dt < threshold$

For Problem 3: Whether $n = M$

No

Yes

$$x(t) \cong \sum_m \mu_m \phi_m(t)$$

Method 2: Basis Pursuit

Change the L_0 norm into the L_1 norm

$$\|c\|_1 = |c_0| + |c_1| + |c_2| + \ldots\ldots$$

(Problem 1) We want to minimize $\|c\|_1$ such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $\|c\|_1$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $\|c\|_1 \leq M$, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Norm (L_α norm): $\|x[n]\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$

$\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha = K$ where K is the number of points such that $x[n] \neq 0$

(Physical meaning: The number of nonzero points)

L_1 norm: $\|x[n]\|_1 = \sum_{n=0}^{N-1} |x[n]|$

(Physical meaning: Sum of Amplitudes)

L_2 norm: $\|x[n]\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$

(Physical meaning: Distance)

Matching Pursuit: Zero order norm $\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha$

Basis Pursuit: First order norm L_1 norm

[Compressive Sensing 參考文獻]

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