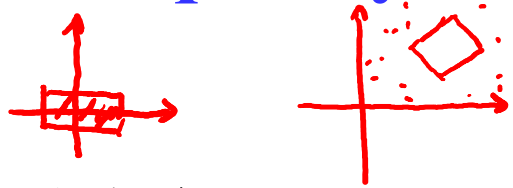
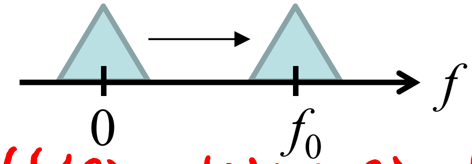
VIII. Motions on the Time-Frequency Distribution



Fourier spectrum 為 1-D form，只有二種可能的運動或變形：

① Modulation $e^{j2\pi f_0 t}x(t) \xrightarrow{FT} X(f-f_0)$

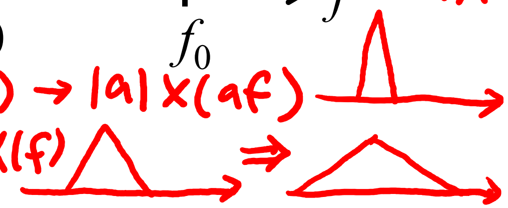
② Scaling $x(t/a) \xrightarrow{FT} |a|X(af)$



|a|X(af)

x(t/a) → |a| x(af)

X(f) ⇒

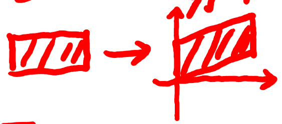


Time-frequency analysis 為 2-D，在 2-D 平面上有多種可能的運動或變形

(1) Horizontal shifting (2) Vertical shifting

(3) Dilation = scaling (4) Shearing

shear (剪羊毛)

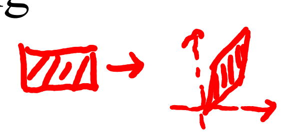


(5) Generalized Shearing (6) Rotation

FrFT

(7) Twisting

LCT 扭曲



8-1 Basic Motions

(1) Horizontal Shifting

$$x(t-t_0) \rightarrow S_x(t-t_0, f)e^{-j2\pi f t_0} \quad \text{,STFT, Gabor}$$

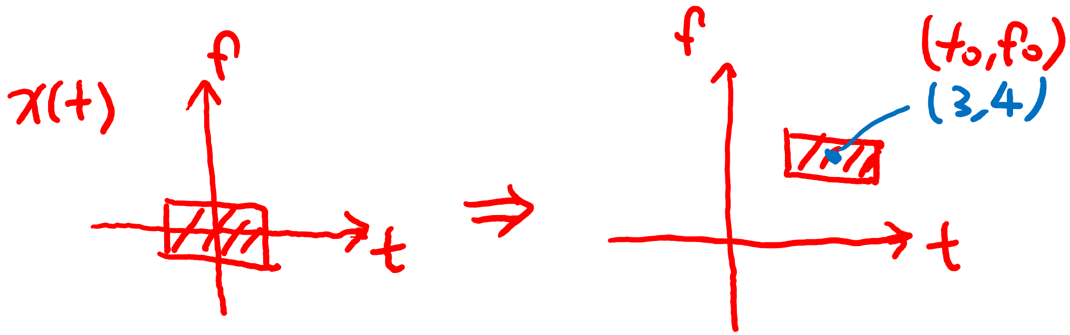
$$\rightarrow W_x(t-t_0, f) \quad \text{,Wigner}$$

(2) Vertical Shifting

$$e^{j2\pi f_0 t}x(t) \rightarrow S_x(t, f-f_0) \quad \text{,STFT,Gabor}$$

modulation

$$\rightarrow W_x(t, f-f_0) \quad \text{,Wigner}$$



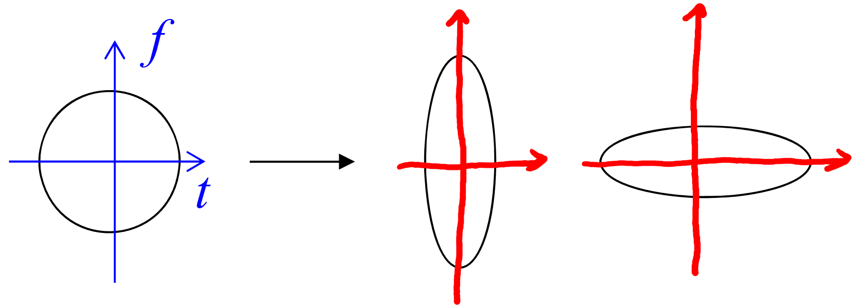
$e^{j8\pi t}x(t-3)$

(3) Dilation (scaling)

$|a|>1$ or $|a|<1$?

$$\frac{1}{\sqrt{|a|}}\,x\!\left(\frac{t}{a}\right) \rightarrow \approx S_x\!\left(\frac{t}{a}, af\right) \text{ ,STFT,Gabor}$$

$$\rightarrow W_x\!\left(\frac{t}{a}, af\right) \text{ ,WDF}$$



instantaneous frequency = at

(4) Shearing

$e^{j\pi a t^2} y(t)$ $e^{j\pi \frac{t^2}{a}} * y(t)$

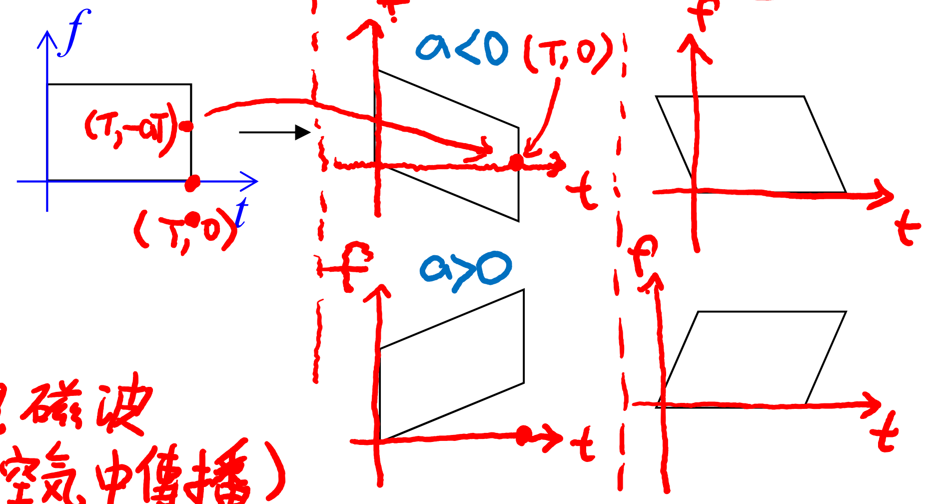
$S_y(t,f)$ $S_x(t,f)$

$$x(t) = e^{j\pi a t^2} y(t)$$

$S_x(t,f) \approx S_y(t, f - \underline{at})$,STFT,Gabor

$W_x(t,f) = W_y(t, f - at)$,WDF

$S_x(T, 0) = S_y(T, -aT)$



$a = \lambda z$ (page 267) Fresnel transform (電磁波在空氣中傳播)

$x(t) = e^{j\pi \frac{t^2}{a}} * y(t)$ (* means convolution)

$S_x(t,f) \approx S_y(t - af, f)$, STFT,Gabor

$W_x(t,f) = W_y(t - af, f)$, WDF

$X(f) = FT\left(e^{j\pi \frac{t^2}{a}}\right) Y(f) = \sqrt{ja}\, e^{-j\pi a f^2} Y(f)$

$e^{-\pi t^2} \to e^{-\pi f^2}$

$e^{-\pi (\frac{t}{c})^2} \to |c| e^{-\pi c^2 f^2}$

if $c^2 = ja$

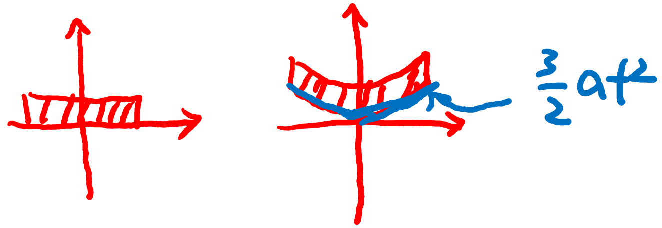
$e^{j\pi \frac{t^2}{a}} \to \sqrt{ja}\, e^{-j\pi a f^2}$

instantaneous freq. $= \frac{3}{2} a t^2$

$a > 0$ or $a < 0$?

If $x(t) = e^{j\pi a t^3} y(t)$

$S_x(t,f) \approx S_y(t, f - \frac{3}{2} a t^2)$



(Proof): When $x(t) = e^{j\pi a t^2} y(t)$,

$$
\begin{aligned}
W_x(t,f) &= \int_{-\infty}^{\infty} x(t+\tau/2)\, x^*(t-\tau/2)\, e^{-j2\pi\tau f} \cdot d\tau \\
&= \int_{-\infty}^{\infty} e^{j\pi a (t+\tau/2)^2} e^{-j\pi a (t-\tau/2)^2} y(t+\tau/2)\, y^*(t-\tau/2)\, e^{-j2\pi\tau f} d\tau \\
&= \int_{-\infty}^{\infty} e^{j2\pi a t\tau} y(t+\tau/2)\, y^*(t-\tau/2)\, e^{-j2\pi\tau f} d\tau \\
&= \int_{-\infty}^{\infty} y(t+\tau/2)\, y^*(t-\tau/2)\, e^{-j2\pi\tau (f-at)} d\tau \\
&= W_y(t, f-at)
\end{aligned}
$$

(5) Generalized Shearing

$$x(t) = e^{j\phi(t)}y(t)$$ 的影響?

