

V. Wigner Distribution Function

V-A Wigner Distribution Function (WDF)

Definition 1: $W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} d\tau$

Definition 2: $W_x(t, \omega) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j\omega\tau} d\tau$ $\omega = 2\pi f$

Another way for computation from the frequency domain

$$\text{Definition 1: } W_x(t, f) = \int_{-\infty}^{\infty} X(f + \eta/2) \cdot X^*(f - \eta/2) e^{j2\pi\eta t} d\eta$$

where $X(f)$ is the Fourier transform of $x(t)$

$$\text{Definition 2: } W_x(t, \omega) = \int_{-\infty}^{\infty} X(\omega + \eta/2) \cdot X^*(\omega - \eta/2) e^{j\eta t} d\eta$$

The **Wigner distribution function** is also called the **Wigner Ville distribution**.

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 5, Prentice Hall, N.J., 1996.

Other References

- [Ref] E. P. Wigner, “On the quantum correlation for thermodynamic equilibrium,” *Phys. Rev.*, vol. 40, pp. 749-759, 1932.
- [Ref] T. A. C. M. Classen and W. F. G. Mecklenbrauker, “The Wigner distribution—A tool for time-frequency signal analysis; Part I,” *Philips J. Res.*, vol. 35, pp. 217-250, 1980.
- [Ref] F. Hlawatsch and G. F. Boudreux–Bartels, “Linear and quadratic time-frequency signal representation,” *IEEE Signal Processing Magazine*, pp. 21-67, Apr. 1992.
- [Ref] R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley-Interscience, NJ, 2004.

The operators that are related to the WDF:

(a) Signal auto-correlation function:

$$C_x(t, \tau) = x(t + \tau/2) \cdot x^*(t - \tau/2)$$

In random process
 $C(\tau) = E(x(t)x^*(t+\tau))$

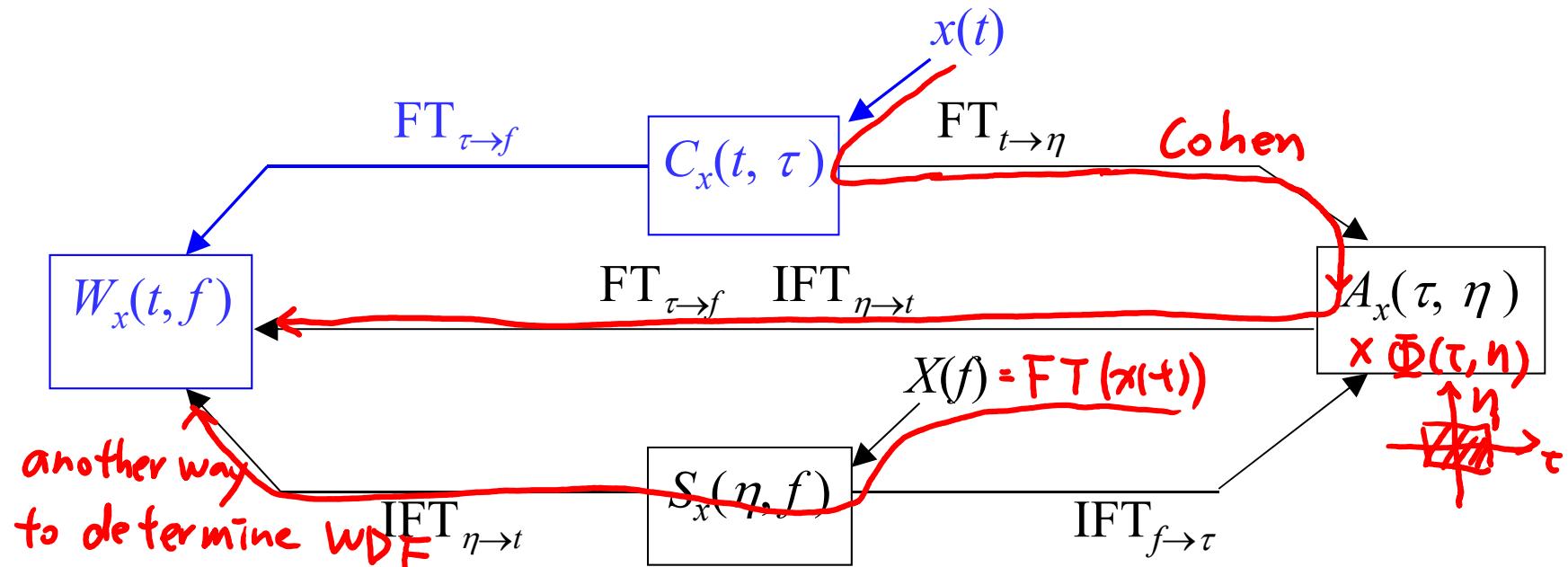
E: expected value

(b) Spectrum auto-correlation function:

$$S_x(\eta, f) = X(f + \eta/2) \cdot X^*(f - \eta/2)$$

(c) Ambiguity function (AF):

$$A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi t\eta} dt$$



V-B Why the WDF Has Higher Clarity?

Due to signal auto-correlation function

$$(1) \text{ If } x(t) = 1 \quad \xleftarrow{h=0} \quad \delta(f)$$

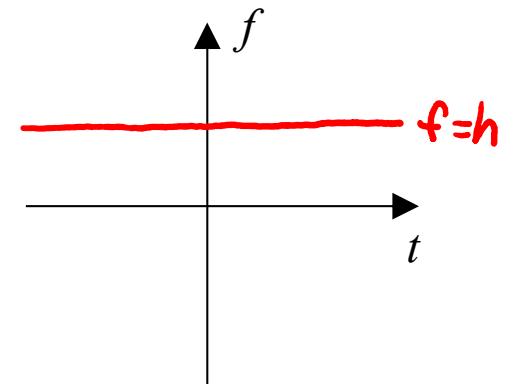
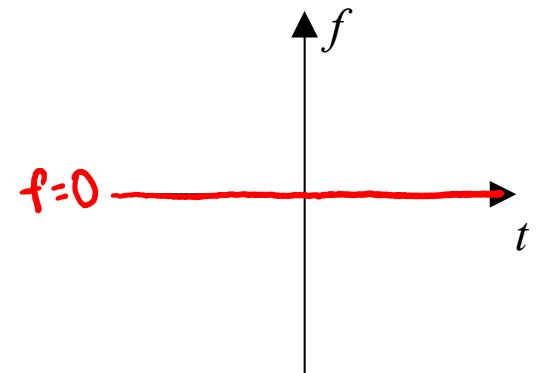
$$(2) \text{ If } x(t) = \exp(j2\pi h t) \quad \xrightarrow{x(t+\tau/2)} \quad \xleftarrow{x^*(t-\tau/2)}$$

$$W_x(t, f) = \int_{-\infty}^{\infty} e^{j2\pi h(t+\tau/2)} e^{-j2\pi h(t-\tau/2)} \cdot e^{-j2\pi \tau f} d\tau$$

$$= \int_{-\infty}^{\infty} e^{j2\pi h\tau} \cdot e^{-j2\pi \tau f} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi \tau(f-h)} d\tau \quad \text{page B8(3)}$$

$$= \delta(f - h) \quad g(f) = f - h$$



Comparing: for the case of the STFT

instantaneous freq. = $2kt$

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$$(3) \text{ If } x(t) = \exp(j2\pi k t^2)$$

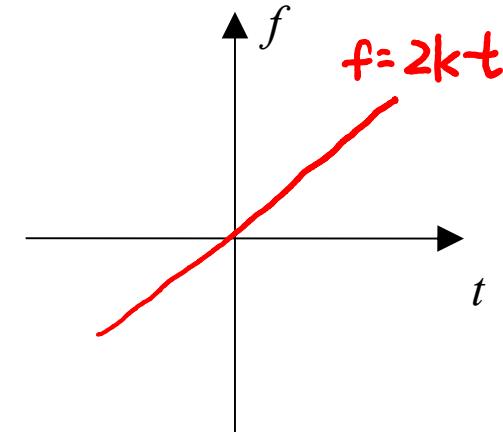
$$W_x(t, f) = \int x(t + \frac{\tau}{2}) \overline{x(t - \frac{\tau}{2})} e^{-j2\pi f \tau} d\tau$$

$$= \int \exp(j2\pi k((t + \frac{\tau}{2})^2 - (t - \frac{\tau}{2})^2)) e^{-j2\pi f \tau} d\tau$$

$$= \int \exp(j4\pi kt\tau) e^{-j2\pi f \tau} d\tau$$

$$= \int e^{-j2\pi \tau(f - 2kt)} d\tau \quad \text{page 138 (3)}$$

$$= \delta(f - 2kt)$$

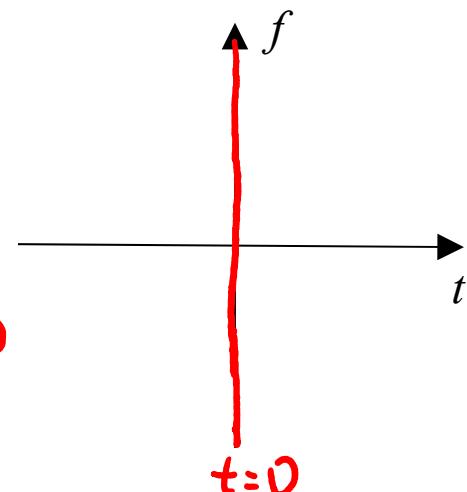


$$(4) \text{ If } x(t) = \delta(t)$$

$$W_x(t, f) = \int_{-\infty}^{\infty} \delta(t + \tau/2) \cdot \delta(t - \tau/2) e^{-j2\pi \tau f} d\tau$$

$$\begin{aligned} \text{Page 138} &= 4 \int_{-\infty}^{\infty} \delta(2t + \tau) \cdot \delta(2t - \tau) e^{-j2\pi \tau f} d\tau \\ \text{公式(2)} &\qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{y(\tau)} \\ &= 4\delta(4t) e^{j4\pi t f} = \delta(t) e^{j4\pi t f} = \delta(t) \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(\tau - \tau_0) \cdot y(\tau) d\tau &\quad \text{Page 138} & \text{Page 138} &\quad \text{Page 138} \\ = y(\tau_0) &\quad \tau_0 = -2t & \text{公式(4)} &\quad \text{公式(2)} & \text{公式(5), } t_0 = 0 \end{aligned}$$



V-C The WDF is not a Linear Distribution

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

If $h(t) = \alpha g(t) + \beta s(t)$

$$\begin{aligned}
 W_h(t, f) &= \int_{-\infty}^{\infty} h(t + \tau/2) \cdot h^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} [\alpha g(t + \tau/2) + \beta s(t + \tau/2)] [\alpha^* g^*(t - \tau/2) + \beta^* s^*(t - \tau/2)] e^{-j2\pi\tau f} d\tau \\
 &= \int_{-\infty}^{\infty} [|\alpha|^2 g(t + \tau/2)g^*(t - \tau/2) + |\beta|^2 s(t + \tau/2)s^*(t - \tau/2) \\
 &\quad + \alpha\beta^* g(t + \tau/2)s^*(t - \tau/2) + \alpha^*\beta g^*(t - \tau/2)s(t + \tau/2)] e^{-j2\pi\tau f} d\tau \\
 &= |\alpha|^2 W_g(t, f) + |\beta|^2 W_s(t, f) \\
 &\quad + \int_{-\infty}^{\infty} [\alpha\beta^* g(t + \tau/2)s^*(t - \tau/2) + \alpha^*\beta g^*(t - \tau/2)s(t + \tau/2)] e^{-j2\pi\tau f} d\tau
 \end{aligned}$$

