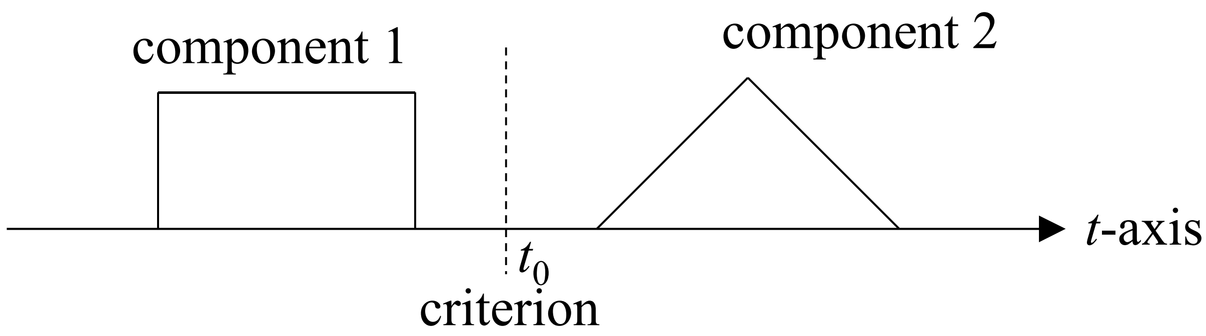
IX. Applications of Time-Frequency Analysis for Filter Design

9-1 Signal Decomposition and Filter Design

Signal Decomposition: Decompose a signal into several components.

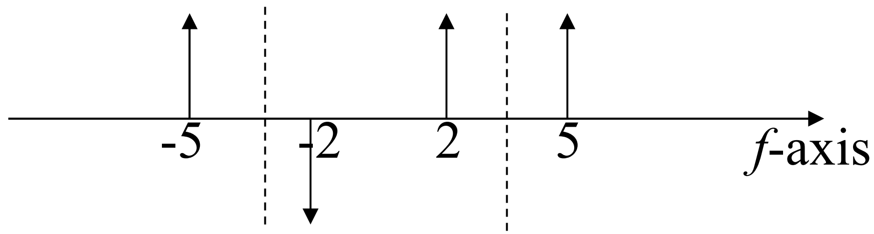
Filter: Remove the undesired component of a signal

(1) Decomposing in the time domain



(2) Decomposing in the frequency domain

$$x(t) = \sin(4\pi t) + \cos(10\pi t)$$



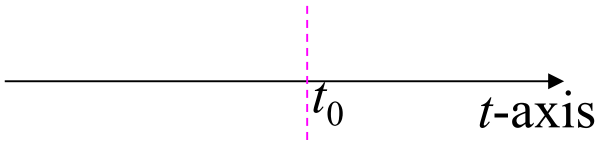
- Sometimes, signal and noise are separable in the time domain →
  (1) without any transform

- Sometimes, signal and noise are separable in the frequency domain →
  (2) using the FT (conventional filter)

$$x_o(t) = IFT\left[FT(x_i(t))H(f)\right]$$

- If signal and noise are not separable in both the time and the frequency domains →
  (3) Using the fractional Fourier transform and time-frequency analysis

以時頻分析的觀點，criterion in the time domain 相當於 cutoff line perpendicular to *t*-axis



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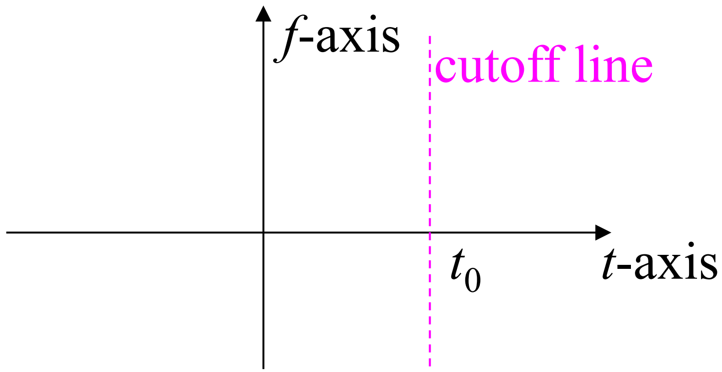
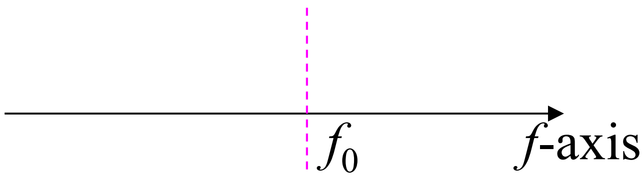


以時頻分析的觀點，criterion in the frequency domain 相當於 cutoff line perpendicular to *f*-axis



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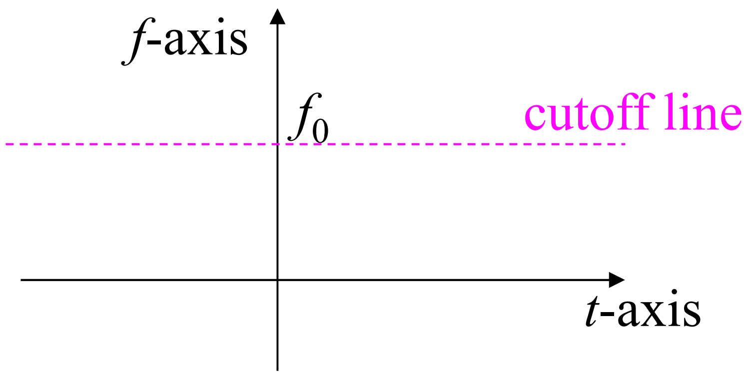
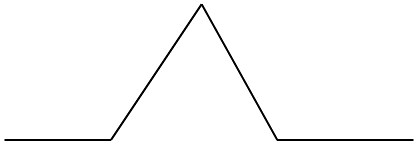
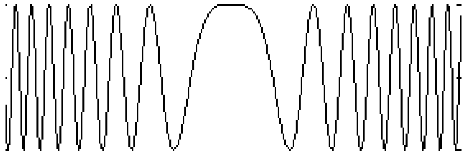
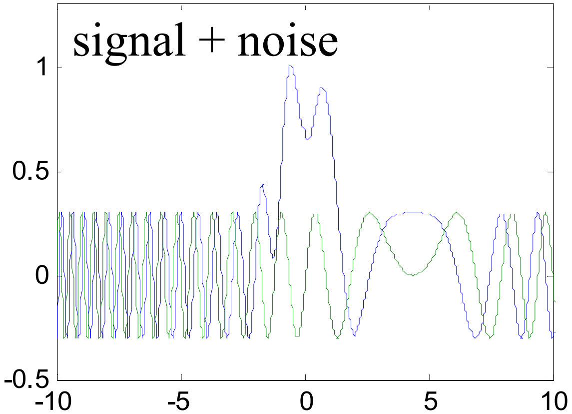
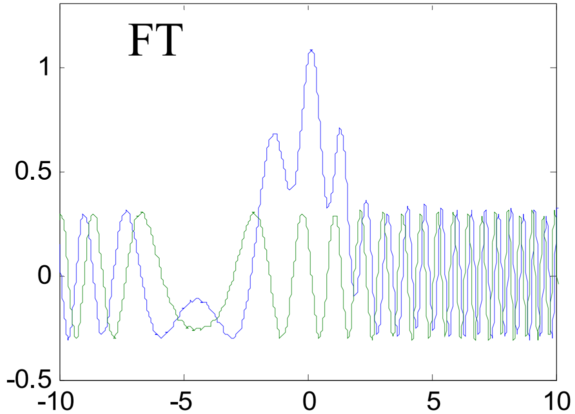


