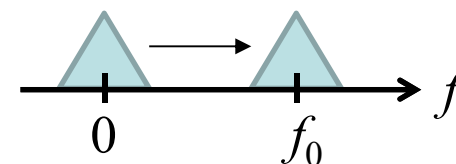


## VIII. Motions on the Time-Frequency Distribution

Fourier spectrum 為 **1-D** form，只有二種可能的運動或變形：

**Modulation**  $e^{j2\pi f_0 t} x(t) \xrightarrow{FT} X(f - f_0)$

**Scaling**  $x(t/a) \xrightarrow{FT} |a| X(af)$



Time-frequency analysis 為 **2-D**，在 2-D 平面上有多種可能的運動或變形

- |                          |                       |
|--------------------------|-----------------------|
| (1) Horizontal shifting  | (2) Vertical shifting |
| (3) Dilation             | (4) Shearing          |
| (5) Generalized Shearing | (6) Rotation          |
| (7) Twisting             |                       |

## 8-1 Basic Motions

### (1) Horizontal Shifting

$$\begin{aligned}x(t - t_0) &\rightarrow S_x(t - t_0, f) e^{-j2\pi f t_0}, \text{STFT, Gabor} \\ &\rightarrow W_x(t - t_0, f), \text{Wigner}\end{aligned}$$

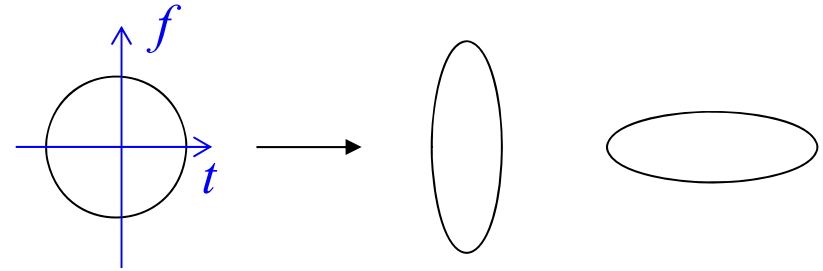
### (2) Vertical Shifting

$$\begin{aligned}e^{j2\pi f_0 t} x(t) &\rightarrow S_x(t, f - f_0), \text{STFT, Gabor} \\ &\rightarrow W_x(t, f - f_0), \text{Wigner}\end{aligned}$$

## (3) Dilation (scaling)

$$\frac{1}{\sqrt{|a|}} x\left(\frac{t}{a}\right) \rightarrow \approx S_x\left(\frac{t}{a}, af\right), \text{STFT, Gabor}$$

$$\rightarrow W_x\left(\frac{t}{a}, af\right), \text{WDF}$$

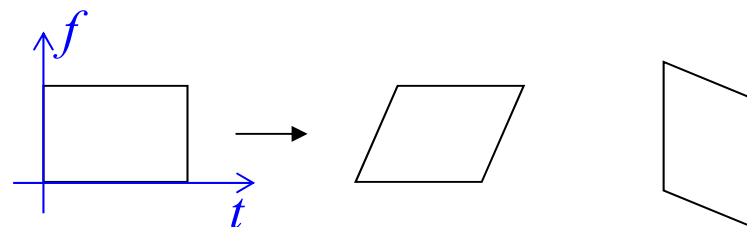


## (4) Shearing

$$x(t) = e^{j\pi at^2} y(t)$$

$$S_x(t, f) \approx S_y(t, f - at), \text{STFT, Gabor}$$

$$W_x(t, f) = W_y(t, f - at), \text{WDF}$$



$$x(t) = e^{j\pi \frac{t^2}{a}} * y(t) \quad (* \text{ means convolution})$$

$$S_x(t, f) \approx S_y(t - af, f), \text{STFT, Gabor}$$

$$W_x(t, f) = W_y(t - af, f), \text{WDF}$$

**(Proof):** When  $x(t) = e^{j\pi at^2} y(t)$ ,

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} e^{j\pi a(t+\tau/2)^2} e^{-j\pi a(t-\tau/2)^2} y(t + \tau/2) y^*(t - \tau/2) e^{-j2\pi\tau f} d\tau \\
 &= \int_{-\infty}^{\infty} e^{j2\pi a t \tau} y(t + \tau/2) y^*(t - \tau/2) e^{-j2\pi\tau f} d\tau \\
 &= \int_{-\infty}^{\infty} y(t + \tau/2) y^*(t - \tau/2) e^{-j2\pi\tau(f-at)} d\tau \\
 &= W_y(t, f - at)
 \end{aligned}$$

## (5) Generalized Shearing

$x(t) = e^{j\phi(t)} y(t)$  的影響?

$$\phi(t) = \sum_{k=0}^n a_k t^k$$

$S_x(t, f) \cong S_y(t, f - \quad)$  ,STFT,Gabor

$W_x(t, f) \cong W_y(t, f - \quad)$  ,WDF

J. J. Ding, S. C. Pei, and T. Y. Ko, “Higher order modulation and the efficient sampling algorithm for time variant signal,” *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

J. J. Ding and C. H. Lee, “Noise removing for time-variant vocal signal by generalized modulation,” *APSIPA ASC*, pp. 1-10, Kaohsiung, Taiwan, Oct. 2013

Q:

If  $x(t) = h(t) * y(t)$  where  $h(t) = IFT \left( \exp \left( j \sum_{k=0}^n a_k f^k \right) \right)$

then

$S_x(t, f) \cong S_y(t, f)$ , STFT, Gabor

$W_x(t, f) \cong W_y(t, f)$ , WDF

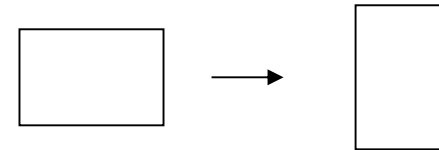
## 8-2 Rotation by $\pi/2$ : Fourier Transform

$$X(f) = FT(x(t))$$

$$|S_X(t, f)| \approx |S_x(-f, t)| \quad , \text{STFT}$$

$$G_X(t, f) = G_x(-f, t)e^{-j2\pi ft} \quad , \text{Gabor}$$

$$W_X(t, f) = W_x(-f, t) \quad , \text{WDF}$$



(clockwise rotation by  $90^\circ$ )

Strictly speaking, the rec-STFT have no rotation property.



