II. Short-time Fourier Transform

II-A Definition

Short-time Fourier transform (STFT)

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Alternative definition

$$X(t,\omega)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j\omega\tau}d\tau$$

參考資料

[1] S. Qian and D. Chen, Section 3-1 in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

[2] S. H. Nawab and T. F. Quatieri, "Short time Fourier transform," in *Advanced Topics in Signal Processing*, pp. 289-337, Prentice Hall, 1987.

STFT $$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$$X(t,\omega)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j\omega\tau}d\tau$$

Inverse of the STFT: To recover $x(t)$,

$$x(t)=w^{-1}(t_1-t)\int_{-\infty}^{\infty} X(t_1,f)e^{j2\pi f t}df$$

where $w(t_1-t)\neq 0$.

For the alternative definition, the inverse transform is:

$$x(t)=\frac{1}{2\pi}w^{-1}(t_1-t)\int_{-\infty}^{\infty} X(t_1,\omega)e^{j\omega t}d\omega$$

The mask function $w(t)$ always has the property of

(a) even: $w(t) = w(-t)$, (通常要求這個條件要滿足)

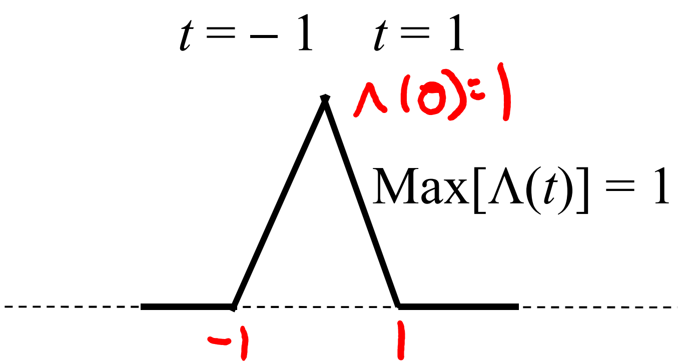
(b) $\max(w(t)) = w(0)$, $w(t_1) \geq w(t_2)$ if $|t_2| > |t_1|$

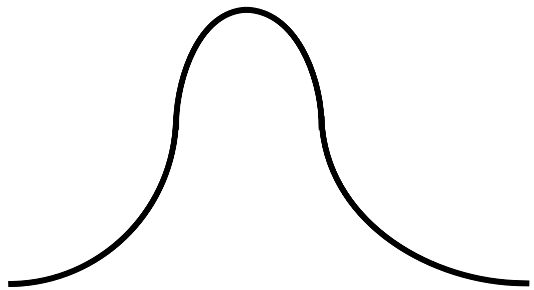
(c) $w(t) \approx 0$ when $|t|$ is large

b is any positive real number

$w(t) = \Lambda(t)$ (triangular function)

$w(t) = \exp(-a|t|^b)$
(hyper-Laplacian function)





How do we make better time-resolution by adjusting a, b?

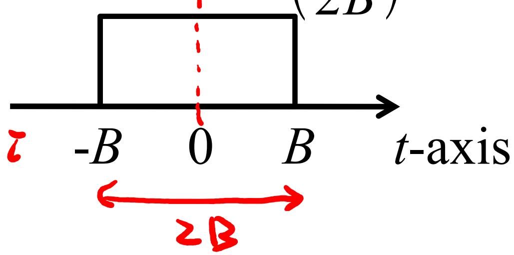
II-B Rec-STFT

Rectangular mask STFT (rec-STFT)

$$X(t,f)=\int_{t-B}^{t+B} x(\tau)e^{-j2\pi f\tau}d\tau$$

$$\int_{-\infty}^{\infty} w(t-\tau)\,w(\tau)\,e^{-j2\pi f\tau}d\tau$$

$$w(t)=rect\left(\frac{t}{2B}\right)$$



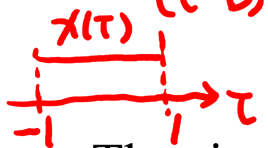
Inverse of the rec-STFT

$$x(t)=\int_{-\infty}^{\infty} X(t_1,f)e^{j2\pi f t}df$$

where $t-B<t_1<t+B$

Ex: if $x(t)=1$ for $-1<t<1$, $B=3$

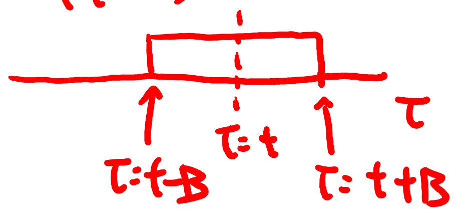
$(t-B,\ t+B)\rightarrow(t-3,\ t+3)$



$\begin{cases}-1<\tau<1\\ t-3<\tau<t+3\end{cases}$

• Case 1: $t+3<-1$, $t<-4$

$X(t,f)=0$



The simplest form of the STFT

• Case 2, $-1<t+3<1$, $-4<t<-2$

$$X(t,f)=\int_{-1}^{t+3}e^{-j2\pi f\tau}d\tau=\frac{e^{-j2\pi f(t+3)}-e^{j2\pi f}}{-j2\pi f}$$

Other types of the STFT may require more computation time than the rec-STFT.

• Case 3 $t+3>1$, $t-3<-1$, $-2<t<2$

$$X(t,f)=\int_{-1}^{1}e^{-j2\pi f\tau}d\tau=FT\left(\Pi\left(\tfrac{\tau}{2}\right)\right)=2\,\mathrm{sinc}(2f)$$

• Case 4 $-1<t-3<1$

• Case 5 $t-3>1$

$X(t,f)=0$

II-C Properties of the Rec-STFT

(1) Integration (recovery):

(a) $\int_{-\infty}^{\infty} X(t,f) e^{j2\pi f v} df = x(v)$ when $v - B < t < v + B$,

$= 0$ otherwise

(b)
$$\int_{-\infty}^{\infty} X(t,f)\, df = \int_{t-B}^{t+B} x(\tau) \int_{-\infty}^{\infty} e^{-j2\pi f \tau} df\, d\tau$$
$$= \int_{t-B}^{t+B} x(\tau)\delta(\tau)\, d\tau$$
$$= \begin{cases} x(0) & \text{when } t - B < 0 < t + B, \quad -B < t < B \\ 0 & \text{otherwise} \end{cases}$$

(2) Shifting property (橫的方向移動)

$$\int_{t-B}^{t+B} x(\tau-\tau_0)e^{-j2\pi f\tau}d\tau = X(t-\tau_0, f)e^{j2\pi f\tau_0}$$

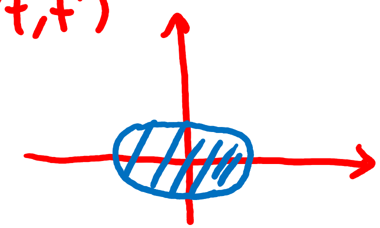
(3) Modulation property (縱的方向移動)

$$\int_{t-B}^{t+B} [x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t, f-f_0)$$

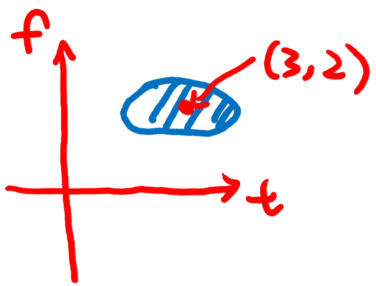
STFT with a rectangular window

If $X(t,f)$ is the rec-STFT of $x(t)$

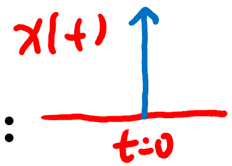
$X(t,f)$

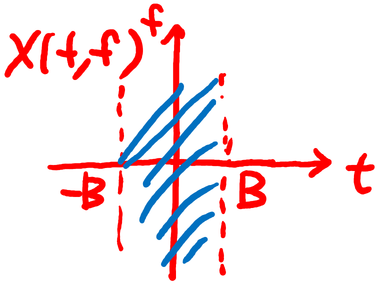


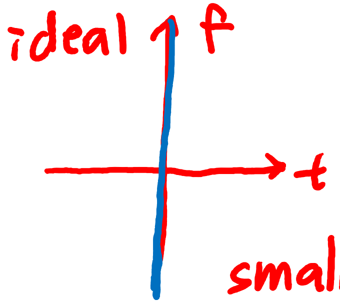
then the rec-STFT of $e^{j4\pi t}x(t-3)$



(4) Special inputs:







(1) When $x(t) = \delta(t)$,

$$X(t,f) = 1 \text{ when } -B < t < B, \quad X(t,f) = 0 \text{ otherwise}$$

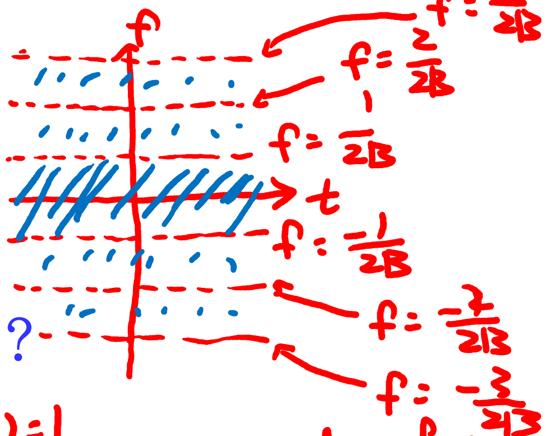
smaller B is better

(2) When $x(t) = 1$

$|X(t,f)| = 0$ if $f = \frac{n}{2B}$
n is a nonzero integer

$$X(t,f) = 2B\,\mathrm{sinc}(2Bf)\,e^{-j2\pi f t}$$

$|X(t,f)| = 2B\,|\mathrm{sinc}(2Bf)|$
$|X(t,f)|$ is independent of t

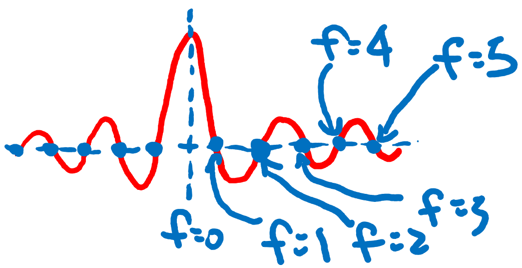


思考： *B* 值的大小，對解析度的影響是什麼？

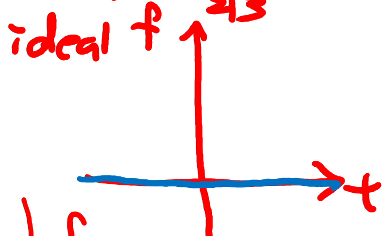
$\mathrm{sinc}(f) = \frac{\sin \pi f}{\pi f}$

$\mathrm{sinc}(f) = 0$ if f is a nonzero integer

$\mathrm{sinc}(0) = 1$



for $x(t) = 1$
larger B is better.



	time resolution	frequency resolution
B ↑	↓	↑
B ↓	↑	↓

uncertainty principle

(5) Linearity property

If $h(t) = \alpha\, x(t) + \beta\, y(t)$ and $H(t, f)$, $X(t, f)$ and $Y(t, f)$ are their rec-STFTs, then

$$H(t,f) = \alpha\, X(t,f) + \beta\, Y(t,f).$$

(6) Power integration property

$$\int_{-\infty}^{\infty} \left|X(t,f)\right|^2 df = \int_{t-B}^{t+B} \left|x(\tau)\right|^2 d\tau$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|X(t,f)\right|^2 df\, dt = 2B\int_{-\infty}^{\infty} \left|x(\tau)\right|^2 d\tau$$

比較：Power integration property of the Fourier transform

if $X(f) = FT(x(t))$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

(7) Energy sum property (Parseval's theorem)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t,f)\, Y^*(t,f)\, df\, dt = 2B\int_{-\infty}^{\infty} x(\tau)\, y^*(\tau)\, d\tau$$

$$\int_{-\infty}^{\infty} X(t,f)\, Y^*(t,f)\, df = \int_{t-B}^{t+B} x(\tau)\, y^*(\tau)\, d\tau$$

思考:

(1) 哪些性質 Fourier transform 也有？

(2) 其他型態的 STFT 是否有類似的性質？

Shifting

$$\int_{-\infty}^{\infty} w(t-\tau)x(\tau-\tau_0)e^{-j2\pi f\tau}d\tau$$

$$=\int_{-\infty}^{\infty} w(t-\tau-\tau_0)x(\tau)e^{-j2\pi f\tau}e^{-j2\pi f\tau_0}d\tau$$

$$=X(t-\tau_0,f)e^{-j2\pi f\tau_0}$$

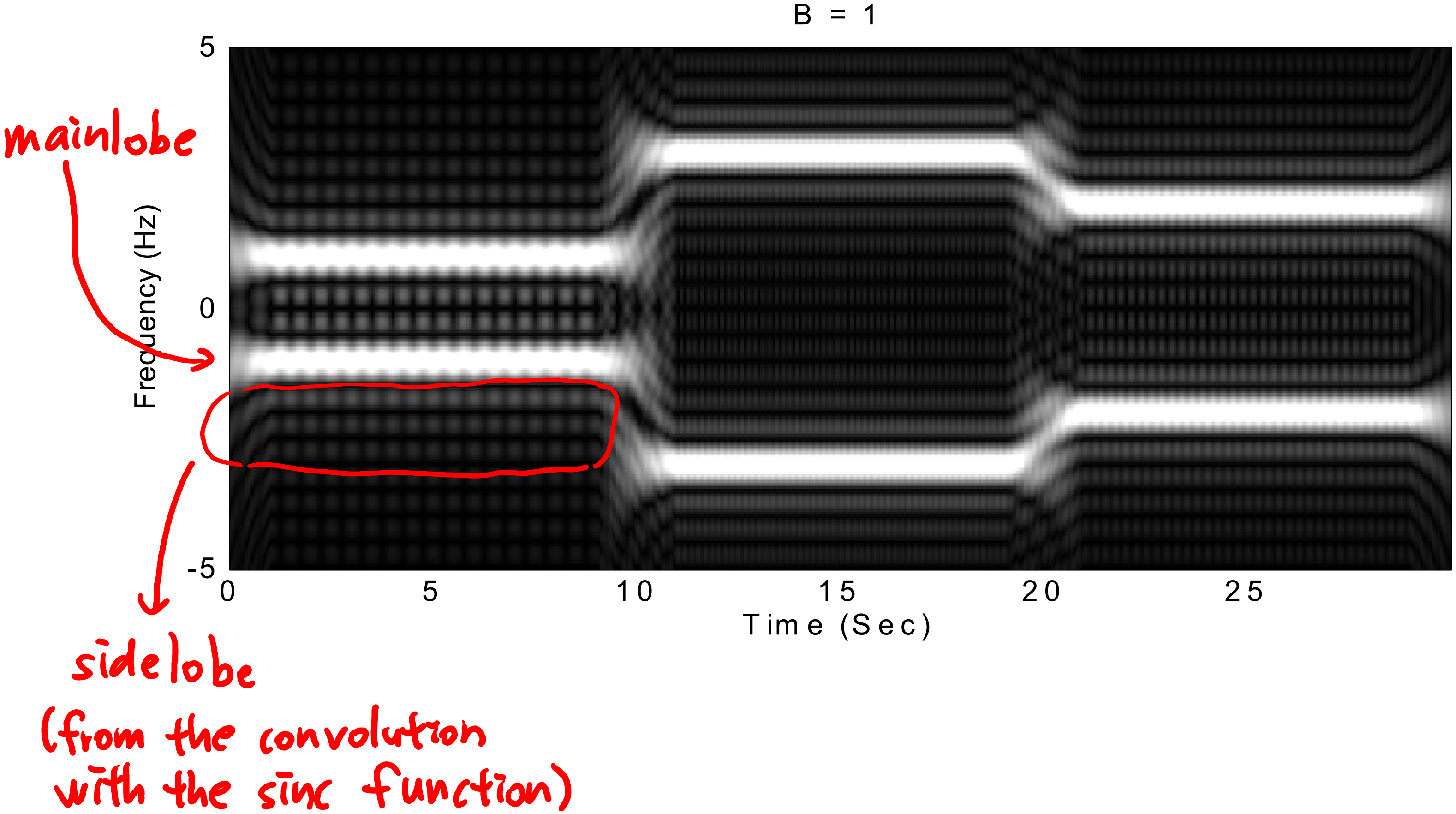
Modulation

$$\int_{-\infty}^{\infty} w(t-\tau)[x(\tau)e^{j2\pi f_0\tau}]e^{-j2\pi f\tau}d\tau = X(t,f-f_0)$$

Example: $x(t) = \cos(2\pi t)$ when $t < 10$, $f = \pm 1$

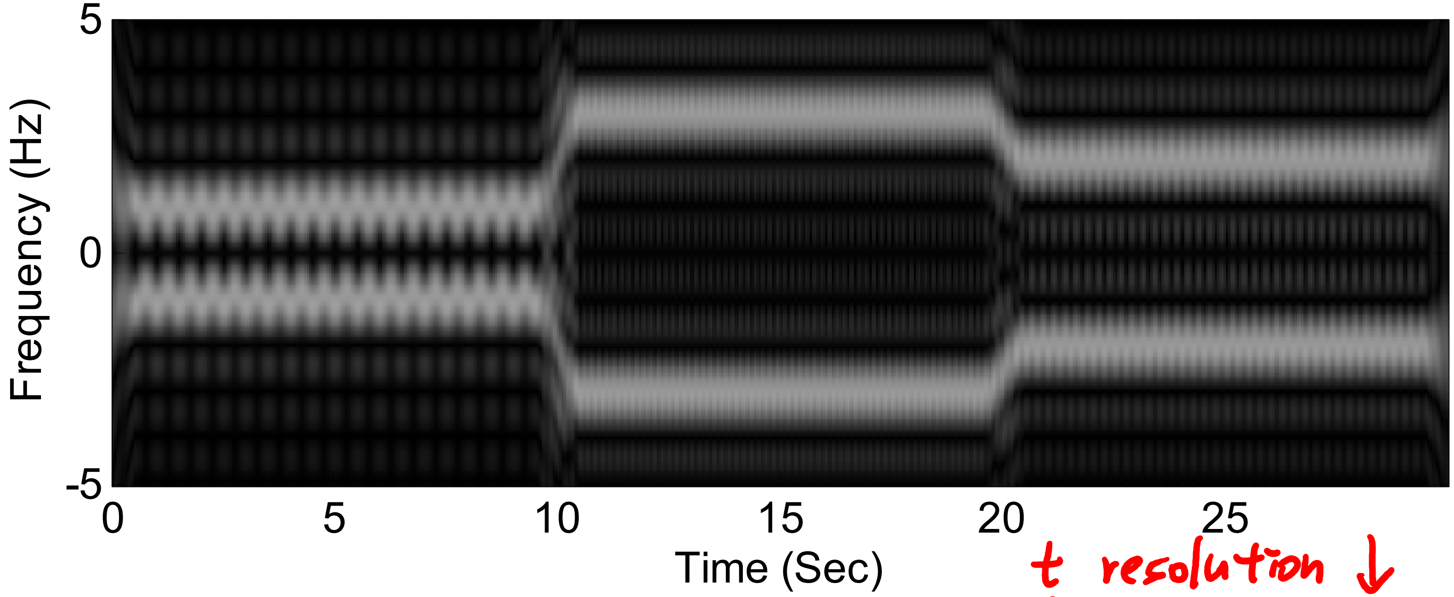
$x(t) = \cos(6\pi t)$ when $10 \le t < 20$, $f = \pm 3$

$x(t) = \cos(4\pi t)$ when $t \ge 20$ $f = \pm 2$



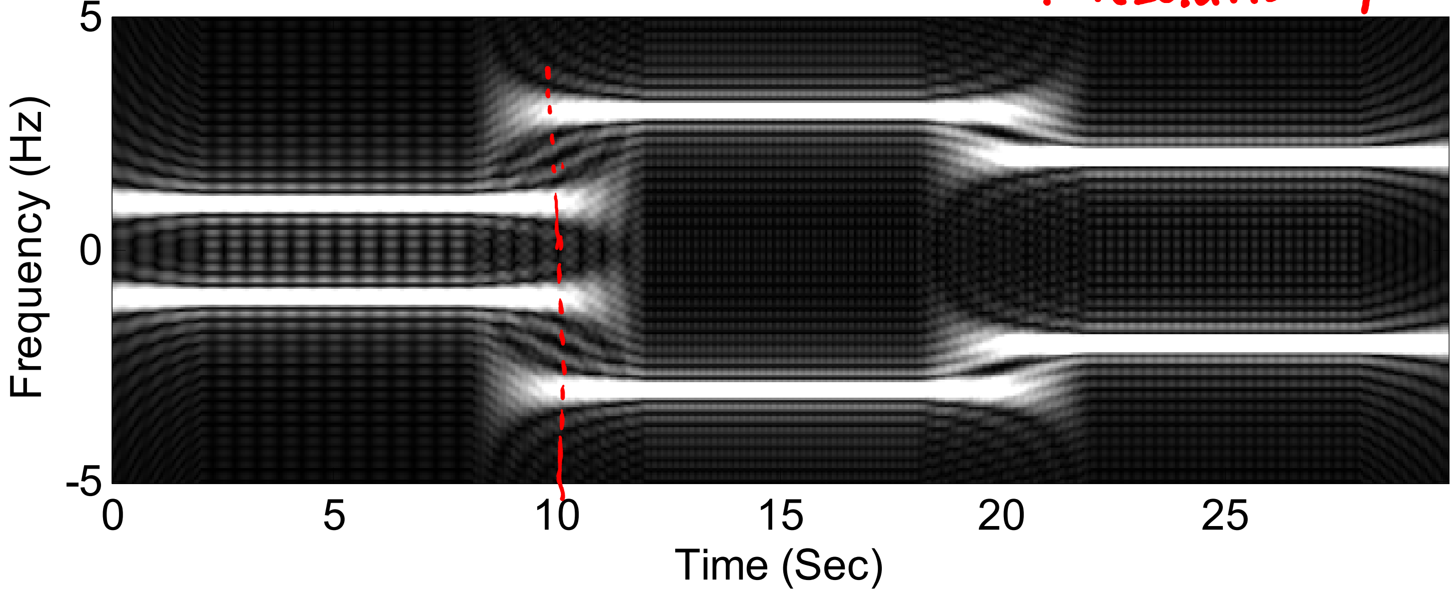
B = 0.5

t resolution ↑
f resolution ↓



B = 2

t resolution ↓
f resolution ↑



II-D Advantage and Disadvantage

• Compared with the Fourier transform:

All the time-frequency analysis methods has the advantage of:

The instantaneous frequency can be observed.

All the time-frequency analysis methods has the disadvantage of:

Higher complexity for computation

- Compared with other types of time-frequency analysis:

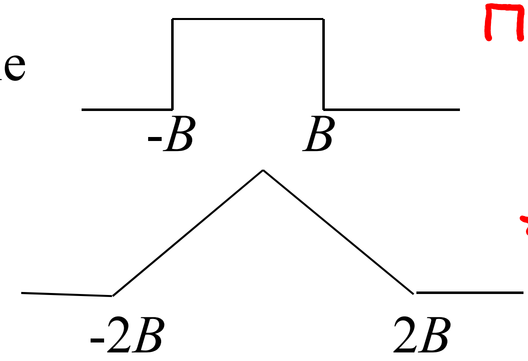
The rec-STFT has an advantage of the least computation time for digital implementation

but its performance is worse than other types of time-frequency analysis.

II-E STFT with Other Windows

(1) Rectangle

(2) Triangle

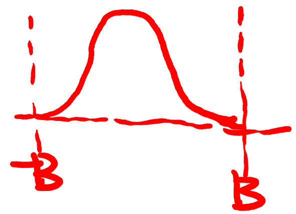


$\Pi\left(\frac{t}{2B}\right) \xrightarrow{FT} 2B\,\mathrm{sinc}(2Bf)$

$= \frac{1}{2B}$ ⊓ $*$ ⊓ $\quad \frac{1}{2B}4B^2\,\mathrm{sinc}^2(2Bf)$

(3) Hanning

$$w(t) = \begin{cases} 0.5 + 0.5\cos(\pi t / B) & \text{when } |t| \le B \\ 0 & \text{otherwise} \end{cases}$$



(4) Hamming

$$w(t) = \begin{cases} 0.54 + 0.46\cos(\pi t / B) & \text{when } |t| \le B \\ 0 & \text{otherwise} \end{cases}$$

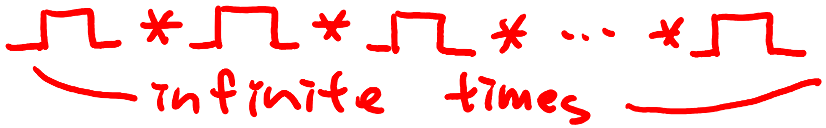
$FT(e^{-\pi t^2}) \to e^{-\pi f^2}$

$FT(e^{-\pi \sigma t^2}) \to \frac{1}{\sqrt{\sigma}} e^{-\pi \frac{f^2}{\sigma}}$

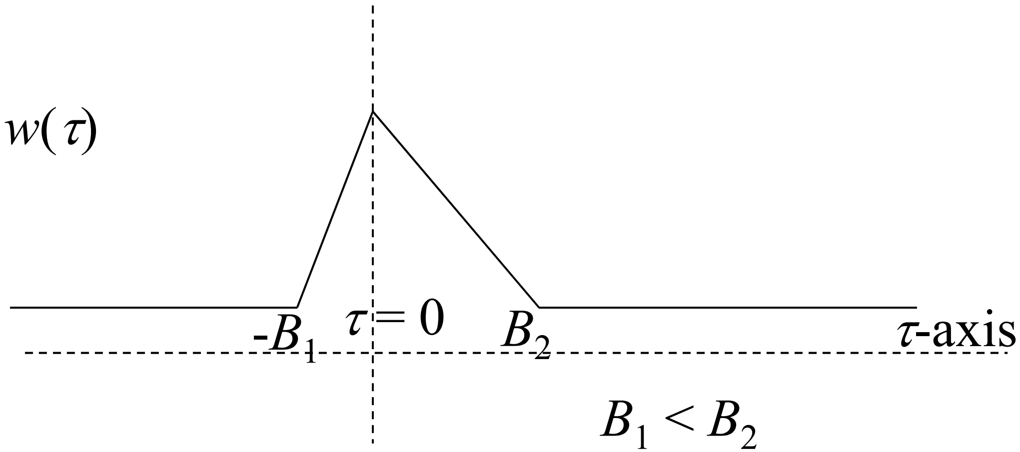
(5) Gaussian

sigma

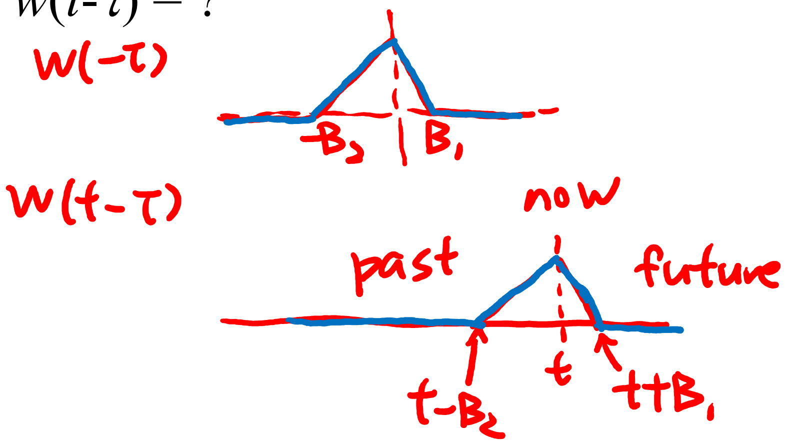
$$w(t) = \exp\left(-\pi\sigma t^2\right)$$



(6) Asymmetric window



$w(t-\tau) = ?$



應用： seismic wave analysis, collision detection

(The applications that require real-time processing)

動腦思考：

(1) Are there other ways to choose the mask of the STFT?