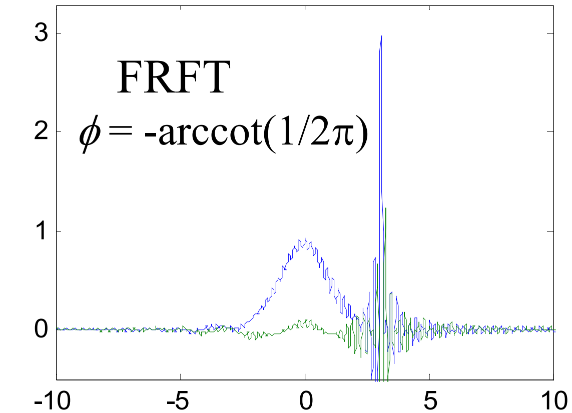
$x(t)$ = triangular signal + chirp noise $0.3\exp[j\,0.5(t-4.4)^2]$

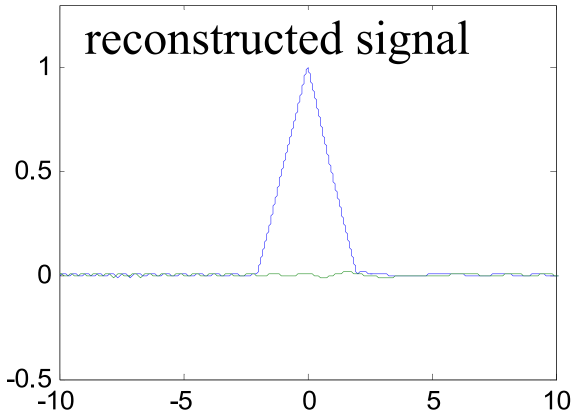




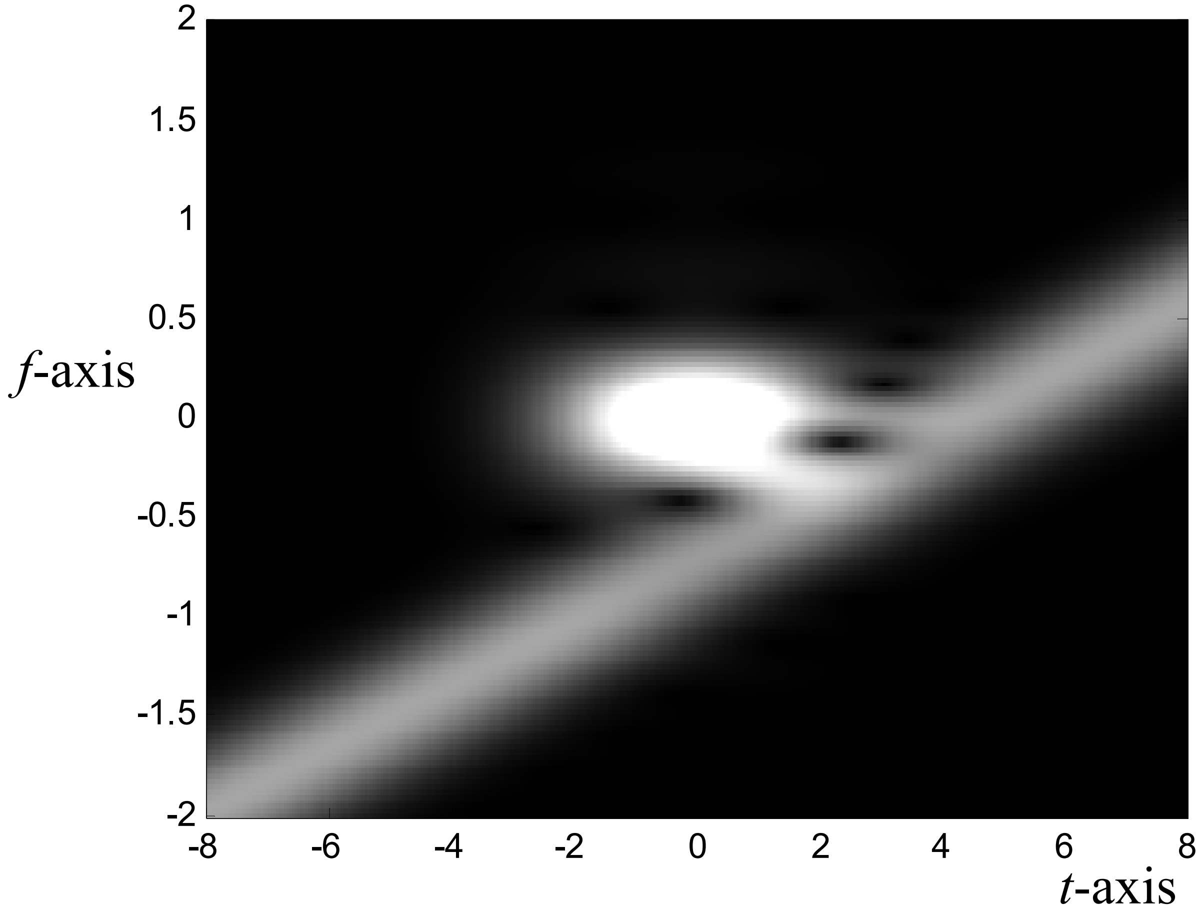








$x(t)$ = triangular signal + chirp noise $0.3\exp[j\,0.5(t-4.4)^2]$



Decomposing in the time-frequency distribution

If $x(t) = 0$ for $t < T_1$ and $t > T_2$

$W_x(t, f) = 0$ for $t < T_1$ and $t > T_2$ (cutoff lines perpendicular to t-axis)

If $X(f) = FT[x(t)] = 0$ for $f < F_1$ and $f > F_2$

$W_x(t, f) = 0$ for $f < F_1$ and $f > F_2$ (cutoff lines parallel to t-axis)

What are the cutoff lines with other directions?