For Gabor transforms, if

$$G_x(t, f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau) d\tau ,$$

$$G_X(t, f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} X(\tau) d\tau \quad X(f) = FT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

then  $G_X(t, f) = G_x(-f, t) e^{-j2\pi t f}$

(clockwise rotation by  $90^\circ$  for amplitude)

If we define the Gabor transform as

$$G_x(t, f) = e^{j\pi f t} \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f \tau} x(\tau) d\tau,$$

and  $G_X(t, f) = e^{j\pi f t} \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f \tau} X(\tau) d\tau$

then  $G_X(t, f) = G_x(-f, t)$

If  $W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$  is the WDF of  $x(t)$ ,

$W_X(t, f) = \int_{-\infty}^{\infty} X(t + \tau/2) \cdot X^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$  is the WDF of  $X(f)$ ,

then  $W_X(t, f) = W_x(-f, t)$

(clockwise rotation by  $90^\circ$ )

還有哪些 time-frequency distribution 也有類似性質？

- If  $X(f) = IFT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{j2\pi ft} dt$  , then

$$W_X(t, f) = W_x(f, -t), \quad G_X(t, f) = G_x(f, -t)e^{j2\pi tf}$$

(counterclockwise rotation by  $90^\circ$ ).

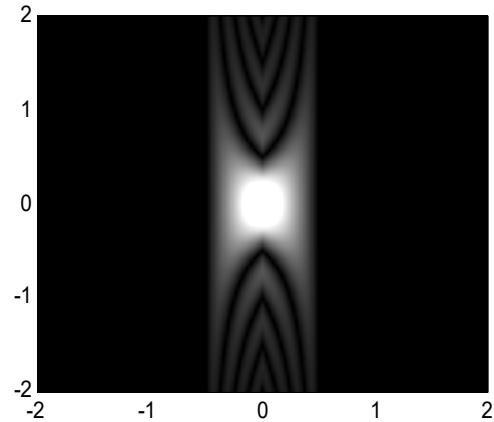
- If  $X(f) = x(-t)$  , then

$$W_X(t, f) = W_x(-t, -f) , \quad G_X(t, f) = G_x(-t, -f).$$

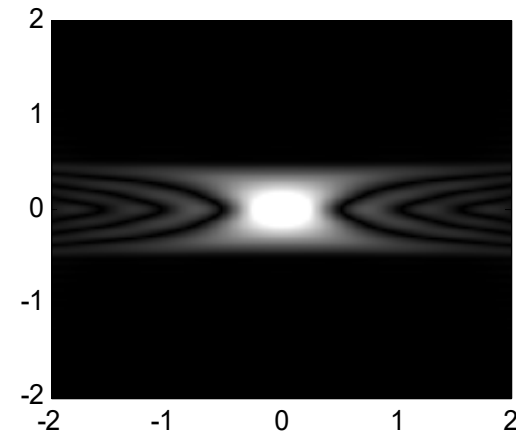
(rotation by  $180^\circ$ ).

Examples:  $x(t) = \Pi(t)$ ,  $X(f) = FT[x(t)] = \text{sinc}(f)$ .

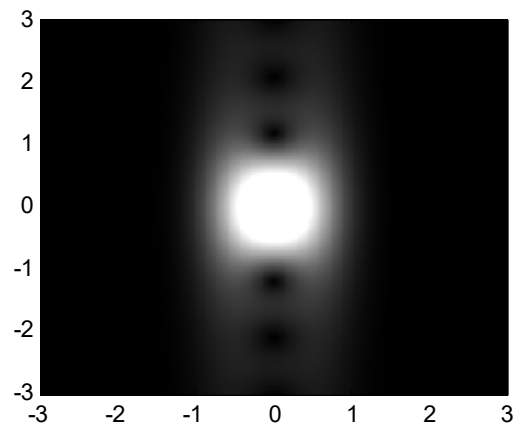
WDF of  $\Pi(t)$



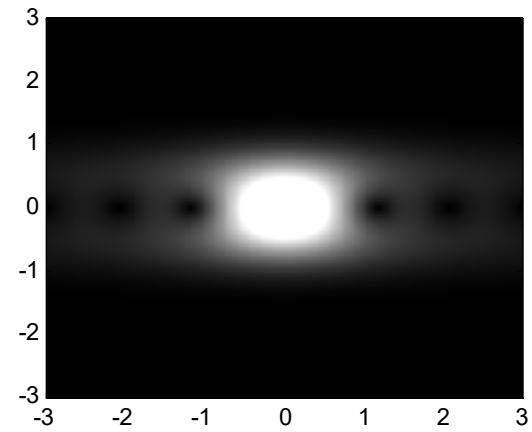
WDF of  $\text{sinc}(f)$



Gabor transform of  $\Pi(t)$



Gabor transform of  $\text{sinc}(f)$



If a function is an **eigenfunction** of the Fourier transform,

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} x(t) dt = \lambda x(f) \quad \lambda = 1, -j, -1, j$$

then its WDF and Gabor transform have the property of

$$W_x(t, f) = W_x(f, -t) \quad |G_x(t, f)| = |G_x(f, -t)|$$

(轉了 90°之後，和原來還是一樣)

Example: **Gaussian function**

$$\exp(-\pi t^2)$$

$$\phi_m(t) = \exp(-\pi t^2) H_m(t)$$

Hermite polynomials:  $H_m(t) = C_m e^{2\pi t^2} \frac{d^m}{dt^m} e^{-2\pi t^2}$ ,  $C_m$  is some constant,

$$H_0(t) = 1 \quad H_1(t) = t \quad H_2(t) = 4\pi t^2 - 1$$

$$H_3(t) = 4\pi t^3 - 3t \quad H_4(t) = 16\pi^2 t^4 - 24\pi t^2 + 3$$

$$\int_{-\infty}^{\infty} e^{-2\pi t^2} H_m(t) H_n(t) dt = D_m \delta_{m,n}, \quad D_m \text{ is some constant,}$$

$$\delta_{m,n} = 1 \quad \text{when } m = n, \quad \delta_{m,n} = 0 \quad \text{otherwise.}$$

[Ref] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 1990.