(2) Which mask is better?

沒有一定的答案

II-F Spectrogram

STFT 的絕對值平方，被稱作 Spectrogram

$$SP_x(t,f)=\left|X(t,f)\right|^2=\left|\int_{-\infty}^{\infty}w(t-\tau)e^{-j2\pi f\tau}x(\tau)d\tau\right|^2$$

比較： spectrum 為 Fourier transform 的絕對值平方

文獻上，spectrogram 這個名詞出現的頻率多於 STFT

但實際上， spectrogram 和 STFT 的本質是相同的

附錄三：使用 Matlab 將時頻分析結果 Show 出來

可採行兩種方式：

(1) 使用 mesh 指令畫出立體圖

(但結果不一定清楚，且執行時間較久)

(2) 將 amplitude 變為 gray-level，用顯示灰階圖的方法將結果表現出來

假設 y 是時頻分析計算的結果

image(abs(y)/max(max(abs(y)))*C)　% C 是一個常數，我習慣選 C=400

或 image(t, f, abs(y)/max(max(abs(y)))*C)

colormap(gray(256))　　% 變成 gray-level 的圖

set(gca,'Ydir','normal')　% 若沒這一行, y-axis 的方向是倒過來的

```
set(gca,'Fontsize',12)        % 改變橫縱軸數值的 font sizes
xlabel('Time (Sec)','Fontsize',12)           % x-axis
ylabel('Frequency (Hz)','Fontsize',12)      % y-axis
title('STFT of x(t)','Fontsize',12)            % title
```

計算程式執行時間的指令：

tic (這指令如同按下碼錶)

toc (show 出碼錶按下後已經執行了多少時間)

註：通常程式執行第一次時，由於要做程式的編譯，所得出的執行時間會比較長

程式執行第二次以後所得出的執行時間，是較為正確的結果

附錄四：使用 Python 將時頻分析的圖畫出來

事前安裝模組

```
pip install numpy
pip install matplotlib
```

假設y為時頻分析結果(應為二維的矩陣數列)，將 y 以灰階方式畫出來

```
import numpy as np
import matplotlib.pyplot as plt
C = 400
y = np.abs(y) / np.max(np.abs(y)) * C
plt.imshow(y, cmap='gray', origin='lower')
# 加上 origin='lower' 避免上下相反
plt.xlabel('Time (Sec)')
plt.ylabel('Frequency (Hz)')
plt.show()
```

感謝2021年擔任助教的蔡昌廷同學

若要加上座標軸數值(在plt.show()之前加上以下程式碼)

```
x_label = ['0', '10', '20', '30']   # 橫軸座標值
y_label = ['-5', '0', '5']          # 縱軸座標值
plt.xticks(np.arange(0, x_max, step=int(x_max/(len(x_label)-1)), x_label)
plt.yticks(np.arange(0, y_max, step=int(y_max/(len(y_label)-1)), y_label)
```

Reference :
https://matplotlib.org/stable/api/_as_gen/matplotlib.pyplot.xticks.html

計算時間

```
import time
start_time = time.time()  #獲取當前時間
end_time = time.time()
total_time = end_time - start_time  #計算時間差來得到總執行時間
```

III. Gabor Transform

III-A Definition

Standard Definition:

$w(\tau) = e^{-\pi\tau^2}$

$$G_x(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

Alternative Definitions:

$$G_{x,1}(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f(\tau-\frac{t}{2})} x(\tau)\, d\tau$$

$G_{x,1}(t,f) = e^{j\pi ft} G_x(t,f)$

$|G_{x,1}(t,f)| = |G_x(t,f)|$

$$G_{x,2}(t,f) = \sqrt[4]{2}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$ ← normalization

$$G_{x,3}(t,\omega) = \int_{-\infty}^{\infty} e^{-(\tau-t)^2/2} e^{-j\omega\tau} x(\tau)\, d\tau$$

$$G_{x,4}(t,\omega) = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(\tau-t)^2}{2}} e^{-j\omega(\tau-\frac{t}{2})} x(\tau)\, d\tau$$

Main Reference

• S. Qian and D. Chen, Sections 3-2 ~ 3-6 in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

Other References

• D. Gabor, “Theory of communication”, *J. Inst. Elec. Eng.*, vol. 93, pp. 429-457, Nov. 1946. (最早提出 Gabor transform)

• M. J. Bastiaans, “Gabor’s expansion of a signal into Gaussian elementary signals,” *Proc. IEEE*, vol. 68, pp. 594-598, 1980.

• R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley- Interscience.

• S. C. Pei and J. J. Ding, “Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing,” *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Note：

許多文獻把 Gabor transform 直接就稱作 short-time Fourier transform (STFT)，實際上，Gabor transform 是 STFT 當中的一個 special case.

III-B Approximation of the Gabor Transform

Although the range of integration is from $-\infty$ to ∞, due to the fact that

$$e^{-\pi a^2} < 0.00001 \qquad \text{when } |a| > 1.9143$$

$$e^{-a^2/2} < 0.00001 \qquad \text{when } |a| > 4.7985$$

the Gabor transform can be simplified as:

$$G_x(t,f) \approx \int_{t-1.9143}^{t+1.9143} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

$$G_{x,3}(t,\omega) = \sqrt{\frac{1}{2\pi}} \int_{t-4.7985}^{t+4.7985} e^{-\frac{(\tau-t)^2}{2}} e^{-j\omega(\tau-\frac{t}{2})} x(\tau)\, d\tau$$

III-C Why Do We Choose the Gaussian Function as a Mask

(1) Among all functions, the Gaussian function has the advantage that the area in time-frequency distribution is minimal.

(和其他的 STFT 相比，比較能夠同時讓 time-domain 和 frequency domain 擁有較好的清晰度)

$w(t)$ 太寬 $\rightarrow$ time domain 的解析度較差

$w(t)$ 太窄 $\rightarrow W(f) = FT[w(t)]$太寬 $\rightarrow$ frequency domain 的解析度較差

(2) Special relation between the Gaussian function and the rectangular function

(Note): 由於 Gaussian function 是 FT 的 eigenfunction，因此 Gabor transform 在 time domain 和 frequency domain 的性質將互相對稱

$$\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi f t} dt = e^{-\pi f^2}$$

$$\int_{-\infty}^{\infty} e^{-t^2/2} e^{-j\omega t} dt = e^{-f^2/2}$$

$a = \pi, \; b = j2\pi f$

according to $$\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi / a} \cdot e^{b^2/4a}$$

$$\sqrt{\frac{\pi}{\pi}}\, e^{\frac{-4\pi^2 f^2}{4\pi}} = e^{-\pi f^2}$$

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

Gaussian function is also an eigenmode in optics, radar system, and other electromagnetic wave systems.

(will be illustrated in the 8th week)

Uncertainty Principle (Heisenberg, 1927)

For a signal $x(t)$, if $\sqrt{t}\,x(t) = 0$ when $|t| \longrightarrow \infty$, then

$$\sigma_t \sigma_f \geq 1/4\pi$$

σ_t: standard deviation in the time axis

where $\sigma_t^2 = \int (t-\mu_t)^2 P_x(t)\,dt$ $\sigma_f^2 = \int (f-\mu_f)^2 P_X(f)\,df$,

$\mu_t = \int t\, P_x(t)\,dt$, $\mu_f = \int f\, P_X(f)\,df$

$X(f) = FT(x(t))$

$P_x(t) = \dfrac{|x(t)|^2}{\int |x(t)|^2\,dt}$, $P_X(f) = \dfrac{|X(f)|^2}{\int |X(f)|^2\,df}$,

$\sigma^2 = E((X-\overline{X})^2)$

↑ variance

E: expected value

In addition to $e^{-\pi t^2}$

① $e^{-\pi(t-t_0)^2}$ shifting

② $e^{j2\pi f_0 t}\, e^{-\pi t^2}$ modulation

③ $A e^{-\pi t^2}$ magnification

④ $e^{-\pi \sigma t^2}$ scaling

all satisfy $\sigma_t \sigma_f = \frac{1}{4\pi}$

ex: if $\sigma = 4$, $\sigma_{t\,new} = \frac{\sigma_t}{2}$

$\sigma_{f\,new} = 2\sigma_f$

(Proof of Henseinberg's uncertainty principle):

From simplification, we consider the case where $\mu_t = \mu_f = 0$

Then, use Parseval's theorem

$$\sigma_t^2 \sigma_f^2 = \frac{1}{4\pi^2} \frac{\int t^2 |x(t)|^2 \, dt}{\int |x(t)|^2 \, dt} \frac{\int |x'(t)|^2 \, dt}{\int |x(t)|^2 \, dt}$$

$$\int |x(t)|^2 \, dt = \int |X(f)|^2 \, df \qquad \text{if } X(f) = FT[x(t)]$$

From Schwarz's inequality $\langle x(t), x(t)\rangle\langle y(t), y(t)\rangle \geq \left|\langle x(t), y(t)\rangle\right|^2$

$$\int t^2 \,|\,x(t)\,|^2\, dt\int |\,x'(t)\,|^2\, dt \geq \left(\left|\int tx^*(t)\frac{d}{dt}x(t)dt\right|^2 + \left|\int tx(t)\frac{d}{dt}x^*(t)dt\right|^2\right)/2$$

$$\geq \left|\int\left(tx^*(t)\frac{d}{dt}x(t) + tx(t)\frac{d}{dt}x^*(t)\right)dt\right|^2/4 \quad \text{(using } |a+b|^2 + |a-b|^2 \geq 2|a|^2\text{)}$$

$$= \left|\int t\frac{d}{dt}\left[x(t)x^*(t)\right]dt\right|^2/4 = \left|tx(t)x^*(t)\Big|_{-\infty}^{\infty} - \int x^*(t)x(t)\,dt\right|^2/4$$

$$= \left|\left[tx(t)x^*(t)\Big|_{t\to\infty} - tx(t)x^*(t)\Big|_{t\to-\infty}\right] - \int x^*(t)x(t)\,dt\right|^2/4$$

$$= \left|\int \left|x(t)\right|^2 dt\right|^2/4$$

$$\sigma_t^2\sigma_f^2 \geq \frac{1}{16\pi^2} \Longrightarrow \sigma_t\sigma_f \geq \frac{1}{4\pi}$$

For Gaussian function

$$x(t) = e^{-\pi t^2} \qquad X(f) = e^{-\pi f^2}$$

$\therefore \sigma_t = \sigma_f$

$\mu_t = 0$, $t = 0$

$$\sigma_t^2 = \int (t-\mu_t)^2 P_x(t)\,dt = \frac{\int (t-\mu_t)^2 \,|x(t)|^2\, dt}{\int |x(t)|^2\, dt} = \frac{\int t^2 \,|x(t)|^2\, dt}{\int |x(t)|^2\, dt}$$

Since $\mu_t = 0$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} e^{-2\pi t^2}\, dt = ?\ \sqrt{\tfrac{1}{2}}$$

$a = 2\pi,\ b = 0$

use $\displaystyle \int_{-\infty}^{\infty} e^{-(at^2+bt)}\, dt = \sqrt{\pi/a}\cdot e^{b^2/4a}$

$\Gamma(n+1) = n!$ gamma function

$$\int_{-\infty}^{\infty} t^2 |x(t)|^2\, dt = \int_{-\infty}^{\infty} t^2 e^{-2\pi t^2}\, dt = 2\int_{0}^{\infty} t^2 e^{-2\pi t^2}\, dt = 2\frac{\Gamma(3/2)}{2(2\pi)^{3/2}} = \frac{1}{2^{5/2}\pi}$$

use $\displaystyle \int_{0}^{\infty} t^m e^{-at^2}\, dt = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

$$\frac{\sqrt{\pi}}{2^{\frac{5}{2}}\pi^{\frac{3}{2}}}$$

$$\Gamma(1/2) = \sqrt{\pi} \qquad \Gamma(n+1) = n\Gamma(n), \qquad \Gamma(3/2) = \sqrt{\pi}/2$$

$n!$ $(n-1)!$

$$\sigma_t^2 = \frac{\int t^2 |x(t)|^2 \, dt}{\int |x(t)|^2 \, dt} = \frac{1}{4\pi}, \qquad \sigma_t = \sqrt{\frac{1}{4\pi}}$$