with the aid of the **FRFT**, the LCT, or the Fresnel transform

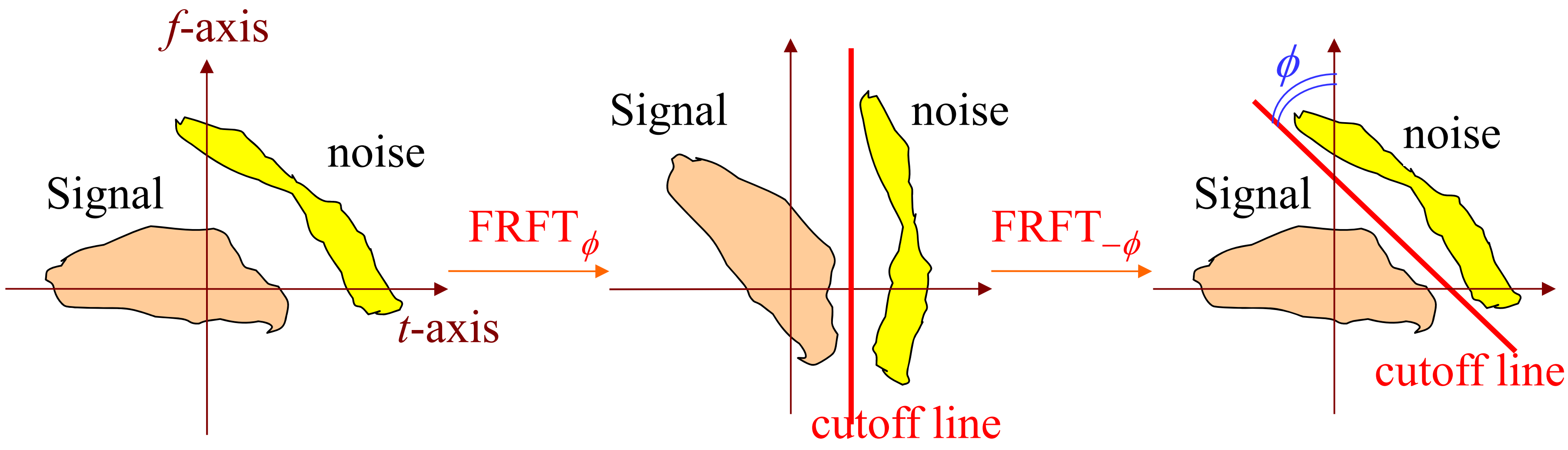
- Filter designed by the fractional Fourier transform

$$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]H(u)\right\}$$

比較： $x_o(t) = IFT\left[FT(x_i(t))H(f)\right]$

O_F^{ϕ} means the fractional Fourier transform:

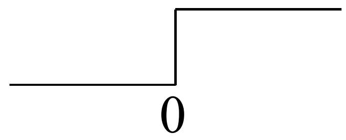
$$O_F^{\phi}\left(x(t)\right) = \sqrt{1 - j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2}\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)\,dt$$



$$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]S(-u+u_0)\right\}$$

or $$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]S(u-u_0)\right\}$$

$S(u)$: Step function



(1) ϕ 由 cutoff line 和 f-axis 的夾角決定

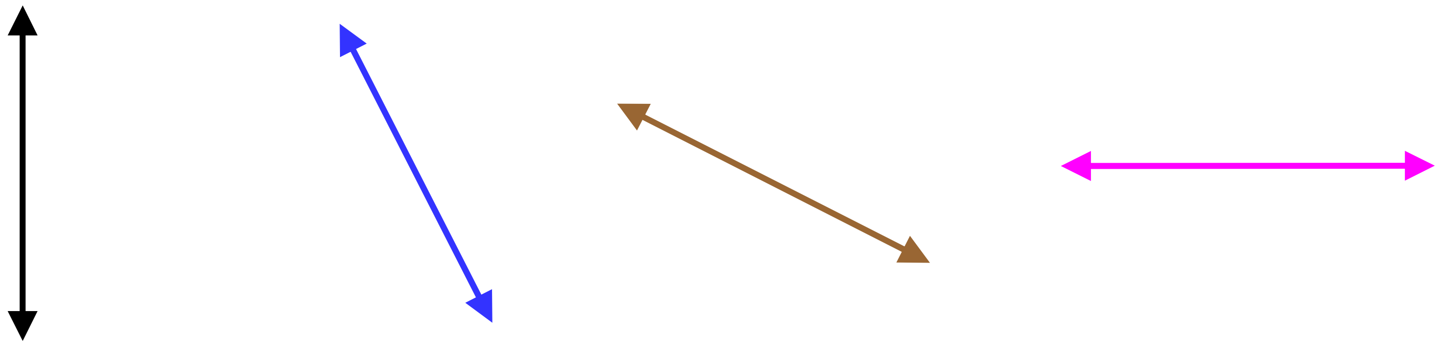
(2) u_0 等於 cutoff line 距離原點的距離

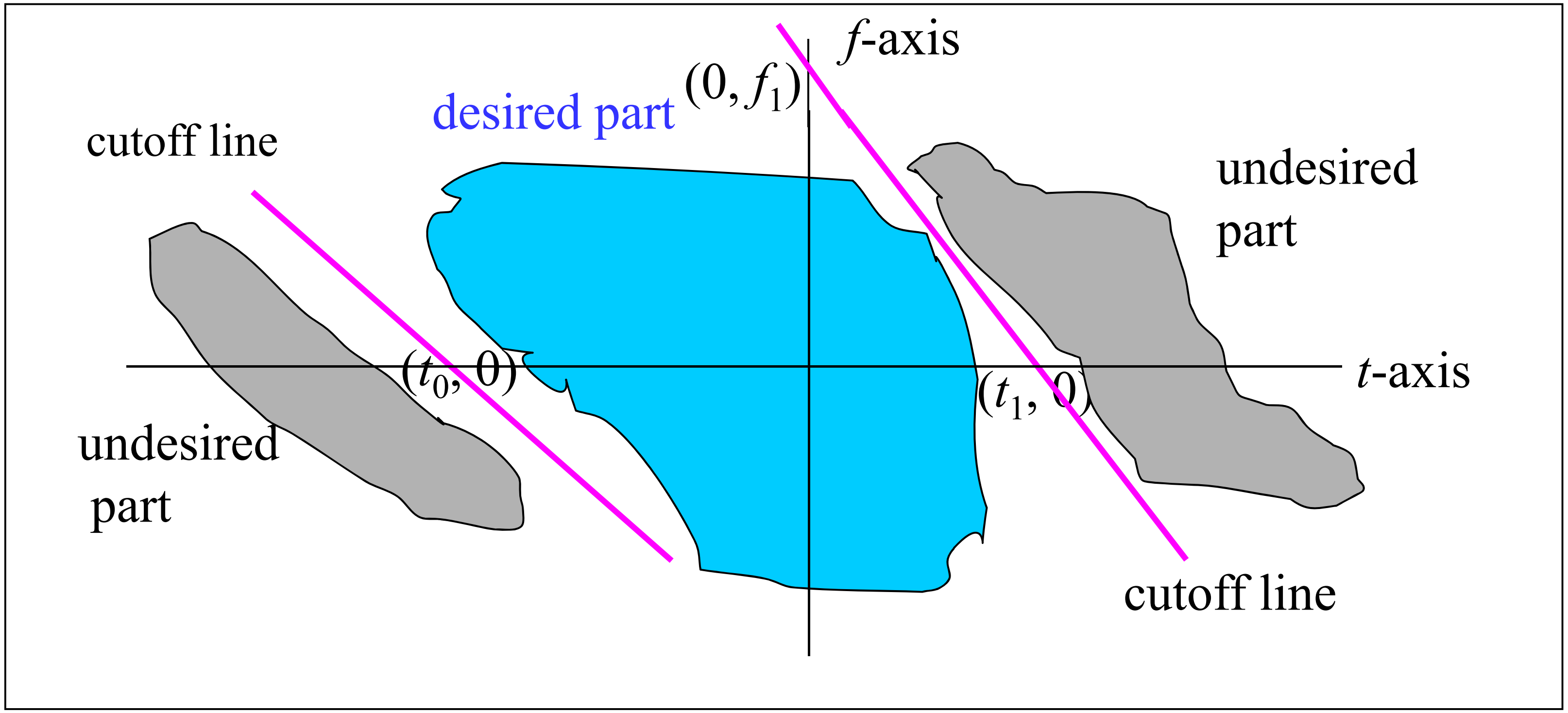
(注意正負號)