Hermite-Gaussian functions are eigenfunctions of the Fourier transform

$$\int_{-\infty}^{\infty} \phi_m(t) e^{-j2\pi ft} dt = (-j)^m \phi_m(f)$$

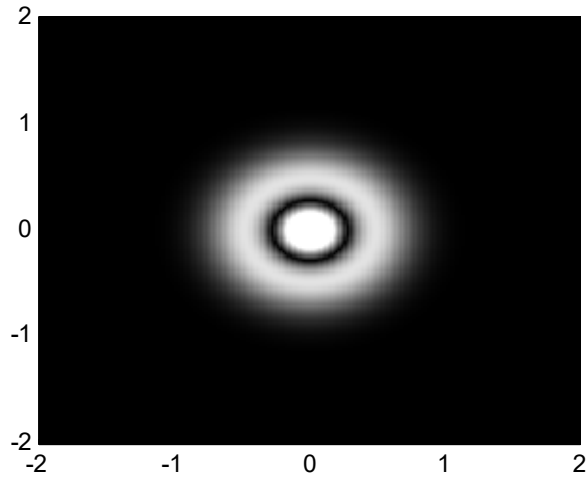
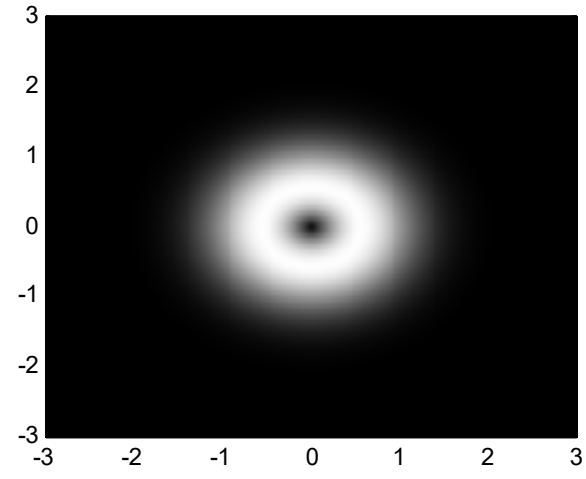
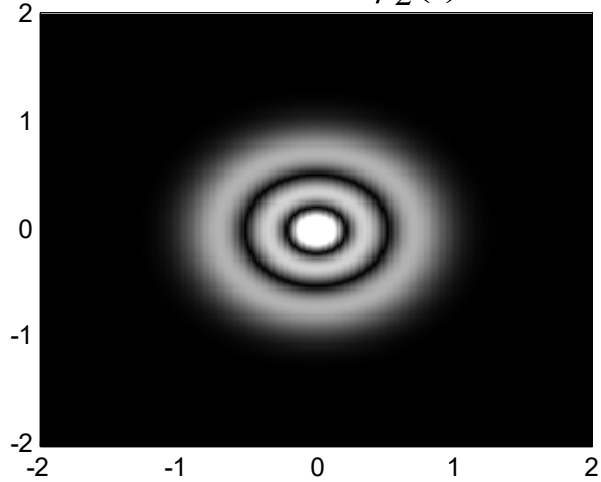
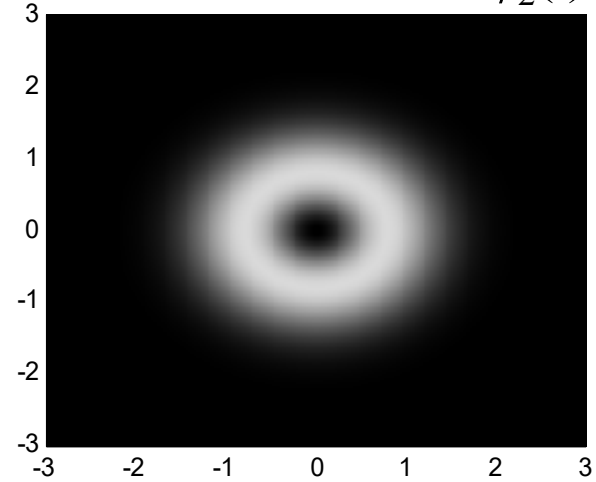
Any eigenfunction of the Fourier transform can be expressed as the form of

$$k(t) = \sum_{q=0}^{\infty} a_{4q+r} \phi_{4q+r}(t) \quad \text{where } r = 0, 1, 2, \text{ or } 3,$$

$a_{4q+r}$  are some constants

$$\int_{-\infty}^{\infty} k(t) e^{-j2\pi ft} dt = (-j)^r k(f)$$



WDF for  $\phi_1(t)$ Gabor transform for  $\phi_1(t)$ WDF for  $\phi_2(t)$ Gabor transform for  $\phi_2(t)$ 

**Problem:** How to rotate the time-frequency distribution by the angle other than  $\pi/2$ ,  $\pi$ , and  $3\pi/2$ ?

## 8-3 Rotation: Fractional Fourier Transforms (FRFTs)

246

$$X_{\phi}(u) = \sqrt{1 - j \cot \phi} e^{j\pi \cot \phi \cdot u^2} \int_{-\infty}^{\infty} e^{-j2\pi \cdot \csc \phi \cdot u t} e^{j\pi \cdot \cot \phi \cdot t^2} x(t) dt, \quad \phi = 0.5a\pi$$

When  $\phi = 0.5\pi$ , the FRFT becomes the FT.