auto terms
 cross terms

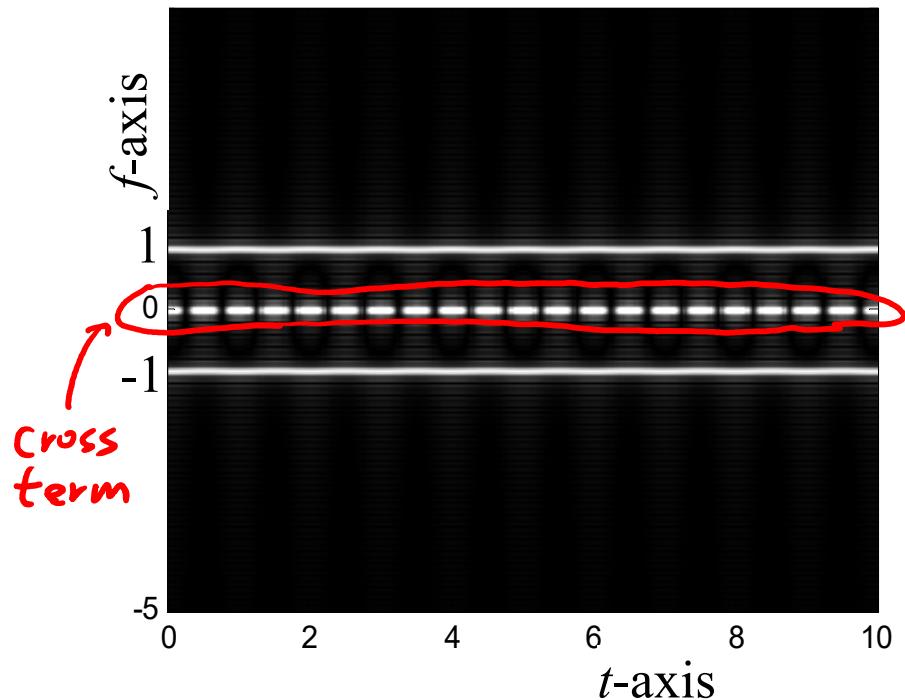
V-D Examples of the WDF

Simulations

$$x(t) = \cos(2\pi t) = 0.5[\exp(j2\pi t) + \exp(-j2\pi t)]$$

$f = \pm 1$
windowed

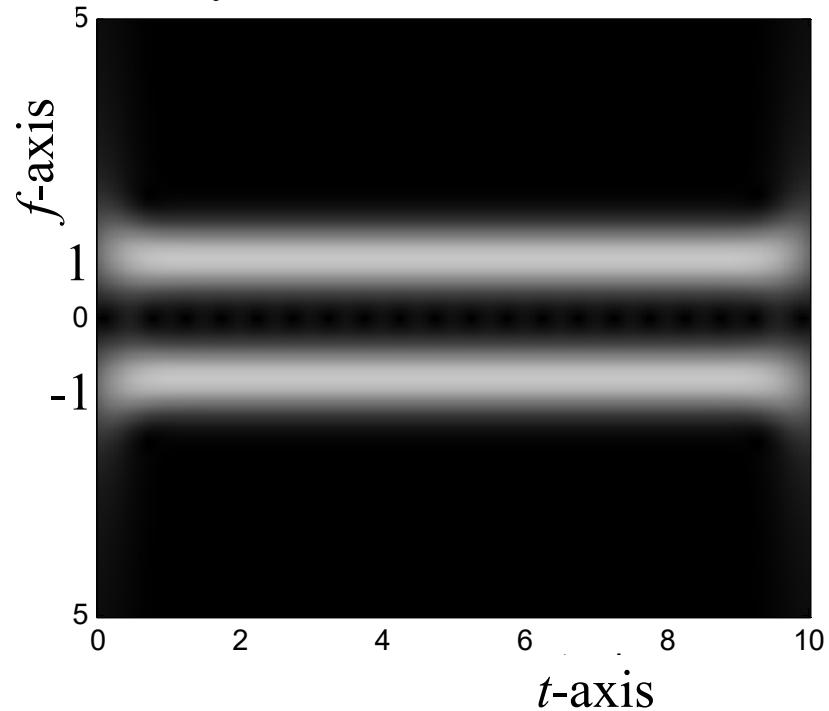
by the WDF



$$f_1 = 1, f_2 = -1$$

$$f_d = 2 \\ t_d = 0$$

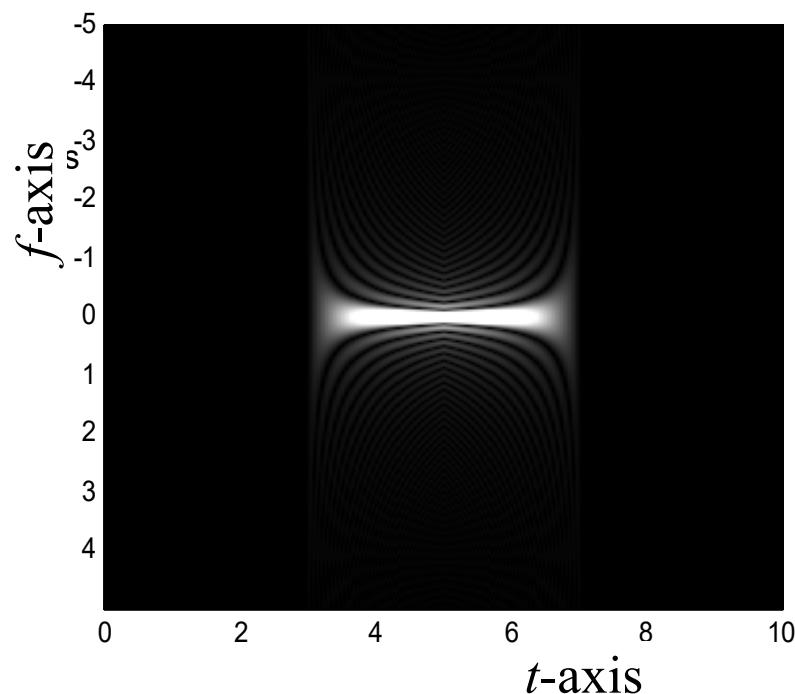
by the Gabor transform



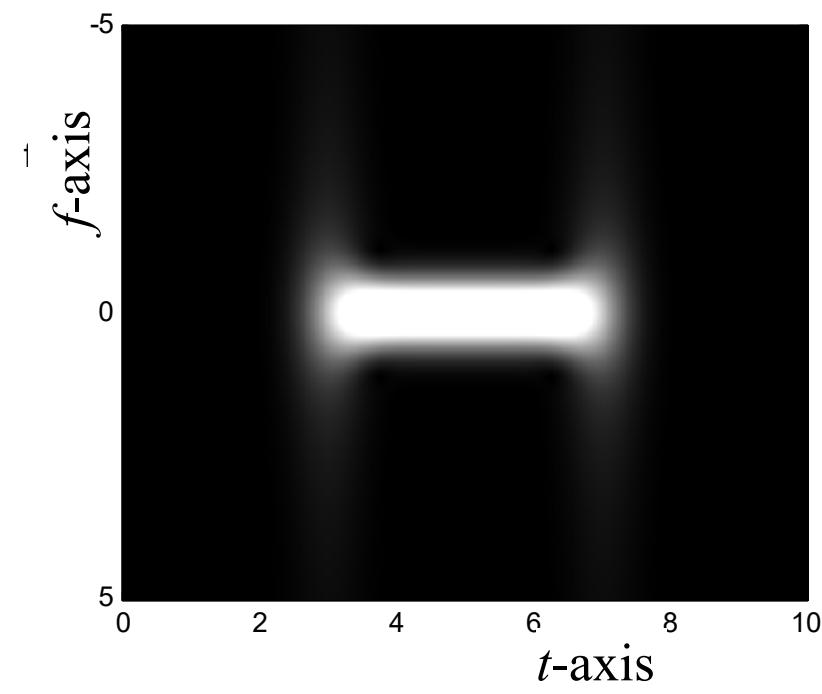
$$x(t) = \Pi((t-5)/4)$$

Π : rectangular function

by the WDF

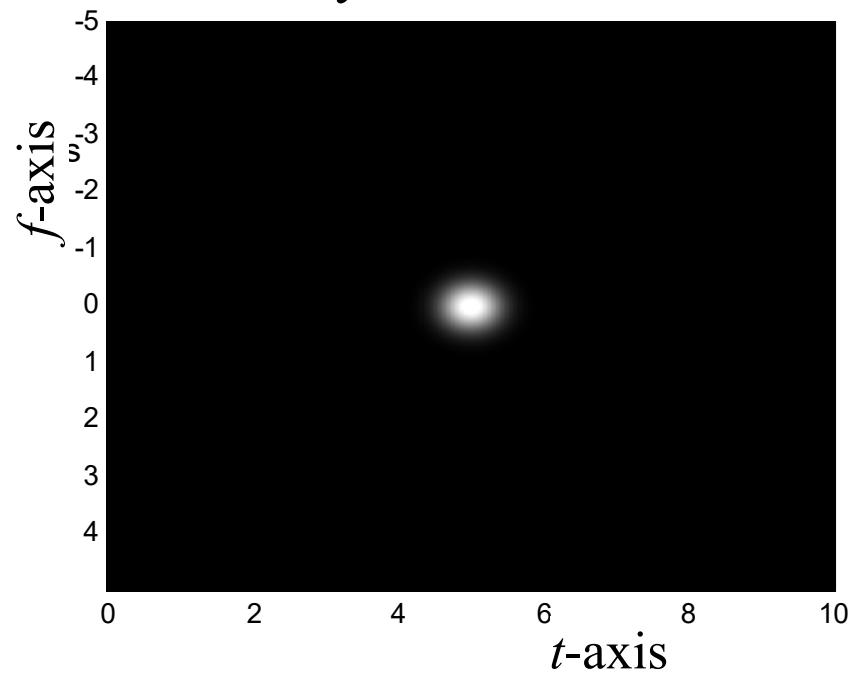


by the Gabor transform

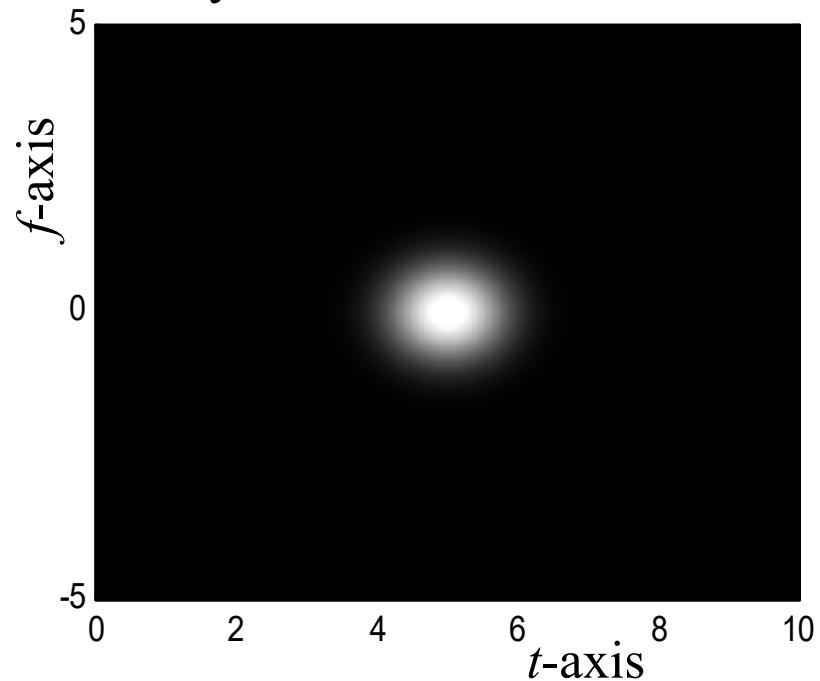


$$x(t) = \exp[-\pi(t-5)^2]$$

by the WDF



by the Gabor transform



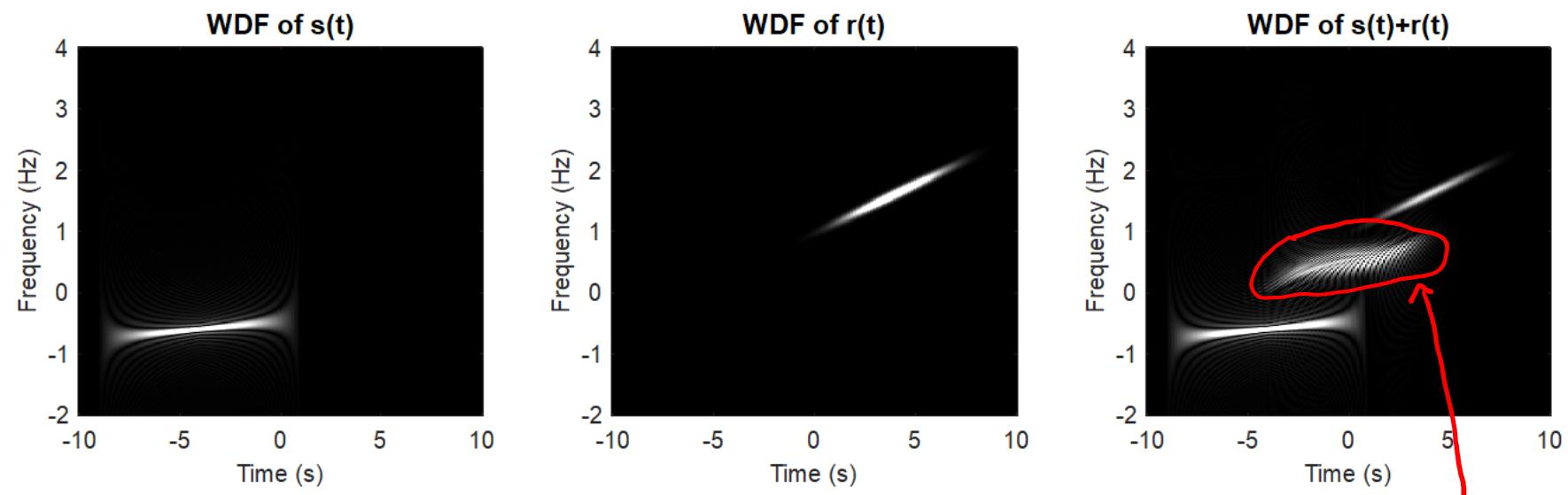
Gaussian function: $e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

Gaussian function's T-F area is minimal.

$$s(t) = \exp\left(jt^2/10 - j3t\right) \quad \text{for } -9 \leq t \leq 1, s(t) = 0 \text{ otherwise,}$$

$$r(t) = \exp\left(jt^2/2 + j6t\right) \exp\left[-(t-4)^2/10\right]$$

$$f(t) = s(t) + r(t)$$



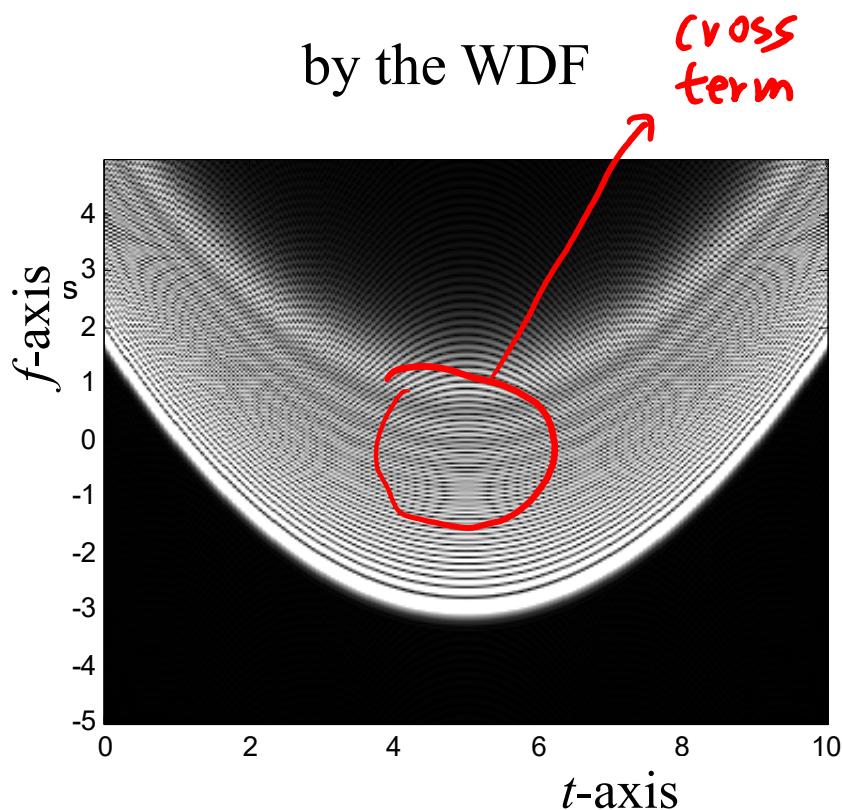
橫軸: t -axis, 縱軸: f -axis

Cross
term

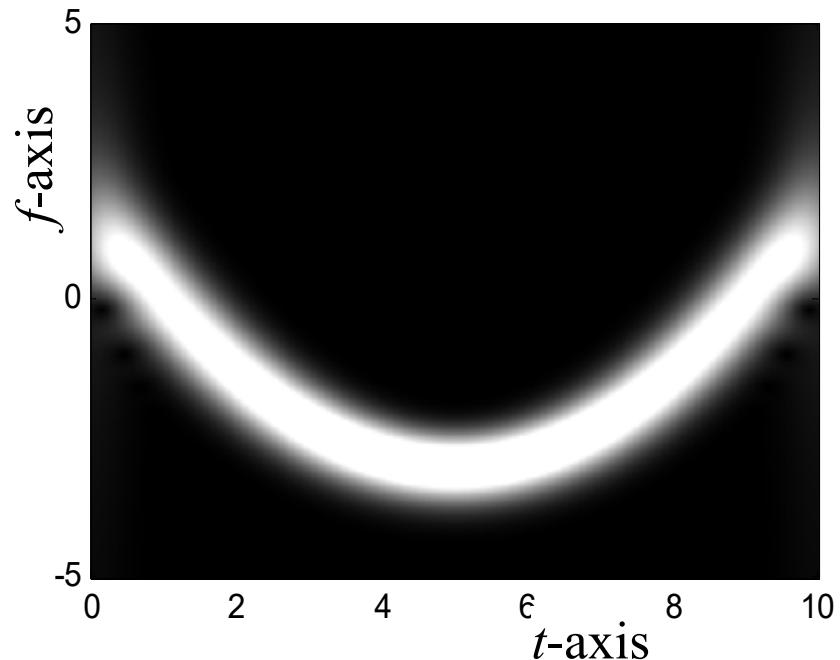
$$x(t) = \exp(j(t-5)^3 - j6\pi t)$$

instantaneous
freq. = ?

by the WDF



by the Gabor transform



If $x(t) = \exp(j\phi(t))$ and $\phi(t)$ is a polynomial with order ≥ 3 , the cross term problem appears.

V-E Digital Implementation of the WDF

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau ,$$

$$W_x(t, f) = 2 \int_{-\infty}^{\infty} x(t + \tau') \cdot x^*(t - \tau') e^{-j4\pi\tau' f} \cdot d\tau' \quad (\text{using } \tau' = \tau/2)$$

unbalanced $t = n\Delta_t$

$$\tau' = p\Delta_t$$

$$\begin{aligned} d\tau' &= d\tau/2 \\ d\tau &= 2d\tau' \end{aligned}$$

Sampling: $t = n\Delta_t$, $f = m\Delta_f$, $\tau' = p\Delta_t$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t \Delta_f} \Delta_t$$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t \Delta_f) \Delta_t$$

When $x(t)$ is not a time-limited signal, it is hard to implement.

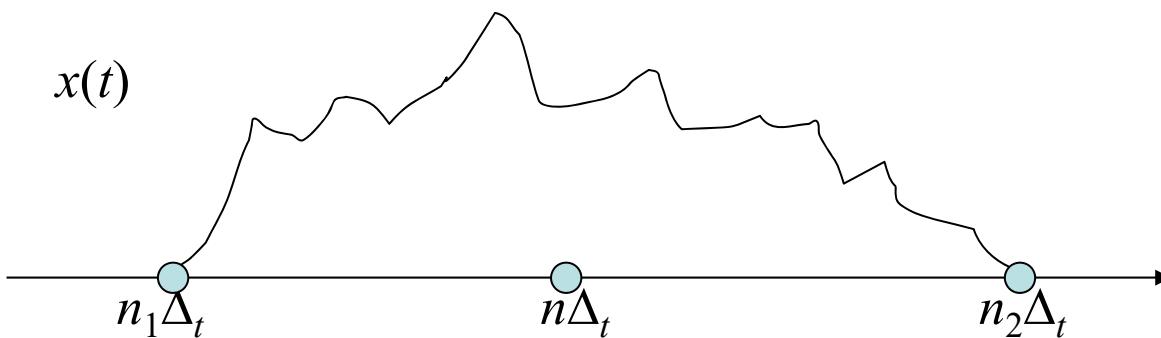
If bandwidth of $x(\tau)$ is Ω_x

$x(\tau/2)$ is $\Omega_x/2$

$x(t + \tau/2)$, $x^*(t - \tau/2)$ $\Omega_x/2$

The bandwidth of $x(t + \tau/2)x^*(t - \tau/2)$ is Ω_x

Suppose that $x(t) = 0$ for $t < n_1 \Delta_t$ and $t > n_2 \Delta_t$



$$x((n+p)\Delta_t)x^*((n-p)\Delta_t) = 0 \quad \text{if } n+p \notin [n_1, n_2] \\ \text{or } n-p \notin [n_1, n_2]$$

• p 的範圍的問題 (當 n 固定時)

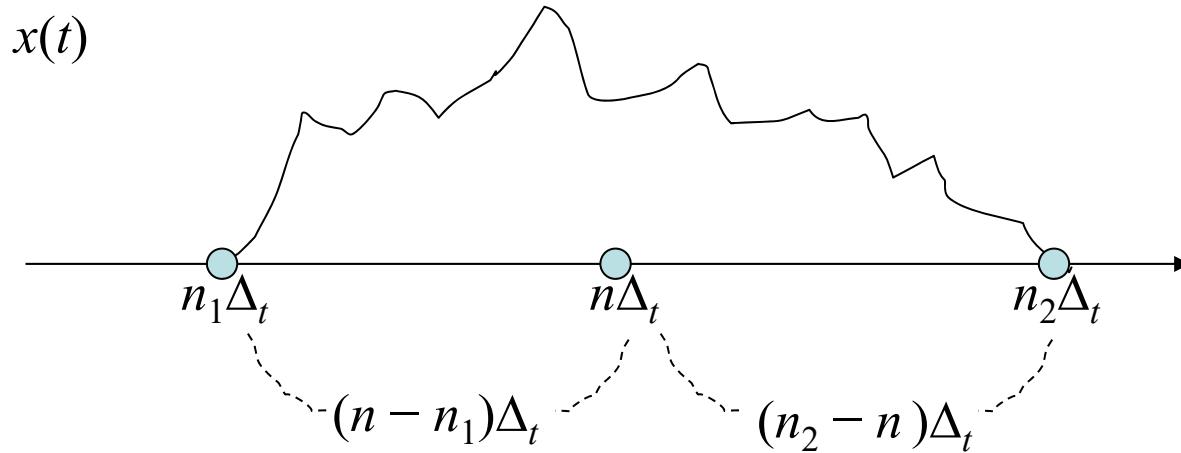
$$\underline{n_1 \leq n + p \leq n_2} \longrightarrow n_1 - n \leq p \leq n_2 - n$$

$$\underline{n_1 \leq n - p \leq n_2} \longrightarrow n_1 - n \leq -p \leq n_2 - n, \quad n - n_2 \leq p \leq n - n_1$$

$n - n_1 \geq p \geq n - n_2$

$$\max(n_1 - n, n - n_2) \leq p \leq \min(n_2 - n, n - n_1) \quad \max(-3, -5)$$

$$-\min(n_2 - n, n - n_1) \leq p \leq \min(n_2 - n, n - n_1) \quad = -\min(3, 5)$$



$$-\min(n_2 - n, n - n_1) \leq p \leq \min(n_2 - n, n - n_1)$$

$\neg Q$ Q

$(n_2 - n)\Delta_t$, $(n - n_1)\Delta_t$: 離兩個邊界的距離

注意：當 $n > n_2$ 或 $n < n_1$ 時，

將沒有 p 能滿足上面的不等式

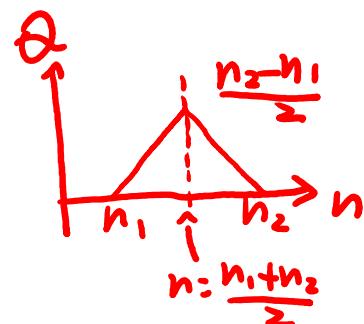
$$W_x(n\Delta_t, m\Delta_f) = 0 \quad \text{for } h < n, \quad 154$$

or $n > n_2$
(since in this case $Q < 0$)

If $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t$$

$\chi((ns+p)\Delta_t) \chi^*((ns-p)\Delta_t)$



$$Q = \min(n_2 - n, n - n_1). \quad (\text{varies with } n)$$

$$p \in [-Q, Q], \quad n \in [n_1, n_2],$$

possible for implementation

Method 1: Direct Implementation (brute force method)

唯一的限制條件？

$$T = n_2 - n_1 + 1$$

$$\text{complexity} = \sum_{n=n_1}^{n_2} (Q(n) + 1) F \quad \Theta(F T^2)$$

$$\approx 2F \sum_{n=n_1}^{n_2} Q(n) = 2F \frac{(n_2 - n_1)}{2} \frac{(n_2 - n_1)}{2} = \frac{F(n_2 - n_1)^2}{2} = \frac{F}{2} T^2$$

Method 2: Using the DFT

When $\Delta_t \Delta_f = \frac{1}{2N}$ and $N \geq 2\text{Max}(Q)+1 = 2(n_2-n_1)/2+1 = n_2-n_1+1 = T$

$$W_x(n\Delta_t, m\Delta_f) = 2\Delta_t \sum_{p=-Q}^Q x((n+p)\Delta_t)x^*((n-p)\Delta_t)e^{-j\frac{2\pi mp}{N}}$$

T點 F點

$$q = p+Q \rightarrow p = q - Q$$

3 大限制條件

$$(iii) \Delta t \leq \frac{1}{2\Omega}$$

Ω = bandwidth of
the auto correlation
function

$$\Delta t < \frac{1}{2\Omega_x}$$

$$W_x(n\Delta_t, m\Delta_f) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{2Q} x((n+q-Q)\Delta_t)x^*((n-q+Q)\Delta_t)e^{-j\frac{2\pi mq}{N}}$$

$$W_x(n\Delta_t, m\Delta_f) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{N-1} c_1(q) e^{-j\frac{2\pi mq}{N}}$$

$$Q = \min(n_2-n, n-n_1). \\ n \in [n_1, n_2],$$

$$c_1(q) = x((n+q-Q)\Delta_t)x^*((n-q+Q)\Delta_t) \quad \text{for } 0 \leq q \leq 2Q$$

$$\text{i.e., } c_1(Q+k) = x((n+k)\Delta_t)x^*((n-k)\Delta_t) \quad \text{for } -Q \leq k \leq Q \quad (k = q-Q)$$

$$c_1(q) = 0 \quad \text{for } 2Q+1 \leq q \leq N-1$$

—

假設 $t = n_0\Delta_t, (n_0+1)\Delta_t, (n_0+2)\Delta_t, \dots, n_1\Delta_t$

$f = m_0\Delta_f, (m_0+1)\Delta_f, (m_0+2)\Delta_f, \dots, m_1\Delta_f$

Step 1: Calculate n_0, n_1, m_0, m_1, N

Step 2: $n = n_0$

Step 3: Determine Q (Q varies with n)

Step 4: Determine $c_1(q)$

Step 5: $C_1(m) = \text{FFT}[c_1(q)]$

Step 6: Convert $C_1(m)$ into $C(n\Delta_t, m\Delta_f)$

$$C(n\Delta_t, m\Delta_f) = C_1(\text{mod}(m, N)) 2\Delta_t e^{\frac{j2\pi m Q}{N}}$$

Step 7: Set $n = n+1$ and return to Step 3 until $n = n_1$.