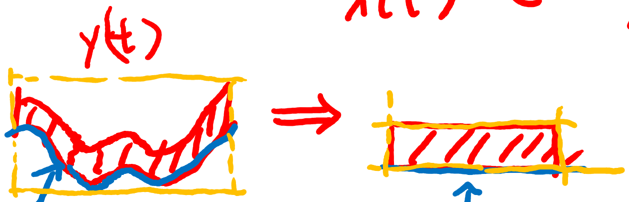
$$\phi(t) = \sum_{k=0}^{n} a_k t^k$$

Instantaneous freq. $= \sum_{k=1}^{n} \frac{k a_k}{2\pi} t^{k-1}$

$f = f_0$

$f = \sum_{k=1}^{n} \frac{k a_k}{2\pi} t^{k-1} + f_0$

$x(t) = e^{j\theta(t)} y(t)$



$f = \sum_{k=0}^{n} b_k t^k$

$f = 0$

$$S_x(t,f) \cong S_y\left(t, f - \sum_{k=1}^{n} \frac{k a_k}{2\pi} t^{k-1}\right) \text{ ,STFT,Gabor}$$

$$W_x(t,f) \cong W_y\left(t, f - \sum_{k=1}^{n} \frac{k a_k}{2\pi} t^{k-1}\right) \text{ ,WDF}$$

$\theta(t) = -2\pi \int \sum_{k=0}^{n} b_k t^k dt$

$\theta(t) = -2\pi \sum_{k=0}^{n} \frac{b_k t^{k+1}}{k+1}$

J. J. Ding, S. C. Pei, and T. Y. Ko, "Higher order modulation and the efficient sampling algorithm for time variant signal," *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

J. J. Ding and C. H. Lee, "Noise removing for time-variant vocal signal by generalized modulation," *APSIPA ASC*, pp. 1-10, Kaohsiung, Taiwan, Oct. 2013

Q:

If $x(t) = h(t) * y(t)$ where $h(t) = IFT\left(\exp\left(j\sum_{k=0}^{n} a_k f^k\right)\right)$

then

$$S_x(t, f) \cong S_y(t + \frac{1}{2\pi}\sum_{k=1}^{n} k a_k f^{k-1}, \quad f)$$,STFT,Gabor

$$W_x(t, f) \cong W_y(t + \frac{1}{2\pi}\sum_{k=1}^{n} k a_k f^{k-1}, \quad f)$$,WDF

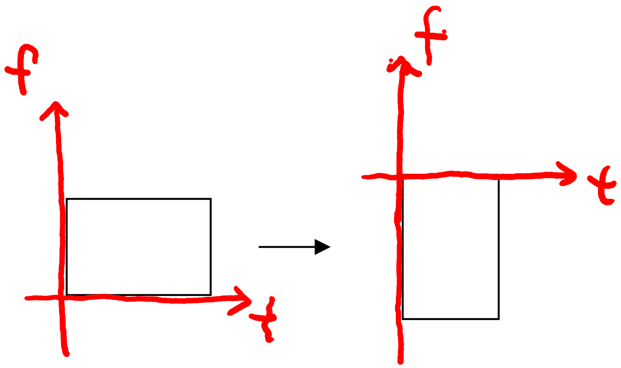
8-2 Rotation by $\pi/2$: Fourier Transform

$$X(f) = FT(x(t))$$

$$|S_X(t,f)| \approx |S_x(-f,t)| \quad \text{,STFT}$$

$$G_X(t,f) = G_x(-f,t)e^{-j2\pi ft} \quad \text{,Gabor}$$

$$W_X(t,f) = W_x(-f,t) \quad \text{,WDF}$$



(clockwise rotation by 90°)

$W_X(1,0) = W_x(0,1)$

$W_X(0,1) = W_x(-1,0)$

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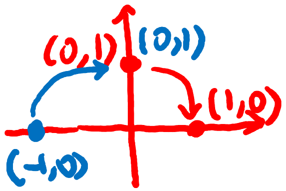
$FT^4(x(t)) = x(t)$

rotation by 360°



Strictly speaking, the rec-STFT have no rotation property.

For Gabor transforms, if

$$\boxed{G_x(t,f)=\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}x(\tau)d\tau}\ ,$$

$$G_X(t,f)=\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}X(\tau)d\tau \qquad X(f)=FT[x(t)]=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$$

then $\boxed{G_X(t,f)=G_x(-f,t)e^{-j2\pi tf}}$

(clockwise rotation by 90° for amplitude)

(Proof): $$G_X(t,f)=\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}\int_{-\infty}^{\infty}x(u)e^{-j2\pi\tau u}du\,d\tau$$

$$=\int_{-\infty}^{\infty}x(u)e^{-\pi(\tau-t)^2}\left(\int_{-\infty}^{\infty}e^{-j2\pi\tau(f+u)}d\tau\right)du$$

$$=\int_{-\infty}^{\infty}x(u)\left(\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi\tau(f+u)}d\tau\right)du=\int_{-\infty}^{\infty}x(u)\left(FT\left(e^{-\pi(\tau-t)^2}\right)\Big|_{f\to f+u}\right)du$$

Since $\quad FT\left(e^{-\pi\tau^2}\right)=e^{-\pi f^2},\quad FT\left(e^{-\pi(\tau-t)^2}\right)=e^{-j2\pi tf}e^{-\pi f^2}$

$$G_X(t,f)=\int_{-\infty}^{\infty}x(u)e^{-j2\pi t(f+u)}e^{-\pi(f+u)^2}du$$

$$=e^{-j2\pi tf}\int_{-\infty}^{\infty}x(u)e^{-j2\pi tu}e^{-\pi(u-(-f))^2}du\ =G_x(-f,t)e^{-j2\pi tf}$$

If we define the Gabor transform as

$$\boxed{G_x(t,f) = e^{j\pi f t}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau},$$

and $G_X(t,f) = e^{j\pi f t}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} X(\tau)\, d\tau$

then $\boxed{G_X(t,f) = G_x(-f,t)}$

If $W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$ is the WDF of $x(t)$,

$W_X(t,f)=\int_{-\infty}^{\infty} X(t+\tau/2)\cdot X^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$ is the WDF of $X(f)$,

then $W_X(t,f)=W_x(-f,t)$

(clockwise rotation by 90°)

還有哪些 time-frequency distribution 也有類似性質？

- If $X(f) = IFT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{j2\pi f t} dt$, then

  $W_X(t,f) = W_x(f,-t)$, $G_X(t,f) = G_x(f,-t) e^{j2\pi t f}$

  (counterclockwise rotation by 90°).

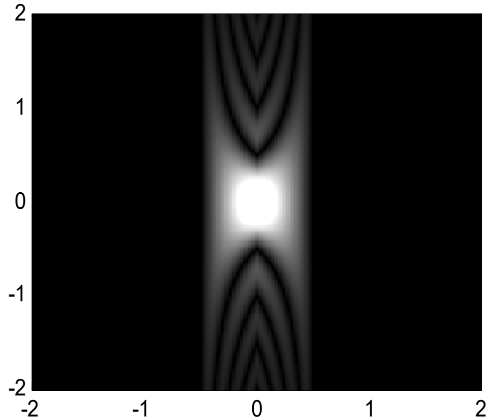
- If $X(f) = x(-t)$, then

  $W_X(t,f) = W_x(-t,-f)$, $G_X(t,f) = G_x(-t,-f)$.

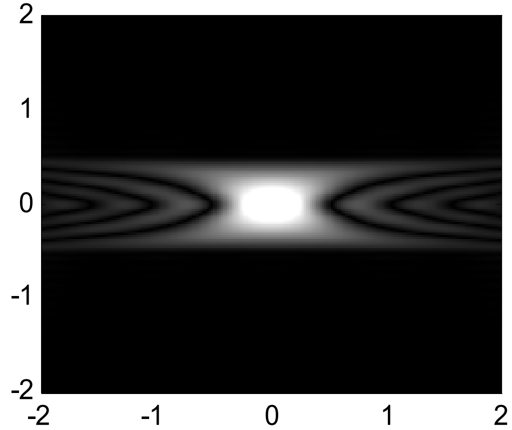
  (rotation by 180°).

Examples: $x(t) = \Pi(t)$, $X(f) = FT[x(t)] = \text{sinc}(f)$.

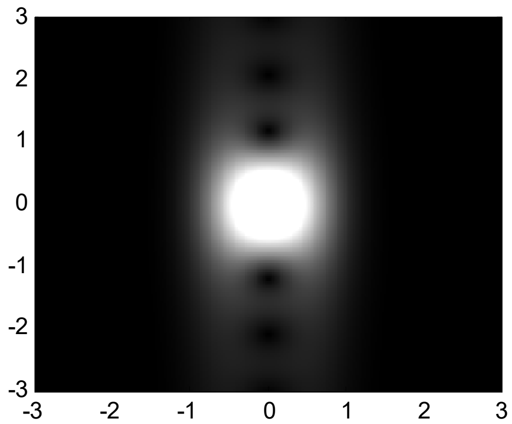
WDF of $\Pi(t)$



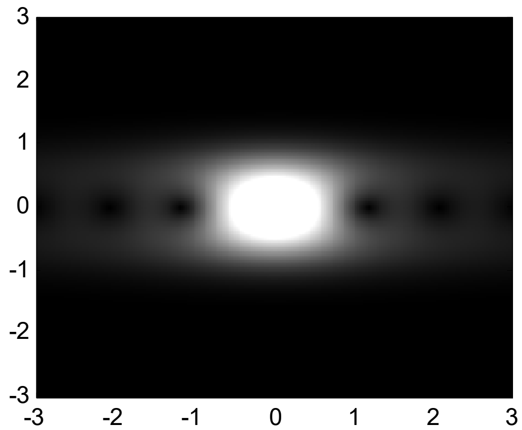
WDF of $\text{sinc}(f)$



Gabor transform of $\Pi(t)$



Gabor transform of $\text{sinc}(f)$



If a function is an eigenfunction of the Fourier transform,

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} x(t)\,dt = \lambda x(f) \qquad \lambda = 1, -j, -1, j$$

then its WDF and Gabor transform have the property of

$$W_x(t,f) = W_x(f,-t) \qquad\qquad \left|G_x(t,f)\right| = \left|G_x(f,-t)\right|$$

(轉了 90°之後，和原來還是一樣）

Example: Gaussian function

$$\exp\left(-\pi t^2\right)$$

Hermite-Gaussian function

$$\phi_m(t) = \exp\left(-\pi t^2\right) H_m(t)$$

Hermite polynomials: $H_m(t) = C_m e^{2\pi t^2} \dfrac{d^m}{dt^m} e^{-2\pi t^2}$, C_m is some constant,

$$H_0(t) = 1 \qquad H_1(t) = t \qquad H_2(t) = 4\pi t^2 - 1$$

$$H_3(t) = 4\pi t^3 - 3t \qquad H_4(t) = 16\pi^2 t^4 - 24\pi t^2 + 3$$

$$\int_{-\infty}^{\infty} e^{-2\pi t^2} H_m(t) H_n(t) = D_m \delta_{m,n}$$, D_m is some constant,

$\delta_{m,n} = 1$ when $m = n$, $\delta_{m,n} = 0$ otherwise.