**Additivity property:**

If we denote the FRFT as  $O_F^{\phi}$  (i.e.,  $X_{\phi}(u) = O_F^{\phi}[x(t)]$ )

then  $O_F^{\sigma} \{O_F^{\phi}[x(t)]\} = O_F^{\phi+\sigma}[x(t)]$

**Physical meaning:** Performing the FT  $a$  times.

Another definition 
$$X_{\phi}(u) = \sqrt{\frac{1 - j \cot \phi}{2\pi}} e^{j \frac{\cot \phi}{2} u^2} \int_{-\infty}^{\infty} e^{-j \csc \phi \cdot u t} e^{j \frac{\cot \phi}{2} t^2} x(t) dt$$

- [Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.
- [Ref] N. Wiener, “Hermitian polynomials and Fourier analysis,” *Journal of Mathematics Physics MIT*, vol. 18, pp. 70-73, 1929.
- [Ref] V. Namias, “The fractional order Fourier transform and its application to quantum mechanics,” *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.
- [Ref] L. B. Almeida, “The fractional Fourier transform and time-frequency representations,” *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.
- [Ref] S. C. Pei and J. J. Ding, “Closed form discrete fractional and affine Fourier transforms,” *IEEE Trans. Signal Processing*, vol. 48, no. 5, pp. 1338-1353, May 2000.

$$FT[x(t)] = X(f)$$

$$FT\{FT[x(t)]\} = x(-t)$$

$$FT(FT\{FT[x(t)]\}) = X(-f) = IFT[f(t)]$$

$$FT[FT(FT\{FT[x(t)]\})] = x(t)$$

What happen if we do the FT non-integer times?

### Physical Meaning:

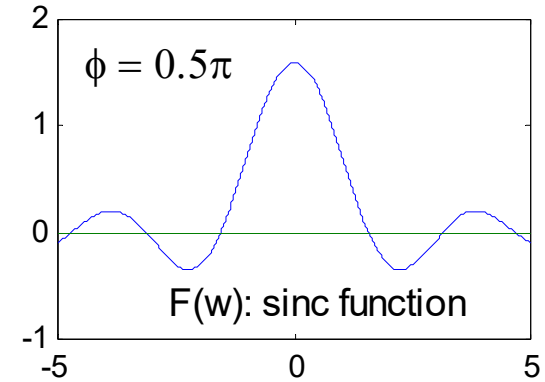
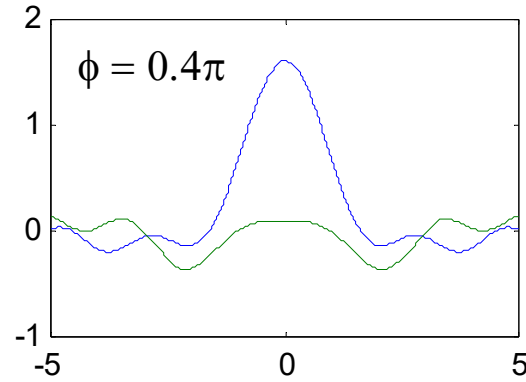
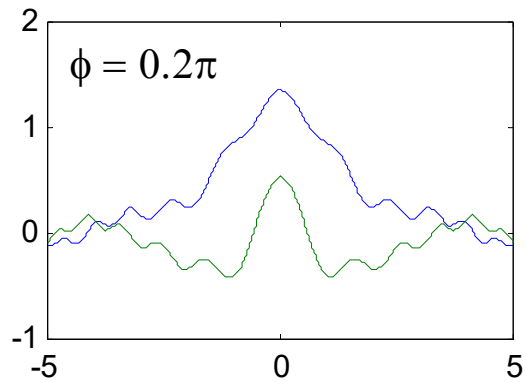
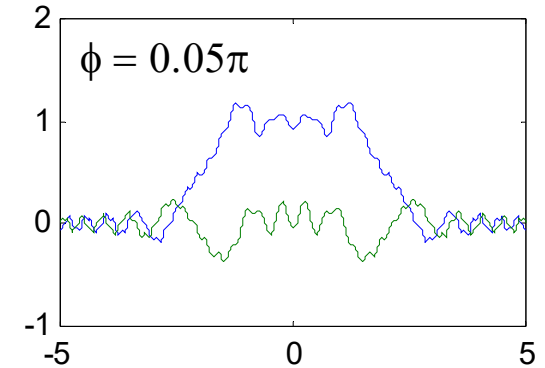
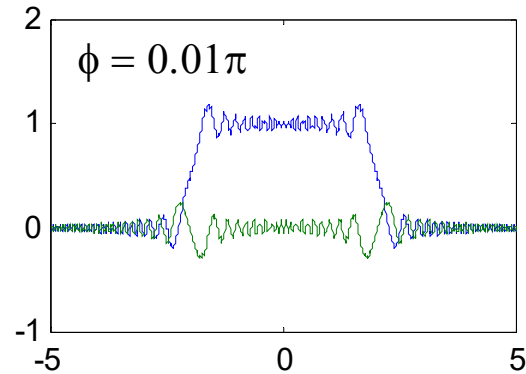
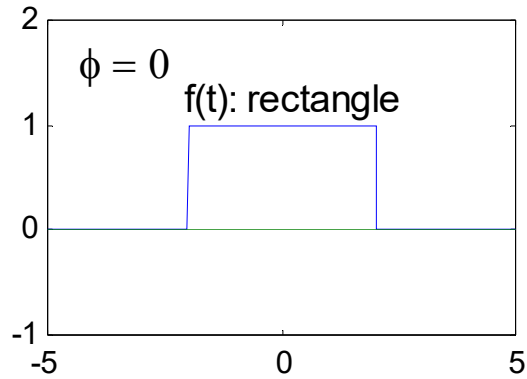
Fourier Transform: time domain  $\rightarrow$  frequency domain