Method 3: Using the Chirp Z Transform

$$\begin{aligned}
 W_x(n\Delta_t, m\Delta_f) &= 2 \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) \exp(-j4\pi mp\Delta_t\Delta_f) \Delta_t \\
 &\quad \text{TRANSFORM} \\
 W_x(n\Delta_t, m\Delta_f) &= 2\Delta_t e^{-j2\pi m^2 \Delta_t \Delta_f} \sum_{p=-Q}^Q x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j2\pi p^2 \Delta_t \Delta_f} e^{j2\pi(p-m)^2 \Delta_t \Delta_f}
 \end{aligned}$$

Step 1 $x_1(n, p) = x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j2\pi p^2 \Delta_t \Delta_f}$

Step 2 $X_2[n, m] = \sum_{p=-Q}^Q x_1[n, p] c[m-p] \quad c[m] = e^{j2\pi m^2 \Delta_t \Delta_f}$

Step 3 $X(n\Delta_t, m\Delta_f) = 2\Delta_t e^{-j2\pi m^2 \Delta_t \Delta_f} X_2[n, m]$

思考：Method 1 的複雜度為多少

(direct) $\Theta(TF^2)$

思考：Method 2 的複雜度為多少

(FFT) $\Theta(TN \log N)$

思考：Method 3 的複雜度為多少

(chirp Z transform) $\Theta(?)$
?

The computation time of the WDF is more than those of the rec-STFT and the Gabor transform.

V-F Properties of the WDF

(1) Projection property	$ x(t) ^2 = \int_{-\infty}^{\infty} W_x(t, f) df \quad X(f) ^2 = \int_{-\infty}^{\infty} W_x(t, f) dt$
(2) Energy preservation property	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f) dt df = \int_{-\infty}^{\infty} x(t) ^2 dt = \int_{-\infty}^{\infty} X(f) ^2 df$
(3) Recovery property	$\int_{-\infty}^{\infty} W_x(t/2, f) e^{j2\pi f t} df = x(t) \cdot x^*(0)$ $x^*(0)$ 已知 $\int_{-\infty}^{\infty} W_x(t, f/2) e^{-j2\pi f t} dt = X(f) \cdot X^*(0)$
(4) Mean condition frequency and mean condition time	If $x(t) = x(t) \cdot e^{j2\pi \phi(t)}$, $X(f) = X(f) \cdot e^{j2\pi \Psi(f)}$ then $\phi'(t) = x(t) ^{-2} \cdot \int_{-\infty}^{\infty} f \cdot W_x(t, f) \cdot df$ $-\Psi'(f) = X(f) ^{-2} \int_{-\infty}^{\infty} t \cdot W_x(t, f) \cdot dt$
(5) Moment properties	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^n W_x(t, f) dt df = \int_{-\infty}^{\infty} t^n x(t) ^2 dt$, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^n W_x(t, f) dt df = \int_{-\infty}^{\infty} f^n X(f) ^2 df$

(6) $W_x(t, f)$ is bound to be real	$\overline{W_x(t, f)} = W_x(t, f)$
(7) Region properties	If $x(t) = 0$ for $t > t_2$ then $W_x(t, f) = 0$ for $t > t_2$ If $x(t) = 0$ for $t < t_1$ then $W_x(t, f) = 0$ for $t < t_1$
(8) Multiplication theory	If $y(t) = x(t)h(t)$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(t, \rho)W_h(t, f - \rho) \cdot d\rho$
(9) Convolution theory	If $y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau) d\tau$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(t - \rho, f) \cdot d\rho$
(10) Correlation theory	If $y(t) = \int_{-\infty}^{\infty} x(t + \tau)h^*(\tau) d\tau$, then $W_y(t, f) = \int_{-\infty}^{\infty} W_x(\rho, f) \cdot W_h(-t + \rho, f) \cdot d\rho$

(11) Time-shifting property	If $y(t) = x(t - t_0)$, then $W_y(t, f) = W_x(t - t_0, f)$
(12) Modulation property	If $y(t) = \exp(j2\pi f_0 t)x(t)$, then $W_y(t, f) = W_x(t, f - f_0)$
(13) Constant multiplication property	If $y(t) = cx(t)$, then $W_y(t, f) = c ^2 W_x(t, f)$
(14) Conjugation property	If $y(t) = x^*(t)$, then $W_y(t, f) = W_x(t, -f)$
(15) Scaling property	If $y(t) = x(ct)$, then $W_y(t, f) = \frac{1}{ c } W_x(ct, \frac{1}{c}f)$

The STFT (including the rec-STFT, the Gabor transform) does not have real region, multiplication, convolution, and correlation properties.

- Why the WDF is always real?

What are the advantages and disadvantages it causes?

related property : If $x(t) = x^*(-t)$, $X(f)$ is real

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (\text{FT}(x(t)))$$

$$X^*(f) = \int_{-\infty}^{\infty} x^*(t) e^{j2\pi f t} dt = - \int_{\infty}^{-\infty} x^*(-t) e^{j2\pi f t} dt, \quad t_1 = -t, \quad dt_1 = -dt$$

$$= \int_{-\infty}^{\infty} x(t_1) e^{-j2\pi f t_1} dt_1, \quad = X(f)$$

- Try to prove of the projection and recovery properties

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \frac{\tau}{2})^* x(t - \frac{\tau}{2}) e^{-j2\pi f \tau} d\tau$$

$$\begin{aligned} \int_{-\infty}^{\infty} W_x(t, f) df &= \int_{-\infty}^{\infty} x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) \left(\int_{-\infty}^{\infty} e^{-j2\pi f \tau} df \right) d\tau \\ &= \int_{-\infty}^{\infty} x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) \delta(\tau) d\tau \quad \text{page 138(1)} \\ &= x(t+0) x^*(t-0) = |x(t)|^2 \quad \text{page 138(4), } t_0=0 \end{aligned}$$

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

- Proof of the region properties

If $x(t) = 0$ for $t < t_0$,

$$x(t + \tau/2) = 0 \quad \text{for } \tau < (t_0 - t)/2 = -(t - t_0)/2,$$

$$x(t - \tau/2) = 0 \quad \text{for } \tau > (t - t_0)/2,$$

Therefore, if $t - t_0 < 0$, the nonzero regions of $x(t + \tau/2)$ and $x(t - \tau/2)$ does not overlap and $x(t + \tau/2) x^*(t - \tau/2) = 0$ for all τ .

The importance of the region property

- Extra Property:

(16) The relation between the WDF and the spectrogram:

Suppose that $x(t)$ is the input function, $w(t)$ is the window function of the STFT, $X(t, f)$ is the STFT of $x(t)$, and $W_x(t, f)$ and $W_w(t, f)$ are the WDFs of $x(t)$ and $w(t)$, respectively, then

$$\left|X(t, f)\right|^2 = \underbrace{W_x(t, f)}_{\text{spectrogram}} * \underbrace{W_w(t, f)}_{\text{spectrogram}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t-u, f-v) W_w(u, v) du dv$$

$$\begin{aligned}
 (\text{Proof}): \quad & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t-u, f-v) W_w(u, v) du dv \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t-u+\tau/2) x^*(t-u-\tau/2) e^{-j2\pi(f-v)\tau} d\tau \\
 &\quad \int_{-\infty}^{\infty} w\left(u + \frac{\eta}{2}\right) w\left(u - \frac{\eta}{2}\right) e^{-j2\pi v \eta} d\eta du dv \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(t-u+\frac{\tau}{2}\right) x^*\left(t-u-\frac{\tau}{2}\right) w\left(u + \frac{\eta}{2}\right) w\left(u - \frac{\eta}{2}\right) e^{-j2\pi f \tau} \\
 &\quad \int_{-\infty}^{\infty} e^{-j2\pi v(\eta-\tau)} dv d\tau d\eta du \tag{Cont.}
 \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(t-u+\frac{\tau}{2}\right) x^{*}\left(t-u-\frac{\tau}{2}\right) w\left(u+\frac{\eta}{2}\right) w\left(u-\frac{\eta}{2}\right) e^{-j 2 \pi f \tau} \\
&\quad \delta(\eta-\tau) d \tau d \eta d u \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(t-u+\frac{\tau}{2}\right) x^{*}\left(t-u-\frac{\tau}{2}\right) w\left(u+\frac{\tau}{2}\right) w\left(u-\frac{\tau}{2}\right) e^{-j 2 \pi f \tau} d \tau d u
\end{aligned}$$

$$\text{Set } \tau_1 = t - u + \frac{\tau}{2}, \quad \tau_2 = t - u - \frac{\tau}{2}$$

$$d\tau_1 d\tau_2 = \left| \det \begin{bmatrix} \partial \tau_1 / \partial \tau & \partial \tau_1 / \partial u \\ \partial \tau_2 / \partial \tau & \partial \tau_2 / \partial u \end{bmatrix} \right| d\tau du = d\tau du$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau_1) x^{*}(\tau_2) w(t-\tau_2) w(t-\tau_1) e^{-j 2 \pi f(\tau_1-\tau_2)} d \tau_1 d \tau_2 \\
&= \int_{-\infty}^{\infty} x(\tau_1) w(t-\tau_1) e^{-j 2 \pi f \tau_1} d \tau_1 \int_{-\infty}^{\infty} x^{*}(\tau_2) w(t-\tau_2) e^{j 2 \pi f \tau_2} d \tau_2 \\
&= X(t, f) X^{*}(t, f) \\
&= |X(t, f)|^2
\end{aligned}$$

V-G Advantages and Disadvantages of the WDF

Advantages: clarity

many good properties

suitable for analyzing the random process

Disadvantages: cross-term problem

not suitable for $\exp(jt^n)$, $n \neq 0, 1, 2$

more time for computation, especial for the signal with long time duration

not one-to-one

V-H Windowed Wigner Distribution Function

When $x(t)$ is not time-limited, its WDF is hard for implementation

$$W_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

↓ with mask

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$$\tau' = \tau/2 \quad \tau = 2\tau'$$

$$d\tau = 2d\tau'$$

$w(\tau)$ is real and time-limited

The windowed WDF is also called the pseudo Wigner-Ville distribution.

Advantages: (1) reduce the computation time

(2) may reduce the cross term problem

Disadvantages: (1) a little poor resolution

(2) lost some mathematical property

$$W_x(t, f) = 2 \int_{-\infty}^{\infty} w(2\tau') x(t + \tau') \cdot x^*(t - \tau') e^{-j4\pi\tau' f} \cdot d\tau'$$

$t = h\Delta_t, \quad \tau' = p\Delta_t, \quad f = m\Delta_f$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-\infty}^{\infty} w(2p\Delta_t) x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t \Delta_f} \Delta_t$$

Suppose that $w(t) = 0$ for $|t| > B$

$w(2p\Delta_t) = 0 \quad \text{for } |p| > Q \text{ and } |p| < -Q$

$2|p|\Delta_t > B \quad Q = \frac{B}{2\Delta_t}$
 $|p| > B/2\Delta_t$

$$W_x(n\Delta_t, m\Delta_f) = 2 \sum_{p=-Q}^{Q} w(2p\Delta_t) x((n+p)\Delta_t) x^*((n-p)\Delta_t) e^{-j4\pi mp\Delta_t \Delta_f} \Delta_t$$

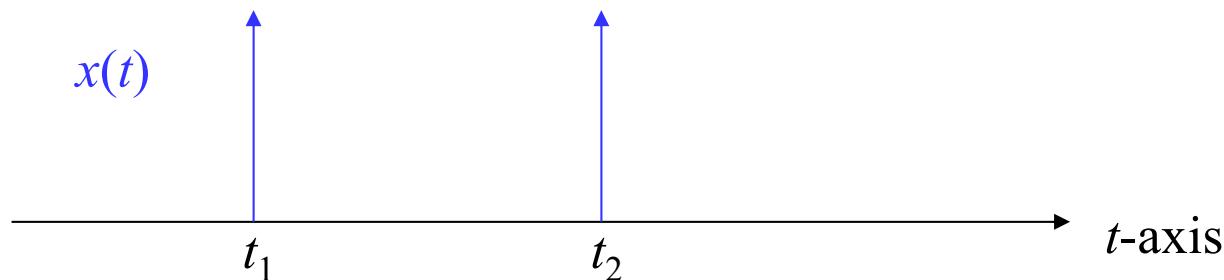
當然，乘上 mask 之後，有一些數學性質將會消失

(B) Why the cross term problem can be avoided ?

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$ is real

Viewing the case where $x(t) = \delta(t - t_1) + \delta(t - t_2)$



理想情形： $W_x(t, f) = 0$ for $t \neq t_1, t_2$

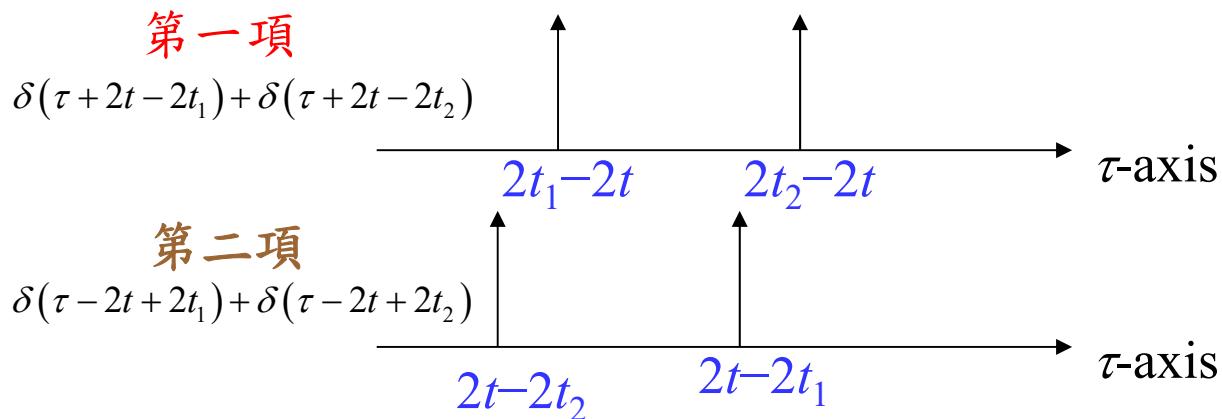
然而，當 mask function $w(\tau) = 1$ 時 (也就是沒有使用 mask function)

$$\underline{x(t)} = \underline{\delta(t - t_1) + \delta(t - t_2)}$$

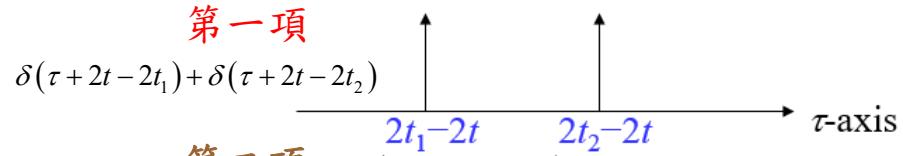
$t_1 < t_2$

$$\begin{aligned} W_x(t, f) &= \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi\tau f} \cdot d\tau \\ &= \int_{-\infty}^{\infty} \left[\delta\left(t + \frac{\tau}{2} - t_1\right) + \delta\left(t + \frac{\tau}{2} - t_2\right) \right] \left[\delta\left(t - \frac{\tau}{2} - t_1\right) + \delta\left(t - \frac{\tau}{2} - t_2\right) \right] e^{-j2\pi\tau f} \cdot d\tau \\ &= 4 \int_{-\infty}^{\infty} \underbrace{[\delta(\tau + 2t - 2t_1) + \delta(\tau + 2t - 2t_2)]}_{\text{第一項}} \underbrace{[\delta(\tau - 2t + 2t_1) + \delta(\tau - 2t + 2t_2)]}_{\text{第二項}} e^{-j2\pi\tau f} \cdot d\tau \end{aligned}$$

from page 138, property 2



3種情形 $W_x(t, f) \neq 0$

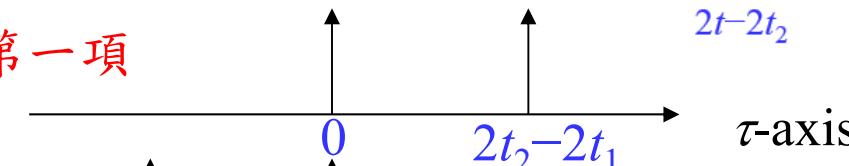


(1) If $t = t_1$

$$\begin{aligned} 2t_1 - 2t &= 2t - 2t_1 \\ 4t &= 4t_1 \\ t &= t_1 \end{aligned}$$

auto term

第一項

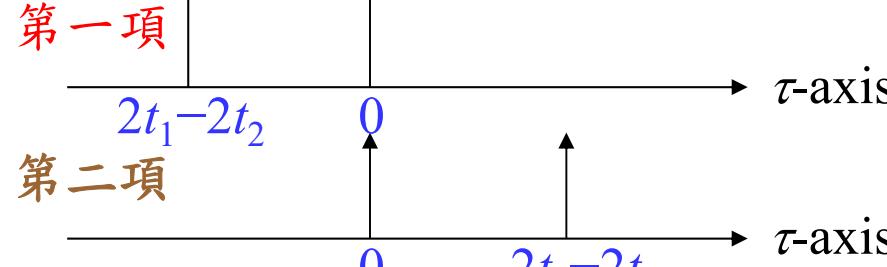


(2) If $t = t_2$

$$\begin{aligned} 2t_2 - 2t &= 2t - 2t_2 \\ 4t &= 4t_2 \\ t &= t_2 \end{aligned}$$