[Ref] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 1990.

Hermite-Gaussian functions are eigenfunctions of the Fourier transform

$$\int_{-\infty}^{\infty} \phi_m(t) e^{-j2\pi f t} dt = (-j)^m \phi_m(f)$$

eigenvalue eigenfunction

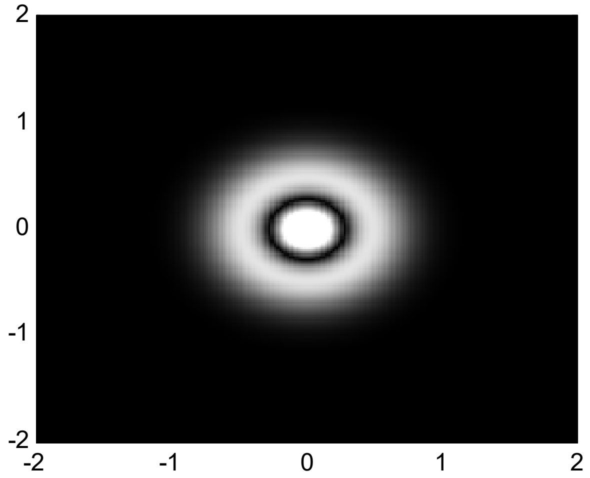
Any eigenfunction of the Fourier transform can be expressed as the form of

$$k(t) = \sum_{q=0}^{\infty} a_{4q+r} \phi_{4q+r}(t)$$

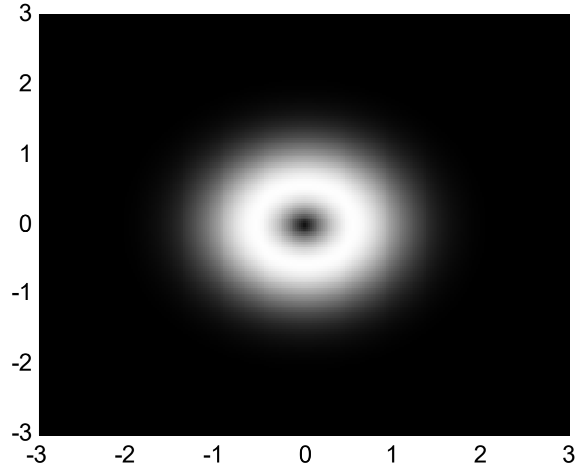
where $r = 0, 1, 2,$ or 3,
a_{4q+r} are some constants

$$\int_{-\infty}^{\infty} k(t) e^{-j2\pi f t} dt = (-j)^r k(f)$$

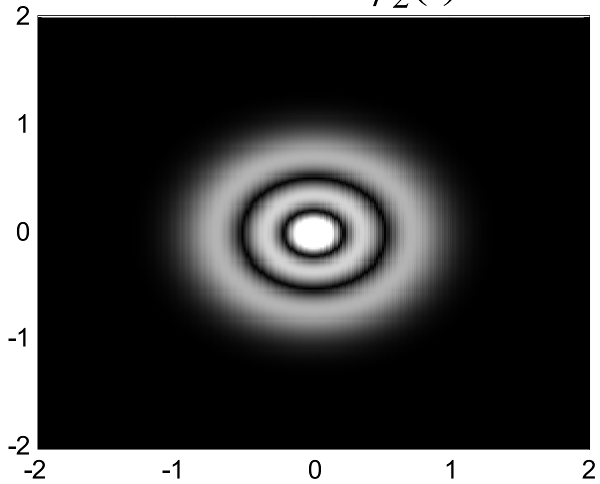
WDF for $\phi_1(t)$



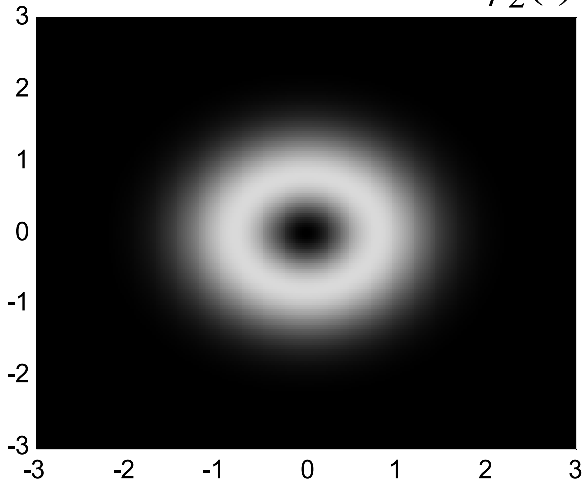
Gabor transform for $\phi_1(t)$



WDF for $\phi_2(t)$



Gabor transform for $\phi_2(t)$



Problem: How to rotate the time-frequency distribution by the angle other than $\pi/2$, π, and $3\pi/2$?

8-3 Rotation: Fractional Fourier Transforms (FRFTs)

$$X_\phi(u) = \sqrt{1 - j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2} \int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)\,dt\,, \qquad \phi = 0.5a\pi$$

chirp

scaled FT

chirp

When $\phi = 0.5\pi$, the FRFT becomes the FT.

For the original FT

$a=1$, $\phi = 0.5\pi$

$\csc\phi = 1$, $\cot\phi = 0$

Additivity property:

If we denote the FRFT as O_F^ϕ (i.e., $X_\phi(u) = O_F^\phi\left[x(t)\right]$)

then $O_F^\sigma\left\{O_F^\phi\left[x(t)\right]\right\} = O_F^{\phi+\sigma}\left[x(t)\right]$

Physical meaning: Performing the FT *a* times.

Another definition $X_{\phi}(u)=\sqrt{\frac{1-j\cot\phi}{2\pi}}\,e^{j\frac{\cot\phi}{2}\cdot u^2}\int_{-\infty}^{\infty}e^{-j\csc\phi\cdot u\,t}\,e^{j\frac{\cot\phi}{2}\cdot t^2}\,x(t)\,dt$

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] N. Wiener, "Hermitian polynomials and Fourier analysis," *Journal of Mathematics Physics MIT*, vol. 18, pp. 70-73, 1929.

[Ref] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

[Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

[Ref] S. C. Pei and J. J. Ding, "Closed form discrete fractional and affine Fourier transforms," *IEEE Trans. Signal Processing*, vol. 48, no. 5, pp. 1338-1353, May 2000.

$$FT\left[x(t)\right] = X(f)$$

$$FT\left\{FT\left[x(t)\right]\right\} = x(-t)$$

$$FT\left(FT\left\{FT\left[x(t)\right]\right\}\right) = X(-f) = IFT\left[f(t)\right]$$

$$FT\left[FT\left(FT\left\{FT\left[x(t)\right]\right\}\right)\right] = x(t)$$

What happen if we do the FT non-integer times?

Physical Meaning:

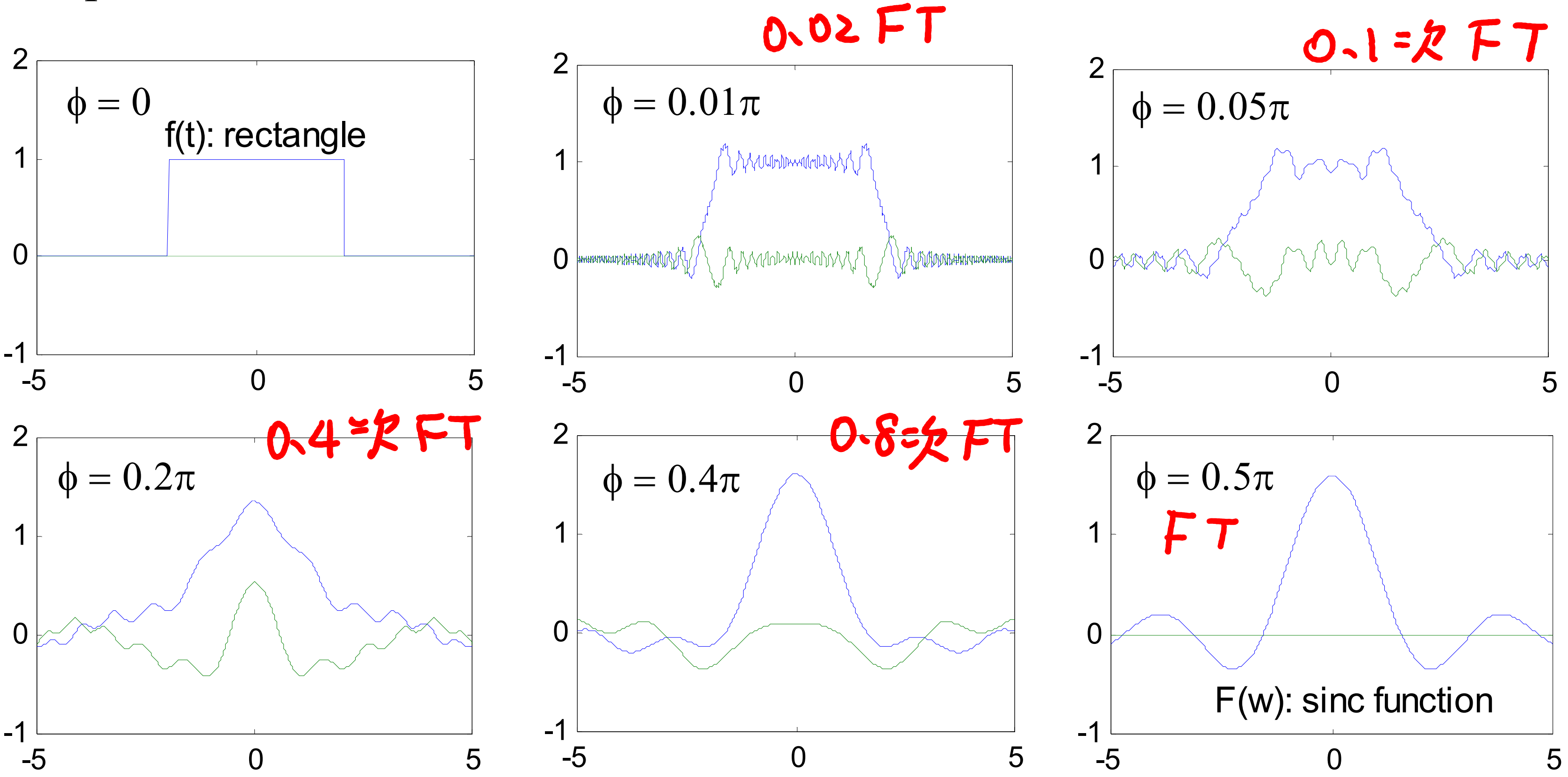
Fourier Transform: time domain $\rightarrow$ frequency domain