- Effect of the filter designed by the fractional Fourier transform (FRFT):

  Placing a cutoff line in the direction of $(-\sin\phi, \cos\phi)$

$\phi = 0$ (time domain) $\phi = 0.15\pi$ $\phi = 0.35\pi$ $\phi = 0.5\pi$ (FT)





f-axis
(0, f_1)
desired part
cutoff line
undesired part
undesired part
(t_0, 0)
(t_1, 0)
t-axis
cutoff line

$\phi = ?$ $\quad u_0 = ?$

- The Fourier transform is suitable to filter out the noise that is a combination of

  sinusoid functions $\exp(jn_1 t)$.

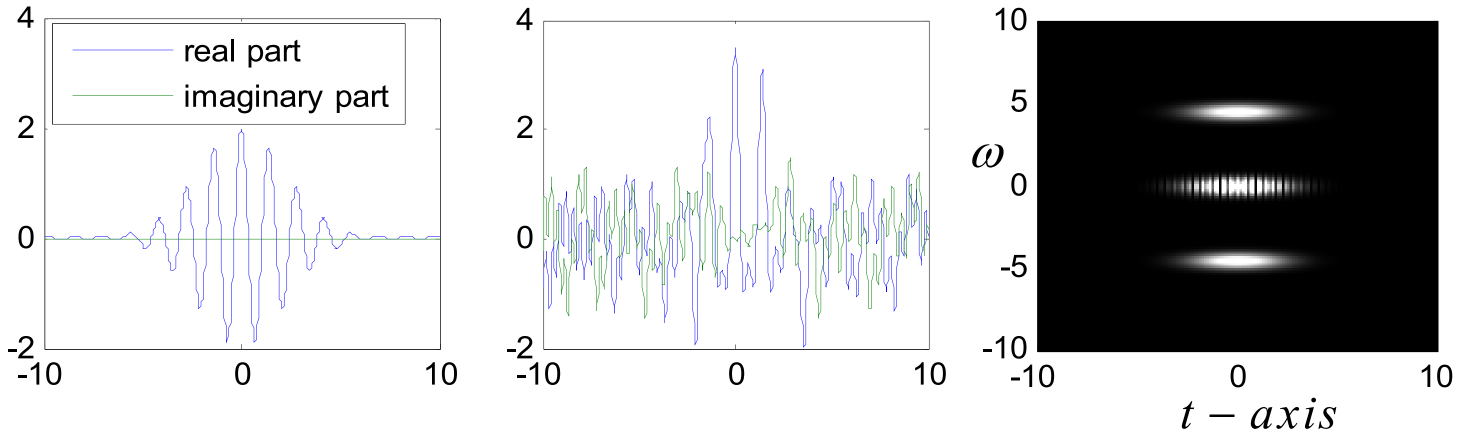
- The fractional Fourier transform (FRFT) is suitable to filter out the noise that is a combination of higher order exponential functions

  $$\exp[j(n_k t^k + n_{k-1} t^{k-1} + n_{k-2} t^{k-2} + \ldots\ldots + n_2 t^2 + n_1 t)]$$

  For example: chirp function $\exp(jn_2 t^2)$

- With the FRFT, many noises that cannot be removed by the FT will be filtered out successfully.

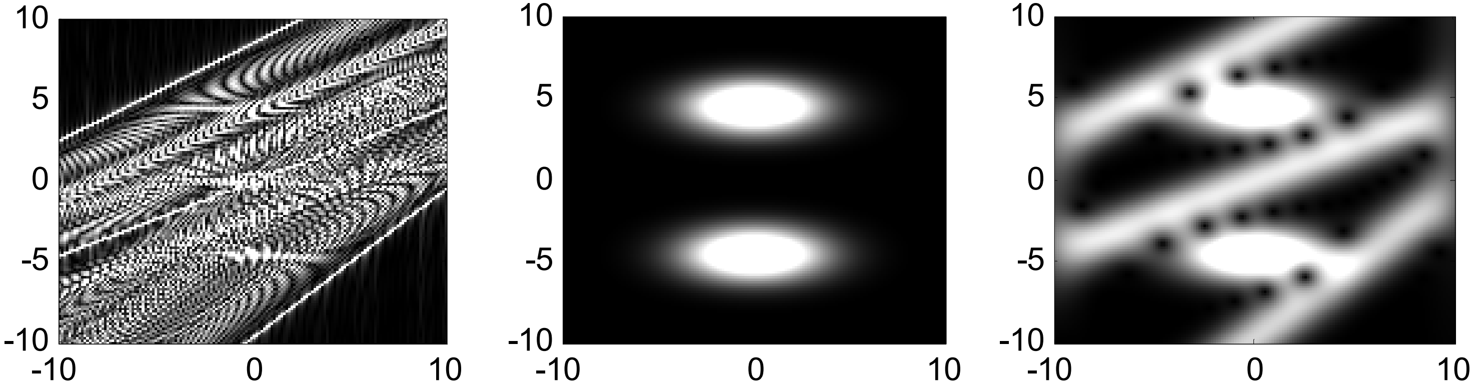
Example (I)



(a) Signal $s(t)$ (b) $f(t) = s(t)$ + noise (c) WDF of $s(t)$

$$s(t) = 2\cos(5t)\exp(-t^2/10)$$

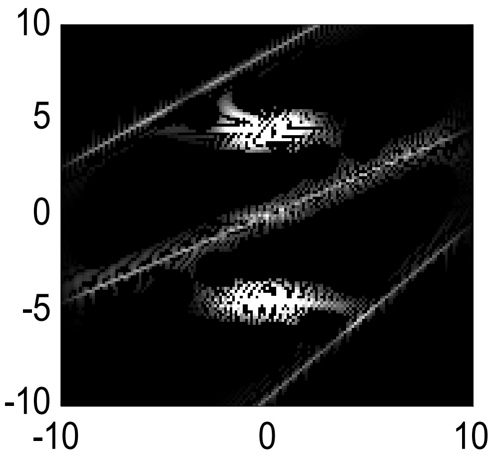
$$n(t) = 0.5e^{j0.23t^2} + 0.5e^{j0.3t^2 + j8.5t} + 0.5e^{j0.46t^2 - j9.6t}$$