同理, $\sigma_f = \sqrt{\frac{1}{4\pi}}$

所以對 Gaussian function 而言，

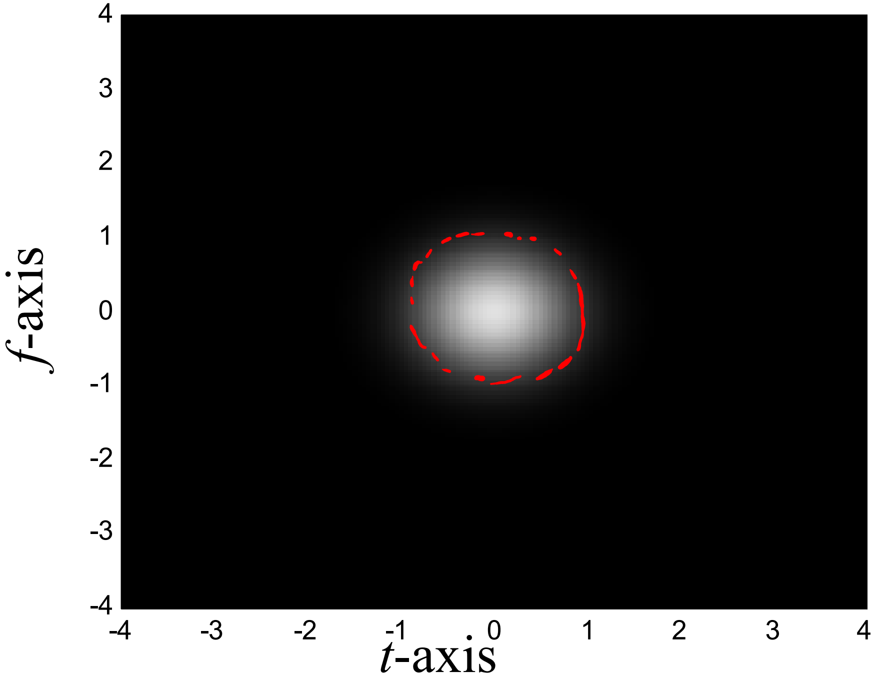
$$\sigma_t \sigma_f = \frac{1}{4\pi}$$

滿足下限

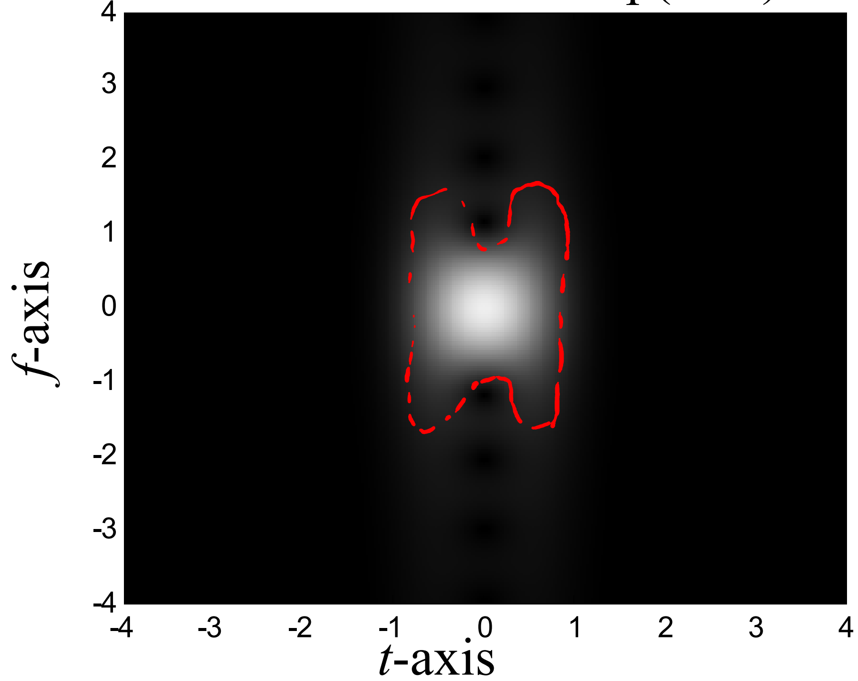
[工具書] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

III-D Simulations

Gabor transform for
Gaussian function $\exp(-\pi t^2)$



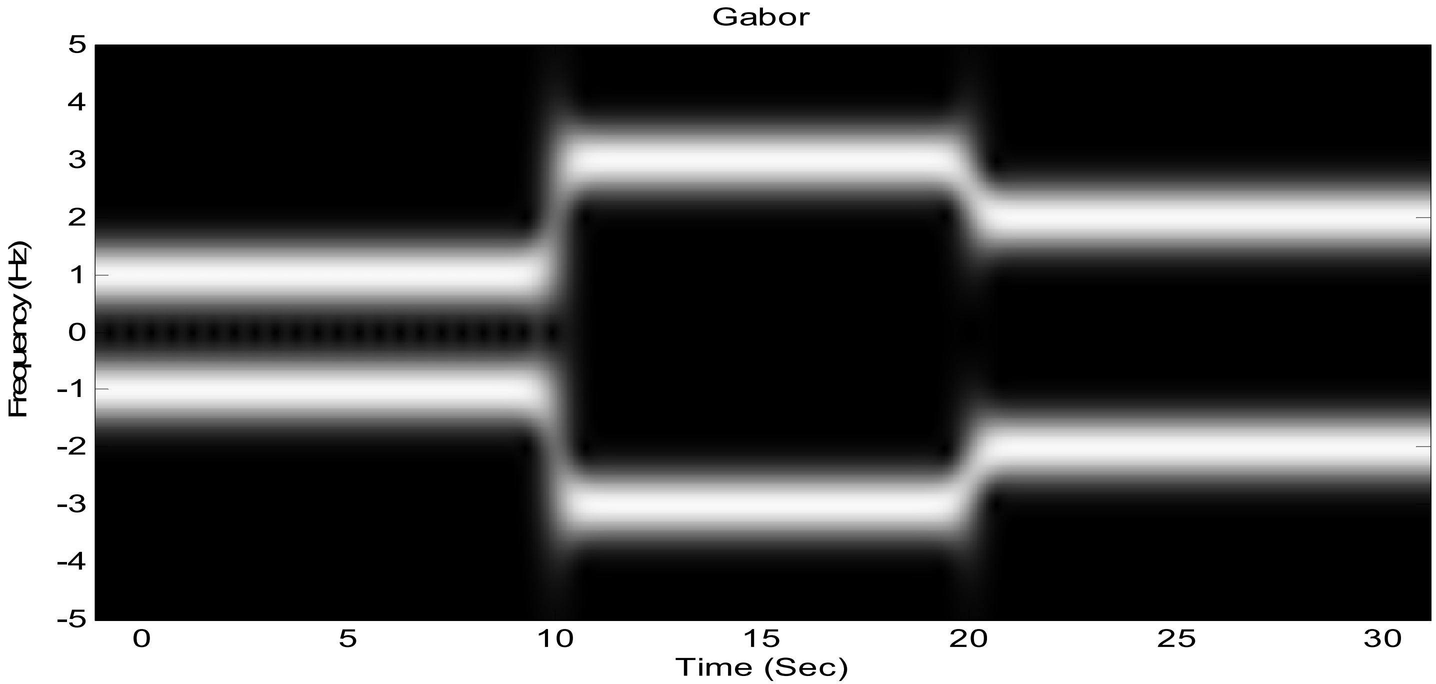
rec-STFT, B = 0.5 for
Gaussian function $\exp(-\pi t^2)$



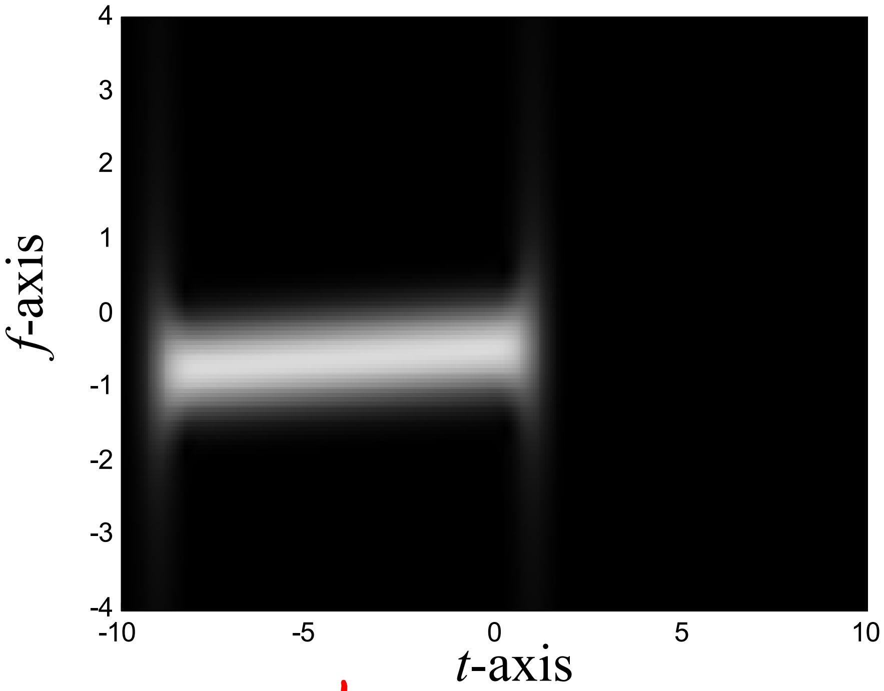
$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

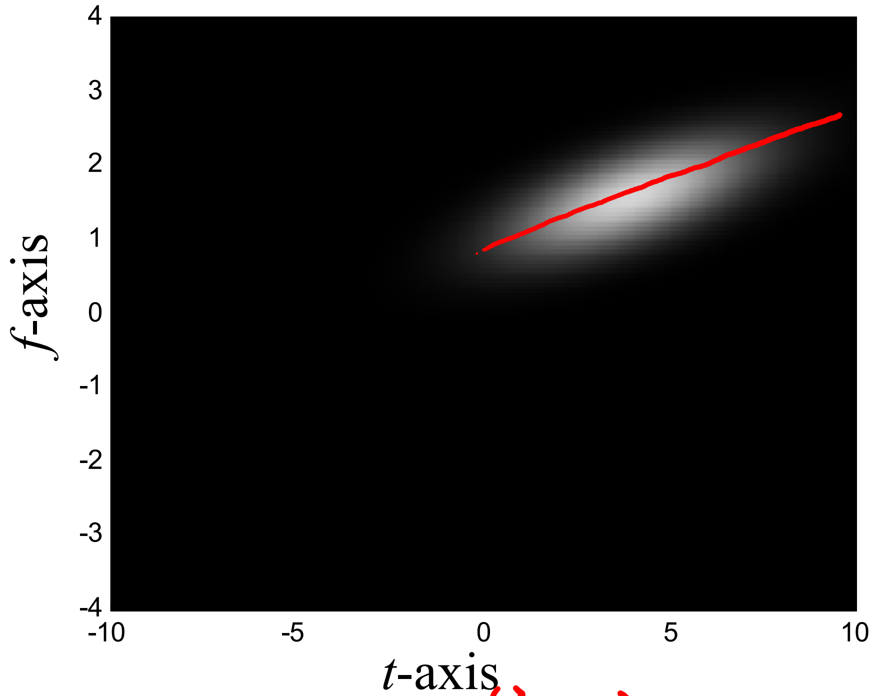
$x(t) = \cos(4\pi t)$ when $t \ge 20$



Gabor transform of $s(t)$



Gabor transform of $r(t)$



chirp

$s(t)=\exp\left(jt^2/10-j3t\right)$ for $-9\le t\le 1$,

$s(t)=0$ otherwise,

instantaneous freq. = ?

$A(t)\,e^{j\left(\frac{t^2}{2}+6t\right)}$

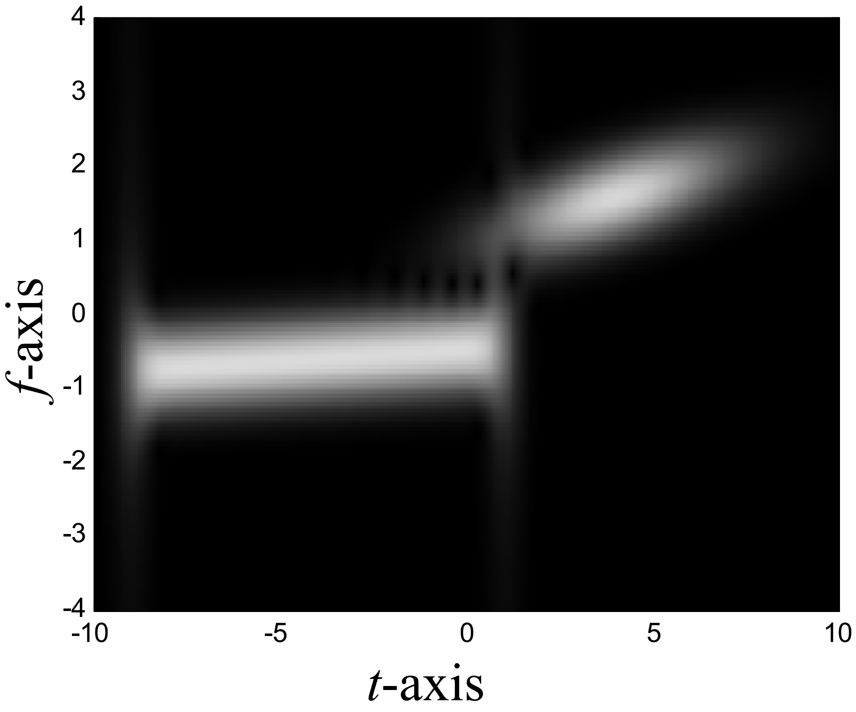
$r(t)=\exp\left(jt^2/2+j6t\right)\exp\left[-(t-4)^2/10\right]$

$A(t)=e^{-\frac{(t-4)^2}{10}}$

instantaneous frequency

$\frac{t}{2\pi}+\frac{3}{\pi}$

Gabor transform for $s(t) + r(t)$



III-E Properties of Gabor Transforms

$$G_x(t,f)=\int_{-\infty}^{\infty} e^{-j2\pi f\tau} e^{-\pi(\tau-t)^2} x(\tau)\,d\tau$$

(1) Integration property

When $k \neq 0$, $\int_{-\infty}^{\infty} G_x(t,f) e^{j2\pi k t f}\, df = e^{-\pi(k-1)^2 t^2} x(kt)$

When $k = 0$, $\int_{-\infty}^{\infty} G_x(t,f)\, df = e^{-\pi t^2} x(0)$

When $k = 1$, $\int_{-\infty}^{\infty} G_x(t,f) e^{j2\pi t f}\, df = x(t)$ (recovery property)

(2) Shifting property

If $y(t) = x(t-t_0)$, then $G_y(t,f) = G_x(t-t_0, f) e^{-j2\pi f t_0}$.