Fractional Fourier transform: time domain  $\rightarrow$  **fractional** domain

**Fractional domain**: the domain between time and frequency

(partially like time and partially like frequency)

## Experiment:



Time domain	Frequency domain	fractional domain
Modulation	Shifting	Modulation + Shifting
Shifting	Modulation	Modulation + Shifting
Differentiation	$\times j2\pi f$	Differentiation and $\times j2\pi f$
$\times -j2\pi f$	Differentiation	Differentiation and $\times -j2\pi f$

$$x(t - t_0) \xrightarrow{FT} \exp(-j2\pi ft_0) X(f)$$

$$x(t - t_0) \xrightarrow{\text{fractional FT}} \exp(j\phi - j2\pi ut_0 \sin \phi) X(u - t_0 \cos \phi)$$

$$\frac{d}{dt} x(t) \xrightarrow{FT} j2\pi f X(f)$$

$$\frac{d}{dt} x(t) \xrightarrow{\text{fractional FT}} j2\pi u X(u) \sin \phi + \frac{d}{du} X(u) \cos \phi$$

**[Theorem]** The fractional Fourier transform (FRFT) with angle  $\phi$  is equivalent to the clockwise rotation operation with angle  $\phi$  for the Wigner distribution function (or for the Gabor transform)

$$\text{FRFT}_\phi = \text{with angle } \phi$$


For the WDF

If  $W_x(t, f)$  is the WDF of  $x(t)$ , and  $W_{X_\phi}(u, v)$  is the WDF of  $X_\phi(u)$ , ( $X_\phi(u)$  is the FRFT of  $x(t)$ ), then

$$W_{X_\phi}(u, v) = W_x(u \cos \phi - v \sin \phi, u \sin \phi + v \cos \phi)$$



For the Gabor transform (with standard definition)

If  $G_x(t, f)$  is the Gabor transform of  $x(t)$ ,  
and  $G_{X_\phi}(u, v)$  is the Gabor transform of  $X_\phi(u)$ , then

$$G_{X_\phi}(u, v) = e^{j[-2\pi uv \sin^2 \phi + \pi(u^2 - v^2) \sin(2\phi)/2]} G_x(u \cos \phi - v \sin \phi, u \sin \phi + v \cos \phi)$$

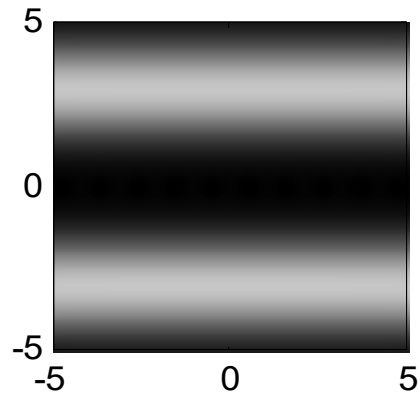
$$|G_{X_\phi}(u, v)| = |G_x(u \cos \phi - v \sin \phi, u \sin \phi + v \cos \phi)|$$

For the Gabor transform (with another definition on page 237)

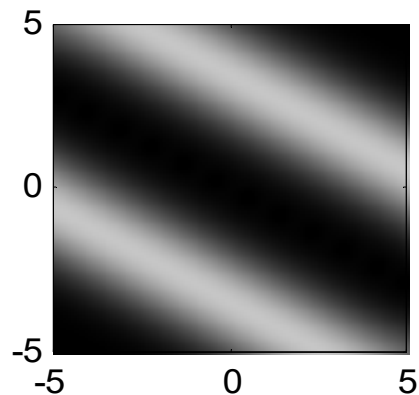
$$G_{X_\phi}(u, v) = G_x(u \cos \phi - v \sin \phi, u \sin \phi + v \cos \phi)$$

The [Cohen's class distribution](#) and the [Gabor-Wigner transform](#) also have the rotation property

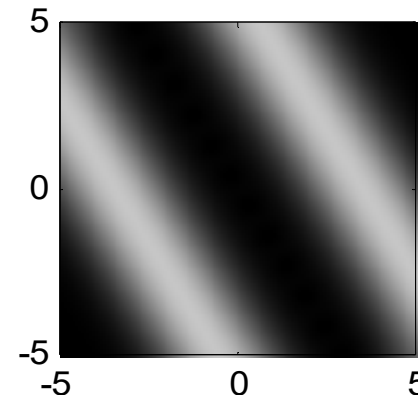
# The Gabor Transform for the FRFT of a cosine function