Fractional Fourier transform: time domain $\rightarrow$ fractional domain

Fractional domain: the domain between time and frequency

(partially like time and partially like frequency)

Experiment:



blue line: real part

green line: imaginary part

Time domain	Frequency domain	fractional domain
Modulation	Shifting	Modulation + Shifting
Shifting	Modulation	Modulation + Shifting
Differentiation	$\times j2\pi f$	Differentiation and $\times j2\pi f$
$\times -j2\pi f$	Differentiation	Differentiation and $\times -j2\pi f$

$$x(t-t_0) \xrightarrow{FT} \exp(-j2\pi f t_0) X(f)$$

$$x(t-t_0) \xrightarrow{fractional\ FT} \exp(j\varphi - j2\pi u t_0 \sin\phi) X(u - t_0 \cos\phi)$$

$$\varphi = \pi t_0^2 \sin\phi \cos\phi$$

$$\frac{d}{dt} x(t) \xrightarrow{FT} j2\pi f\, X(f)$$

$$\frac{d}{dt} x(t) \xrightarrow{fractional\ FT} j2\pi u\, X(u) \sin\phi + \frac{d}{du} X(u) \cos\phi$$

[Theorem] The fractional Fourier transform (FRFT) with angle ϕ is equivalent to the **clockwise rotation operation** with angle ϕ for the Wigner distribution function (or for the Gabor transform)

FRFT with parameter ϕ = ↷ with angle ϕ

For the WDF

If $W_x(t, f)$ is the WDF of $x(t)$, and $W_{X\phi}(u, v)$ is the WDF of $X_\phi(u)$, ($X_\phi(u)$ is the FRFT of $x(t)$), then

$$\boxed{W_{X_\phi}(u,v) = W_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

page 264 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$

For the Gabor transform (with standard definition)

If $G_x(t, f)$ is the Gabor transform of $x(t)$,

and $G_{X\phi}(u, v)$ is the Gabor transform of $X_\phi(u)$, then

$$\boxed{G_{X_\phi}(u,v) = e^{j[-2\pi uv\sin^2\phi + \pi(u^2-v^2)\sin(2\phi)/2]} G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

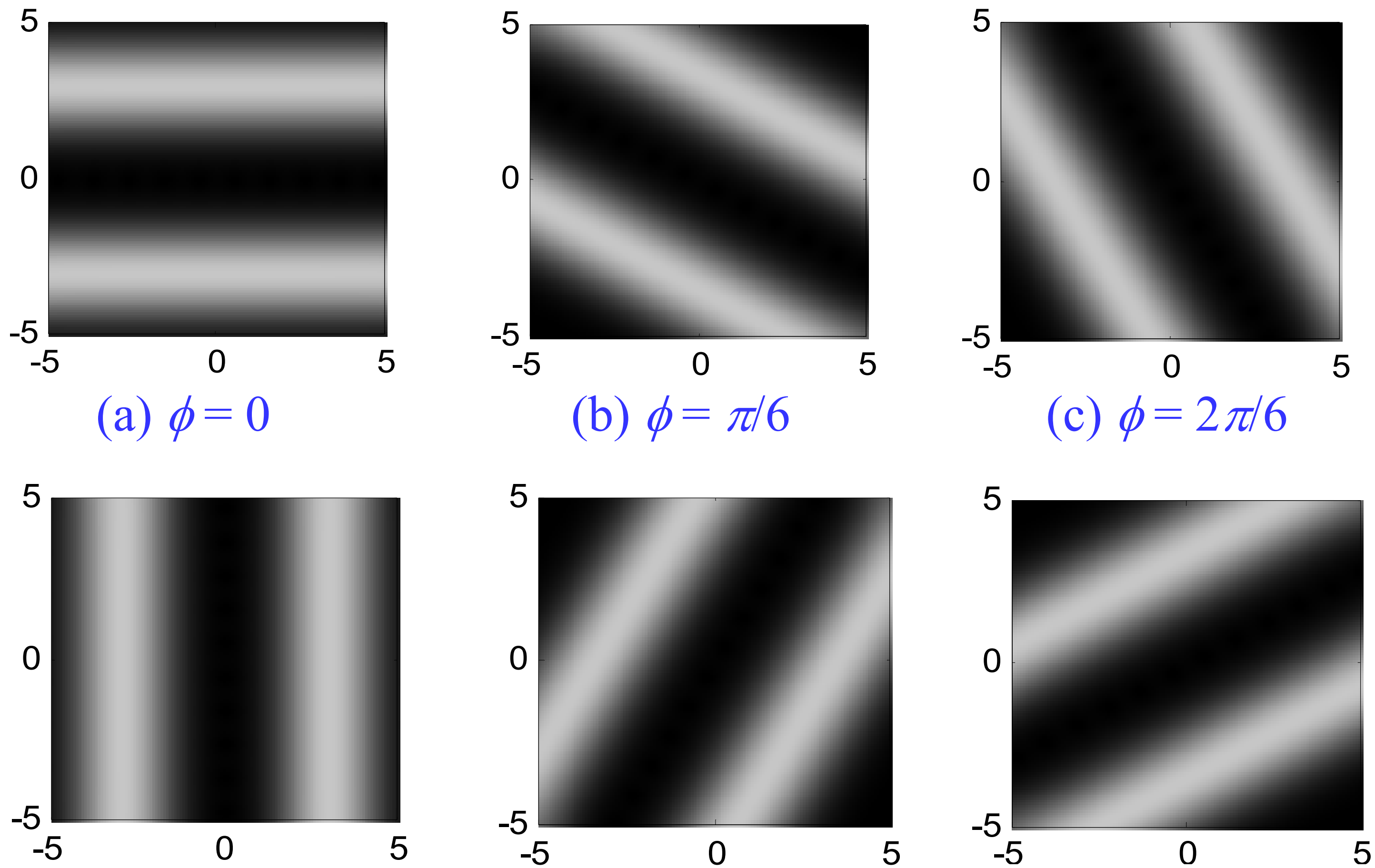
$$\boxed{\left|G_{X_\phi}(u,v)\right| = \left|G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)\right|}$$

For the Gabor transform (with another definition on page 244)

$$\boxed{G_{X_\phi}(u,v) = G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

The Cohen's class distribution and the Gabor-Wigner transform also have the rotation property

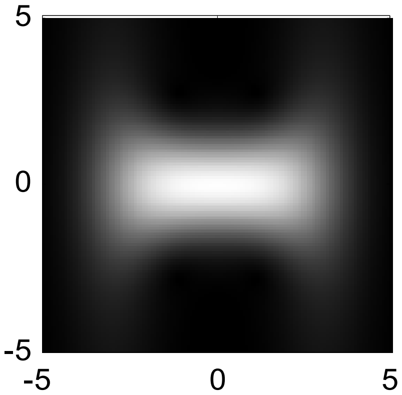
The Gabor Transform for the FRFT of a cosine function



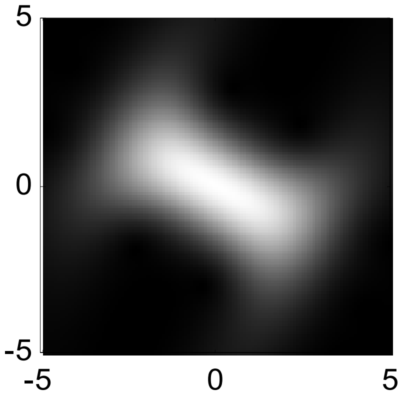
(a) $\phi = 0$ (b) $\phi = \pi/6$ (c) $\phi = 2\pi/6$

(d) $\phi = 3\pi/6$ (e) $\phi = 4\pi/6$ (f) $\phi = 5\pi/6$

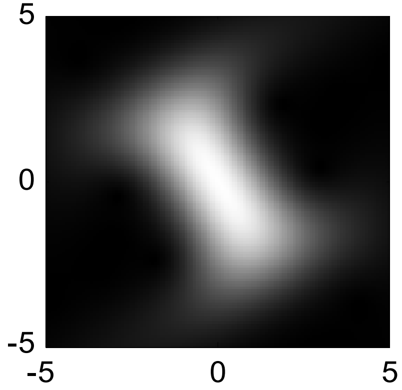
The Gabor Transform for the FRFT of a rectangular function.



(a) $\phi = 0$

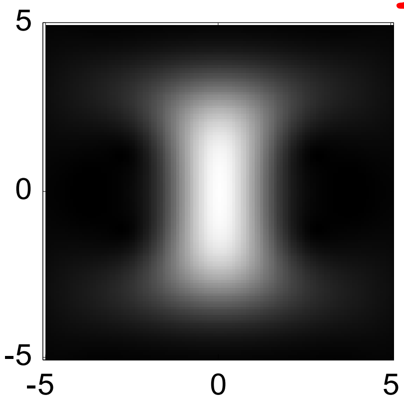


$\frac{1}{3}$FT (b) $\phi = \pi/6$ 30°



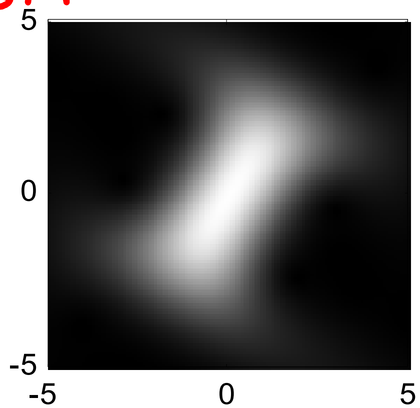
$\frac{2}{3}$FT

(c) $\phi = 2\pi/6$ 60°



(d) $\phi = 3\pi/6$

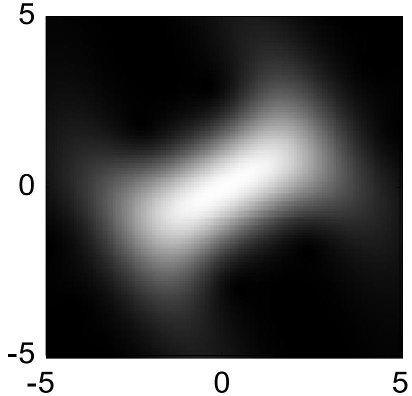
FT 90°



(e) $\phi = 4\pi/6$

120°

$1\frac{1}{3}$FT



(f) $\phi = 5\pi/6$

150°

$1\frac{2}{3}$FT

完整

8-4 Twisting: Linear Canonical Transform (LCT)