(3) Modulation property

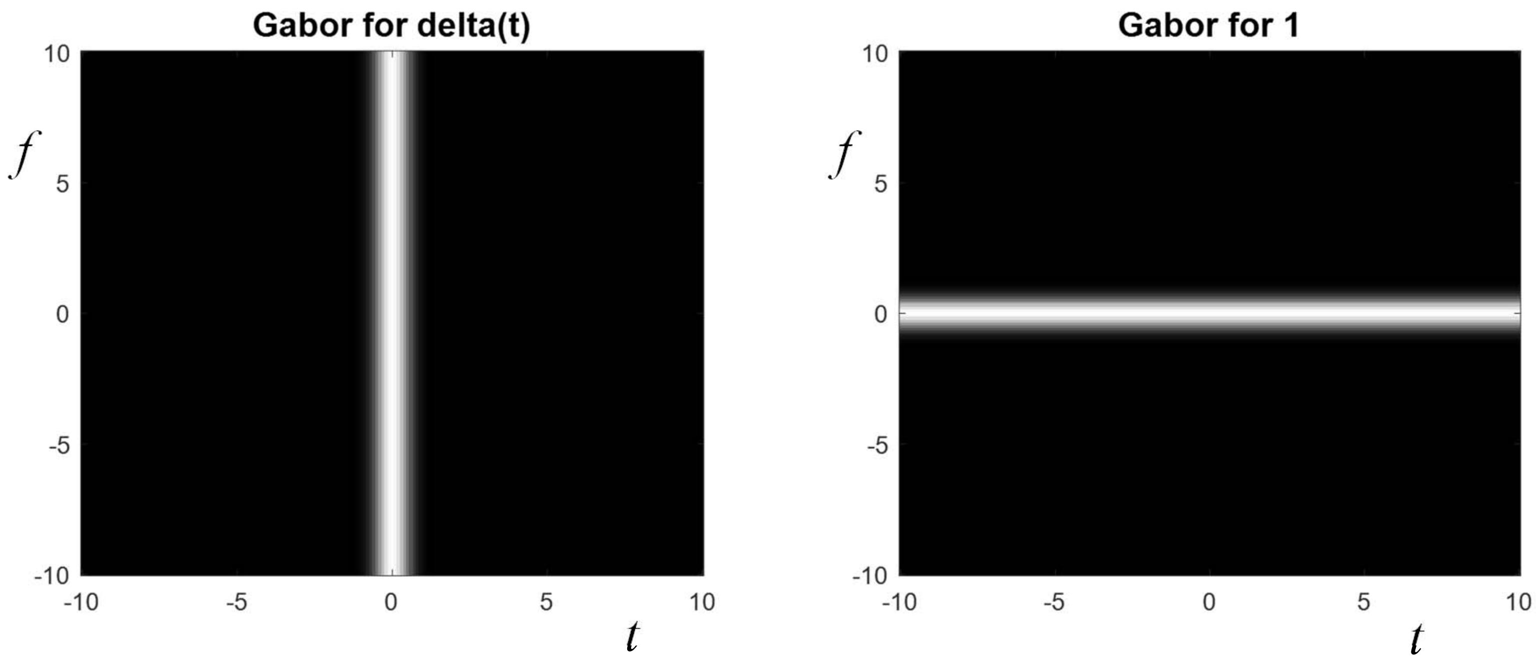
If $y(t) = x(t)\exp(j2\pi f_0 t)$, then $G_y(t,f) = G_x(t, f-f_0)$

(4) Special inputs:

(a) When $x(\tau) = \delta(\tau)$, $G_x(t,f) = e^{-\pi t^2}$

(b) When $x(\tau) = 1$, $G_x(t,f) = e^{-j2\pi f t} e^{-\pi f^2}$ $|G_x(t,f)| = e^{-\pi f^2}$

(symmetric for the time and frequency domains)



(5) Power decayed property

- If $x(t) = 0$ for $t > t_0$, then .

$$\int_{-\infty}^{\infty}\left|G_x(t,f)\right|^2 df < e^{-2\pi(t-t_0)^2}\int_{-\infty}^{\infty}\left|G_x(t_0,f)\right|^2 df$$

i.e., $\underset{(\text{fix } t,\text{ vary } f)}{\text{average of}} \left|G_x(t,f)\right|^2 < e^{-2\pi(t-t_0)^2} \times \underset{(\text{fix } t_0,\text{ vary } f)}{\text{average of}} \left|G_x(t_0,f)\right|^2$ for $t > t_0$.

(Proof):

$$G_x(t,f) = \int_{-\infty}^{t_0} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau \quad G_x(t_0,f) = \int_{-\infty}^{t_0} e^{-\pi(\tau-t_0)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau$$

Since $(\tau-t)^2 > (\tau-t_0)^2 + (t_0-t)^2$ $\quad e^{-\pi(t-\tau)^2} < e^{-\pi(t-t_0)^2} e^{-\pi(t_0-\tau)^2}$

$$G_x(t,f) < e^{-\pi(t-t_0)^2} G_x(t_0,f)$$

- If $X(f) = FT\left[x(t)\right] = 0$ for $f > f_0$, then

$$\underset{(\text{fix } f,\text{ vary } t)}{\text{average of}} \left|G_x(t,f)\right|^2 < e^{-2\pi(f-f_0)^2} \times \underset{(\text{fix } f_0,\text{ vary } t)}{\text{average of}} \left|G_x(t,f_0)\right|^2 \text{ for } f > f_0.$$

(6) Linearity property

If $z(\tau) = \alpha\, x(\tau) + \beta\, y(\tau)$ and $G_z(t, f)$, $G_x(t, f)$ and $G_y(t, f)$ are their Gabor transforms, then

$$G_z(t,f) = \alpha\, G_x(t,f) + \beta\, G_y(t,f)$$

(7) Power integration property:

$$\int_{-\infty}^{\infty} \left|G_x(t,f)\right|^2 df = \int_{-\infty}^{\infty} e^{-2\pi(\tau-t)^2} \left|x(\tau)\right|^2 d\tau \approx \int_{u-1.9143}^{u+1.9143} e^{-2\pi(\tau-u)^2} \left|x(\tau)\right|^2 d\tau$$

(8) Energy sum property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_x(t,f)\, G_y^*(t,f)\, df\, dt = \int_{-\infty}^{\infty} x(\tau)\, y^*(\tau)\, d\tau$$

where $G_x(t, f)$ and $G_y(t, f)$ are the Gabor transforms of $x(\tau)$ and $y(\tau)$, respectively.

III-F Scaled Gabor Transforms

$w(\tau) = e^{-\sigma\pi\tau^2}$

$$G_x(t,f) = \sqrt[4]{\sigma}\int_{-\infty}^{\infty} e^{-\sigma\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau$$

⇓ (finite interval form)

$e^{-\sigma\pi a^2} < 10^{-5}$ if $|a| > \frac{1.9143}{\sqrt{\sigma}}$

$$G_x(t,f) = \sqrt[4]{\sigma}\int_{t-\frac{1.9143}{\sqrt{\sigma}}}^{t+\frac{1.9143}{\sqrt{\sigma}}} e^{-\sigma\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau$$

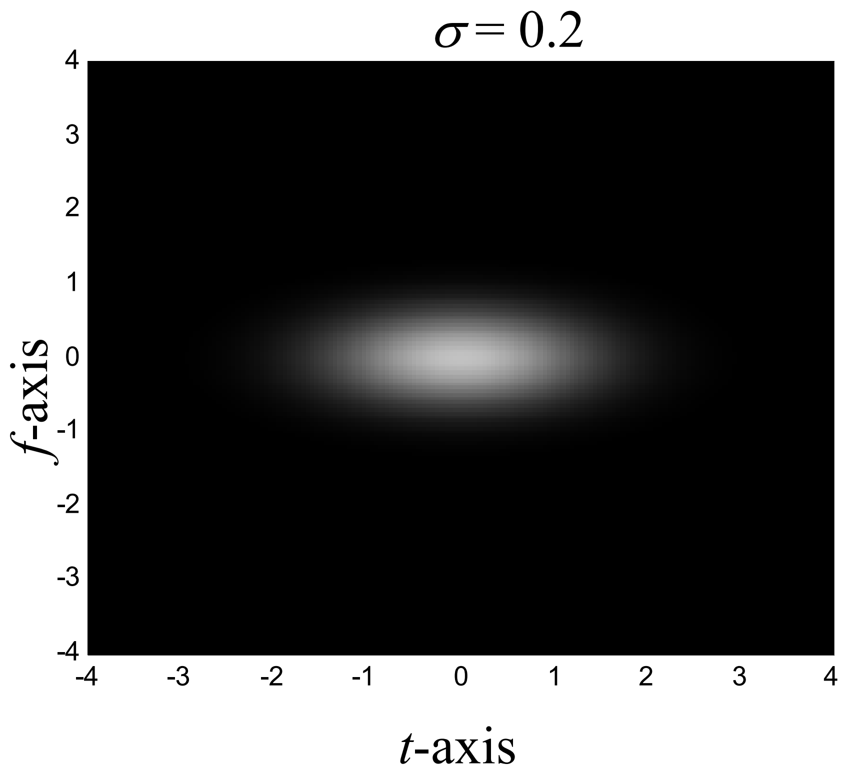
larger σ: higher resolution in the time domain

lower resolution in the frequency domain

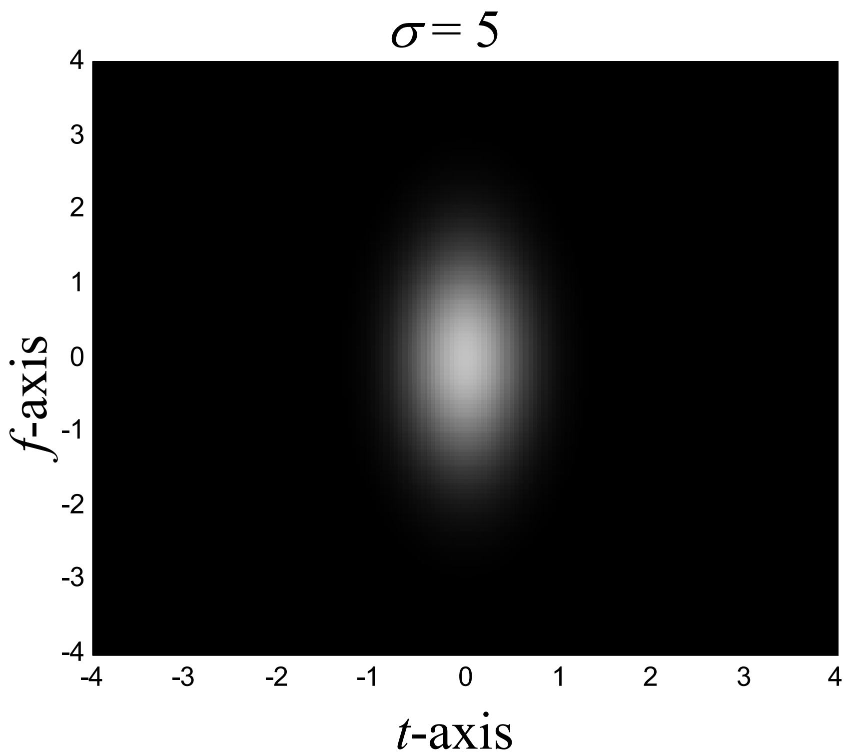
smaller σ: higher resolution in the frequency domain

lower resolution in the time domain

Gabor transform for Gaussian function $\exp(-\pi t^2)$



Gabor transform for Gaussian function $\exp(-\pi t^2)$



處理對 time resolution 相對上比 frequency resolution 敏感的信號

(1) Using the generalized Gabor transform with larger σ

(2) Using other time unit instead of second

例如，原本　t (單位：sec)　　f (單位： Hz)

對聲音信號可以改成

t (單位： 0.1 sec)　f (單位：10 Hz)

III-G Gabor Transforms with Adaptive Window Width

For a signal,

when the instantaneous frequency varies fast $\rightarrow$ larger σ

when instantaneous frequency varies slowly $\rightarrow$ smaller σ

$$G_x(t,f) = \sqrt[4]{\sigma(t)} \int_{-\infty}^{\infty} e^{-\sigma(t)\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

$\sigma(t)$ is a function of t

S. C. Pei and S. G. Huang, "STFT with adaptive window width based on the chirp rate," *IEEE Trans. Signal Processing*, vol. 60, issue 8, pp. 4065-4080, 2012.