auto term

第一項



(3) If $t = (t_1 + t_2)/2$

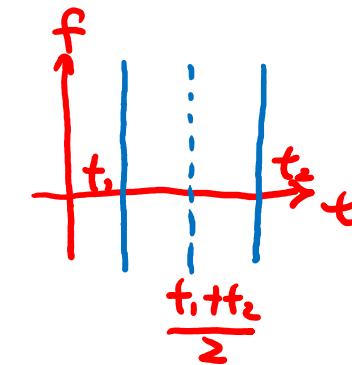
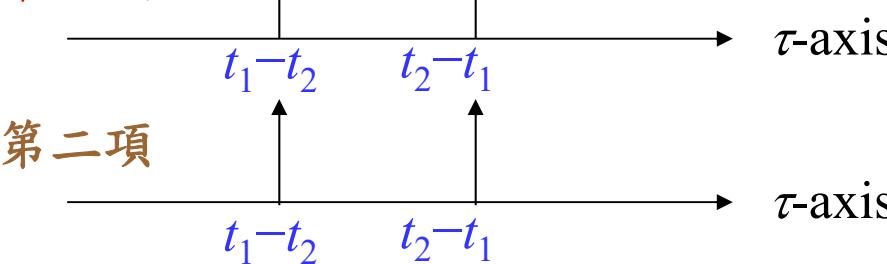
$$2t_1 - 2t = 2t - 2t_2$$

$$4t = 2t_1 + t_2$$

$$t = (t_1 + t_2)/2$$

cross term

第一項

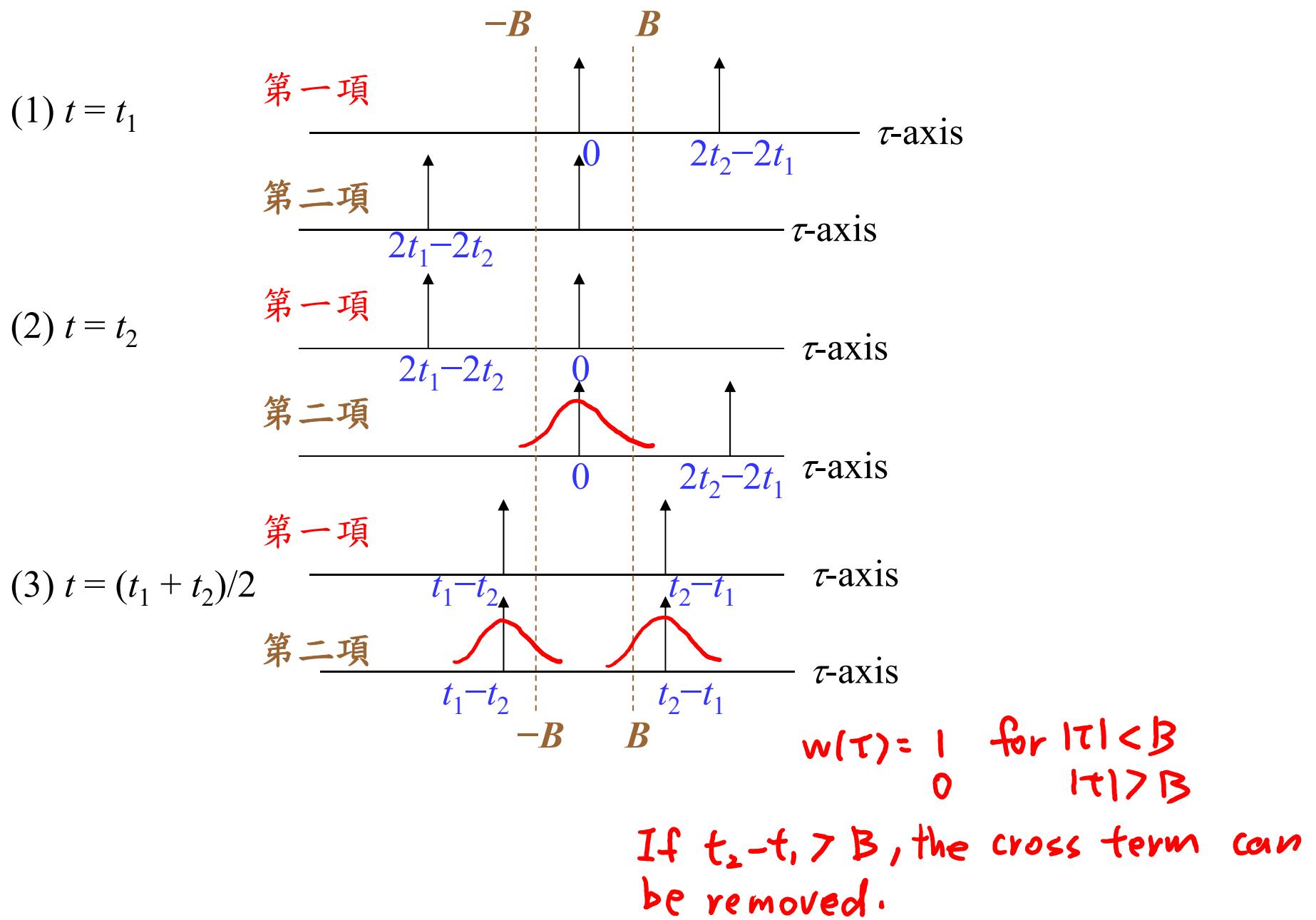


With mask function

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} w(\tau) x(t + \tau/2) x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} w(\tau) [\delta(\tau + 2t - 2t_1) + \delta(\tau + 2t - 2t_2)] \\
 &\quad \times [\delta(\tau - 2t + 2t_1) + \delta(\tau - 2t + 2t_2)] e^{-j2\pi\tau f} \cdot d\tau
 \end{aligned}$$

Suppose that $w(\tau) = 0$ for $|\tau| > B$, B is positive.

If $B < t_2 - t_1$



附錄八：研究所學習新知識把握的要點

- (1) Concepts: 這個方法的核心概念、基本精神是什麼
- (2) Comparison: 這方法和其他方法之間，有什麼相同的地方？
有什麼相異的地方
- (3) Advantages: 這方法的優點是什麼
(3-1) Why? 造成這些優點的原因是什麼
- (4) Disadvantages: 這方法的缺點是什麼
(4-1) Why? 造成這些缺點的原因是什麼
- (5) Applications: 這個方法要用來處理什麼問題，有什麼應用
- (6) Innovations: 這方法有什麼可以改進的地方
或是可以推廣到什麼地方

看過一篇論文或一個章節之後，若能夠回答(1)-(5)的問題，就代表你已經學通了這個方法

如果你的目標是發明創造出新的方法，可試著回答(3-1), (4-1), 和(6)的問題

每個領域每個月至少都有100篇以上的新論文，閱讀能力再厲害的人，也不可能都像大學讀書一樣篇篇都逐字讀過，所以，要選讀哪幾篇論文，哪些要詳讀，哪些可以只抓重點，這種擇要的能力，是研究所學生們應該練習的。

VI. Other Time Frequency Distributions

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 6, Prentice Hall, N.J., 1996.

Requirements for time-frequency analysis:

- (1) higher clarity (2) avoid cross-term
 ← tradeoff →
(3) less computation time (4) good mathematical properties

VI-A Cohen's Class Distribution 柯恩

VI-A-1 Ambiguity Function

模糊

$\eta / \text{ite} /$

$$A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt$$

(1) If $x(t) = \exp[-\alpha\pi(t - t_0)^2 + j2\pi f_0 t]$ scaling, shifting, and modulation of a Gaussian function

$$\begin{aligned} A_x(\tau, \eta) &= \int_{-\infty}^{\infty} e^{-\alpha\pi(t+\tau/2-t_0)^2+j2\pi f_0(t+\tau/2)} e^{-\alpha\pi(t-\tau/2-t_0)^2-j2\pi f_0(t-\tau/2)} \cdot e^{-j2\pi t\eta} \cdot dt \\ &= \int_{-\infty}^{\infty} e^{-\alpha\pi[2(t-t_0)^2+\tau^2/2]+j2\pi f_0\tau} \cdot e^{-j2\pi t\eta} \cdot dt \\ &= \int_{-\infty}^{\infty} e^{-\alpha\pi[2t^2+\tau^2/2]+j2\pi f_0\tau} \cdot e^{-j2\pi t\eta} e^{-j2\pi t_0\eta} \cdot dt \end{aligned}$$

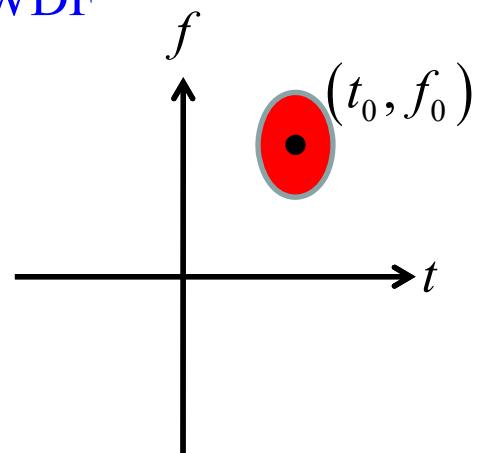
$$A_x(\tau, \eta) = \sqrt{\frac{1}{2\alpha}} \exp\left[-\pi\left(\frac{\alpha\tau^2}{2} + \frac{\eta^2}{2\alpha}\right)\right] \exp[j2\pi(f_0\tau - t_0\eta)]$$

$$|A_x(\tau, \eta)| = \sqrt{\frac{1}{2\alpha}} \exp\left(-\pi\left(\frac{\alpha\tau^2}{2} + \frac{\eta^2}{2\alpha}\right)\right)$$

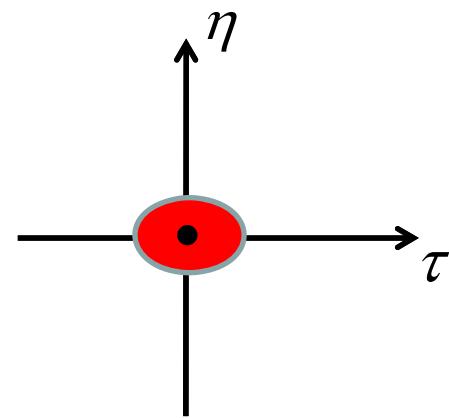
$|A_x(\tau, \eta)|$ is maximal at $(\tau, \eta) = (0, 0)$

WDF and AF for the signal with **only 1 term**

WDF



AF



$$(2) \text{ If } x(t) = \underbrace{\exp[-\alpha_1 \pi (t-t_1)^2 + j2\pi f_1 t]}_{x_1(t)} + \underbrace{\exp[-\alpha_2 \pi (t-t_2)^2 + j2\pi f_2 t]}_{x_2(t)}$$

$$\begin{aligned} A_x(\tau, \eta) &= \int_{-\infty}^{\infty} x_1(t + \tau/2) \cdot x_1^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \xleftarrow{} A_{x_1}(\tau, \eta) \\ &\quad \int_{-\infty}^{\infty} x_2(t + \tau/2) \cdot x_2^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \xleftarrow{} A_{x_2}(\tau, \eta) \\ &\quad \int_{-\infty}^{\infty} x_1(t + \tau/2) \cdot x_2^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \xleftarrow{} A_{x_1 x_2}(\tau, \eta) \\ &\quad \int_{-\infty}^{\infty} x_2(t + \tau/2) \cdot x_1^*(t - \tau/2) \cdot e^{-j2\pi t\eta} \cdot dt + && \xleftarrow{} A_{x_2 x_1}(\tau, \eta) \end{aligned}$$

$$A_x(\tau, \eta) = A_{x_1}(\tau, \eta) + A_{x_2}(\tau, \eta) + A_{x_1 x_2}(\tau, \eta) + A_{x_2 x_1}(\tau, \eta)$$

$$\begin{aligned} A_{x_1}(\tau, \eta) &= \sqrt{\frac{1}{2\alpha_1}} \exp\left[-\pi\left(\frac{\alpha_1 \tau^2}{2} + \frac{\eta^2}{2\alpha_1}\right)\right] \exp[j2\pi(f_1 \tau - t_1 \eta)] \\ A_{x_2}(\tau, \eta) &= \sqrt{\frac{1}{2\alpha_2}} \exp\left[-\pi\left(\frac{\alpha_2 \tau^2}{2} + \frac{\eta^2}{2\alpha_2}\right)\right] \exp[j2\pi(f_2 \tau - t_2 \eta)] \end{aligned} \quad \left. \begin{array}{l} \text{auto} \\ \text{term} \end{array} \right\}$$

Maximums of amplitude are at $(\tau, \eta) = (0, 0)$

When $\alpha_1 = \alpha_2$

$|A_{x_1 x_2}(\tau, \eta)|$ is maximal at $(\tau, \eta) = (t_d, f_d)$ 180°

$$A_{x_1 x_2}(\tau, \eta) = \sqrt{\frac{1}{2\alpha_\mu}} \exp \left[-\pi \left(\alpha_\mu \frac{(\tau - t_d)^2}{2} + \frac{(\eta - f_d)^2}{2\alpha_\mu} \right) \right] \\ \times \exp \left[j2\pi(f_\mu \tau - t_\mu \eta + f_d t_\mu) \right]$$

$$t_\mu = (t_1 + t_2)/2, \quad f_\mu = (f_1 + f_2)/2, \quad \alpha_\mu = (\alpha_1 + \alpha_2)/2,$$

$$\underline{t_d = t_1 - t_2}, \quad \underline{f_d = f_1 - f_2}, \quad \alpha_d = \alpha_1 - \alpha_2$$

$$A_{x_2 x_1}(\tau, \eta) = A_{x_1 x_2}^*(-\tau, -\eta)$$

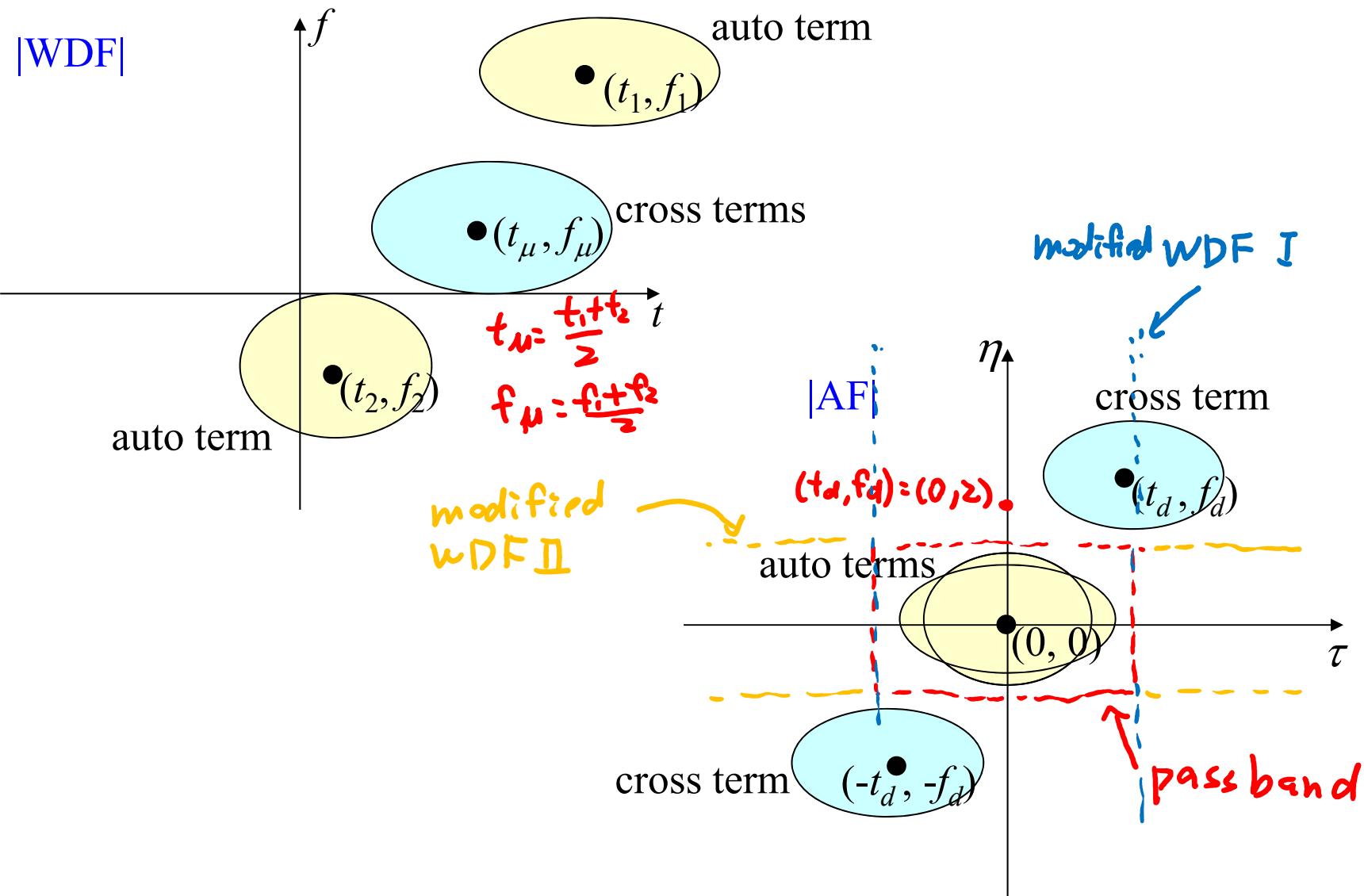
$|A_{x_2 x_1}(\tau, \eta)|$ is maximal at $(\tau, \eta) = (-t_d, -f_d)$

When $\alpha_1 \neq \alpha_2$

$$A_{x_1 x_2}(\tau, \eta) = \sqrt{\frac{1}{2\alpha_\mu}} \exp \left[-\pi \frac{[(\eta - f_d) + j(\alpha_1 t_1 + \alpha_2 t_2) - j\alpha_d \tau / 2]^2}{2\alpha_\mu} \right] \\ \exp \left[-\pi \left(\alpha_1 (t_1 - \frac{\tau}{2})^2 + \alpha_2 (t_2 + \frac{\tau}{2})^2 \right) \right] \exp \left[j2\pi f_\mu \tau \right]$$

$$A_{x_2 x_1}(\tau, \eta) = A_{x_1 x_2}^*(-\tau, -\eta)$$

WDF and AF for the signal with 2 terms



For the ambiguity function

The **auto term** is always near to the origin

The **cross-term** is always far from the origin

VI-A-2 Definition of Cohen's Class Distribution

The Cohen's Class distribution is a further generalization of the Wigner distribution function

$$C_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_x(\tau, \eta) \Phi(\tau, \eta) \exp(j2\pi(\eta t - \tau f)) d\eta d\tau$$

$$\text{where } A_x(\tau, \eta) = \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi t\eta} dt$$

is the ambiguity function (AF).

$$\Phi(\eta, \tau) = 1 \rightarrow \text{WDF}$$

$$C_x(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(u + \tau/2) x^*(u - \tau/2) \phi(t - u, \tau) du e^{-j2\pi f\tau} d\tau$$

$$\text{where } \phi(t, \tau) = \int_{-\infty}^{\infty} \Phi(\tau, \eta) \exp(j2\pi \eta t) d\eta \quad \eta_1 := -\eta \quad \tau_1 := -\tau$$

$$\begin{aligned} C_x^*(t, f) &= \iint A_x^*(\tau, \eta) \Phi^*(\tau, \eta) e^{j2\pi(-\eta t + \tau f)} d\eta d\tau \\ &= \iint A_x^*(-\tau_1, \eta_1) \Phi^*(\tau_1, \eta_1) e^{j2\pi(\eta_1 t - \tau_1 f)} (-1)^2 d\eta_1 d\tau_1, \end{aligned}$$

How does the Cohen's class distribution avoid the cross term?