2 chirps + 1 scaled FT

$$X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{jb}}\,e^{j\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-j2\pi\frac{1}{b}ut}e^{j\pi\frac{a}{b}t^2}x(t)dt \quad \text{when } b\neq 0$$

$$X_{(a,0,c,d)}(u)=\sqrt{d}\cdot e^{j\pi cdu^2}x(du) \quad \text{when } b=0$$

If $b=0$, $d=-1$, $|X_{abcd}(u)|=|x(-u)|$

$ad-bc=1$ should be satisfied

Four parameters a, b, c, d

When $a=d=1$, $b=\lambda z$

$$X(u)=\sqrt{\frac{1}{\lambda z}}\,e^{j\pi\frac{1}{\lambda z}u^2}\int e^{-j\frac{2\pi}{\lambda z}ut}e^{j\frac{\pi}{\lambda z}t^2}x(t)dt$$

$$=\sqrt{\frac{1}{\lambda z}}\int e^{j\frac{\pi}{\lambda z}(u-t)^2}x(t)dt$$

FT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

IFT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

FrFT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$
(page 253)

$\frac{a}{b}=\frac{d}{b}=\cot\phi$

Additivity property of the WDF

If we denote the LCT by $O_F^{(a,b,c,d)}$, i.e., $X_{(a,b,c,d)}(u) = O_F^{(a,b,c,d)}[x(t)]$

then $$O_F^{(a_2,b_2,c_2,d_2)}\left\{O_F^{(a_1,b_1,c_1,d_1)}[x(t)]\right\} = O_F^{(a_3,b_3,c_3,d_3)}[x(t)]$$

where $$\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

[Ref] K. B. Wolf, "*Integral Transforms in Science and Engineering*," Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

If $W_{X_{(a,b,c,d)}}(u,v)$ is the WDF of $X_{(a,b,c,d)}(u)$, where $X_{(a,b,c,d)}(u)$ is the LCT of $x(t)$, then

$$W_{X_{(a,b,c,d)}}(u,v) = W_x(du - bv\ ,\ -cu + av)$$

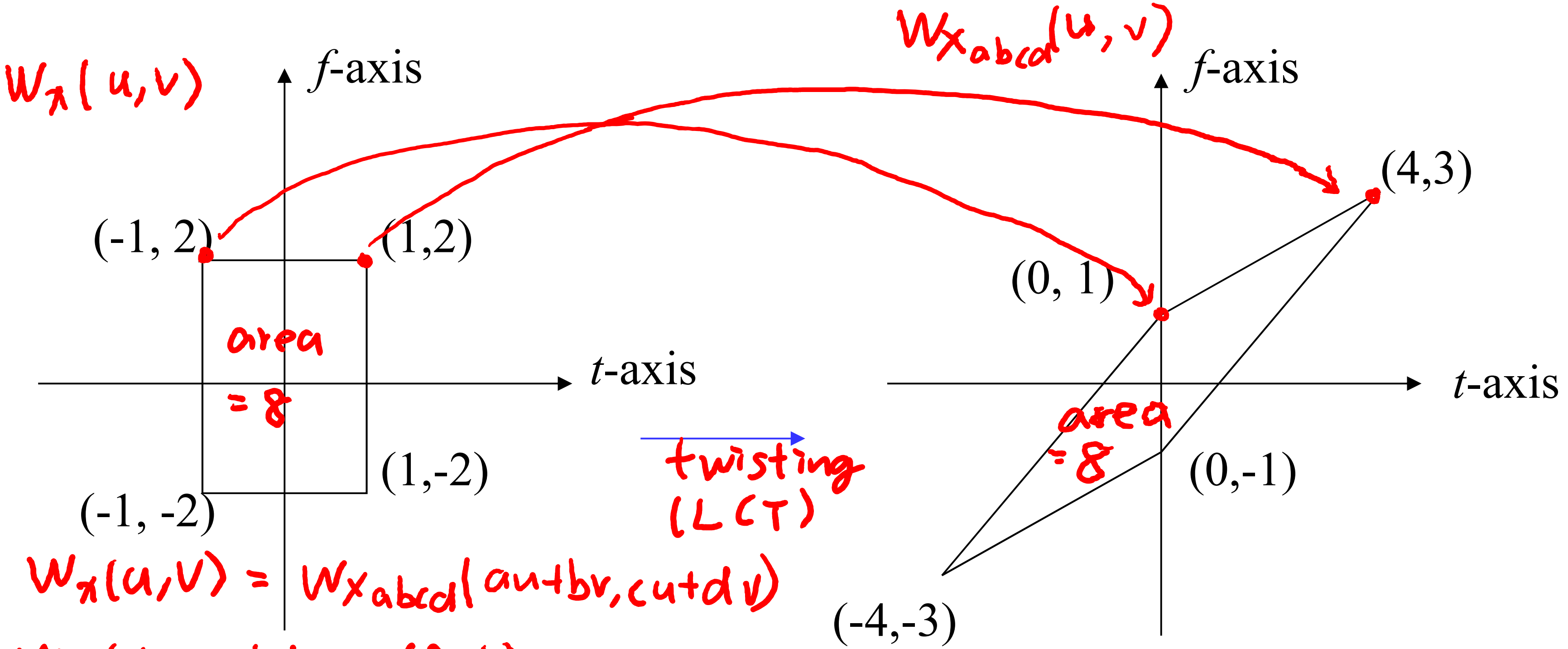
$$W_{X_{(a,b,c,d)}}(au + bv\ ,\ cu + dv) = W_x(u,v)$$

LCT == twisting operation for the WDF

$W_{x_{abcd}}(0,0) = W_x(0,0)$
for any a, b, c, d

The Cohen's class distribution also has the twisting operation.

我們可以自由的用 LCT 將一個中心在 (0, 0) 的平行四邊形的區域，扭曲成另外一個面積一樣且中心也在 (0, 0) 的平行四邊形區域。



$$W_x(u,v) = W_{x_{abcd}}(au+bv, cu+dv)$$

$$W_x(-1,2) = W_{x_{abcd}}(0,1)$$

$$W_x(1,2) = W_{x_{abcd}}(4,3)$$

$$\begin{cases} -a+2b=0 \\ a+2b=4 \end{cases} \qquad \begin{cases} -c+2d=1 \\ c+2d=3 \end{cases}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}}\, e^{j\pi \frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-j2\pi\frac{1}{b}ut}\, e^{j\pi\frac{a}{b}t^2}\, x(t)\,dt \qquad \text{when } b \neq 0$$

$$X_{(a,0,c,d)}(u) = \sqrt{d}\cdot e^{j\pi c d u^2}\, x(du) \qquad \text{when } b = 0$$

$ad - bc = 1$ should be satisfied

linear canonical transform

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ fractional Fourier transform
  - $\phi = \pi/2$ Fourier transform
  - $\phi = 0$ identity operation
  - $\phi = -\pi/2$ inverse Fourier transform
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$ Fresnel transform (convolution with a chirp)
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}$ chirp multiplication

  $$X_{(a,0,c,d)}(u) = e^{j\pi\tau u^2}\, x(u)$$
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/\sigma & 0 \\ 0 & \sigma \end{bmatrix}$ scaling

附錄十一　Linear Canonical Transform 和光學系統的關係

(1) Fresnel Transform (電磁波在空氣中的傳播)

$$U_o(x,y) = -\frac{i}{\lambda}\frac{e^{ikz}}{z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}\left[(x-x_i)^2+(y-y_i)^2\right]} U_i(x_i,y_i)\,dx_i dy_i$$

$k = 2\pi/\lambda$: wave number　　λ: wavelength　　z: distance of propagation

$$U_o(x,y) = e^{ikz}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(y-y_i)^2}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(x-x_i)^2} U_i(x_i,y_i)\,dx_i dy_i$$

(2 個 1-D 的 LCT)

$U_i(x,y) * e^{j\frac{k}{2z}x^2}$

convolution

$= U_i(x,y) * e^{j\frac{\pi}{\lambda z}x^2}$

Fresnel transform 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

(2) Spherical lens, refractive index = n

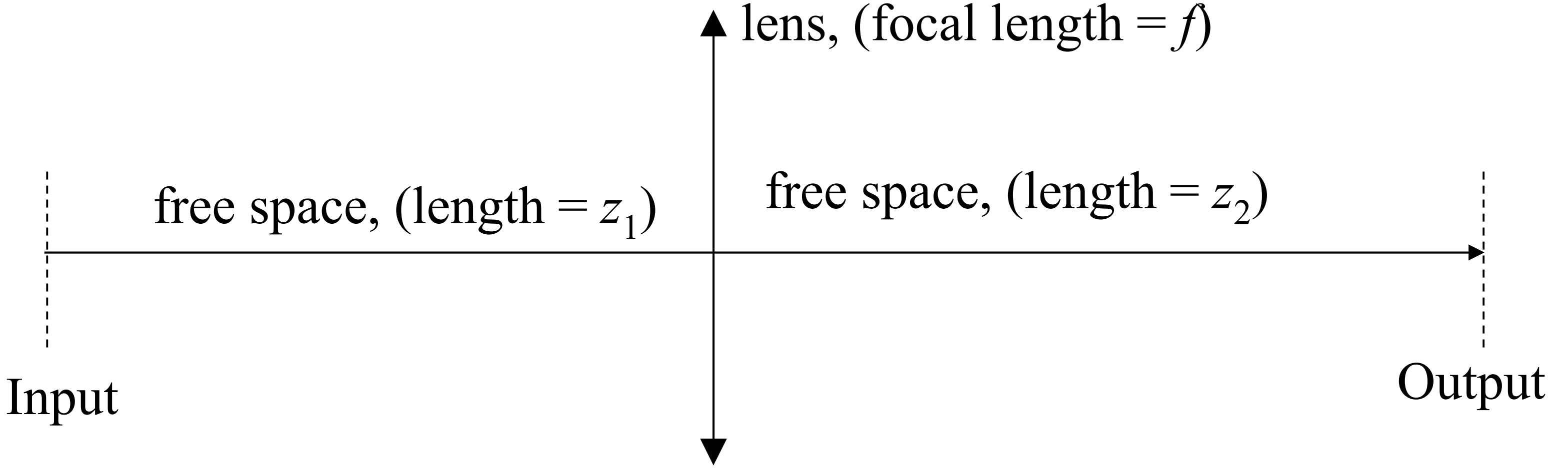
$$U_o(x,y) = e^{ikn\Delta} e^{-j\frac{k}{2f}\left[x^2+y^2\right]} U_i(x,y)$$

f: focal length Δ: thickness of lens

$\frac{k}{2f} = \frac{\pi}{\lambda f}$

經過 lens 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}$ 的情形

(3) Free space 和 Spherical lens 的綜合



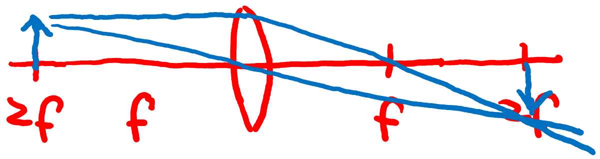
Input 和 output 之間的關係，可以用 LCT 表示

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_2}{f} & \lambda(z_1+z_2)-\dfrac{\lambda z_1 z_2}{f} \\ -\dfrac{1}{\lambda f} & 1-\dfrac{z_1}{f} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_2}{f} & \lambda(z_1+z_2)-\dfrac{\lambda z_1 z_2}{f} \\ -\dfrac{1}{\lambda f} & 1-\dfrac{z_1}{f} \end{bmatrix}$$