附錄五：Matlab 寫程式的原則以及部分常用的指令

(1) 迴圈能避免就儘量避免

(2) 儘可能使用 Matrix 及 Vector operation

(3) 能夠不在迴圈內做的運算，則移到迴圈外

(4) 寫一部分即測試，不要全部寫完再測試 (縮小範圍比較容易 debug)

(5) 先測試簡單的例子，成功後再測試複雜的例子

註：作業 Matlab Program (or Python program) 鼓勵各位同學儘量用精簡而快速的方式寫。Program 執行速度越快，分數就越高。

一些重要的 Matlab 指令

(1) function: 放在第一行，可以將整個程式函式化

(2) tic, toc: 計算時間

tic 為開始計時，toc 為顯示時間

(3) find: 找尋一個 vector 當中不等於 0 的entry 的位置

範例：find([1 0 0 1]) = [1, 4]

find(abs([-5:5])<=2) = [4, 5, 6, 7, 8]

(因為 abs([-5:5])<=2 = [0 0 0 1 1 1 1 1 0 0 0])

(4) ′ : Hermitian (transpose + conjugation)， .′ : transpose

(5) imread: 讀圖

(註： 較老的 Matlab 版本 imread 要和 double 並用

A=double(imread('Lena.bmp'));

(6) image: 將圖顯示出來，

(i) 顯示灰階圖

image(A)   % A has the size of *M*x*N*x1

colormap(gray(256))

(ii) 顯示彩色圖，整數的情形

image(A)   % A has the size of *M*x*N*x3 and the entries are integer

(iii) 顯示彩色圖，非整數的情形

image(A)   % A has the size of *M*x*N*x3 and the entries are non-integer

(7) imshow, imagesc : 也可用來顯示圖

(8) imwrite: 製做圖檔

(9) aviread: 讀取 video 檔，限副檔名為 avi

(10) VideoReader: 讀取 video 檔

(11) VideoWriter: 製作 video 檔

(12) xlsread 或 readmatrix 或 readcell : 由 Excel 檔讀取資料

(13) xlswrite: 將資料寫成 Excel 檔

(14) dlmread: 讀取 *.txt 或其他類型檔案的資料

(15) dlmwrite: 將資料寫成 *.txt 或其他類型檔案

附錄六：寫 Python 版本程式可能會用到的重要指令

建議必安裝模組

```
pip install numpy
pip install scipy
pip install opencv-python
pip install matplotlib
```

(1) 定義函式：使用def

(2) 計算時間

```
import time
start_time = time.time()   #獲取當前時間
end_time = time.time()
total_time = end_time - start_time   #計算時間差來得到總執行時間
```

感謝2021年擔任助教的蔡昌廷同學

(3) 讀取圖檔、顯示圖檔、輸出圖檔

(方法一)

```
import cv2
image = cv2.imread(file_name)  #預設color channel為BGR
cv2.imshow('test', image)
# 若 image 的值非整數，要改成 cv2.imshow('test', image/255)
cv2.waitKey(0)
cv2.destroyAllWindows()
cv2.imwrite(file_name, image)   #需將color channel轉為BGR
```

(方法二)

```
import matplotlib.pyplot as plt
image = plt.imread(file_name)  #預設color channel為RGB
plt.imshow(image)
# 若 image 的值非整數，要改成 plt.imshow(image/255)
plt.show()
plt.imsave(file_name, image)   #需將color channel轉為RGB
```

(4) 尋找array中滿足特定條件的值的位置

(相當於 Matlab 的 find 指令)

```
import numpy as np
a = np.array([0, 1, 2, 3, 4, 5])
index = np.where(a > 3) # 回傳array([4, 5])
print(index)
    (array([4, 5], dtype=int64),)
index[0][0]
    4
index[0][1]
    5
```

$$\mathbf{A_1} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix}$$

```
A1= np.array([[1,3,6],[2,4,5]])
index = np.where(A1 > 3)
print(index)
    (array([0, 1, 1], dtype=int64), array([2, 1, 2], dtype=int64))
```

(代表滿足 A1 > 3 的點的位置座標為 [0, 2], [1, 1], [1, 2]

```
[index[0][0], index[1][0]]
   [0, 2]
[index[0][1], index[1][1]]
   [1, 1]
[index[0][2], index[1][2]]
   [1, 2]
```

(5) Hermitian、transpose

```
import numpy as np
result = np.conj(matrix.T)     # Hermitian
result = matrix.T    # transpose
```

(6) 在 Python 當中讀取 Matlab 當中的 mat 檔

```
data = scipy.io.loadmat('***.mat')
y = np.array(data['y'])   # 假設 y 是 ***.mat 當中儲存的資料
```

IV. Implementation

IV-A Method 1: Direct Implementation

以 STFT 為例

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Converting into the Discrete Form

$t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_t$

$$X(n\Delta_t, m\Delta_f)=\sum_{p=-\infty}^{\infty} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

Suppose that $w(t) \cong 0$ for $|t| > B$, $B/\Delta_t = Q$

$$X(n\Delta_t, m\Delta_f)=\sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

Problem：對 scaled Gabor transform 而言，$Q = ?$

Suppose that there are
T samples along t-axis
F samples along f-axis
complexity : $TF(2Q+1)$
$O(TFQ)$
from $d\tau$

$w((n-p)\Delta_t) \cong 0$ for $|(n-p)\Delta_t| > B$ $\quad |n-p| > \frac{B}{\Delta_t} = Q$

$-Q < n-p < Q$
$n-Q < p < n+Q$

T F

$\frac{1.9143}{\sqrt{\sigma}\,\Delta_t}$

if $\sigma = 200$
$\Delta_t = \frac{1}{40000}$
$5200 < Q < 5600$

• **Constraint for Δ_t** (The only constraint for the direct implementation method)

If $w(\tau) \xrightarrow{FT} W(f)$

$w(-\tau) \xrightarrow{FT} W(-f)$

To avoid the aliasing effect,

$\Delta_t < 1/2\Omega$, Ω is the bandwidth of ? $w(t-\tau)x(\tau)$

If the bandwidth of $x(\tau)$ is Ω_x, bandwidth of $w(\tau)$ is Ω_w

then the bandwidth of $w(t-\tau)$ is also Ω_w

$\Omega = \Omega_x + \Omega_w$

$\Delta t < \frac{1}{2(\Omega_x + \Omega_w)}$

There is no constraint for Δ_f when using the direct implementation method.

Four Implementation Methods

4 (1) Direct implementation

Complexity: $O(TFQ)$

假設 t-axis 有 T 個 sampling points, f-axis 有 F 個 sampling points

2 (2) FFT-based method

Complexity: $O(TN\log N)$

unbalanced form $O(\frac{T}{S} N\log N)$

1 (3) FFT-based method with recursive formula

Complexity: $O(TF)$

3 (4) Chirp-Z transform method

Complexity: $O(TN\log N)$

(A) Direct Implementation

Advantage： simple, flexible

Disadvantage ： higher complexity

(B) DFT-Based Method

Advantage ： lower complexity

Disadvantage ： with some constraints

(C) Recursive Method

Advantage ： least complexity

Disadvantage ： (i) only suitable for the rectangular window
(ii) the error may be culmulated

(D) Chirp Z Transform

Advantage ： flexible (the only constraint is $\Delta t < \frac{1}{2(\Omega_x+\Omega_w)}$

Disadvantage ： more computation loading than methods (B) (C)

IV-B Method 2: FFT-Based Method

(i) $\Delta_t < \frac{1}{2(\Omega_x + \Omega_w)}$

Constraints：(ii) $\Delta_t\Delta_f = 1/N$,

(iii) $N = 1/(\Delta_t\Delta_f) \geqq 2Q + 1$: ($\Delta_t\Delta_f$是整數的倒數)

Standard form of the DFT $Y[m] = \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi mn}{N}}$ complexity : $O(N\log N)$

$$X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

if $\Delta_t = \frac{1}{40000}$ $\Delta_f = 1$

$N = 40000$

$\Delta_t\Delta_f = 1/N$ $N = \frac{1}{\Delta_t\Delta_f}$

$$X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x(p\Delta_t)e^{-j\frac{2\pi pm}{N}}\Delta_t$$

p	n−Q	n+Q
q	0	2Q

$q = p-(n-Q) \rightarrow p = (n-Q)+q$

$$X(n\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{2Q} w((Q-q)\Delta_t)x((q+n-Q)\Delta_t)e^{-j\frac{2\pi qm}{N}}$$

Note that the input of the N-point FFT should have N points (others are set to zero).

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1\left(q\right) e^{-j\frac{2\pi qm}{N}}$$

, $q = p-(n-Q) \rightarrow p = (n-Q)+q$

where $x_1\left(q\right) = w\left((Q-q)\Delta_t\right) x\left((n-Q+q)\Delta_t\right)$ for $0 \le q \le 2Q$,

$x_1(q) = 0$ for $2Q < q < N$.

$n\text{-}Q \le n\text{-}Q+q \le n+Q$

$Q \ge Q - q \ge -Q$

$k = q\text{-}Q$

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1\left(q\right)\right)$$

$w(-k\Delta_t)$

where $x_1\left(q\right) = w\left(k\Delta_t\right) x\left((n+k)\Delta_t\right)$ for $0 \le q \le 2Q$, $-Q \le k \le Q$ $(k = q\text{-}Q)$

$x_1\left(q\right) = 0$ for $2Q < q < N$.

(Suppose that $w(t) = w(-t)$)

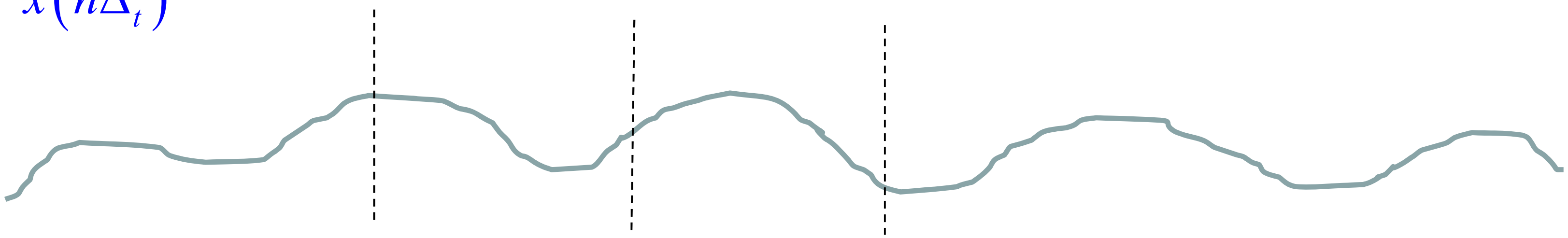
$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1\left(q\right)\right)$$

where $x_1\left(q\right) = w\left(k\Delta_t\right) x\left((n+k)\Delta_t\right)$ for $-Q \le k \le Q$ $(k = q-Q)$

$x_1\left(q\right) = 0$ for $2Q < q < N$. (Suppose that $w(t) = w(-t)$)

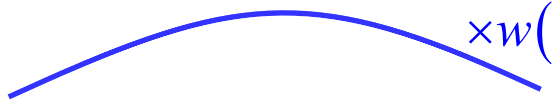
center

$x\left(n\Delta_t\right)$



$x\left((n-Q)\Delta_t\right)$ $x\left(n\Delta_t\right)$ $x\left((n+Q)\Delta_t\right)$

take a segment of the signal (2Q+1) points



$\times w\left(k\Delta_t\right)$ $-Q \le k \le Q$

$-B \le k\Delta_t \le B$

multiplied by the window

$x_1\left(q\right)$: $w\left(k\Delta_t\right) x\left((n+k)\Delta_t\right)$ padding N-2Q-1 zeros

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1\left(q\right)\right)$$

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} DFT\left(x_1(q)\right)$$

T F

注意：

if F < N , for every n only one DFT is required

(1) 可以使用 Matlab 的 FFT 指令來計算 $\sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}} = X_1(m)$

but $f = m\Delta f$, m can be negative

ex: $-5 \le f \le 5$, $\Delta f = 0.05$

$-100 \le m \le 100$

$0 \le m \le N-1$

$X_1(m+N) = \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi q(m+N)}{N}}$

$= \sum_{q=0}^{N-1} x_1(q) e^{-j2\pi q} e^{-j\frac{2\pi}{N}qm}$

$X_1(m+N) = X_1(m)$

(2) 對每一個固定的 n，都要計算一次下方的式子

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}} = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} X_1(m)$$

(fixed n)

F　　N log N

(3) Complexity = ?

$T(F + N\log N) \cong TN\log N$

假設 $t = n_0\Delta_t,\ (n_0+1)\,\Delta_t,\ (n_0+2)\,\Delta_t,\ \cdots\cdots,\ (n_0+T\text{-}1)\Delta_t$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F\text{-}1)\Delta_f$

$\frac{1}{\Delta_t \Delta_f} = N$ 122

N should be an integer.