Choose $\Phi(\tau, \eta)$ low pass function.

$$\Phi(\tau, \eta) \approx 1 \quad \text{for small } |\eta|, |\tau|$$

$$\Phi(\tau, \eta) \approx 0 \quad \text{for large } |\eta|, |\tau|$$

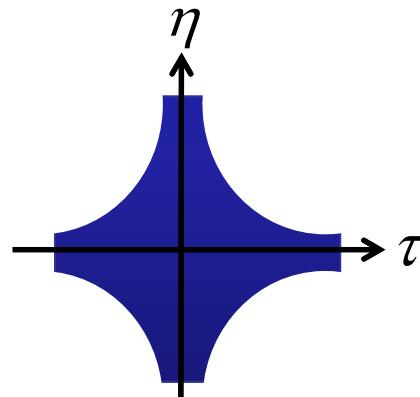
[Ref] L. Cohen, “Generalized phase-space distribution functions,” *J. Math. Phys.*, vol. 7, pp. 781-806, 1966.

[Ref] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, New York, 1995.

VI-A-3 Several Types of Cohen's Class Distribution

Choi-Williams Distribution (One of the Cohen's class distribution)

$$\Phi(\tau, \eta) = \exp[-\alpha(\eta\tau)^2]$$

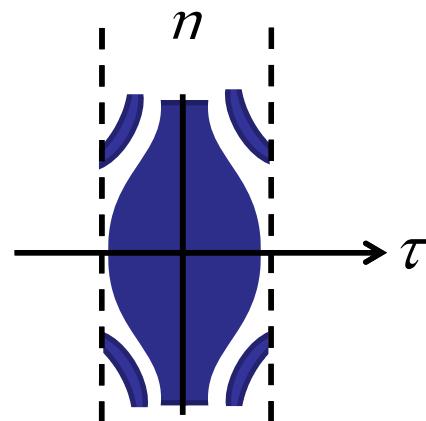


[Ref] H. Choi and W. J. Williams, “Improved time-frequency representation of multicomponent signals using exponential kernels,” *IEEE. Trans. Acoustics, Speech, Signal Processing*, vol. 37, no. 6, pp. 862-871, June 1989.

Cone-Shape Distribution (One of the Cohen's class distribution)

$$\phi(t, \tau) = \frac{1}{|\tau|} \exp(-2\pi\alpha\tau^2) \Pi\left(\frac{t}{\tau}\right)$$

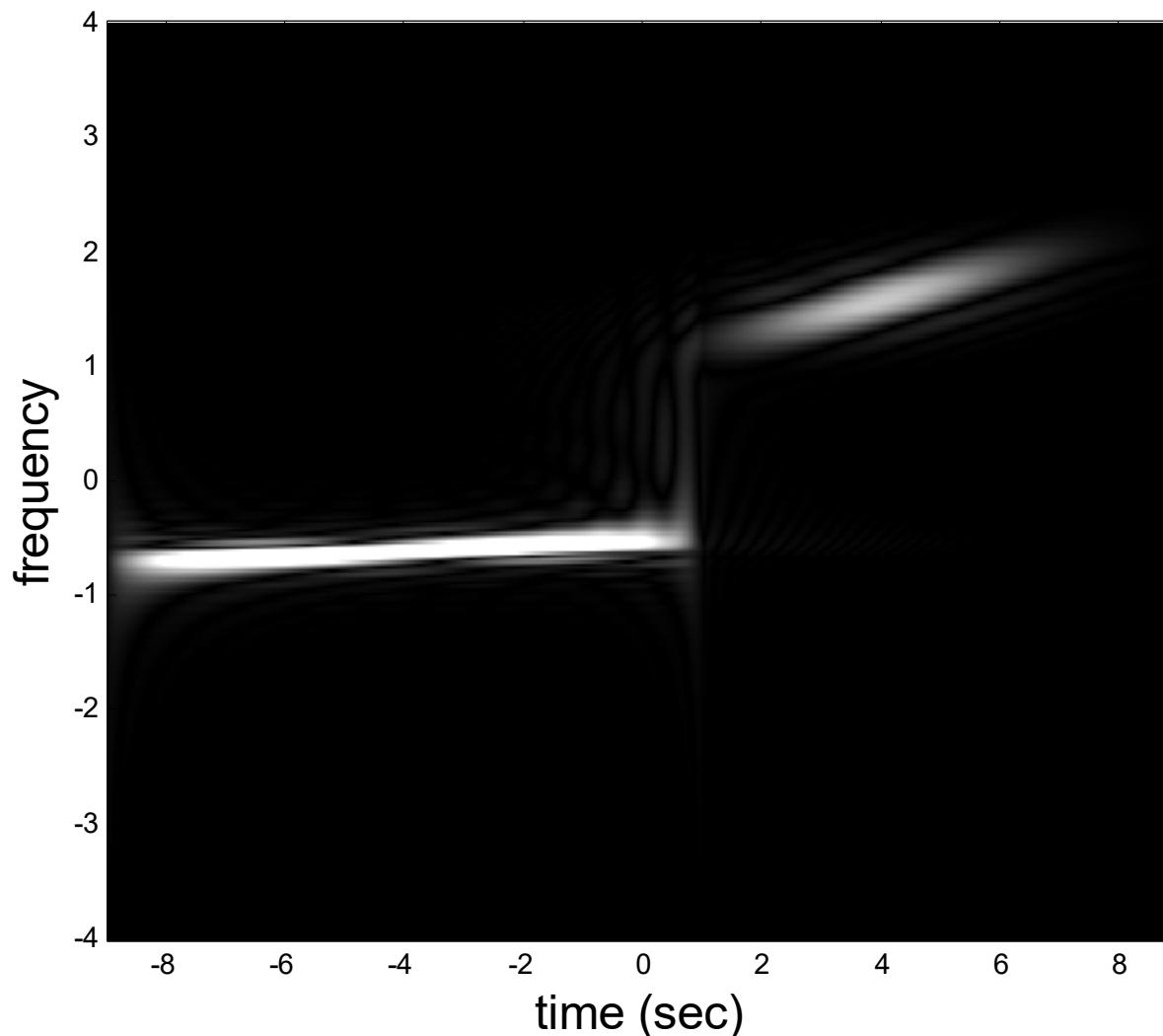
$$\Phi(\tau, \eta) = \text{sinc}(\eta\tau) \exp(-2\pi\alpha\tau^2)$$



[Ref] Y. Zhao, L. E. Atlas, and R. J. Marks, “The use of cone-shape kernels for generalized time-frequency representations of nonstationary signals,” *IEEE Trans. Acoustics, Speech, Signal Processing*, vol. 38, no. 7, pp. 1084-1091, July 1990.

Cone-Shape distribution for the example on pages 97, 149

$$(\alpha = 1)$$



Distributions	$\Phi(\tau, \eta)$
Wigner	1
Cohen (circular)	$\Phi(\tau, \eta) = 1$ for $\sqrt{\eta^2 + \tau^2} < r$ $\Phi(\tau, \eta) = 0$ otherwise
Cohen (rectangular)	$\Phi(\tau, \eta) = 1$ for $\text{Max}(\eta , \tau) < T$ $\Phi(\tau, \eta) = 0$ otherwise
Choi-Williams	$\exp[-\alpha(\eta\tau)^2]$
Cone-Shape	$\text{sinc}(\eta\tau)\exp(-2\pi\alpha\tau^2)$
Page	$\exp(j\pi\eta \tau)$
Levin (Margenau-Hill)	$\cos(\pi\eta\tau)$
Born-Jordan	$\text{sinc}(\eta\tau)$

註：感謝 2007 年修課的王文阜同學

VI-A-4 Advantages and Disadvantages of Cohen's Class Distributions

The Cohen's class distribution may avoid the cross term and has [higher clarity](#).

However, it requires more computation time and lacks of well mathematical properties.

Moreover, there is a tradeoff between [the quality of the auto term](#) and [the ability of removing the cross terms](#).

VI-A-5 Implementation for the Cohen's Class Distribution

$$\begin{aligned}
 C_x(t, f) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_x(\tau, \eta) \Phi(\tau, \eta) \exp(j2\pi(\eta t - \tau f)) d\eta d\tau \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \Phi(\tau, \eta) e^{-j2\pi u\eta + j2\pi(\eta t - \tau f)} du d\eta d\tau
 \end{aligned}$$

簡化法 1：不是所有的 $A_x(\eta, \tau)$ 的值都需要算出

If $\Phi(\tau, \eta) = 0$ for $|\eta| > B$ or $|\tau| > C$

$$C_x(t, f) = \int_{-C}^C \int_{-B}^B \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \Phi(\tau, \eta) e^{-j2\pi u\eta + j2\pi(\eta t - \tau f)} du d\eta d\tau$$

簡化法 2：注意， η 這個參數和 input 及 output 都無關

$$\begin{aligned} C_x(t, f) &= \int_{-C}^C \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot [\int_{-B}^B \Phi(\tau, \eta) e^{j2\pi\eta(t-u)} d\eta] e^{-j2\pi\tau f} du d\tau \\ &= \int_{-C}^C \int_{-\infty}^{\infty} x\left(u + \frac{\tau}{2}\right) x^*\left(u - \frac{\tau}{2}\right) \cdot \Psi(\tau, t-u) e^{-j2\pi\tau f} du d\tau \end{aligned}$$

$$\Psi(\tau, t) = \int_{-B}^B \Phi(\tau, \eta) e^{j2\pi\eta t} d\eta$$

由於 $\Psi(\tau, t)$ 和 input 無關，可事先算出，所以只剩 2 個積分式

VI-B Modified Wigner Distribution Function

$$\begin{aligned}
 W_x(t, f) &= \int_{-\infty}^{\infty} x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau \\
 &= \int_{-\infty}^{\infty} X(f + \eta/2) \cdot X^*(f - \eta/2) e^{j2\pi t\eta} \cdot d\eta \\
 \text{where } X(f) &= FT[x(t)]
 \end{aligned}$$

Modified Form I

$$W_x(t, f) = \int_{-B}^B w(\tau) x(t + \tau/2) \cdot x^*(t - \tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

Modified Form II

$$\begin{aligned}
 W_x(t, f) &= \int_{-B}^B w(\eta) X(f + \eta/2) \cdot X^*(f - \eta/2) e^{j2\pi t\eta} \cdot d\eta \\
 &\quad \text{--- } |w(\eta)| \geq 0 \text{ for } |\eta| > B
 \end{aligned}$$

Modified Form III (Pseudo L-Wigner Distribution)

$$W_x(t, f) = \int_{-\infty}^{\infty} w(\tau) x^L\left(t + \frac{\tau}{2L}\right) \cdot \overline{x^L\left(t - \frac{\tau}{2L}\right)} e^{-j2\pi\tau f} \cdot d\tau$$

增加 L 可以減少 cross term 的影響 (但是不會完全消除)

[Ref] L. J. Stankovic, S. Stankovic, and E. Fakultet, “An analysis of instantaneous frequency representation using time frequency distributions-generalized Wigner distribution,” *IEEE Trans. on Signal Processing*, pp. 549-552, vol. 43, no. 2, Feb. 1995

P.S.: 感謝2006年修課的林政豪同學

Modified Form IV (Polynomial Wigner Distribution Function)

$$W_x(t, f) = \int_{-\infty}^{\infty} \left[\prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) \right] e^{-j2\pi\tau f} d\tau$$

\prod : 連乘

$$\prod_{n=1}^3 n = 1 \times 2 \times 3$$

When $q = 2$ and $d_1 = d_{-1} = 0.5$, it becomes the original Wigner distribution function.

ex: $q=4 \quad \prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) = x(t+d_1\tau)x^*(t-d_{-1}\tau)x(t+d_2\tau)x^*(t-d_{-2}\tau)$

It can avoid the cross term when the order of phase of the exponential function is no larger than $q/2 + 1$.

$q=2$	$q/2+1=2$	$q=6$	$q/2+1=4$
$q=4$	$q/2+1=3$	$q=8$	$q/2+1=5$

However, the cross term between two components cannot be removed.

[Ref] B. Boashash and P. O'Shea, "Polynomial Wigner-Ville distributions & their relationship to time-varying higher order spectra," *IEEE Trans. Signal Processing*, vol. 42, pp. 216–220, Jan. 1994.

[Ref] J. J. Ding, S. C. Pei, and Y. F. Chang, "Generalized polynomial Wigner spectrogram for high-resolution time-frequency analysis," *APSIPA ASC*, Kaohsiung, Taiwan, Oct. 2013.

d_l should be chosen properly such that

$$\star \quad \prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) = \exp \left(j2\pi \sum_{n=1}^{q/2+1} n a_n t^{n-1} \tau \right)$$

when $x(t) = \exp \left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n \right)$

then

$$\text{Instantaneous frequency} = \frac{q/2+1}{\sum_{n=1}^{q/2+1} n a_n t^{n-1}}$$

$$W_x(t, f) = \int_{-\infty}^{\infty} \exp \left(-j2\pi(f - \sum_{n=1}^{q/2+1} n a_n t^{n-1}) \tau \right) d\tau \cong \delta \left(f - \sum_{n=1}^{q/2+1} n a_n t^{n-1} \right)$$

(from page 138(1))

page 138(3)

$$t \rightarrow \tau$$

$$g(f) = f - \sum_{n=1}^{q/2+1} n a_n \tau^{n-1}$$

$$\prod_{l=1}^{q/2} x(t + d_l \tau) x^*(t - d_{-l} \tau) = \exp \left(j2\pi \sum_{n=1}^{q/2+1} n a_n t^{n-1} \tau \right)$$

$$x(t) = \exp \left(j2\pi \sum_{n=1}^{q/2+1} a_n t^n \right)$$

when $q = 2$ $x(t) = \exp(j2\pi(a_1 t + a_2 t^2))$

$$x(t + d_1 \tau) x^*(t - d_{-1} \tau) = \exp(j2\pi(a_1 + 2a_2 t) \tau) \quad \text{---} \quad -a_2(t - d_{-1} \tau)^2$$

$$a_2(t + d_1 \tau)^2 + a_1(t + d_1 \tau) - a_2(t - d_{-1} \tau)^2 - a_1(t - d_{-1} \tau) = 2a_2 t \tau + a_1 \tau$$

$$\frac{2a_2(d_1 + d_{-1})t\tau + a_2(d_1^2 - d_{-1}^2)\tau^2}{2a_2} + \frac{a_1(d_1 + d_{-1})\tau}{a_1} = 2a_2 t \tau + a_1 \tau$$

$$\frac{\cancel{2a_2}}{\cancel{2a_2}} \rightarrow d_1 + d_{-1} = 1 \quad d_1 - d_{-1} = 0$$

$$\rightarrow \underline{\underline{d_1 = d_{-1} = 1/2}}$$

When $q = 4$, $x(t) = \exp(j2\pi(a_1t + a_2t^2 + a_3t^3))$

$$\prod_{l=1}^2 x(t + d_l \tau) x^*(t - d_{-l} \tau) = \exp\left(j2\pi \sum_{n=1}^3 n a_n t^{n-1} \tau\right)$$

$$x(t + d_1 \tau) x^*(t - d_{-1} \tau) x(t + d_2 \tau) x^*(t - d_{-2} \tau) = \exp\left(j2\pi \sum_{n=1}^3 n a_n t^{n-1} \tau\right)$$

$$a_3(t + d_1 \tau)^3 + a_2(t + d_1 \tau)^2 + a_1(t + d_1 \tau)$$

$$+ a_3(t + d_2 \tau)^3 + a_2(t + d_2 \tau)^2 + a_1(t + d_2 \tau)$$

$$- a_3(t - d_{-1} \tau)^3 - a_2(t - d_{-1} \tau)^2 - a_1(t - d_{-1} \tau)$$

$$- a_3(t - d_{-2} \tau)^3 - a_2(t - d_{-2} \tau)^2 - a_1(t - d_{-2} \tau)$$

$$= 3a_3 t^2 \tau + 2a_2 t \tau + a_1 \tau$$

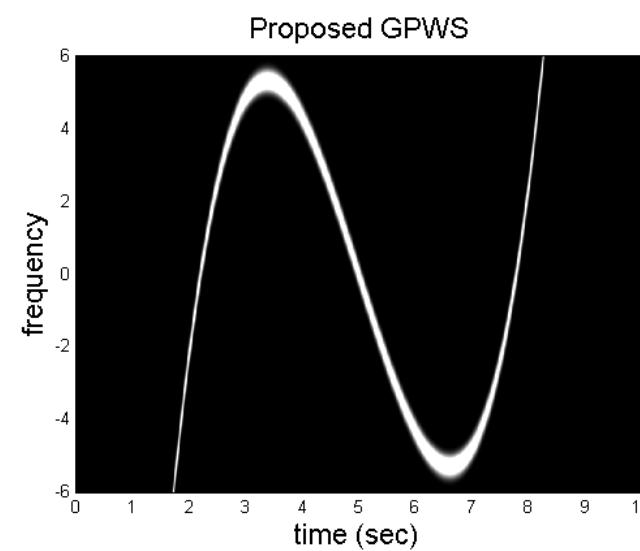
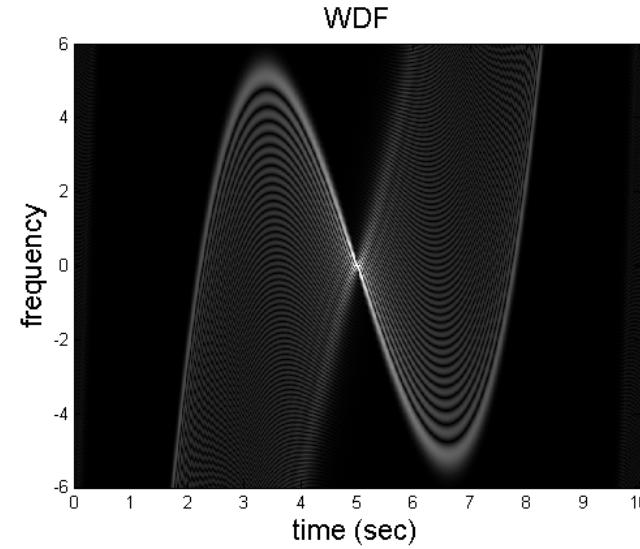
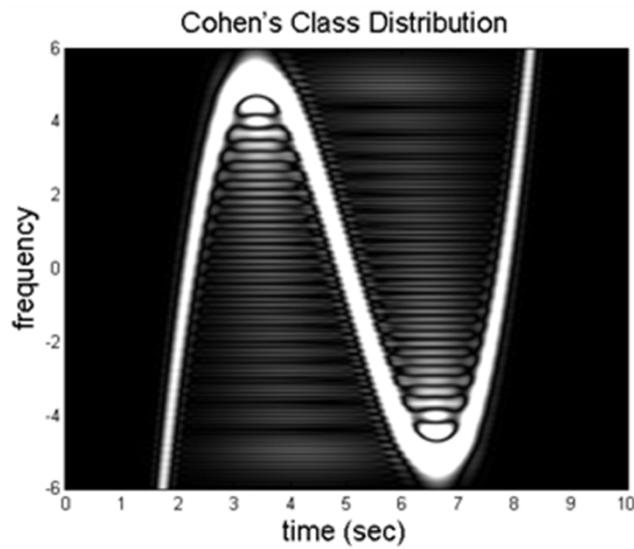
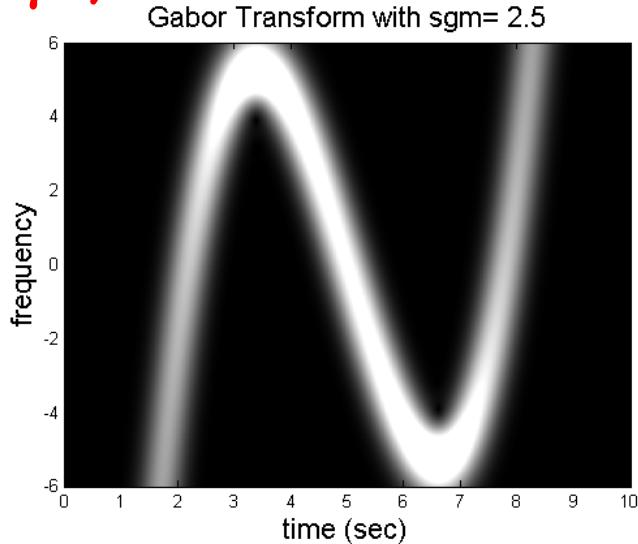
$$\Rightarrow \begin{cases} d_1 + d_2 + d_{-1} + d_{-2} = 1 \\ d_1^2 + d_2^2 - d_{-1}^2 - d_{-2}^2 = 0 \\ d_1^3 + d_2^3 + d_{-1}^3 + d_{-2}^3 = 0 \end{cases}$$

Cohen's class distribution cannot remove the cross term
when the order of phase > 2

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$$x(t) = \exp(j(t-5)^4 - j5\pi(t-5)^2)$$

but the polynomial WDF can.