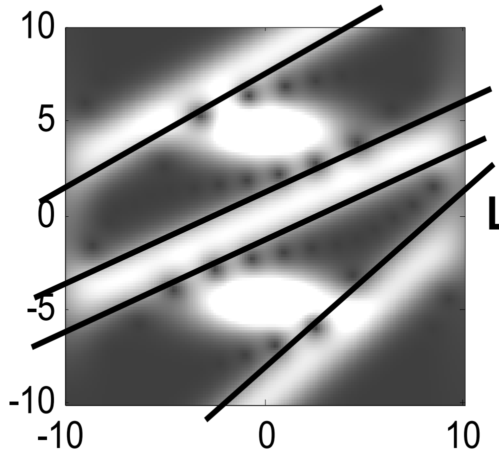
(d) WDF of $f(t)$ (e) GT of $s(t)$ (f) GT of $f(t)$

GT: Gabor transform, WDF: Wigner distribution function

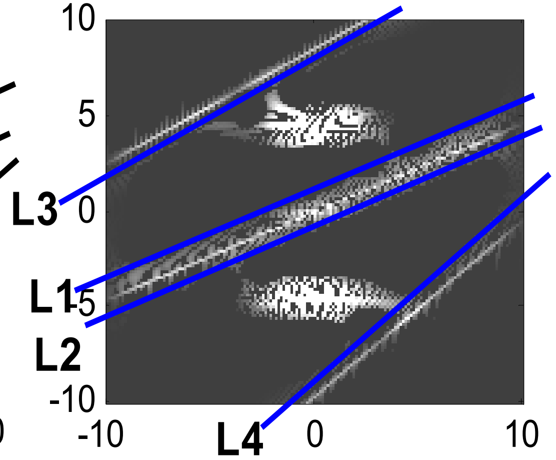
horizontal: t-axis, vertical: ω-axis



(g) GWT of $f(t)$

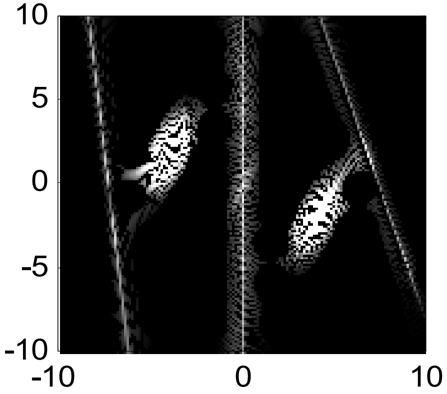


(h) Cutoff lines on GT

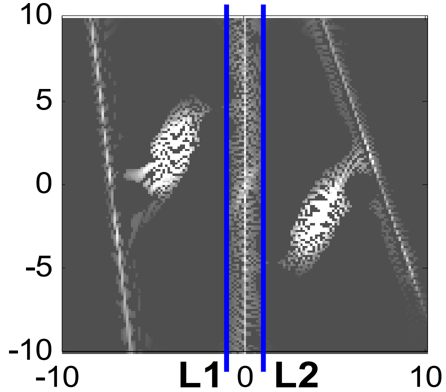


(i) Cutoff lines on GWT

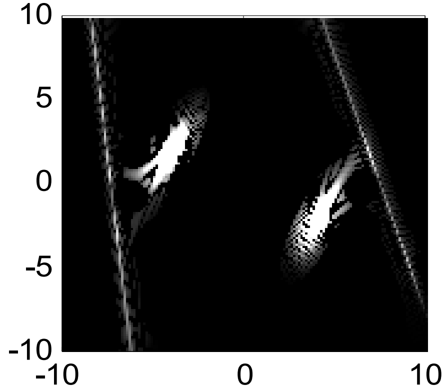
根據斜率來決定 FrFT 的 order



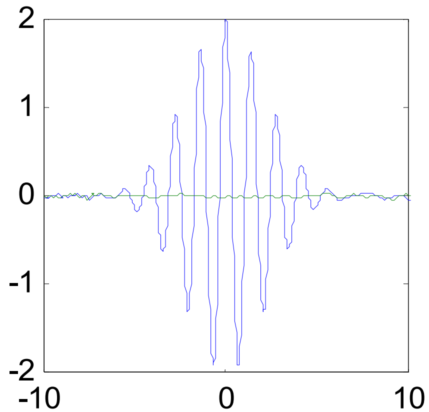
(j) performing the FRFT and calculate the GWT



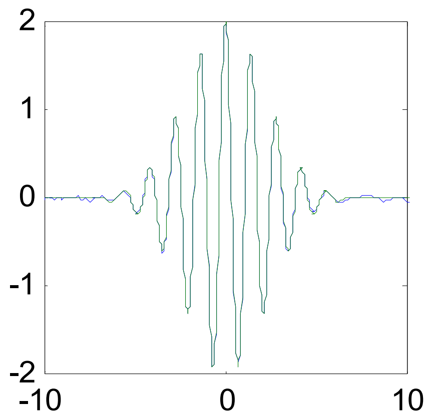
(k) High pass filter



(l) GWT after filter



(m) recovered signal

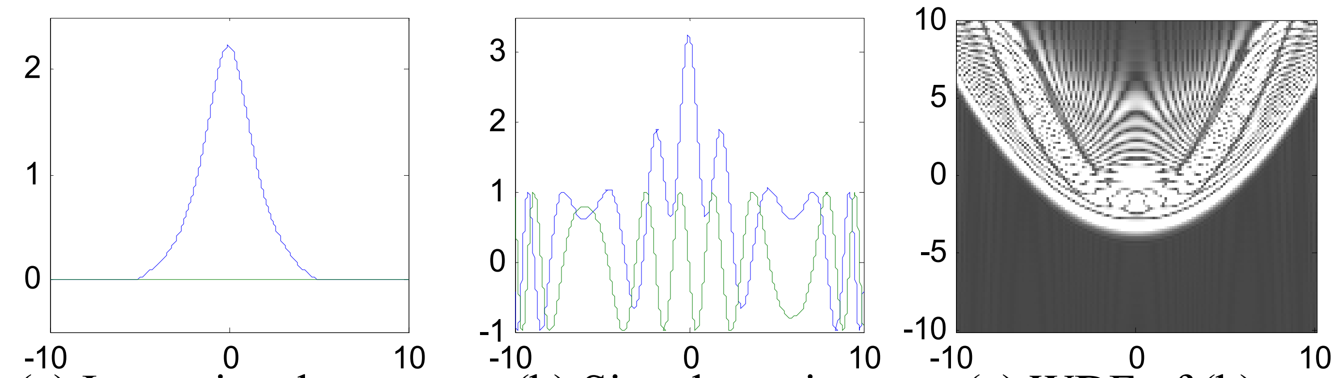


(n) recovered signal (green) and the original signal (blue)