Step 1: Calculate n_0, m_0, T, F, N, Q $\quad Q = \frac{B}{\Delta_t}$

Step 2: $n = n_0$

loop

Step 3: Determine $x_1(q)$ ← 超出範圍添0

Step 4: $X_1(m) = \text{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(n\Delta_t, m\Delta_f)$

(Steps 3–5: page 119)

$m = f/\Delta_f$

$m_1 = \text{mod}(m, N)+1$ for Matlab

$m_1 = m \% N$ for Python

$$X\left(n\Delta_t, m\Delta_f\right) = X_1(?)\times ?$$

$$\Delta_t\, e^{j\frac{2\pi(Q-n)m}{N}} X_1\left(((m))_N\right)$$

$((m))_N = m \bmod N$

ex: $m=-99$, $N=400$

$((-99))_{400} = 301$

$$X_1[m] = \sum_{q=0}^{N-1} x_1(q)\, e^{-j\frac{2\pi qm}{N}} \qquad X_1[m] = X_1[m+N]$$

Step 6: Set $n = n+1$ and return to Step 3 until $n = n_0+T\text{-}1$.

IV-C Method 3: Recursive Method

constraint (iv): only suitable for the rectangular window

- **A very fast way for implementing the rec-STFT**

(n 和 $n-1$ 有recursive的關係)

$$X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} x\left(p\Delta_t\right) e^{-j\frac{2\pi pm}{N}} \Delta_t$$

$$X\left((n-1)\Delta_t, m\Delta_f\right) = \sum_{p=n-1-Q}^{n-1+Q} x(p\Delta_t) e^{-j\frac{2\pi pm}{N}} \Delta_t$$

(1) Calculate $X(\min(n)\Delta_t, m\Delta_f)$ by the N-point FFT

$$X\left(n_0\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n_0)m}{N}} \sum_{q=0}^{N-1} x_1\left(q\right) e^{-j\frac{2\pi qm}{N}}, \qquad n_0 = \min(n),$$

$$x_1\left(q\right) = x\left((n_0 - Q + q)\Delta_t\right) \quad \text{for } q \le 2Q, \qquad x_1(q) = 0 \quad \text{for } q > 2Q$$

(2) Applying the recursive formula to calculate $X(n\Delta_t, m\Delta_f)$,

$n = n_0 + 1 \sim \max(n)$ $\qquad N\log N + (T-1)2F \cong 2TF \qquad O(TF)$

$$X\left(n\Delta_t, m\Delta_f\right) = X\left((n-1)\Delta_t, m\Delta_f\right) - x\left((n-Q-1)\Delta_t\right) e^{-j2\pi(n-Q-1)m/N} \Delta_t + x\left((n+Q)\Delta_t\right) e^{-j2\pi(n+Q)m/N} \Delta_t$$

T點 F點

$p = n-1-Q$

2F points (when n is fixed)

$p = n+Q$

F points

IV-D Method 4: Chirp Z Transform

$$\exp\left(-j2\pi\, pm\Delta_t\Delta_f\right) = \exp\left(-j\pi\, p^2\Delta_t\Delta_f\right)\exp\left(j\pi(p-m)^2\Delta_t\Delta_f\right)\exp\left(-j\pi m^2\Delta_t\Delta_f\right)$$

ifft (fft(y) fft(h))

For the STFT

y[m] * h[m]

= Σ y[p] h[m−p]

$$X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} w\left((n-p)\Delta_t\right)x\left(p\Delta_t\right)e^{-j2\pi\, pm\Delta_t\Delta_f}\Delta_t$$

y[p]

$h[p] = e^{j\pi p^2\Delta_t\Delta_f}$

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{-j\pi m^2\Delta_t\Delta_f}\sum_{p=n-Q}^{n+Q} w\left((n-p)\Delta_t\right)x\left(p\Delta_t\right)e^{-j\pi p^2\Delta_t\Delta_f}e^{j\pi(p-m)^2\Delta_t\Delta_f}$$

T F

Step 1 multiplication 2Q+1

for a fixed n

Step 2 convolution

Step 3 multiplication

two chirp multiplications

+ one convolution

3 N log N (3 FFTs are required)

T(2Q+1+3 N log N + F) ≈ 3 T N log N

Θ(T N log N)

Step 1 $x_1[p] = w((n-p)\Delta_t)\, x(p\Delta_t)\, e^{-j\pi p^2 \Delta_t \Delta_f}$ $\qquad n\text{-}Q \le p \le n\text{+}Q$

Step 2 $X_2[n,m] = \sum_{p=n-Q}^{n+Q} x_1[p]\, c[m-p]$ $\qquad c[m] = e^{j\pi m^2 \Delta_t \Delta_f}$

Step 3 $X(n\Delta_t, m\Delta_f) = \Delta_t\, e^{-j\pi m^2 \Delta_t \Delta_f} X_2[n,m]$

Step 2 在計算上，需要用到 linear convolution 的技巧

Question: Step 2 要用多少點的 DFT?

- **Illustration for the Question on Page 124**

$$y[n] = \sum_k x[n-k]h[k]$$

- Case 1

When length($x[n]$) = N, length($h[n]$) = K, N and K are finite,

$\longrightarrow$ length($y[n]$) = $N+K-1$,

Using the ($N+K-1$)-point DFTs (學信號處理的人一定要知道的常識)

- Case 2

$x[n]$ has finite length but $h[n]$ has infinite length ????

$$y[n] = \sum_k x[n-k]h[k]$$

- Case 2

$x[n]$ has finite length but $h[n]$ has infinite length

$x[n]$ 的範圍為 $n \in [n_1, n_2]$，範圍大小為 $N = n_2 - n_1 + 1$

$h[n]$ 無限長

$y[n] = \sum_k x[n-k]h[k]$ $y[n]$ 每一點都有值 (範圍無限大)

但我們只想求出 $y[n]$ 的其中一段

希望算出的 $y[n]$ 的範圍為 $n \in [m_1, m_2]$，範圍大小為 $M = m_2 - m_1 + 1$

$h[n]$ 的範圍？

要用多少點的 FFT？

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

改寫成 $y[n]=x[n]*h[n]=\sum_{s=n_1}^{n_2}x[s]h[n-s]$

$$y[n]=x[n_1]h[n-n_1]+x[n_1+1]h[n-n_1-1]+x[n_1+2]h[n-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[n-n_2]$$

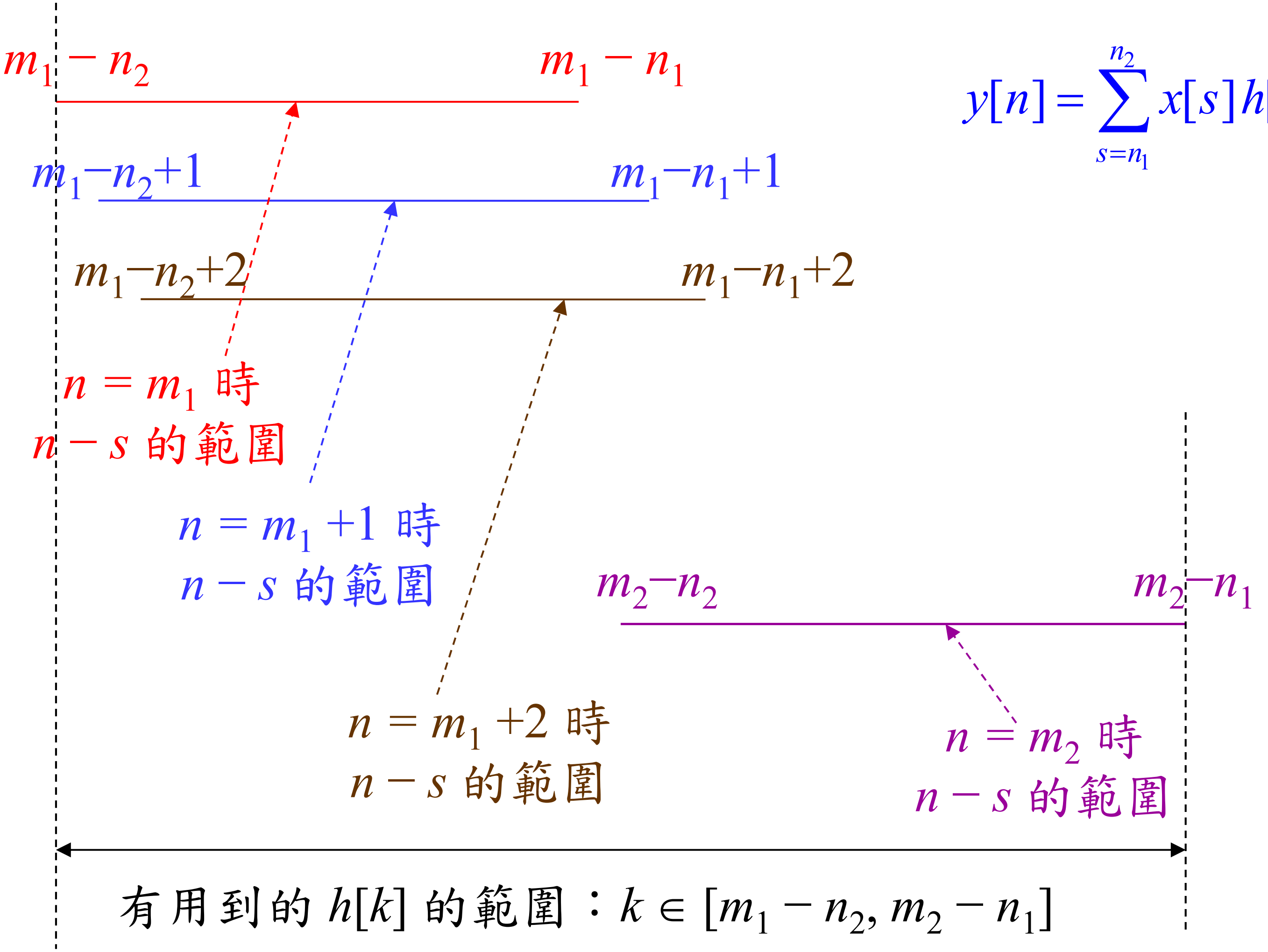
當$n=m_1$

$$y[m_1]=x[n_1]h[m_1-n_1]+x[n_1+1]h[m_1-n_1-1]+x[n_1+2]h[m_1-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_1-n_2]$$

當$n=m_2$

$$y[m_2]=x[n_1]h[m_2-n_1]+x[n_1+1]h[m_2-n_1-1]+x[n_1+2]h[m_2-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_2-n_2]$$

$$y[n]=\sum_{s=n_1}^{n_2}x[s]h[n-s]$$



所以，有用到的 $h[k]$ 的範圍是 $k \in [m_1 - n_2 , m_2 - n_1]$

範圍大小為 $m_2 - n_1 - m_1 + n_2 + 1 = N + M - 1$

FFT implementation for Case 2

$x_1[n] = x[n + n_1]$ for $n = 0, 1, 2, \dots, N-1$

$x_1[n] = 0$ for $n = N, N+1, N+2, \dots\dots, L-1$ $L = N + M - 1$

$h_1[n] = h[n + m_1 - n_2]$ for $n = 0, 1, 2, \dots, L-1$

$$y_1[n] = IFFT_L\left(FFT_L\{x_1[n]\}\, FFT_L\{h_1[n]\}\right)$$

$y[n] = y_1[n - m_1 + N - 1]$ for $n = m_1, m_1+1, m_1+2, \dots, m_2$

IV-E Unbalanced Sampling for STFT and WDF

將 pages 114 and 118 的方法作修正

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$t-\tau = n\Delta_t - p\Delta_\tau$
$= nS\Delta_\tau - p\Delta_\tau$
$= (nS-p)\Delta_\tau$

$$X(n\Delta_t, m\Delta_f)=\sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j2\pi pm\Delta_\tau\Delta_f}\Delta_\tau$$

where $t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_\tau$, $B = Q\Delta_\tau$ (假設 $w(t) \cong 0$ for $|t| > B$),

$S = \Delta_t / \Delta_\tau$ $\qquad\Delta_t \neq \Delta_\tau$

ex: $\Delta_\tau = \frac{1}{44100}$ $\quad \Delta_t = \frac{1}{100}$

$S=441$

註：Δ_τ (sampling interval for the input signal)

Δ_t (sampling interval for the output t-axis) can be different.

However, it is better that $S = \Delta_t / \Delta_\tau$ is an integer.

input sampling interval

When (1) $\Delta_\tau\Delta_f = 1/N$, (2) $N = 1/(\Delta_\tau\Delta_f) > 2Q+1$: ($\Delta_\tau\Delta_f$只要是整數的倒數即可)

(3) $\Delta_\tau < 1/2\Omega$, Ω is the bandwidth of $w(\tau - t)x(\tau)$

i.e., $|FT\{w(\tau-t)x(\tau)\}| = |X(t,f)| \approx 0$ when $|f| > \Omega$

$$X(n\Delta_t, m\Delta_f) = \sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j\frac{2\pi pm}{N}}\Delta_\tau$$

令 $q = p - (nS-Q) \rightarrow p = (nS-Q) + q$

Q-n is replaced by Q-nS

$$X(n\Delta_t, m\Delta_f) = \Delta_\tau e^{j\frac{2\pi(Q-nS)m}{N}} \sum_{q=0}^{N-1} x_1(q)e^{-j\frac{2\pi qm}{N}}$$

$x_1(q) = w((Q-q)\Delta_\tau)x((nS-Q+q)\Delta_\tau)$ for $0 \le q \le 2Q$,

$x_1(q) = 0$ for $2Q < q < N$.