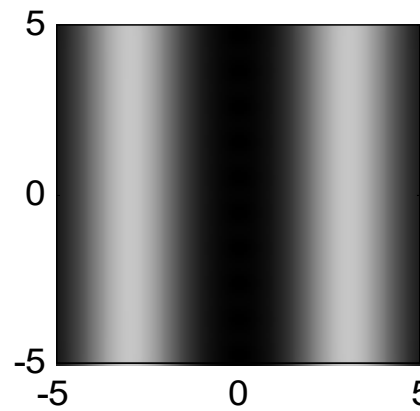
(a)  $\phi = 0$



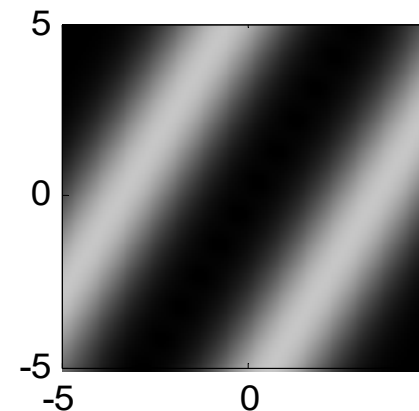
(b)  $\phi = \pi/6$



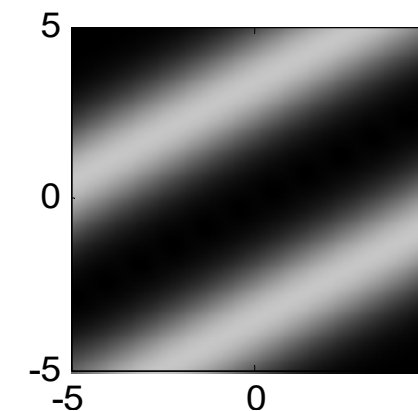
(c)  $\phi = 2\pi/6$



(d)  $\phi = 3\pi/6$

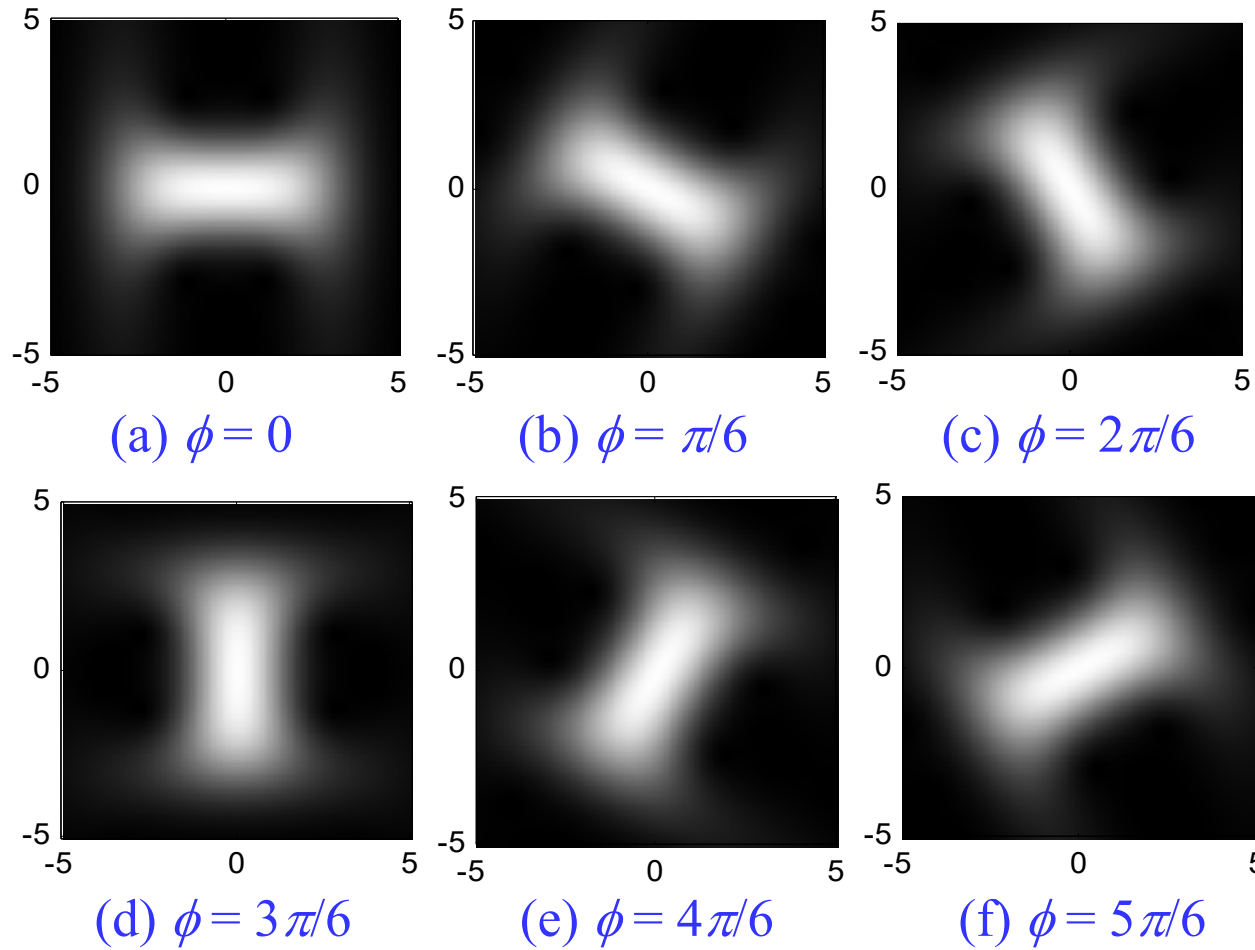


(e)  $\phi = 4\pi/6$



(f)  $\phi = 5\pi/6$

The Gabor Transform for the FRFT of a rectangular function.



## 8-4 Twisting: Linear Canonical Transform (LCT)

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}} e^{j\pi \frac{d}{b} u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b} ut} e^{j\pi \frac{a}{b} t^2} x(t) dt \quad \text{when } b \neq 0$$

$$X_{(a,0,c,d)}(u) = \sqrt{d} \cdot e^{j\pi cd u^2} x(du) \quad \text{when } b = 0$$

$ad - bc = 1$  should be satisfied

Four parameters  $a, b, c, d$

## Additivity property of the WDF

If we denote the LCT by  $O_F^{(a,b,c,d)}$ , i.e.,  $X_{(a,b,c,d)}(u) = O_F^{(a,b,c,d)}[x(t)]$

then  $O_F^{(a_2,b_2,c_2,d_2)} \{ O_F^{(a_1,b_1,c_1,d_1)} [x(t)] \} = O_F^{(a_3,b_3,c_3,d_3)} [x(t)]$

where  $\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

[Ref] K. B. Wolf, “*Integral Transforms in Science and Engineering*,” Ch. 9:  
Canonical transforms, New York, Plenum Press, 1979.