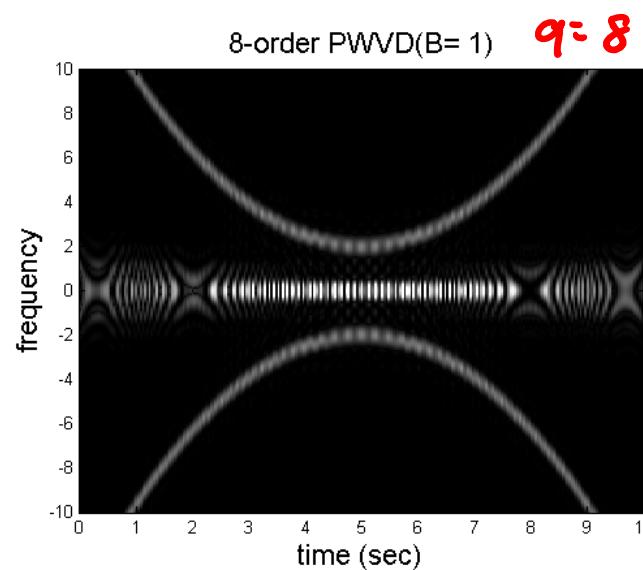
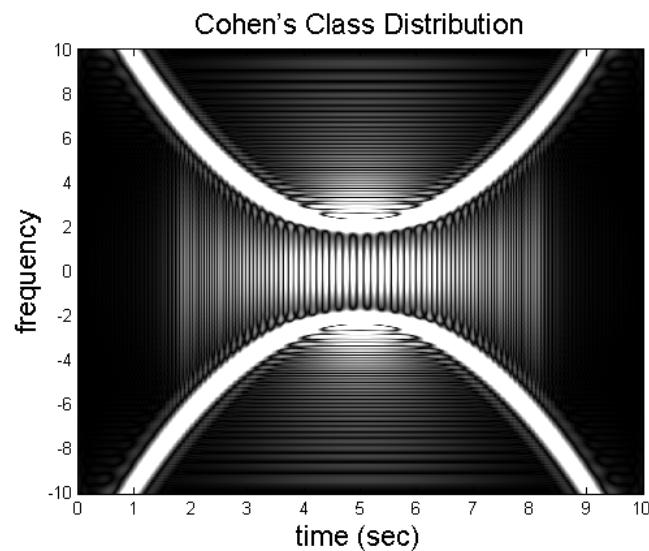
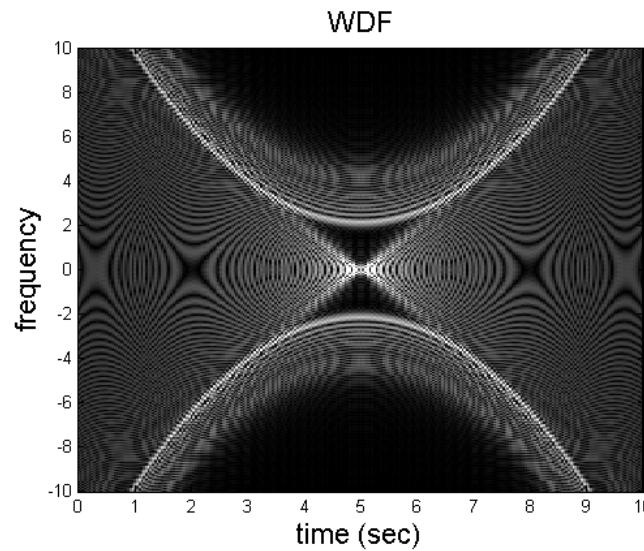
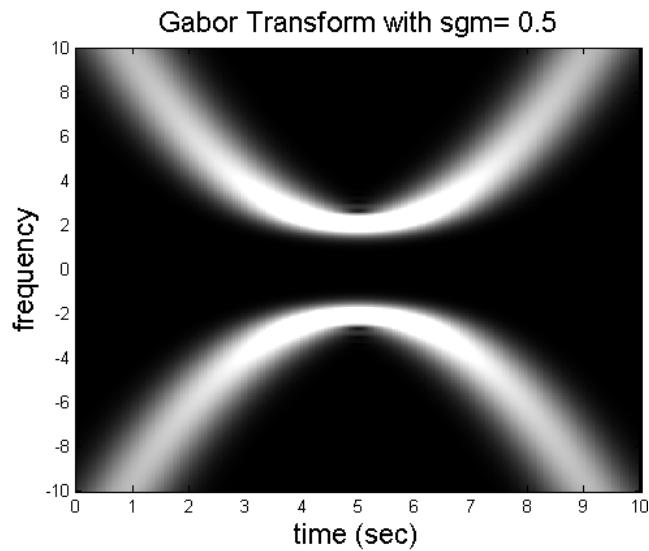


$$\begin{aligned} q &= ? \\ q &\geq 6 \\ q/2+1 &\geq 4 \end{aligned}$$

$q \geq 4$ The polynomial WDF cannot remove the cross terms
 $q/2+1 \geq 3$

$$x(t) = 2\cos((t-5)^3 + 4\pi t)$$

caused by multiple components. 199



VI-C Gabor-Wigner Transform

[Ref] S. C. Pei and J. J. Ding, “Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing,” *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Advantages:

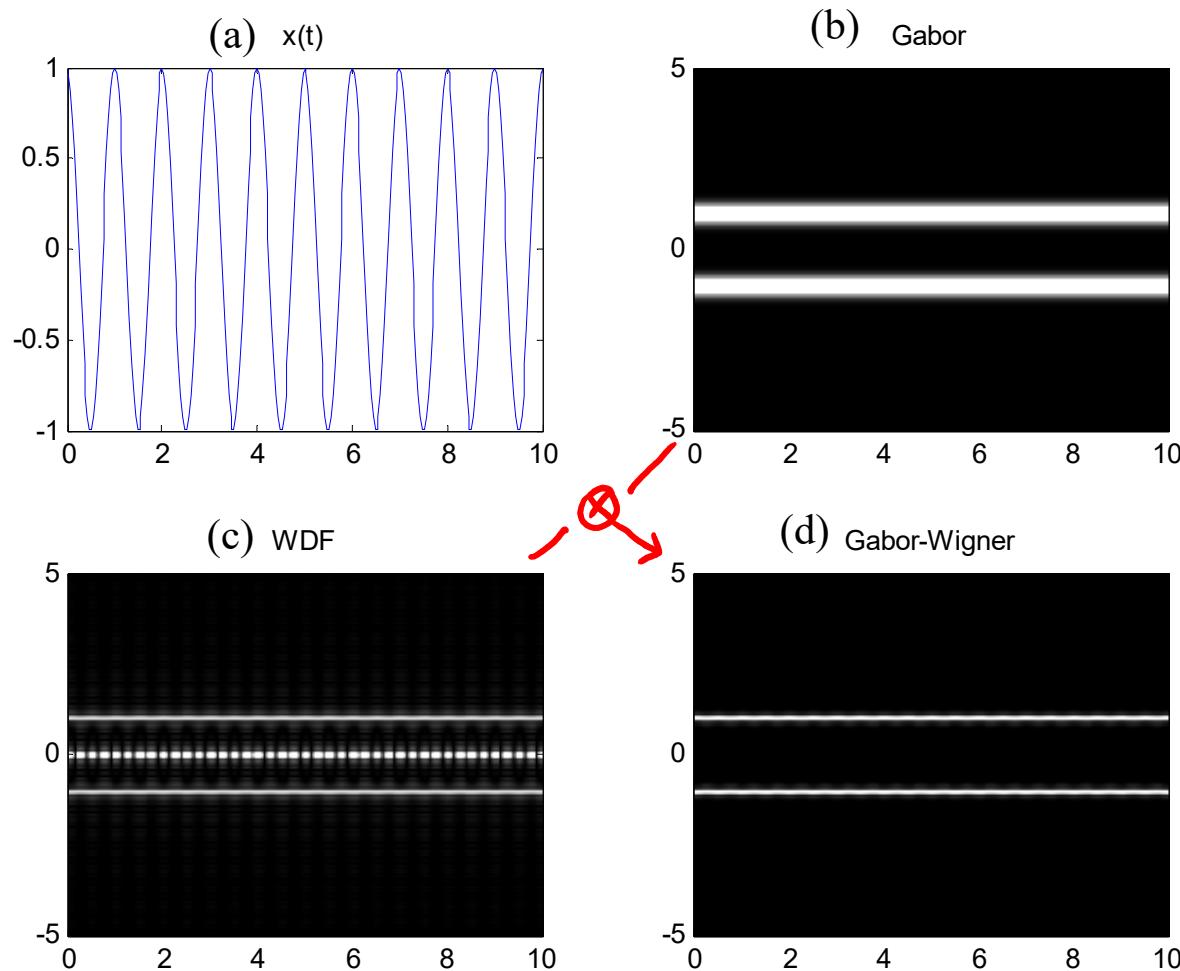
combine the advantage of the WDF and the Gabor transform

advantage of the WDF → higher clarity

advantage of the Gabor transform → no cross-term

$$D_x(t, f) = G_x^2(t, f)W_x(t, f)$$

$$x(t) = \cos(2\pi t)$$

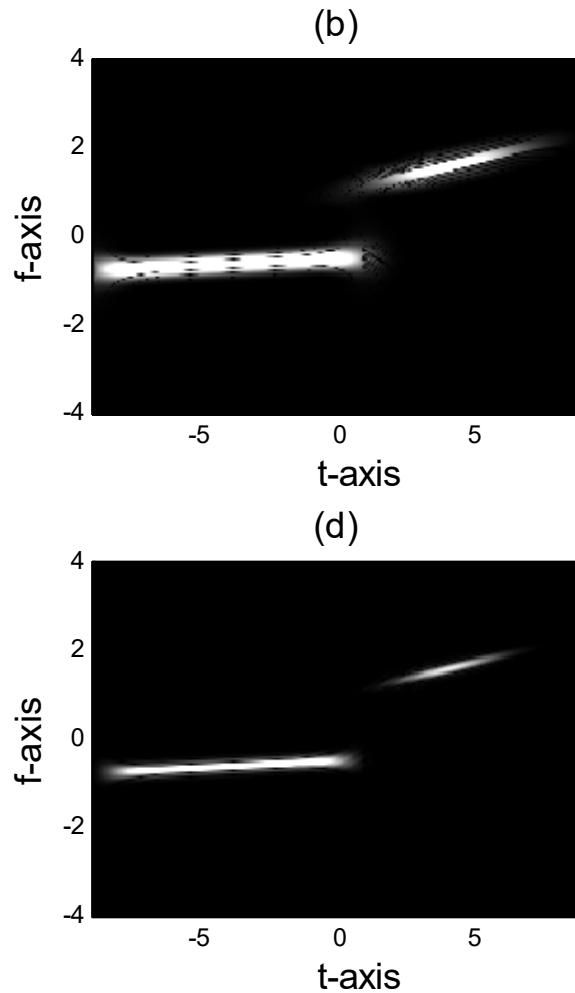
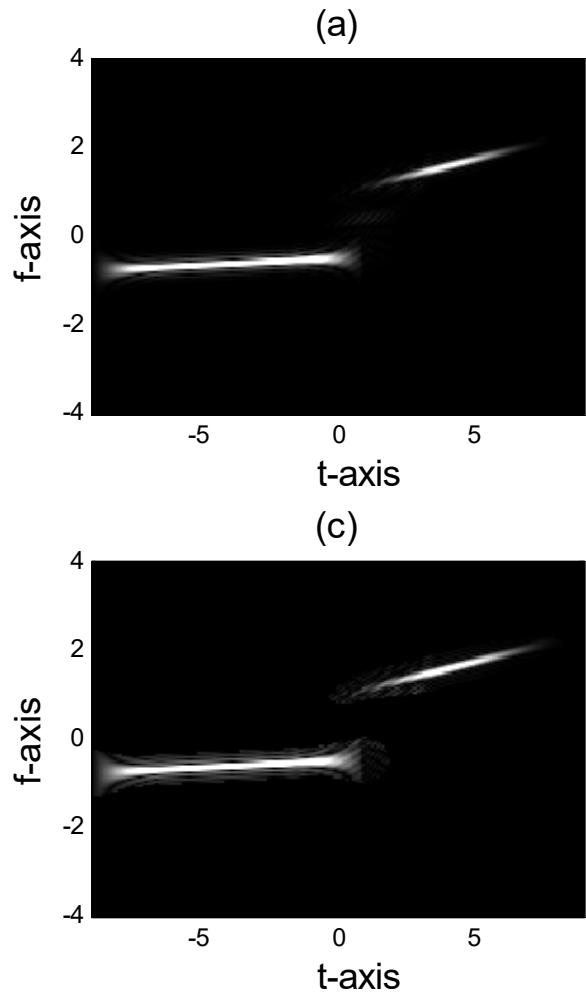


(a) $D_x(t, f) = G_x(t, f)W_x(t, f)$

(b) $D_x(t, \omega) = \min(|G_x(t, f)|^2, |W_x(t, f)|)$ ²⁰²

(c) $D_x(t, f) = W_x(t, f) \times \{|G_x(t, f)| > 0.25\}$

(d) $D_x(t, f) = G_x^{2.6}(t, f)W_x^{0.7}(t, f)$



(b)、(c) are real

思考：

- (1) Which type of the Gabor-Wigner transform is better?
- (2) Can we further generalize the results?

Implementation of the Gabor-Wigner Transform : 簡化技巧

(1) When $G_x(t, f) \approx 0$, $D_x(t, f) = G_x^\alpha(t, f)W_x^\beta(t, f) \approx 0$

先算 $G_x(t, f)$

$W_x(t, f)$ 只需算 $G_x(t, f)$ 不近似於 0 的地方

(2) When $x(t)$ is real, 對 Gabor transform 而言

$$X(f) = X^*(-f) \quad \text{if } x(t) \text{ is real, where } X(f) = FT[x(t)]$$

附錄九： Fourier Transform 常用的性質

$$X(f) = FT[x(t)] = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi f t) dt$$

(1) Recovery (inverse Fourier transform)	$x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t) df$
(2) Integration	$x(0) = \int_{-\infty}^{\infty} X(f) df$
(3) Modulation	$FT[x(t)e^{j2\pi f_0 t}] = X(f - f_0)$
(4) Time Shifting	$FT[x(t - t_0)] = X(f)e^{-j2\pi f t_0}$
(5) Scaling	$FT[x(at)] = \frac{1}{ a } X\left(\frac{f}{a}\right)$
(6) Time Reverse	$FT[x(-t)] = X(-f)$
(7) Real Output	If $x(t) = x^*(-t)$, then $X(f)$ is real.