← 傅氏光學 (Fourier optics)

幾何光學



$z_1 = z_2 = 2f \rightarrow$ 即高中物理所學的「倒立成像」

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -\dfrac{1}{\lambda f} & -1 \end{bmatrix}$$

$z_1 = z_2 = f \rightarrow$ Fourier Transform + Scaling

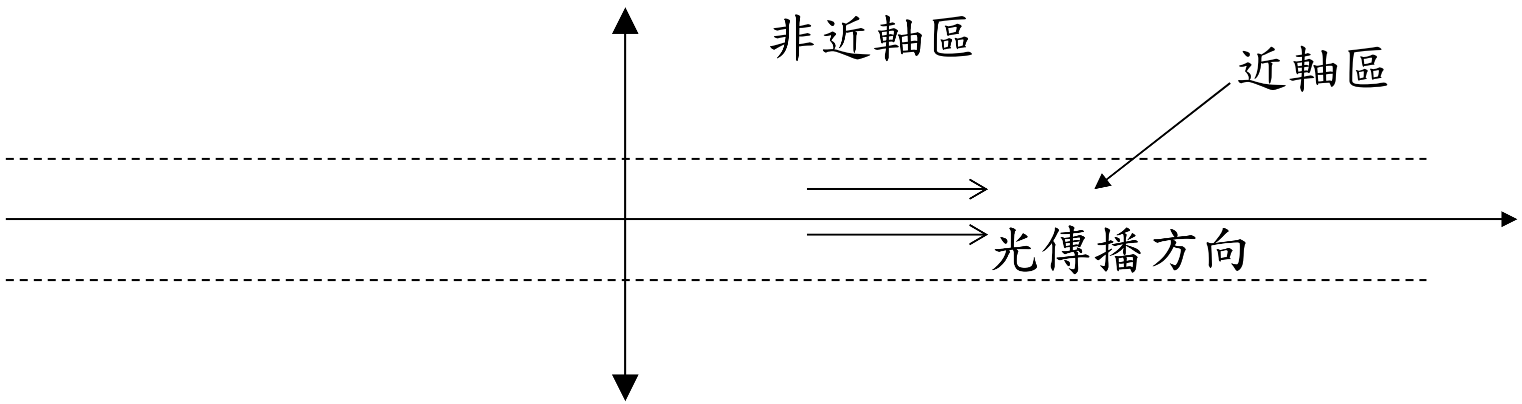
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & \lambda f \\ -\dfrac{1}{\lambda f} & 0 \end{bmatrix}$$

$z_1 = z_2 \rightarrow$ fractional Fourier Transform + Scaling

用 LCT 來分析光學系統的好處：

只需要用到 2×2 的矩陣運算，避免了複雜的物理理論和數學積分

但是 LCT 來分析光學系統的結果，只有在「近軸」的情形下才準確



參考資料：

[1] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *J. Opt. Soc. Am. A*, vol.12, 743-751, 1995.

[2] L. M. Bernardo, "ABCD matrix formalism of fractional Fourier optics," *Optical Eng.*, vol. 35, no. 3, pp. 732-740, March 1996.

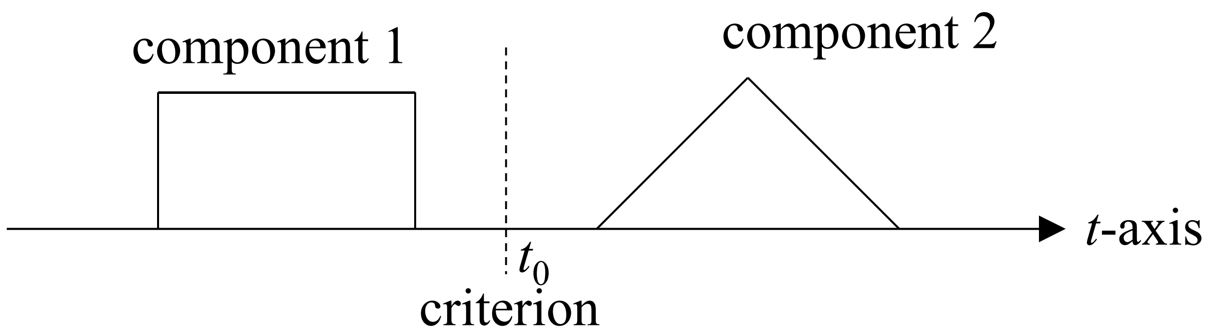
IX. Applications of Time-Frequency Analysis for Filter Design

9-1 Signal Decomposition and Filter Design

Signal Decomposition: Decompose a signal into several components.

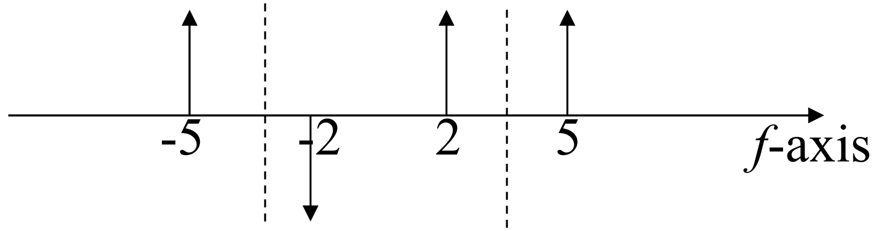
Filter: Remove the undesired component of a signal

(1) Decomposing in the time domain



(2) Decomposing in the frequency domain

$$x(t) = \sin(4\pi t) + \cos(10\pi t)$$



- Sometimes, signal and noise are separable in the time domain $\rightarrow$

  (1) without any transform

- Sometimes, signal and noise are separable in the frequency domain $\rightarrow$

  (2) using the FT (conventional filter)

$$x_o(t) = IFT\left[FT(x_i(t))H(f)\right]$$

IFT: $FrFT_{-\phi}$ FT: $FrFT_{\phi}$

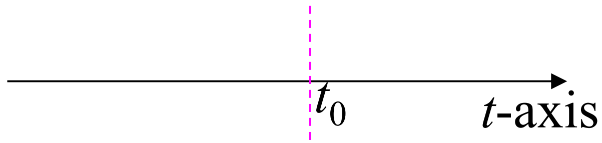
$H(f)=1$ for $|f| < 3.5$

$H(f)=0$ otherwise

- If signal and noise are not separable in both the time and the frequency domains $\rightarrow$

  (3) Using the fractional Fourier transform and time-frequency analysis

以時頻分析的觀點，criterion in the time domain 相當於 cutoff line perpendicular to t-axis



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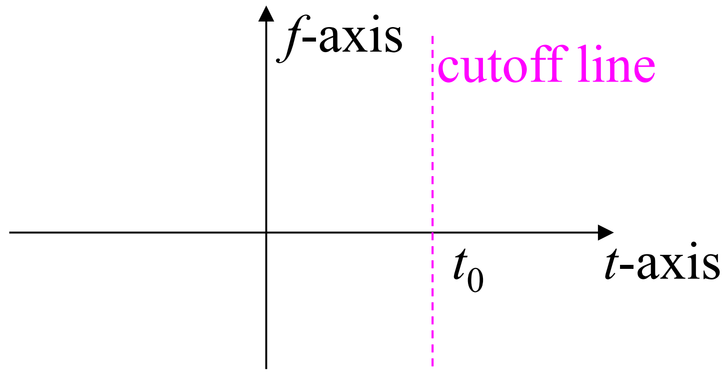
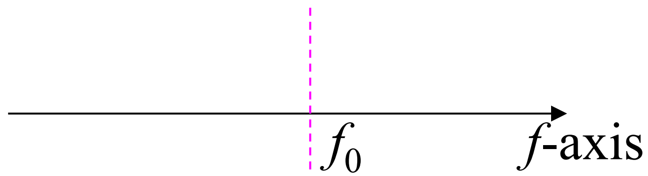


以時頻分析的觀點，criterion in the frequency domain 相當於 cutoff line perpendicular to f-axis