If $w(t) = w(-t)$)

n is replaced by nS

$x_1(q) = w(k\Delta_\tau)x((nS+k)\Delta_\tau)$ for $0 \le q \le 2Q$, $k = q-Q$, $-Q \le k \le Q$

比較 page119

$x_1(q) = 0$ for $2Q < q < N$.

假設 $t = c_0\Delta_t,\ (c_0+1)\,\Delta_t,\ (c_0+2)\,\Delta_t,\ \cdots\cdots,\ (c_0+C-1)\,\Delta_t$

$= c_0S\Delta_\tau,\ (c_0S+S)\,\Delta_\tau,\ (c_0S+2S)\,\Delta_\tau,\ \cdots\cdots,\ [c_0S+(C-1)S]\,\Delta_\tau$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F-1)\,\Delta_f$

$\tau = n_0\,\Delta_\tau,\ (n_0+1)\,\Delta_\tau,\ (n_0+2)\,\Delta_\tau,\ \cdots\cdots,\ (n_0+T-1)\,\Delta_\tau,$ $\quad S = \Delta_t/\,\Delta_\tau$

Step 1: Calculate c_0, m_0, n_0, C, F, T, N, Q

Step 2: $n = c_0$

Step 3: Determine $x_1(q)$

Step 4: $X_1(m) = \mathrm{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(\,n\Delta_t,\ m\Delta_f)$

Step 6: Set $n = n+1$ and return to Step 3 until $n = c_0+C-1$.

Complexity = ?

IV-F Non-Uniform Δ_t

(A) 先用較大的 Δ_t

(B) 如果發現 $\left|X\left(n\Delta_t, m\Delta_f\right)\right|$ 和 $\left|X\left((n+1)\Delta_t, m\Delta_f\right)\right|$ 之間有很大的差異

則在 $n\Delta_t$, $(n+1)\,\Delta_t$ 之間選用較小的 sampling interval Δ_{t1}

($\Delta_\tau < \Delta_{t1} < \Delta_t$, $\Delta_t/\,\Delta_{t1}$ 和$\Delta_{t1}/\,\Delta_\tau$ 皆為整數)

再用 page 131 的方法算出

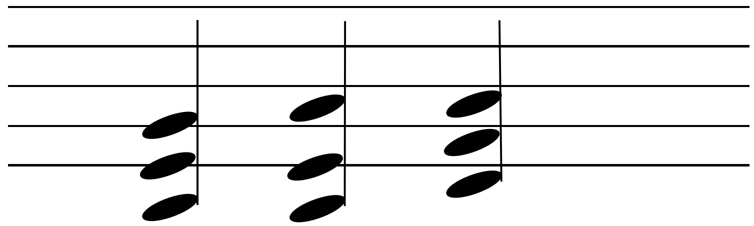
$$X\left(n\Delta_t + \Delta_{t1}, m\Delta_f\right),\quad X\left(n\Delta_t + 2\Delta_{t1}, m\Delta_f\right),\quad \cdots\cdots,\quad X\left((n+1)\Delta_t - \Delta_{t1}, m\Delta_f\right)$$

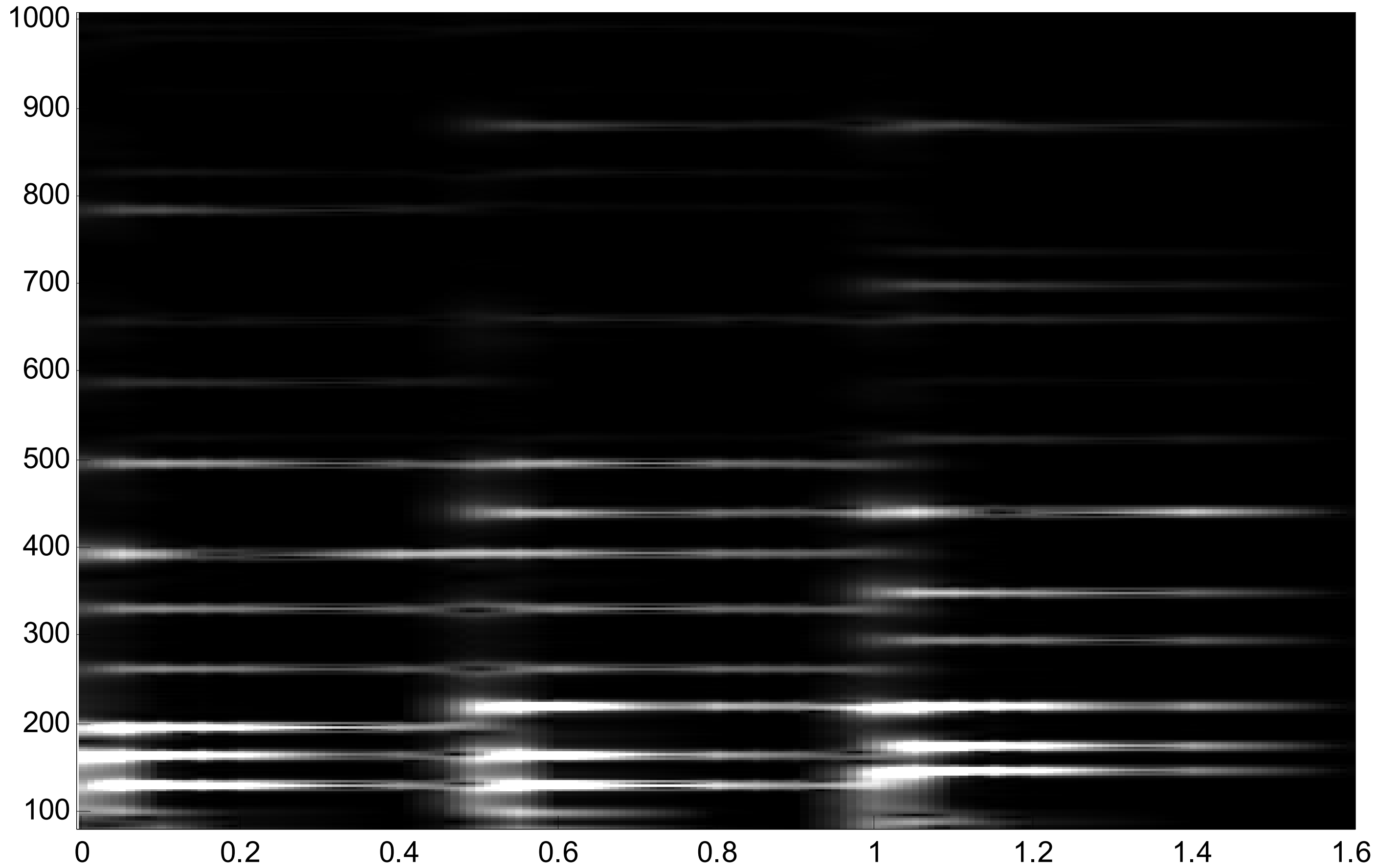
(C) 以此類推，如果 $\left|X\left(n\Delta_t + k\Delta_{t1}, m\Delta_f\right)\right|$, $\left|X\left(n\Delta_t + (k+1)\Delta_{t1}, m\Delta_f\right)\right|$

的差距還是太大，則再選用更小的 sampling interval Δ_{t2}

($\Delta\tau < \Delta t_2 < \Delta t_1$, $\Delta t_1/\,\Delta t_2$ 和$\Delta t_2/\,\Delta\tau$ 皆為整數)

Gabor transform of a music signal





$\Delta_\tau = 1/44100$ (總共有 44100×1.6077 sec $+ 1 = 70902$ 點)

(A) Choose $\Delta_t = \Delta_\tau$

running time = out of memory (2008年) 15.262140 sec (2022年)

(B) Choose $\Delta_t = 0.01 = 441\Delta_\tau$ (1.6/0.01 + 1 = 161 points)

running time = 1.0940 sec (2008年) 0.041053 sec (2022年)

(C) Choose the non-uniform sampling points on the *t*-axis as

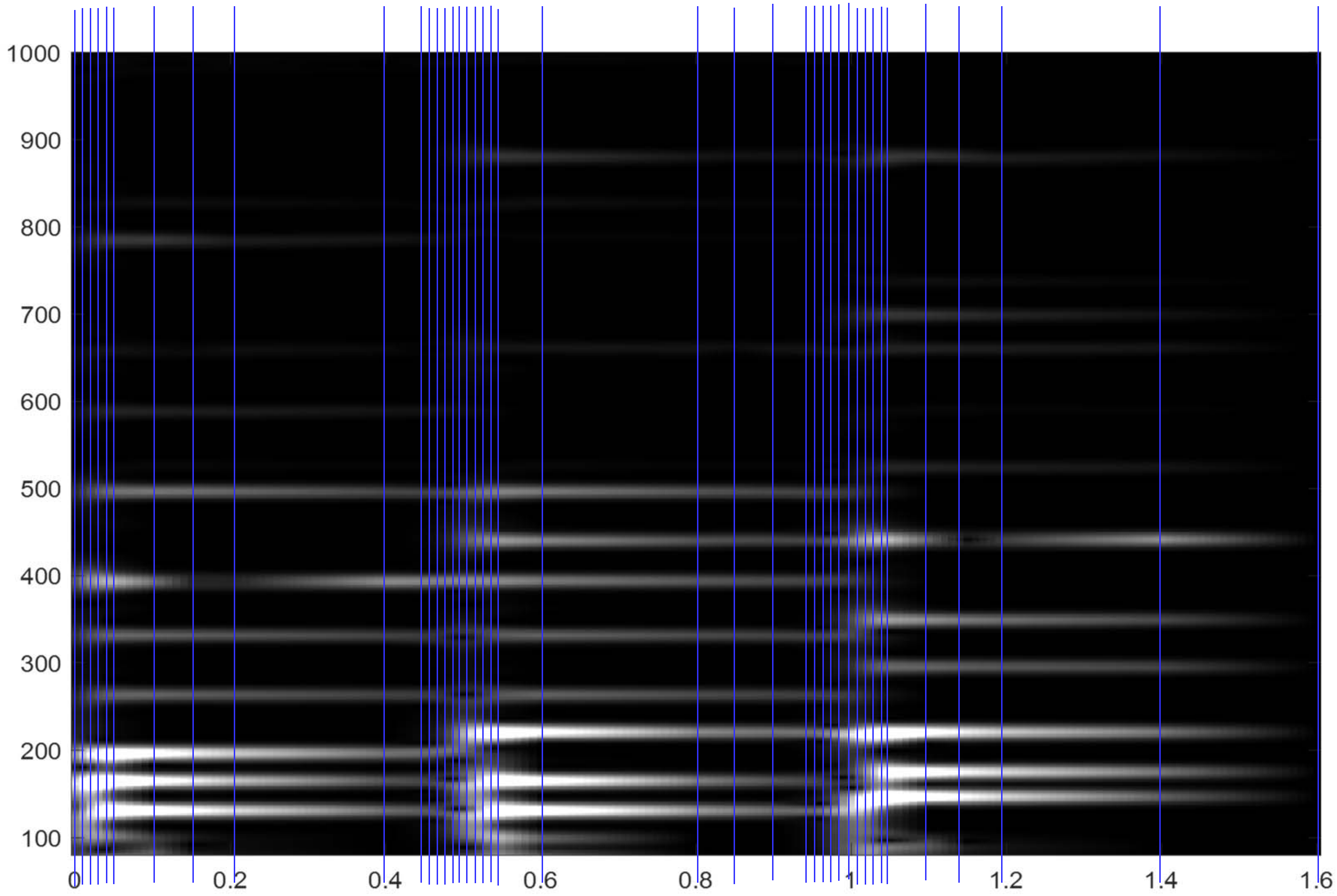
t = 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.15, 0.2, 0.4, 0.45, 046, 0.47, 0.48, 0.49, 0.5, 0.51, 0.52, 0.53, 0.54, 0.55, 0.6, 0.8, 0.85, 0.9, 0.95, 0.96, 0.97, 0.98, 0.99, 1, 1.01, 1.02, 1.03, 1.04, 1.05, 1.1, 1.15, 1.2, 1.4, 1.6

(41 points)

intervals: 0.2 → 0.05 → 0.01

running time = 0.2970 sec (2008年) 0.010594 sec (2022年)

with adaptive output sampling intervals



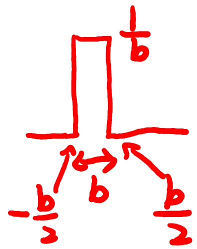
附錄七 和 Dirac Delta Function 相關的常用公式

(1) $\int_{-\infty}^{\infty} e^{-j2\pi t f} dt = \delta(f)$

(2) $\delta(t) = |a|\delta(at)$ (scaling property)

$\delta_b(t)$

$\int \delta_b(t)dt = 1$

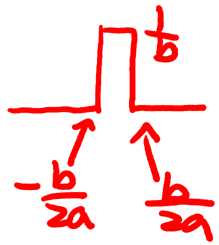


$\delta(t) = \lim_{b \to 0} \delta_b(t)$

(3) $\int_{-\infty}^{\infty} e^{-j2\pi t g(f)} dt = \delta(g(f)) = \sum_n |g'(f_n)|^{-1} \delta(f - f_n)$

where f_n are the zeros of $g(f)$

$\delta_b(at)$



(4) $\int_{-\infty}^{\infty} \delta(t - t_0) y(t, \cdots\cdots) dt = y(t_0, \cdots\cdots)$ (sifting property I)

$\int_{-\infty}^{\infty} a\delta_b(at)dt = 1$

(5) $\delta(t - t_0) y(t, \cdots\cdots) = \delta(t - t_0) y(t_0, \cdots\cdots)$ (sifting property II)