If  $W_{X_{(a,b,c,d)}}(u, v)$  is the WDF of  $X_{(a,b,c,d)}(u)$ , where  $X_{(a,b,c,d)}(u)$  is the LCT of  $x(t)$ , then

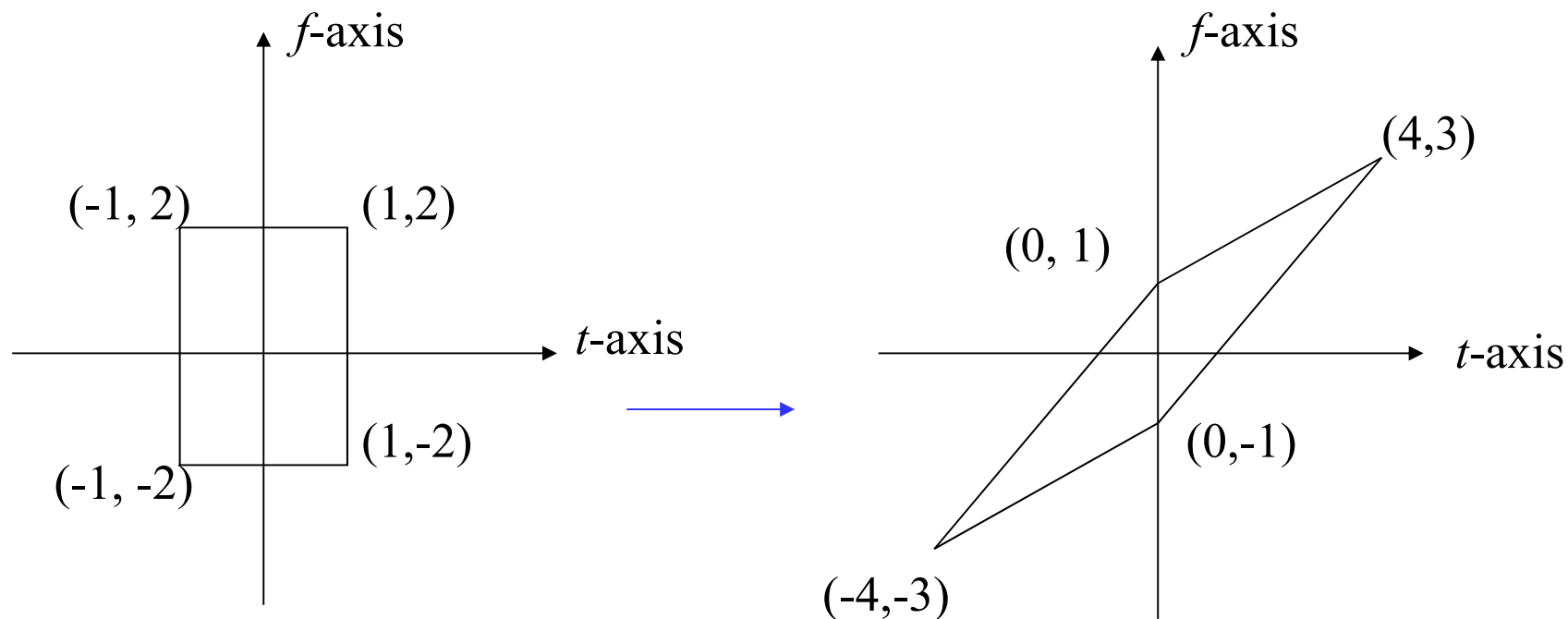
$$W_{X_{(a,b,c,d)}}(u, v) = W_x(du - bv, -cu + av)$$

$$W_{X_{(a,b,c,d)}}(au + bv, cu + dv) = W_x(u, v)$$

LCT == twisting operation for the WDF

The Cohen's class distribution also has the twisting operation.

我們可以自由的用 LCT 將一個中心在  $(0, 0)$  的平行四邊形的區域，扭曲成另外一個面積一樣且中心也在  $(0, 0)$  的平行四邊形區域。



$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}} e^{j\pi \frac{d}{b} u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b} ut} e^{j\pi \frac{a}{b} t^2} x(t) dt \quad \text{when } b \neq 0$$

$$X_{(a,0,c,d)}(u) = \sqrt{d} \cdot e^{j\pi cd u^2} x(du) \quad \text{when } b = 0$$

$ad - bc = 1$  should be satisfied

linear canonical transform

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}$$

fractional Fourier transform

$\phi = \pi/2$  Fourier transform

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$$

Fresnel transform

$\phi = 0$  identity operation

$\phi = -\pi/2$  inverse Fourier transform

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}$$

chirp multiplication

$$X_{(a,0,c,d)}(u) = e^{j\pi \tau u^2} x(u)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/\sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

scaling



## 附錄十一 Linear Canonical Transform 和光學系統的關係

(1) Fresnel Transform (電磁波在空氣中的傳播)

$$U_o(x, y) = -\frac{i}{\lambda} \frac{e^{ikz}}{z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\frac{k}{2z}[(x-x_i)^2+(y-y_i)^2]} U_i(x_i, y_i) dx_i dy_i$$

$k = 2\pi/\lambda$ : wave number       $\lambda$ : wavelength       $z$ : distance of propagation

$$U_o(x, y) = e^{ikz} \sqrt{\frac{1}{j\lambda z}} \int_{-\infty}^{\infty} e^{j\frac{k}{2z}(y-y_i)^2} \sqrt{\frac{1}{j\lambda z}} \int_{-\infty}^{\infty} e^{j\frac{k}{2z}(x-x_i)^2} U_i(x_i, y_i) dx_i dy_i$$

(2 個 1-D 的 LCT)