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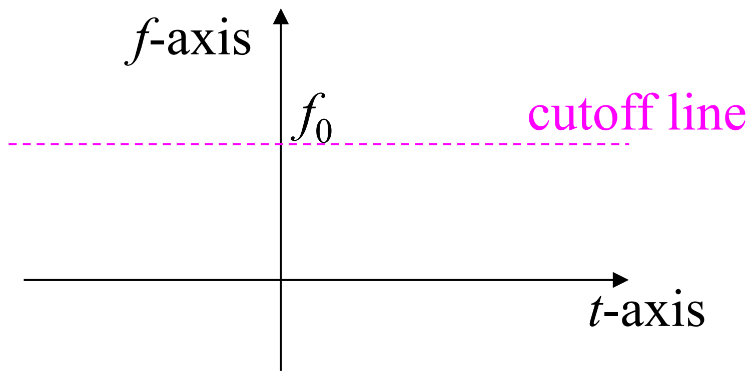
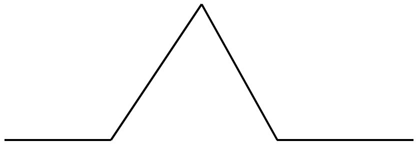
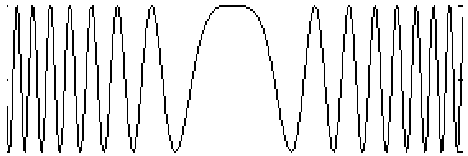
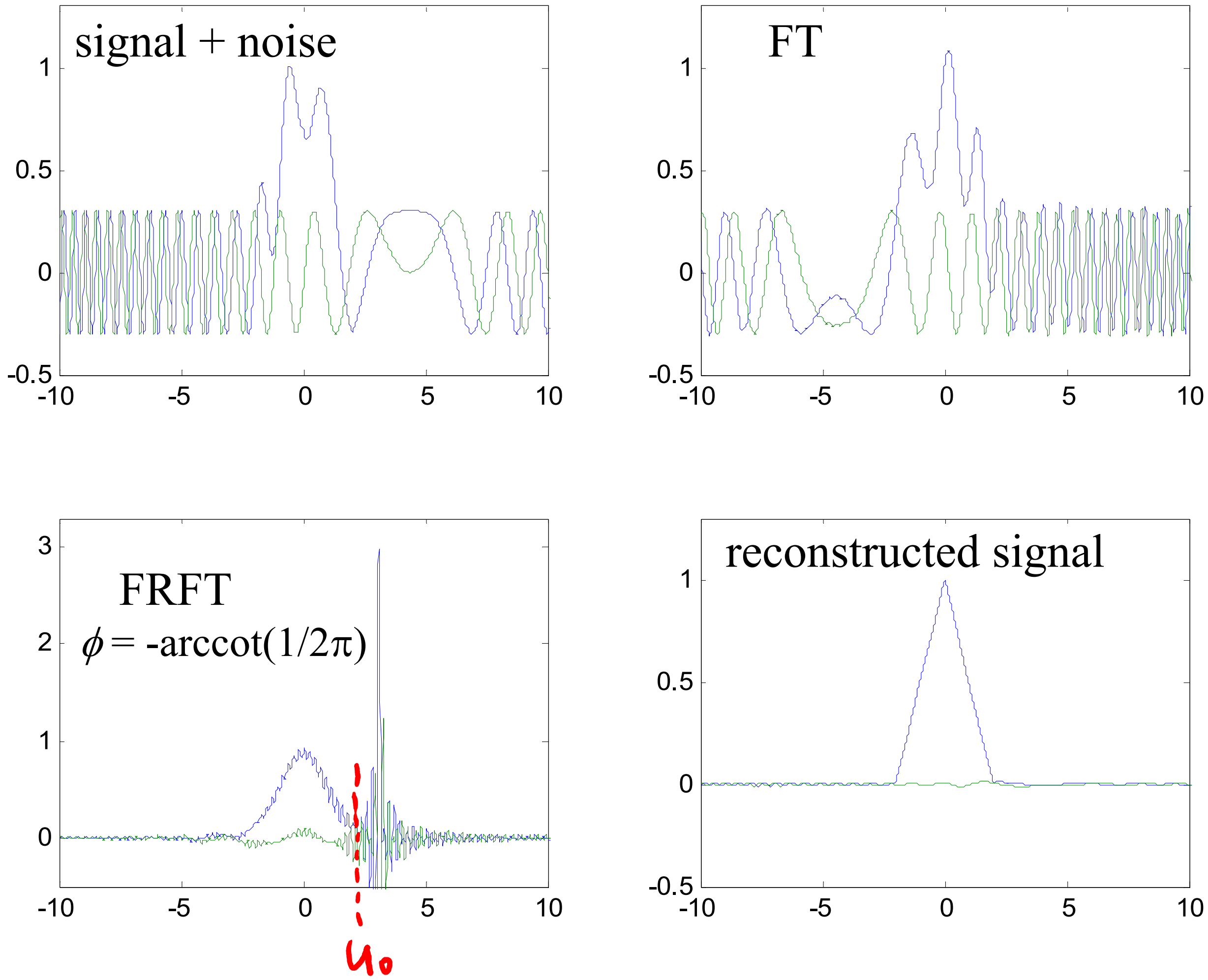


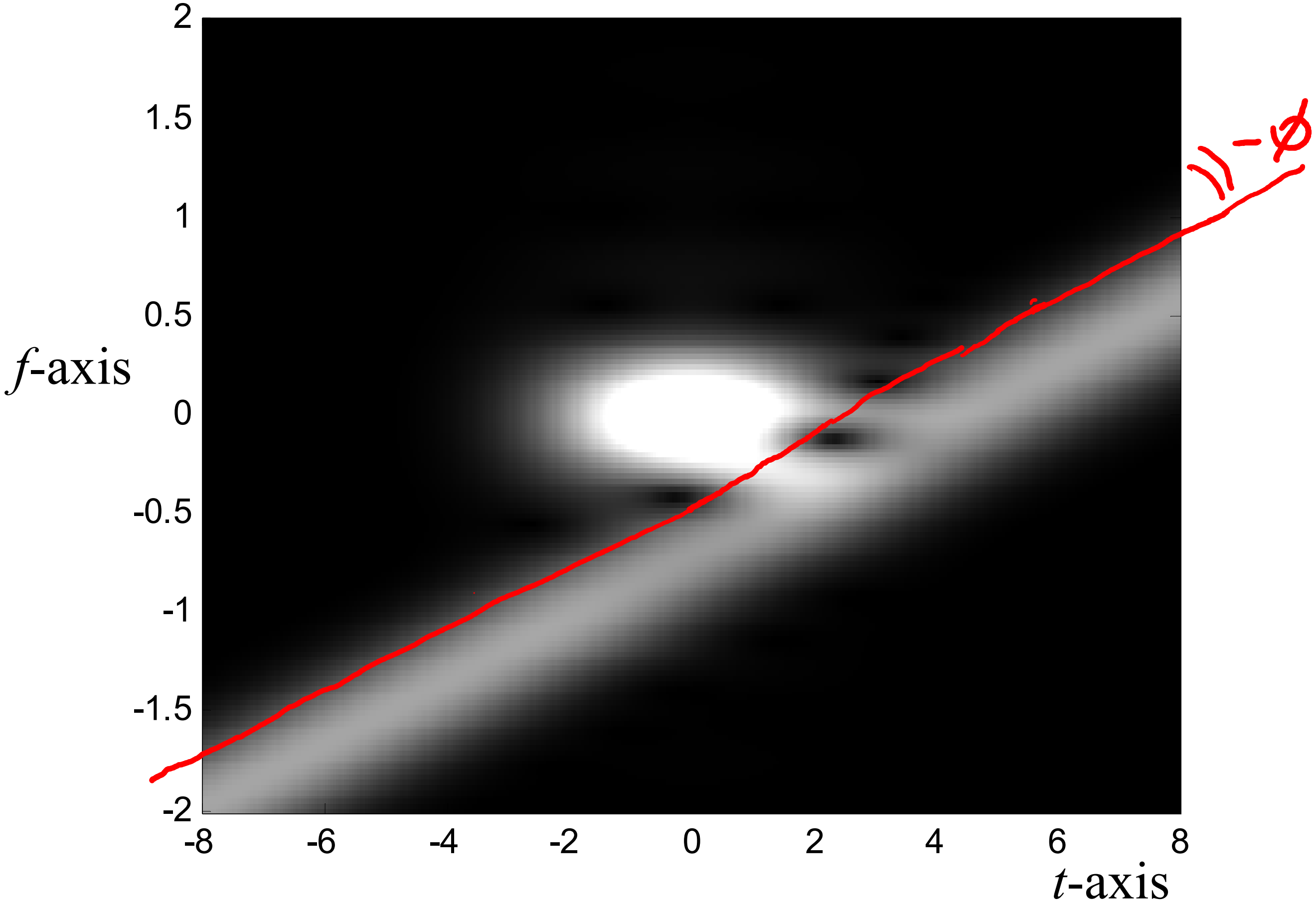
$x(t)$ = triangular signal + chirp noise $0.3\exp[j\,0.5(t-4.4)^2]$







$x(t)$ = triangular signal + chirp noise $0.3\exp[j\, 0.5(t-4.4)^2]$



Decomposing in the time-frequency distribution

If $x(t) = 0$ for $t < T_1$ and $t > T_2$

$W_x(t, f) = 0$ for $t < T_1$ and $t > T_2$ (cutoff lines perpendicular to *t*-axis)

If $X(f) = FT[x(t)] = 0$ for $f < F_1$ and $f > F_2$

$W_x(t, f) = 0$ for $f < F_1$ and $f > F_2$ (cutoff lines parallel to *t*-axis)

What are the cutoff lines with other directions?

with the aid of the **FRFT**, the LCT, or the Fresnel transform

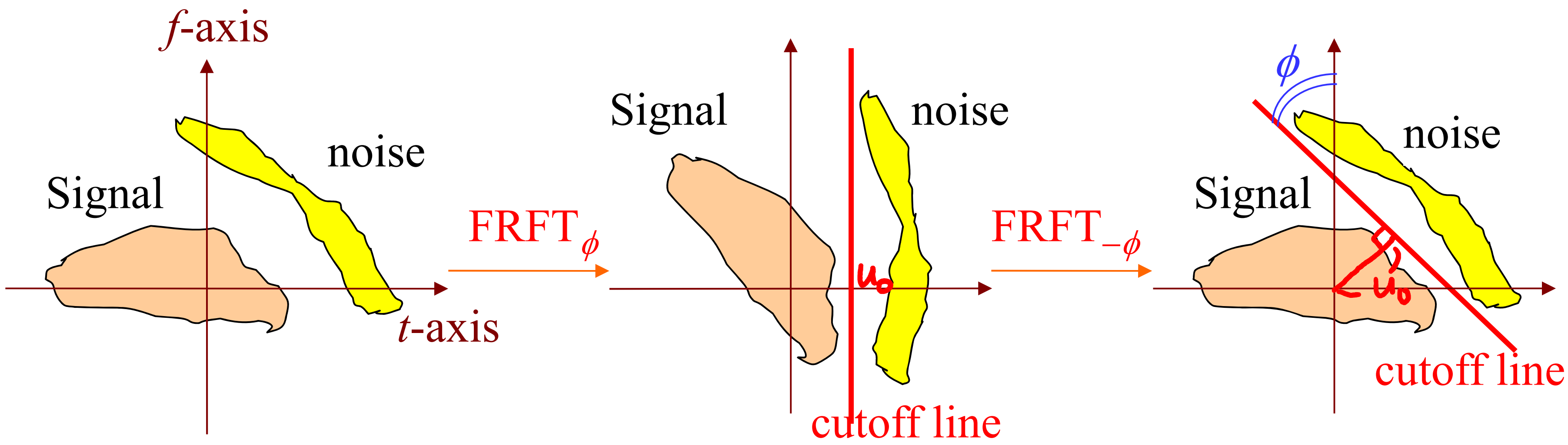
- Filter designed by the fractional Fourier transform

$$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]H(u)\right\}$$

比較： $x_o(t) = IFT\left[FT(x_i(t))H(f)\right]$

O_F^{ϕ} means the fractional Fourier transform:

$$O_F^{\phi}\left(x(t)\right) = \sqrt{1-j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2}\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)\,dt$$