(8) Real / Imaginary Input	If $x(t)$ is real, then $X(f) = X^*(-f)$; If $x(t)$ is pure imaginary, then $X(f) = -X^*(-f)$
(9) Even / Odd Input	If $x(t) = x(-t)$, then $X(f) = X(-f)$; If $x(t) = -x(-t)$, then $X(f) = -X(-f)$;
(10) Conjugation	$FT[x^*(t)] = X^*(-f)$
(11) Differentiation	$FT[x'(t)] = j2\pi f X(f)$
(12) Multiplication by t	$FT[tx(t)] = \frac{j}{2\pi} X'(f)$
(13) Division by t	$FT\left[\frac{x(t)}{t}\right] = -j2\pi \int_{-\infty}^f X(\mu) d\mu$
(14) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} x(t) ^2 dt = \int_{-\infty}^{\infty} X(f) ^2 df$
(15) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$

(16) Linearity	$FT[ax(t) + by(t)] = aX(f) + bY(f)$
(17) Convolution	If $z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$, then $Z(f) = X(f)Y(f)$
(18) Multiplication	If $z(t) = x(t)y(t)$, then $Z(f) = X(f)*Y(f) = \int_{-\infty}^{\infty} X(\mu)Y(f - \mu) d\mu$
(19) Correlation	If $z(t) = \int_{-\infty}^{\infty} x(\tau) y^*(\tau - t) d\tau$, then $Z(f) = X(f)Y^*(f)$
(20) Two Times of Fourier Transforms	$FT\{FT[x(t)]\} = x(-t)$ $\text{FT(FT(FT(x(t))))} = X(-f)$
(21) Four Times of Fourier Transforms	$FT[FT(FT\{FT[x(t)]\})] = x(t)$

VII. Other Time Frequency Distributions (II)

The trend of time-frequency analysis in recent years:

- (1) S transform and its generalization
- (2) Time-variant signal expansion (Compressive sensing)
- (3) Improvement for the Hilbert-Huang transform
- (4) time-frequency reassignment

VII-A S Transform

(S transform \neq Laplace transform)

(Modification from the Gabor transform)

$$S_x(t, f) = |f| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi(t-\tau)^2 f^2\right] \exp(-j2\pi f\tau) d\tau$$

page 83

Gabor: window $e^{-\pi\tau^2}$

S transform: window $e^{-\pi\tau^2 f^2} |f|$

compared to scaled Gabor

$$\sigma = f^2$$

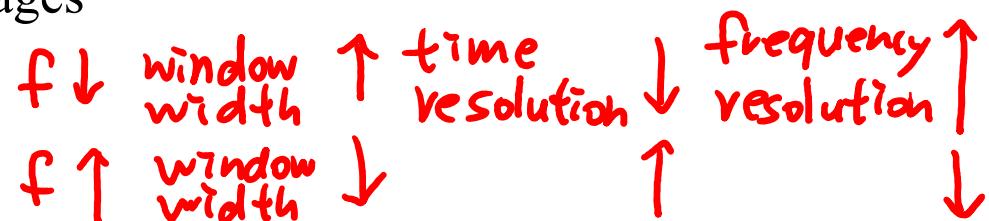
closely related to the wavelet transform

advantages and disadvantages

Set A 100 200 Hz

Set B 1000 1100 Hz

$$261.63 \cdot 2^{\frac{k}{12}} \text{ Hz}$$



[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

S transform 和 Gabor transform 相似。

但是 Gaussian window 的寬度會隨著 f 而改變

$$w(t) = \exp[-\pi t^2]$$

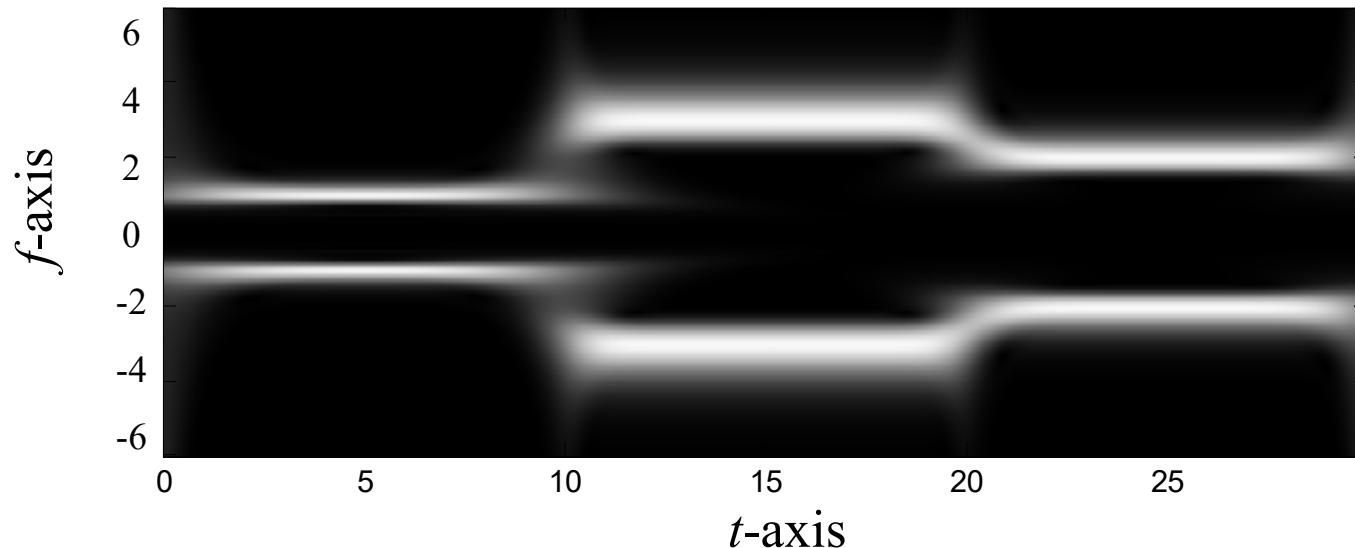
$$w(t) = |f| \exp[-\pi t^2 f^2]$$

低頻： worse time resolution, better frequency resolution

高頻： better time resolution, worse frequency resolution

$$\int_{t - \frac{1.914\zeta}{|f|}}^{t + \frac{1.914\zeta}{|f|}}$$

The result of the S transform (compared with page 95)



replace f by $s(f)$

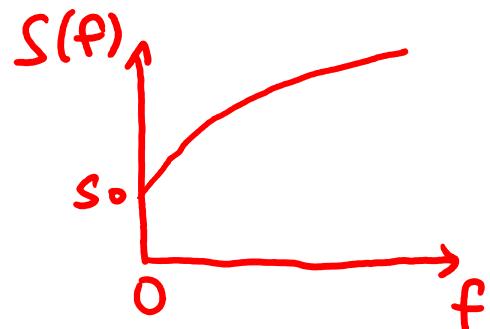
$$w(\tau) = e^{-\pi \tau^2 s^2(f)}$$

- General form

$$S_x(t, f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi(t-\tau)^2 s^2(f)\right] \exp(-j2\pi f\tau) d\tau$$

$s(f)$ increases with f

$$\text{ex: } S(f) = S_0 + |f|^{0.5}$$



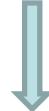
$$\int_{t - \frac{1.9143}{S(f)}}^{t + \frac{1.9143}{S(f)}}$$

C. R. Pinnegar and L. Mansinha, "The S-transform with windows of arbitrary and varying shape," *Geophysics*, vol. 68, pp. 381-385, 2003.

Fast algorithm of the S transform

When f is fixed, the S transform can be expressed as a convolution form:

$$S_x(t, f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi(t-\tau)^2 s^2(f)\right] \exp(-j2\pi f\tau) d\tau$$



$$S_x(t, f) = |s(f)| \left(x(t) \exp(-j2\pi f t) *_{\substack{\text{convolution} \\ \text{along } t\text{-axis}}} \exp\left[-\pi t^2 s^2(f)\right] \right)$$

(for every fixed f)

Remember: $g(t) * h(t) = \int g(\tau)h(t-\tau)d\tau$

Q: Can we use the FFT-based method on page 120 to implement the S transform?

VII-B Generalized Spectrogram

[Ref] P. Boggiatto, G. De Donno, and A. Oliaro, "Two window spectrogram and their integrals," *Advances and Applications*, vol. 205, pp. 251-268, 2009.

Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)G_{x,w_2}^*(t,f)$

$$G_{x,w_1}(t,f) = \int_{-\infty}^{\infty} w_1(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$$G_{x,w_2}(t,f) = \int_{-\infty}^{\infty} w_2(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Original spectrogram: $w_1(t) = w_2(t)$

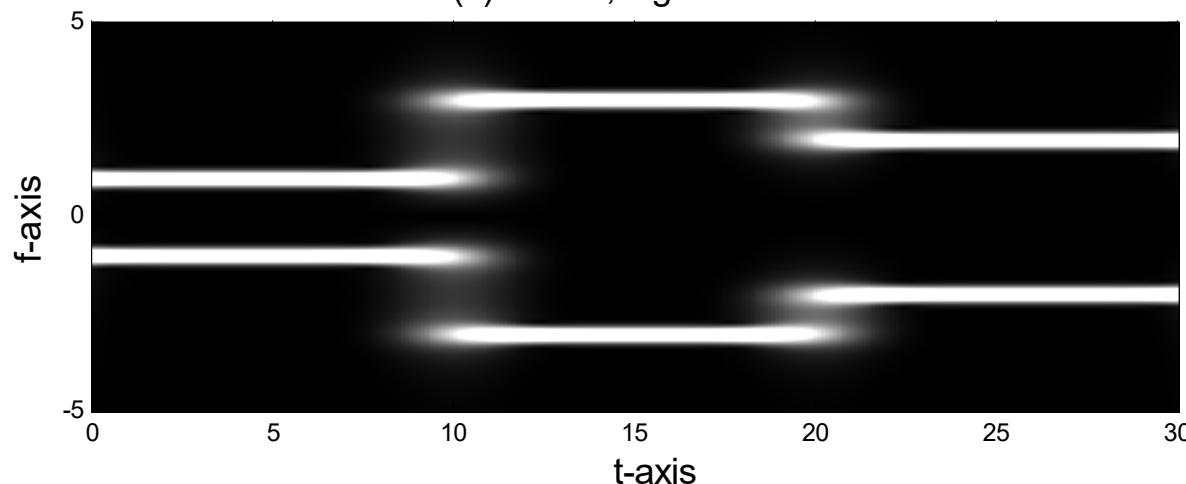
To achieve better clarity, $w_1(t)$ can be chosen as a **wider window**,
 $w_2(t)$ can be chosen as a **narrower window**.

$x(t) = \cos(2\pi t)$ when $t < 10$,

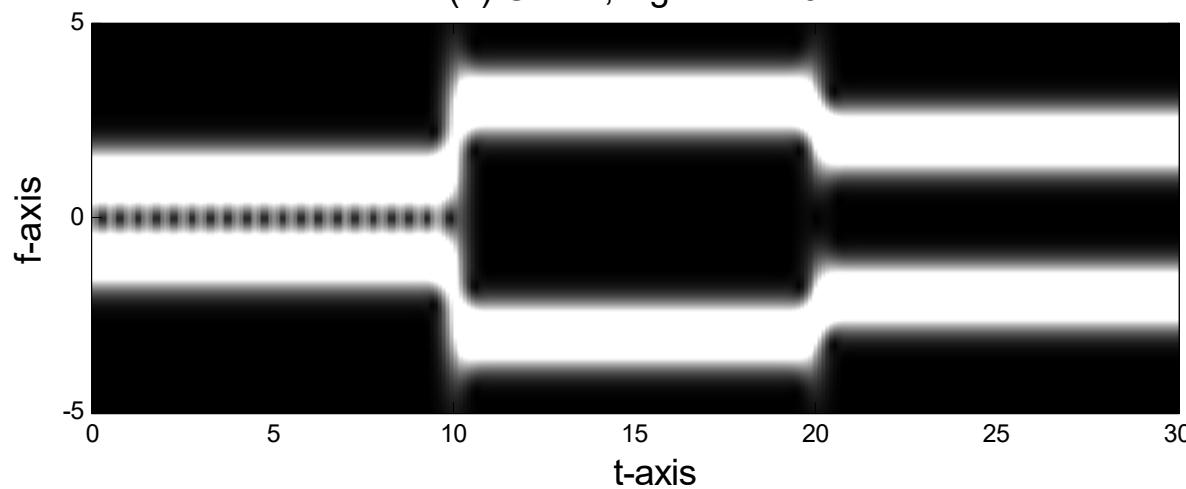
$x(t) = \cos(6\pi t)$ when $10 \leq t < 20$,

$x(t) = \cos(4\pi t)$ when $t \geq 20$

(a) Gabor, sigma = 0.1



(b) Gabor, sigma = 1.6

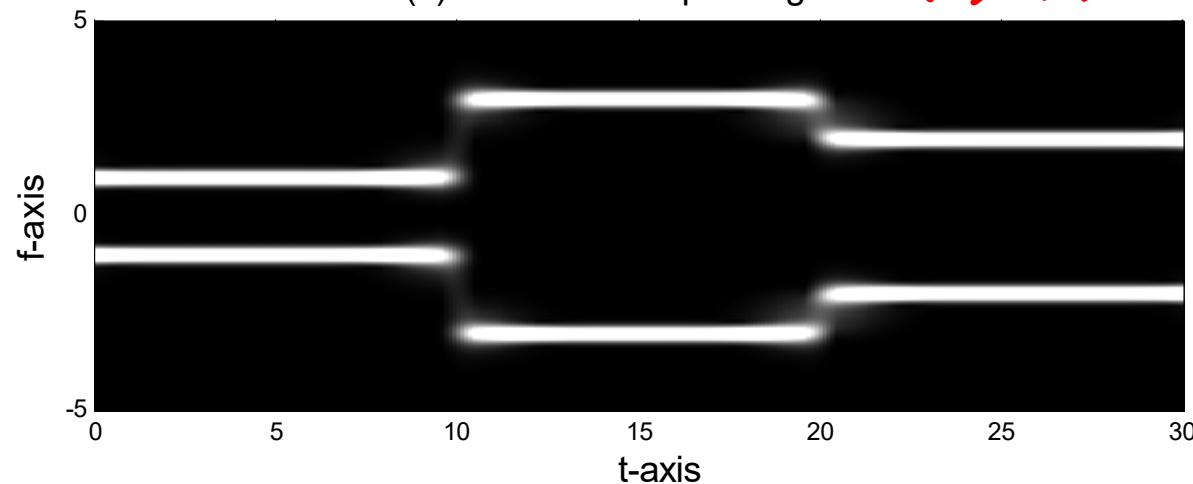


$x(t) = \cos(2\pi t)$ when $t < 10$,

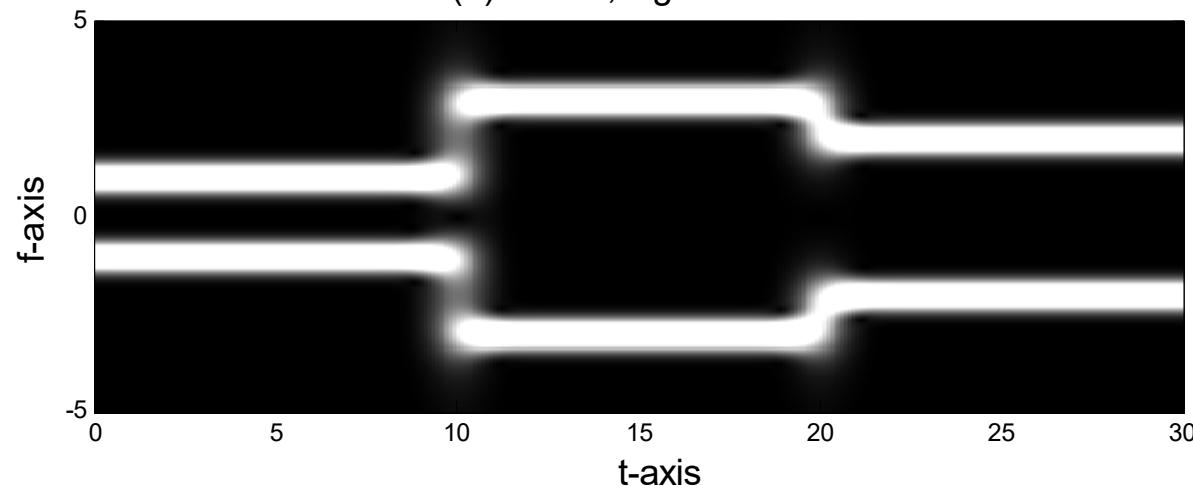
$x(t) = \cos(6\pi t)$ when $10 \leq t < 20$,

$x(t) = \cos(4\pi t)$ when $t \geq 20$

(c) Generalized spectrogram $= (a)X(b)$



(d) Gabor, sigma = 0.4



Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)G_{x,w_2}^*(t,f)$

Further Generalization for the spectrogram:

$$SP_{x,w_1,w_2}(t,f) = G_{x,w_1}^\alpha(t,f)\overline{G_{x,w_2}^\beta(t,f)}$$

or

$$SP_{x,w_1,w_2}(t,f) = |G_{x,w_1}(t,f)|^\alpha |G_{x,w_2}(t,f)|^\beta$$

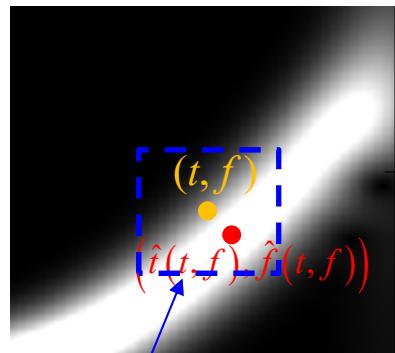
VII-C Reassignment Method

重分配

post processing

After computing the time-frequency distribution, we can use the following way to make the energy even more concentrated.

(1) First, estimate the offset.



$\varphi(u - t, v - f)$
*patch
window*

$$\hat{t}(t, f) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u \cdot \varphi(u - t, v - f) \cdot X(u, v) dudv}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(u - t, v - f) \cdot X(u, v) dudv}$$

$$\hat{f}(t, f) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v \cdot \varphi(u - t, v - f) \cdot X(u, v) dudv}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(u - t, v - f) \cdot X(u, v) dudv}$$

$X(t, f)$: time-frequency analysis (STFT, WDF...) of $x(t)$,
 $\varphi(u, v) = 1$ when $|u|, |v| < B$
 $\varphi(u, v) = 0$ otherwise

(2) Then, shift the time frequency distribution at (t, f) to $(\hat{t}(t, f), \hat{f}(t, f))$

(2) Then, shift the time frequency distribution at (t, f) to $(\hat{t}(t, f), \hat{f}(t, f))$

$$\hat{X}(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(t_1, f_1) \delta(t - \hat{t}(t_1, f_1)) \delta(f - \hat{f}(t_1, f_1)) dt_1 df_1$$

*move $X(t_1, f_1)$ to the location of
 $(\hat{t}(t_1, f_1), \hat{f}(t_1, f_1))$*

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VII-D Basis Expansion Time-Frequency Analysis

Since the amplitude $A(t)=1$ and frequency $f=f_m$ are fixed
就如同 Fourier series is time-invariant

- Fourier series: $\varphi_m(t) = \exp(j2\pi f_m t)$, $x(t) \approx \sum_{m=1}^M a_m \exp(j2\pi f_m t)$

$$a_m = \frac{\langle x(t), \varphi_m(t) \rangle}{\langle \varphi_m(t), \varphi_m(t) \rangle} = \frac{1}{T} \int_0^T x(t) \exp(-j2\pi f_m t) dt \quad f_m = m/T$$

部分的 Time-Frequency Analysis 也是意圖要將 signal 表示成如下的型態

$$x(t) \approx \sum_{m=1}^M a_m \varphi_m(t)$$

並且要求在 M 固定的情形下，

$$\text{approximation error} = \int_{-\infty}^{\infty} \left| x(t) - \sum_{m=1}^M a_m \varphi_m(t) \right|^2 dt \quad \text{為最小}$$

將 $\varphi_m(t)$ 一般化，不同的 basis 之間不只是有 frequency 的差異

(1) Three Parameter Atoms

$$x(t) \approx \sum a_{t_0, f_0, \sigma} \varphi_{t_0, f_0, \sigma}(t)$$

amplitude: $A(t) = \exp\left[-\frac{\pi(t-t_0)^2}{\sigma^2}\right]$

$$\varphi_{t_0, f_0, \sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp(j2\pi f_0 t) \exp\left(-\frac{\pi(t-t_0)^2}{\sigma^2}\right)$$

3 parameters: t_0 controls the central time
 f_0 controls the frequency
 σ controls the scaling factor

[Ref] S. G. Mallat and Z. Zhang, “Matching pursuits with time-frequency dictionaries,” *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397-3415, Dec. 1993.

Since $\varphi_{t_0, f_0, \sigma}(t)$ are not orthogonal, $a_{t_0, f_0, \sigma}$ should be determined by a matching pursuit process.