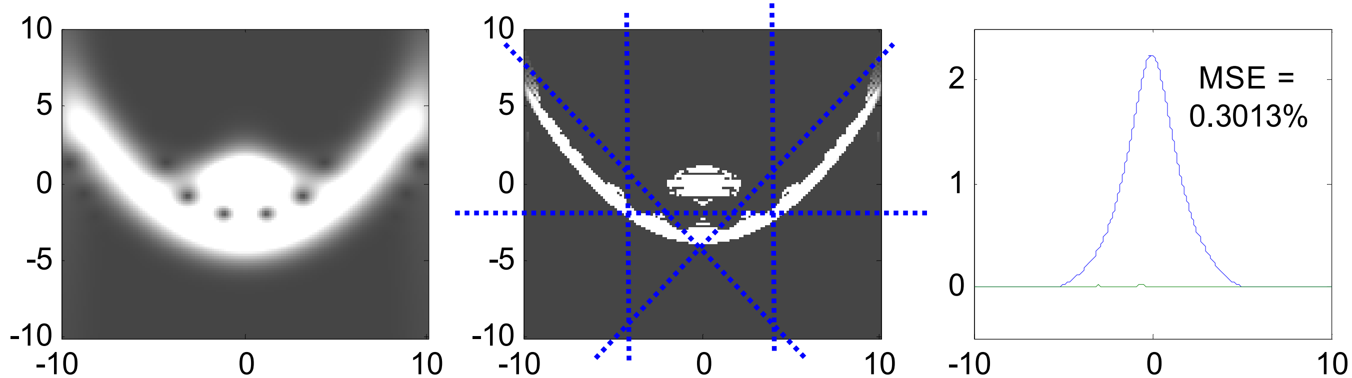
mean square error (MSE) = 0.1128%

Example (II)

Signal + $0.7\exp(j0.032t^3 - j3.4t)$



(a) Input signal (b) Signal + noise (c) WDF of (b)



(d) Gabor transform of (b) (e) GWT of (b) (f) Recovered signal

[Important Theory]:

Using the **FT** can only filter the noises that do not overlap with the signals **in the frequency domain** (1-D)

In contrast, using the **FRFT** can filter the noises that do not overlap with the signals **on the time-frequency plane** (2-D)

[思考]

Q1: 哪些 time-frequency distribution 比較適合處理 filter 或 signal decomposition 的問題？

Q2: Cutoff lines 有可能是非直線的嗎？

[Ref] Z. Zalevsky and D. Mendlovic, "Fractional Wiener filter," *Appl. Opt.*, vol. 35, no. 20, pp. 3930-3936, July 1996.

[Ref] M. A. Kutay, H. M. Ozaktas, O. Arikan, and L. Onural, "Optimal filter in fractional Fourier domains," *IEEE Trans. Signal Processing*, vol. 45, no. 5, pp. 1129-1143, May 1997.

[Ref] B. Barshan, M. A. Kutay, H. M. Ozaktas, "Optimal filters with linear canonical transformations," *Opt. Commun.*, vol. 135, pp. 32-36, 1997.

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

9-2 TF analysis and Random Process

For a random process $x(t)$, we cannot find the explicit value of $x(t)$.
The value of $x(t)$ is expressed as a probability function.

- Auto-covariance function $R_x(t,\tau)$

$$R_x(t,\tau) = E\left[x(t+\tau/2)x^*(t-\tau/2)\right]$$

In usual, we suppose that $E[x(t)] = 0$ for any t

$$E\left[x(t+\tau/2)x^*(t-\tau/2)\right]$$
$$= \int\int x(t+\tau/2,\zeta_1)x^*(t-\tau/2,\zeta_2)P(\zeta_1,\zeta_2)\,d\zeta_1 d\zeta_2$$

(alternative definition of the auto-covariance function:

$$\hat{R}_x(t,\tau) = E\left[x(t)x(t+\tau)\right]$$

- Power spectral density (PSD) $S_x(t,f)$

$$S_x(t,f) = \int_{-\infty}^{\infty} R_x(t,\tau)e^{-j2\pi f\tau}d\tau$$

- Relation between the WDF and the random process

$$E\left[W_x(t,f)\right]=\int_{-\infty}^{\infty}E\left[x(t+\tau/2)x^*(t-\tau/2)\right]\cdot e^{-j2\pi f\tau}\cdot d\tau$$

$$=\int_{-\infty}^{\infty}R_x(t,\tau)\cdot e^{-j2\pi f\tau}\cdot d\tau$$

$$=\int_{-\infty}^{\infty}R_x(t,\tau)\cdot e^{-j2\pi f\tau}\cdot d\tau$$

$$=S_x(t,f)$$

- Relation between the ambiguity function and the random process

$$E\left[A_x(\eta,\tau)\right]=\int_{-\infty}^{\infty}E\left[x(t+\tau/2)x^*(t-\tau/2)\right]e^{-j2\pi t\eta}\,dt=\int_{-\infty}^{\infty}R_x(t,\tau)e^{-j2\pi t\eta}\,dt$$

- Stationary random process:

the statistical properties do not change with t.

Auto-covariance function $R_x(t_1,\tau) = R_x(t_2,\tau) = R_x(\tau)$ for any t,

$$R_x(\tau) = E\left[x(\tau/2)x^*(-\tau/2)\right]$$

$$= \int\int x(\tau/2,\zeta_1)x^*(-\tau/2,\zeta_2)P(\zeta_1,\zeta_2)\,d\zeta_1 d\zeta_2$$

PSD: $S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j2\pi f\tau}d\tau$

White noise: $S_x(f) = \sigma$ where σ is some constant.

- When $x(t)$ is stationary,

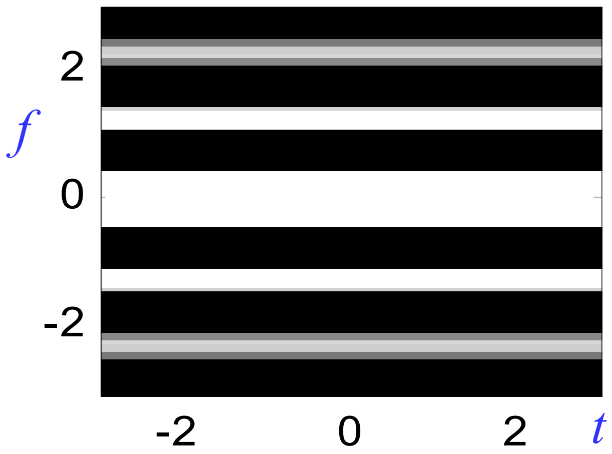
$$E\left[W_x(t,f)\right] = S_x(f)$$

(invariant with t)

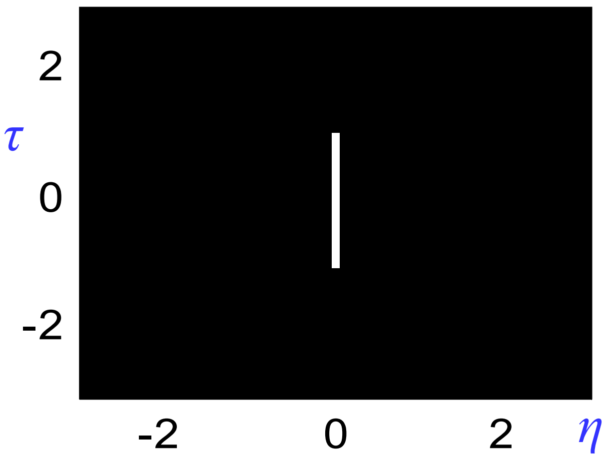
$$E\left[A_x(\eta,\tau)\right] = \int_{-\infty}^{\infty} R_x(\tau)\cdot e^{-j2\pi t\eta}\cdot dt = R_x(\tau)\delta(\eta)$$