Fresnel transform 相當於 LCT  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

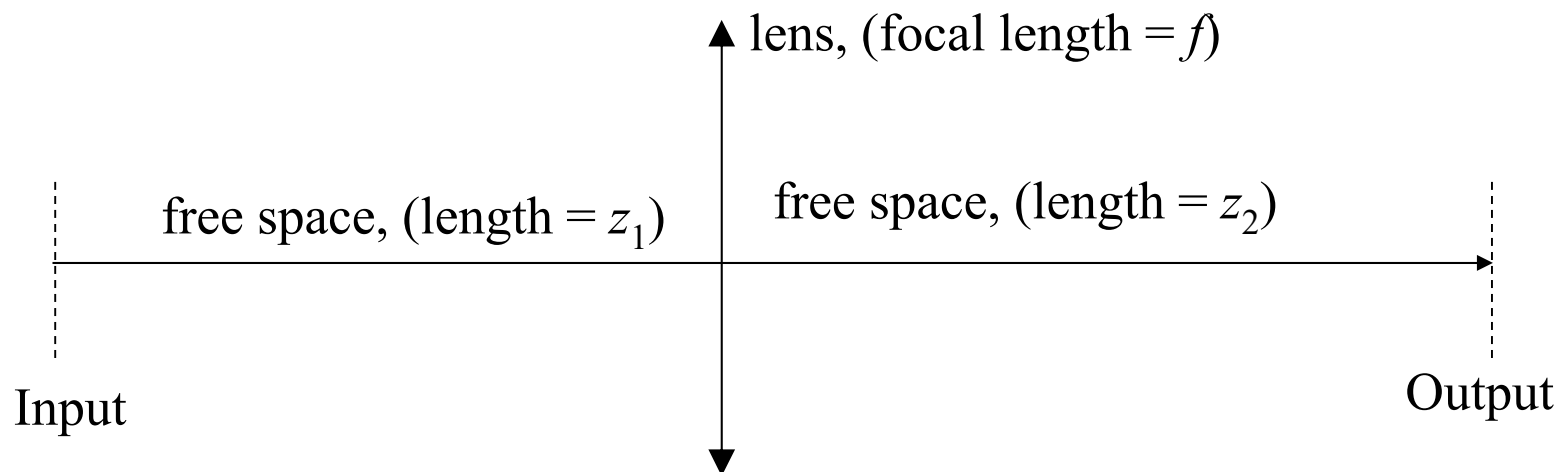
(2) Spherical lens, refractive index =  $n$

$$U_o(x, y) = e^{ikn\Delta} e^{-j\frac{k}{2f}[x^2+y^2]} U_i(x, y)$$

$f$ : focal length     $\Delta$ : thickness of lens

經過 lens 相當於 LCT  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}$  的情形

## (3) Free space 和 Spherical lens 的綜合



Input 和 output 之間的關係，可以用 LCT 表示

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{z_2}{f} & \lambda(z_1 + z_2) - \frac{\lambda z_1 z_2}{f} \\ -\frac{1}{\lambda f} & 1 - \frac{z_1}{f} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 - \frac{z_2}{f} & \lambda(z_1 + z_2) - \frac{\lambda z_1 z_2}{f} \\ -\frac{1}{\lambda f} & 1 - \frac{z_1}{f} \end{bmatrix}$$

$z_1 = z_2 = 2f \rightarrow$  即高中物理所學的「倒立成像」

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -\frac{1}{\lambda f} & -1 \end{bmatrix}$$

$z_1 = z_2 = f \rightarrow$  Fourier Transform + Scaling

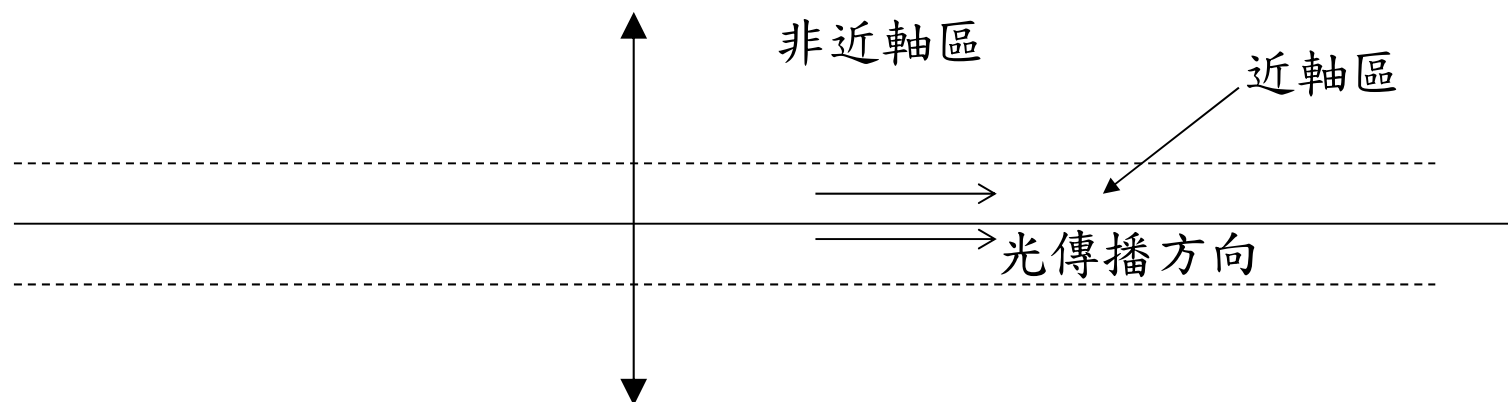
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & \lambda f \\ -\frac{1}{\lambda f} & 0 \end{bmatrix}$$

$z_1 = z_2 \rightarrow$  fractional Fourier Transform + Scaling

用 LCT 來分析光學系統的好處：

只需要用到  $2 \times 2$  的矩陣運算，避免了複雜的物理理論和數學積分

但是 LCT 來分析光學系統的結果，只有在「近軸」的情形下才準確



參考資料：

- [1] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *J. Opt. Soc. Am. A*, vol.12, 743-751, 1995.
- [2] L. M. Bernardo, "ABCD matrix formalism of fractional Fourier optics," *Optical Eng.*, vol. 35, no. 3, pp. 732-740, March 1996.