(2) Four Parameter Atoms (Chirplet)

示

$$x(t) \approx \sum a_{t_0, f_0, \sigma, \eta} \varphi_{t_0, f_0, \sigma, \eta}(t)$$

$$\varphi_{t_0, f_0, \sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp\left(j2\pi(f_0 t + \frac{\eta}{2} t^2) - \frac{\pi(t-t_0)^2}{\sigma^2}\right)$$

instantaneous freq = $\eta t + f_0$

4 parameters: t_0 controls the central time

f_0 controls the initial frequency

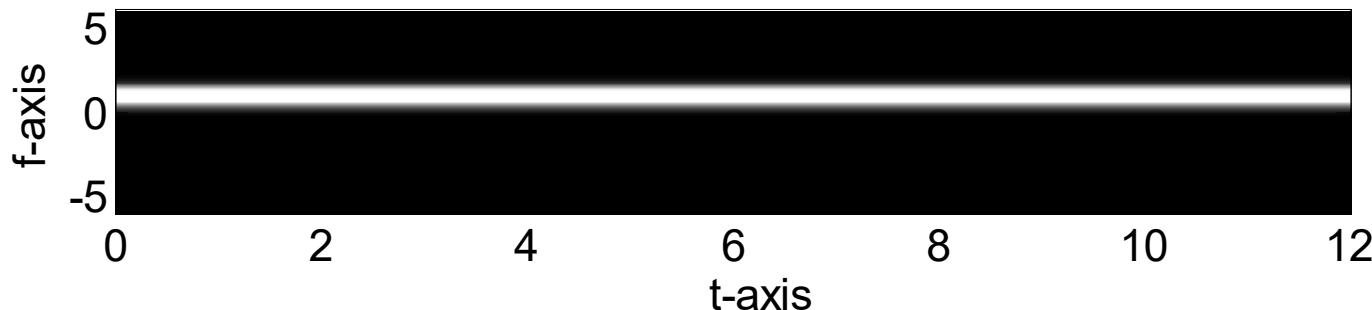
σ controls the scaling factor

η controls the chirp rate

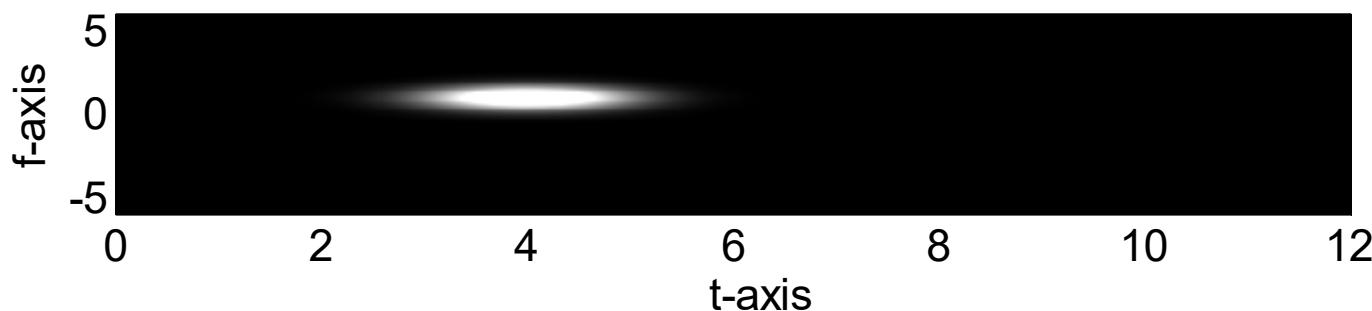
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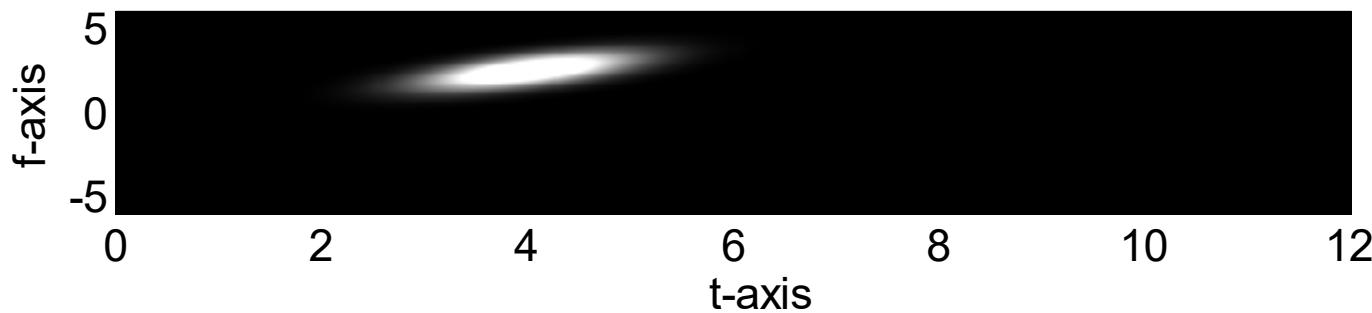
(a) STFT of a Fourier basis



(b) STFT of a 3-parameter atom



(c) STFT of a chirplet (4-parameter atom)



(3) Prolate Spheroidal Wave Function (PSWF)

$$x(t) \cong \sum_{n,T,\Omega,t_0,f_0} a_{n,T,\Omega,t_0,f_0} \psi_{n,T,\Omega}(t-t_0) \exp(j2\pi f_0 t)$$

where $\psi_{n,T,\Omega}(t)$ is the prolate spheroidal wave function

[Ref] D. Slepian and H. O. Pollak, “Prolate spheroidal wave functions, Fourier analysis and uncertainty-I,” *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, 1961.

Concept of the prolate spheroidal wave function (PSWF):

- FT: $X(f) = \int_{-\infty}^{\infty} \exp(-j2\pi f t) x(t) dt$, $x, f \in (-\infty, \infty)$.

energy preservation property (Parseval's property)

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- finite Fourier transform (fi-FT):

$$X_{fi}(f) = \int_{-T}^{T} \exp(-j2\pi f t) x(t) dt$$

space interval: $t \in [-T, T]$, frequency interval: $f \in [-\Omega, \Omega]$

$$0 < \text{energy preservation ratio} = \frac{\int_{-\Omega}^{\Omega} |X_{fi}(f)|^2 df}{\int_{-T}^{T} |x(t)|^2 dt} < 1$$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\frac{\int_{-\Omega}^{\Omega} |X_{fi}(f)|^2 df}{\int_{-T}^{T} |x(t)|^2 dt}$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\frac{\int_{-\Omega}^{\Omega} |X_{f_i}(f)|^2 df}{\int_{-T}^T |x(t)|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$

$\psi_{1,T,\Omega}(t)$ can maximize $\frac{\int_{-\Omega}^{\Omega} |X_{f_i}(f)|^2 df}{\int_{-T}^T |x(t)|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$ and $\psi_{1,T,\Omega}$

$\psi_{2,T,\Omega}(t)$ can maximize $\frac{\int_{-\Omega}^{\Omega} |X_{f_i}(f)|^2 df}{\int_{-T}^T |x(t)|^2 dt}$

and so on.

- Prolate spheroidal wave functions (PSWFs) are the continuous functions that satisfy: $\int_{-T}^T K_{F,\Omega}(t_1, t) \psi_{n,T,\Omega}(t) dt = \lambda_{n,T,\Omega} \psi_{n,T,\Omega}(t_1)$,

where $K_{F,\Omega}(t_1, t) = \frac{\sin[2\pi\Omega(t_1 - t)]}{\pi(t_1 - t)}$

PSWFs are orthonormal and can be sorted according to the values of $\lambda_{n,T,\Omega}$'s:

$$\int_{-T}^T \psi_{m,T,\Omega}(t) \psi_{n,T,\Omega}(t) dt = \delta_{m,n}$$

$$1 > \lambda_{0,T,\Omega} > \lambda_{1,T,\Omega} > \lambda_{2,T,\Omega} > \dots > 0. \quad (\text{All of } \lambda_{n,T,\Omega} \text{'s are real})$$

附錄十：Compressive Sensing and Matching Pursuit 的觀念

壓縮感知

Different from orthogonal basis expansion, which applies a complete and orthogonal basis set, compressive sensing is to use an **over-complete** and **non-orthogonal basis set** to expand a signal.

Example:

Fourier series expansion is an orthogonal basis expansion method:

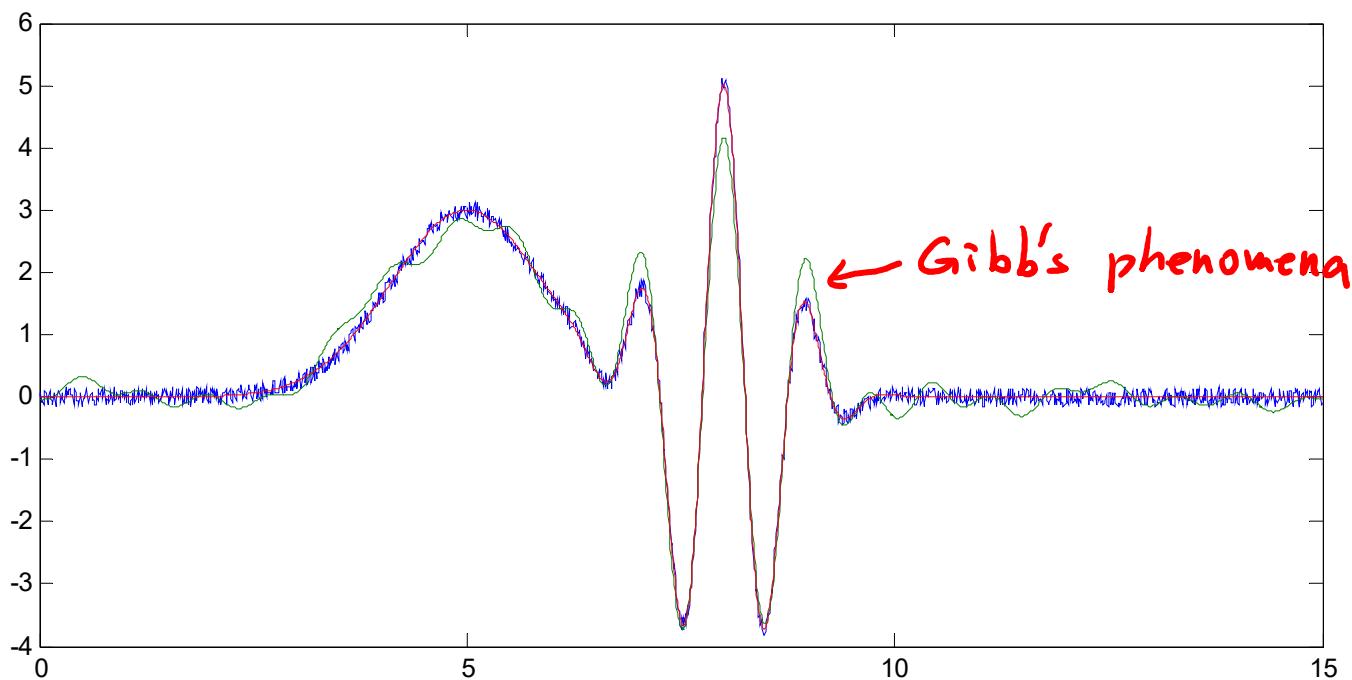
$$x(t) \approx \sum_{m=1}^M a_m \exp(j2\pi f_m t)$$

$$\int \exp(j2\pi f_m t) \overline{\exp(j2\pi f_n t)} dt = 0 \quad \text{if } f_m \neq f_n$$

Three-parameter atom expansion, Four-parameter atom (chirplet) expansion, and PSWF expansion are over-complete and non-orthogonal basis expansion methods.

$$x(t) \approx \sum a_{t_0, f_0, \sigma} \varphi_{t_0, f_0, \sigma}(t)$$

$\varphi_{t_0, f_0, \sigma}(t)$ do not form a complete and orthogonal set.



For example, in the above figure, the blue line is the original signal

- When using three-parameter atoms, the expansion result is the red line

$$x(t) = 3e^{-0.2\pi(t-5)^2} + 2.5e^{-0.4\pi(t-8)^2+j2\pi t} + 2.5e^{-0.4\pi(t-8)^2-j2\pi t}$$

Only 3 terms are used and the normalized root square error is 0.39%

$$t_0=5, G^2=5, f_0=0 \quad t_0=8, G^2=2.5, f_0=1 \quad t_0=8, G^2=2.5, f_0=-1$$

- When using Fourier basis, if 31 terms are used, the expansion result is the green line and the normalized root square error is 3.22%

The problems that compressive sensing deals with:

Suppose that $b_0(t), b_1(t), b_2(t), b_3(t) \dots$ form an over-complete and non-orthogonal basis set.

(Problem 1) We want to minimize $\|c\|_0$ ($\|\cdot\|_0$ is L_0 norm, $\|c\|_0$ 意指 c_m 的值不為 0 的個數) such that

$$x(t) = \sum_m c_m b_m(t)$$

L_0 norm
= sparse

(Problem 2) We want to minimize $\|c\|_0$ such that

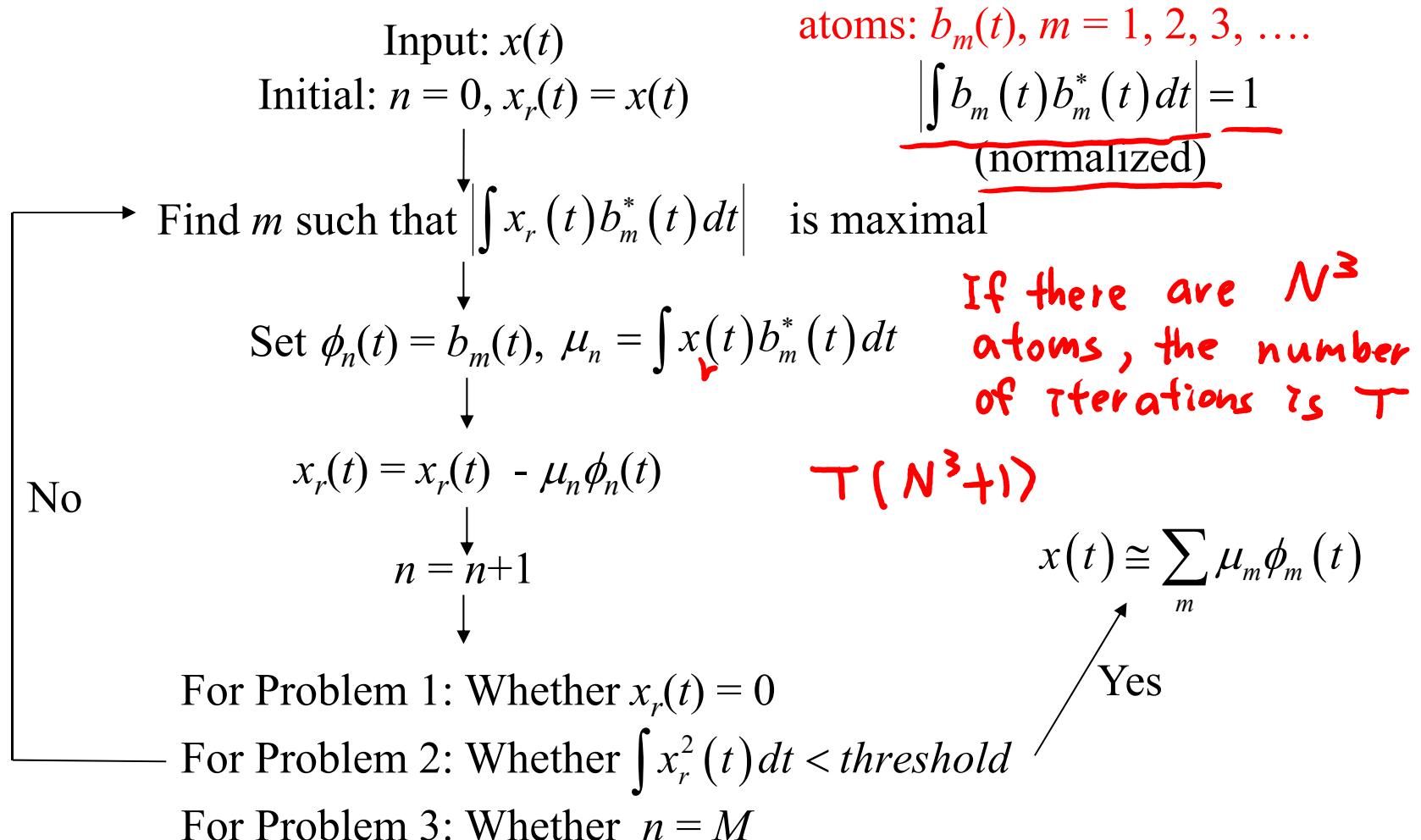
$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < \text{threshold}$$

(Problem 3) When $\|c\|_0$ is fixed to M , we want to minimize

$$\int \left(x(t) - \sum_{m=1}^M c_m b_m(t) \right)^2 dt$$

Question: How do we solve the optimization problems on page 229?

Method 1: Matching Pursuit (Greedy Algorithm)



Method 2: Basis Pursuit

Change the L_0 norm into the L_1 norm

$$\|c\|_1 = |c_0| + |c_1| + |c_2| + \dots$$

(Problem 1) We want to minimize $\|c\|_1$ such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $\|c\|_1$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < \text{threshold}$$

(Problem 3) When $\|c\|_1 \leq M$, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

$$|x[n]|^0 = \begin{cases} 1 & \text{if } x[n] \neq 0 \\ 0 & \text{if } x[n] = 0 \end{cases}$$

Norm (L_α norm): $\|x[n]\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$

$$\lim_{\alpha \rightarrow 0} (L_\alpha \text{ norm})^\alpha = K \quad \text{where } K \text{ is the number of points such that } x[n] \neq 0$$

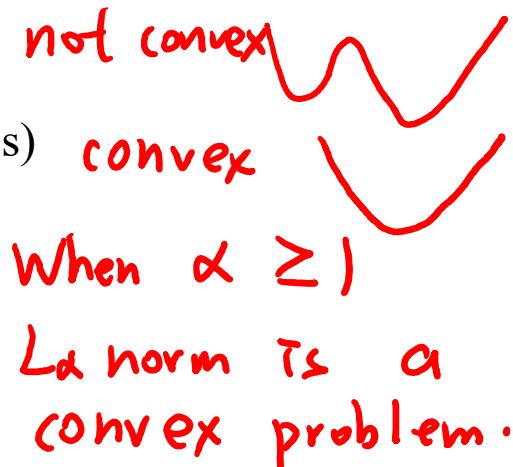
(Physical meaning: The number of nonzero points)

L_1 norm: $\|x[n]\|_1 = \sum_{n=0}^{N-1} |x[n]|$

(Physical meaning: Sum of Amplitudes)

L_2 norm: $\|x[n]\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$

(Physical meaning: Distance)



Matching Pursuit: Zero order norm $\lim_{\alpha \rightarrow 0} (L_\alpha \text{ norm})^\alpha$

Basis Pursuit: First order norm L_1 norm

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