(nonzero only when $\eta = 0$)

a typical $E[W_x(t,f)]$ for stationary random process



a typical $E[A_x(\eta, \tau)]$ for stationary random process



- For white noise,

$$E\left[W_g(t,f)\right] = \sigma$$

$$E\left[A_x(\eta,\tau)\right] = \sigma\delta(\tau)\delta(\eta)$$

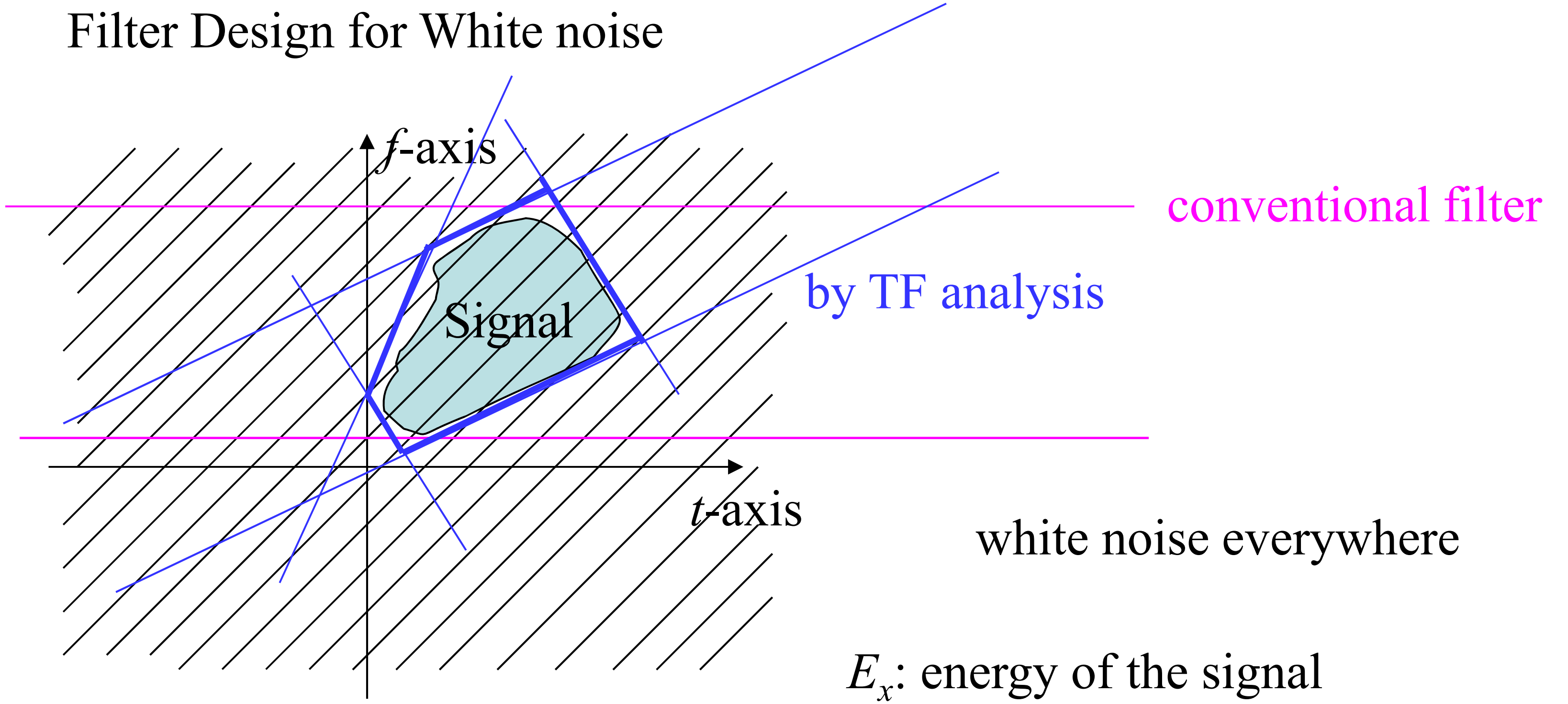
[Ref 1] W. Martin, “Time-frequency analysis of random signals”, *ICASSP’82*, pp. 1325-1328, 1982.

[Ref 2] W. Martin and P. Flandrin, “Wigner-Ville spectrum analysis of nonstationary processed”, *IEEE Trans. ASSP*, vol. 33, no. 6, pp. 1461-1470, Dec. 1983.

[Ref 3] P. Flandrin, “A time-frequency formulation of optimum detection” , *IEEE Trans. ASSP*, vol. 36, pp. 1377-1384, 1988.

[Ref 4] S. C. Pei and J. J. Ding, “Fractional Fourier transform, Wigner distribution, and filter design for stationary and nonstationary random processes,” *IEEE Trans. Signal Processing*, vol. 58, no. 8, pp. 4079-4092, Aug. 2010.

Filter Design for White noise



white noise everywhere

$$SNR \approx \log_{10} \frac{E_x}{\iint_{\substack{(t,f)\in \\ \text{signal part}}} S_x(t,f)\,dt\,df}$$

E_x: energy of the signal

A: area of the time frequency distribution of the signal

The PSD of the white noise is $S_n(f) = \sigma$

$$SNR \approx \log_{10} \frac{E_x}{\sigma A}$$

- If $E\left[W_x\left(t,f\right)\right]$ varies with t and $E\left[A_x\left(\eta,\tau\right)\right]$ is nonzero when $\eta \neq 0$, then $x(t)$ is a non-stationary random process.

- If ① $h\left(t\right)=x_1\left(t\right)+x_2\left(t\right)+x_3\left(t\right)+\cdots\cdots+x_k\left(t\right)$

  ② $x_n(t)$'s have zero mean for all t's

  ③ $x_n(t)$'s are mutually independent for all t's and τ's

  $$E\left[x_m(t+\tau/2)x_n^*(t-\tau/2)\right]=E\left[x_m(t+\tau/2)\right]E\left[x_n^*(t-\tau/2)\right]=0$$

  if $m \neq n$, then

  $$E\left[W_h\left(t,f\right)\right]=\sum_{n=1}^{k}E\left[W_{x_n}\left(t,f\right)\right], \quad E\left[A_h\left(\eta,\tau\right)\right]=\sum_{n=1}^{k}E[A_{x_n}\left(\eta,\tau\right)]$$

(1) Random process for the STFT

$E[x(t)] \neq 0$ should be satisfied.

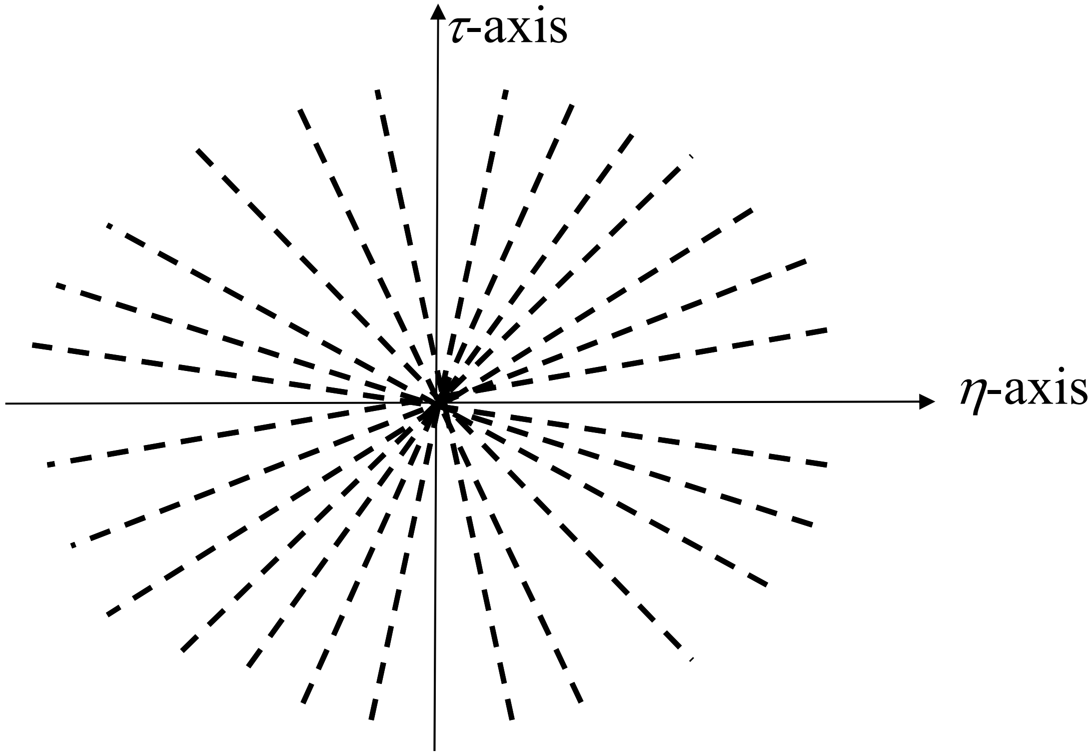
Otherwise,

$$E[X(t,f)] = E[\int_{t-B}^{t+B} x(\tau)w(t-\tau)e^{-j2\pi f\tau}d\tau] = \int_{t-B}^{t+B} E[x(\tau)]w(t-\tau)e^{-j2\pi f\tau}d\tau$$

for zero-mean random process, $E[X(t,f)] = 0$

(2) Decompose by the AF and the FRFT

Any non-stationary random process can be expressed as a summation of the fractional Fourier transform (or chirp multiplication) of stationary random process.



An ambiguity function plane can be viewed as a combination of infinite number of radial lines.

Each radial line can be viewed as the fractional Fourier transform of a stationary random process.

信號處理小常識

$$S(f) = \sigma$$ white noise

$$S(f) = \frac{\sigma}{f}$$

$$S(f) = \sigma f$$

$$S(f) = \sigma f^{\alpha}$$ $\alpha \neq 0$ color noise

附錄十二　Time-Frequency Analysis 理論發展年表

AD 1785　The Laplace transform was invented

AD 1812　The Fourier transform was invented

AD 1822　The work of the Fourier transform was published

AD 1910　The Haar Transform was proposed

AD 1927　Heisenberg discovered the uncertainty principle

AD 1929　The fractional Fourier transform was invented by Wiener

AD 1932　The Wigner distribution function was proposed

AD 1946　The short-time Fourier transform and the Gabor transform was proposed.
In the same year, the computer was invented

AD 1961　Slepian and Pollak found the prolate spheroidal wave function

AD 1965　The Cooley-Tukey algorithm (FFT) was developed

註：沒列出發明者的，指的是 transform / distribution 的名稱和發明者的名字相同

AD 1966 Cohen's class distribution was invented

AD 1970s VLSI was developed

AD 1971 Moshinsky and Quesne proposed the linear canonical transform

AD 1980 The fractional Fourier transform was re-invented by Namias

AD 1981 Morlet proposed the wavelet transform

AD 1982 The relations between the random process and the Wigner distribution function was found by Martin and Flandrin

AD 1988 Mallat and Meyer proposed the multiresolution structure of the wavelet transform;
In the same year, Daubechies proposed the compact support orthogonal wavelet

AD 1989 The Choi-Williams distribution was proposed; In the same year, Mallat proposed the fast wavelet transform

註：沒列出發明者的，指的是 transform / distribution 的名稱和發明者的名字相同

AD 1990 The cone-Shape distribution was proposed by Zhao, Atlas, and Marks

AD 1990s The discrete wavelet transform was widely used in image processing

AD 1992 The generalized wavelet transform was proposed by Wilson et. al.

AD 1993 Mallat and Zhang proposed the matching pursuit;
In the same year, the rotation relation between the WDF and the fractional Fourier transform was found by Lohmann

AD 1994 The applications of the fractional Fourier transform in signal processing were found by Almeida, Ozaktas, Wolf, Lohmann, and Pei;
Boashash and O'Shea developed polynomial Wigner-Ville distributions

AD 1995 Auger and Flandrin proposed time-frequency reassignment

L. J. Stankovic, S. Stankovic, and Fakultet proposed the pseudo Wigner distribution

AD 1996 Stockwell, Mansinha, and Lowe proposed the S transform

Daubechies and Maes proposed the synchrosqueezing transform.

AD 1998 N. E. Huang proposed the Hilbert-Huang transform

Chen, Donoho, and Saunders proposed the basis pursuit

AD 1999 Bultan proposed the four-parameter atom (i.e., the chirplet)

AD 2000 The standard of JPEG 2000 was published by ISO

Another wavelet-based compression algorithm, SPIHT, was proposed by Kim, Xiong, and Pearlman

The curvelet was developed by Donoho and Candes

AD 2000s The applications of the Hilbert Huang transform in signal processing, climate analysis, geology, economics, and speech were developed

AD 2002 The bandlet was developed by Mallet and Peyre;
Stankovic proposed the time frequency distribution with complex arguments

AD 2003 Pinnegar and Mansinha proposed the general form of the S transform

Liebling et al. proposed the Fresnel.

AD 2005 The contourlet was developed by Do and Vetterli;

The shearlet was developed by Kutyniok and Labate

The generalized spectrogram was proposed by Boggiatto, et al.

AD 2006 Donoho proposed compressive sensing

AD 2006~ Accelerometer signal analysis becomes a new application.

AD 2007 The Gabor-Wigner transform was proposed by Pei and Ding

AD 2007 The multiscale STFT was proposed by Zhong and Zeng.

AD 2007~ Many theories about compressive sensing were developed by Donoho, Candes, Tao, Zhang ….

AD 2010~ Many applications about compressive sensing are found.

AD 2012 The generalized synchrosqueezing transform was proposed by Li and Liang

AD 2015~ Time-frequency analysis was widely combined with the deep learning technique for signal identification

AD 2017 The wavelet convolutional neural network was proposed by Kang et al.

The higher order synchrosqueezing transform was proposed by Pham and Meignen

AD 2018~ With the fast development of hardware and software, the time-frequency distribution of a 10^6-point data can be analyzed efficiently within 0.1 Second

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