$H(u) = 1 \quad u < u_0$

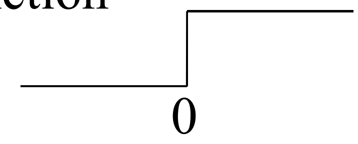
$H(u) = 0 \quad u > u_0$

$$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]H(u)\right\}$$

If $H(u) = S(-u+u_0)$ $\qquad H(u) = \begin{cases} 1 & u < u_0 \\ 0 & u > u_0 \end{cases}$

If $H(u) = S(u-u_0)$ $\qquad H(u) = \begin{cases} 1 & u > u_0 \\ 0 & u < u_0 \end{cases}$

$S(u)$: Step function



(1) ϕ 由 cutoff line 和 f-axis 的夾角決定

(2) u_0 等於 cutoff line 距離原點的距離

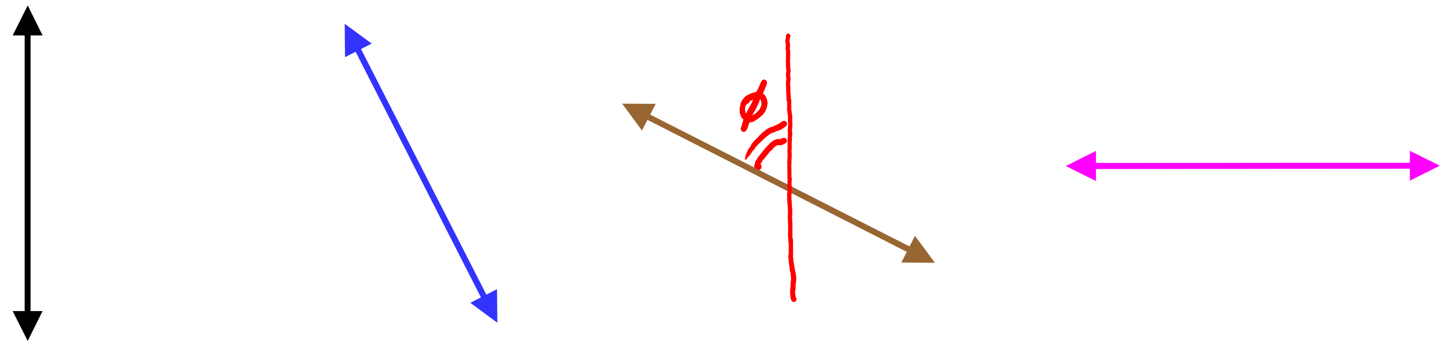
(注意正負號)

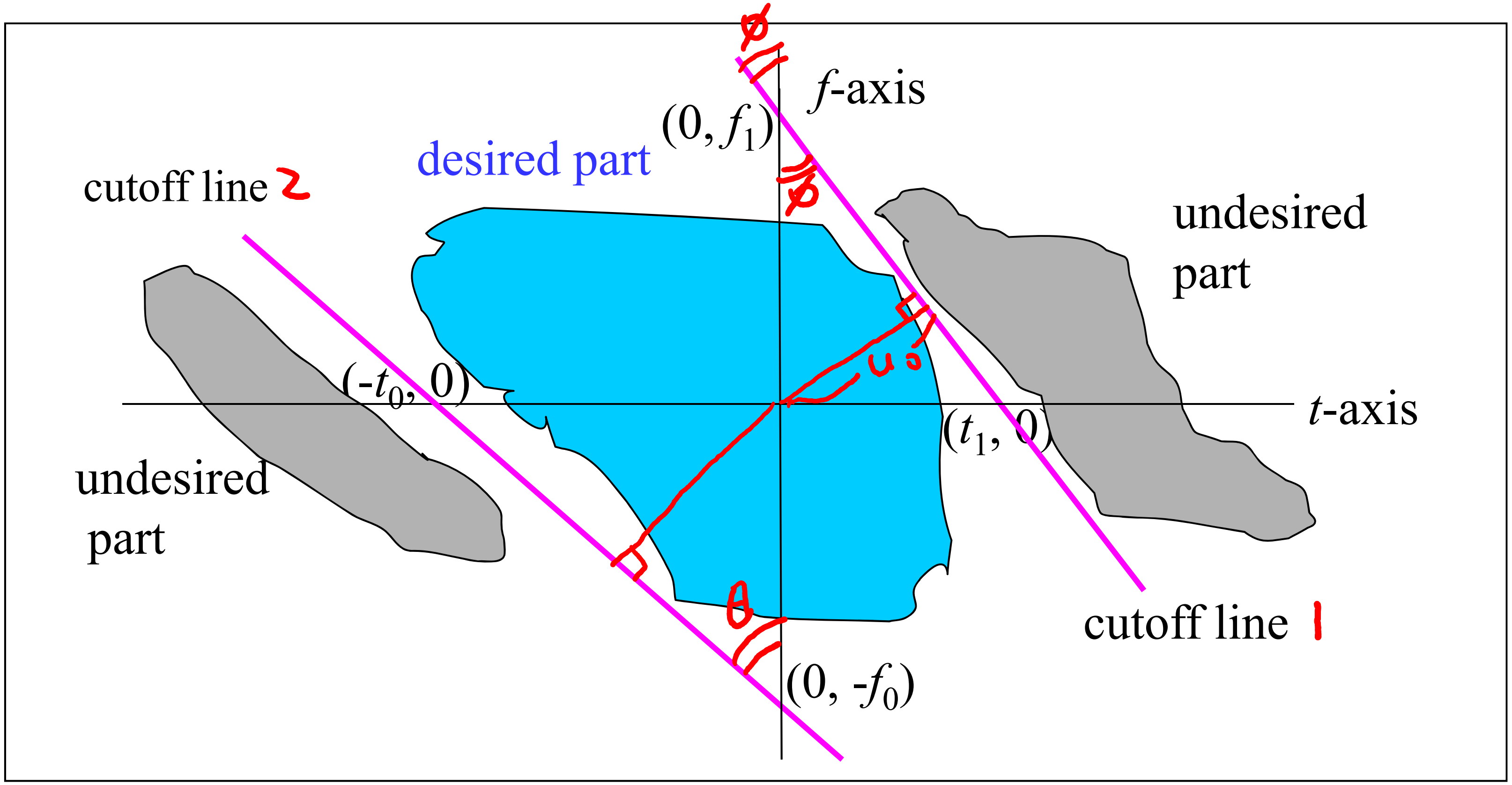
- Effect of the filter designed by the fractional Fourier transform (FRFT):

  Placing a cutoff line in the direction of $(-\sin\phi, \cos\phi)$

$\phi = 0$ (time domain) $\phi = 0.15\pi$ $\phi = 0.35\pi$ $\phi = 0.5\pi$ (FT)

分數次 FT





$\phi = ?$ $u_0 = ?$

2 times of FrFT filter

for cutoff line 1

$\phi = \arctan\left(\frac{t_1}{f_1}\right)$

$\frac{u_0\sqrt{t_1^2+f_1^2}}{2} = \frac{t_1 f_1}{2}$, $u_0 = \frac{t_1 f_1}{\sqrt{t_1^2+f_1^2}}$

cutoff line 2

$\theta = \arctan\left(\frac{t_0}{f_0}\right)$

$u_1 = -\frac{t_0 f_0}{\sqrt{f_0^2+f_0^2}}$

- The Fourier transform is suitable to filter out the noise that is a combination of

  sinusoid functions $\exp(jn_1 t)$.

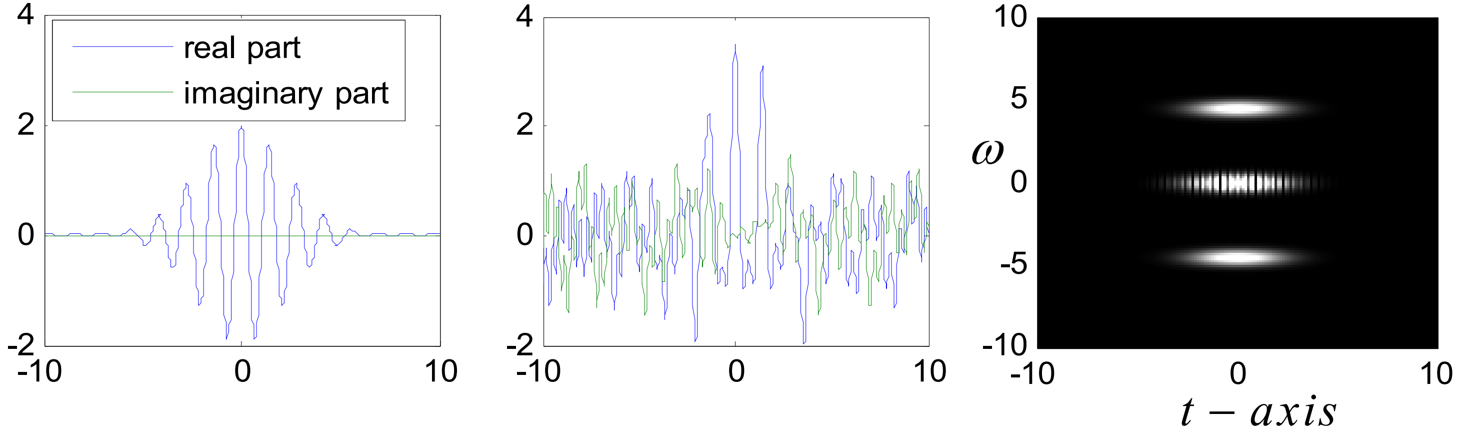
- The fractional Fourier transform (FRFT) is suitable to filter out the noise that is a combination of higher order exponential functions

  $$\exp[j(n_k t^k + n_{k-1} t^{k-1} + n_{k-2} t^{k-2} + \ldots\ldots + n_2 t^2 + n_1 t)]$$

  For example: chirp function $\exp(jn_2 t^2)$

- With the FRFT, many noises that cannot be removed by the FT will be filtered out successfully.

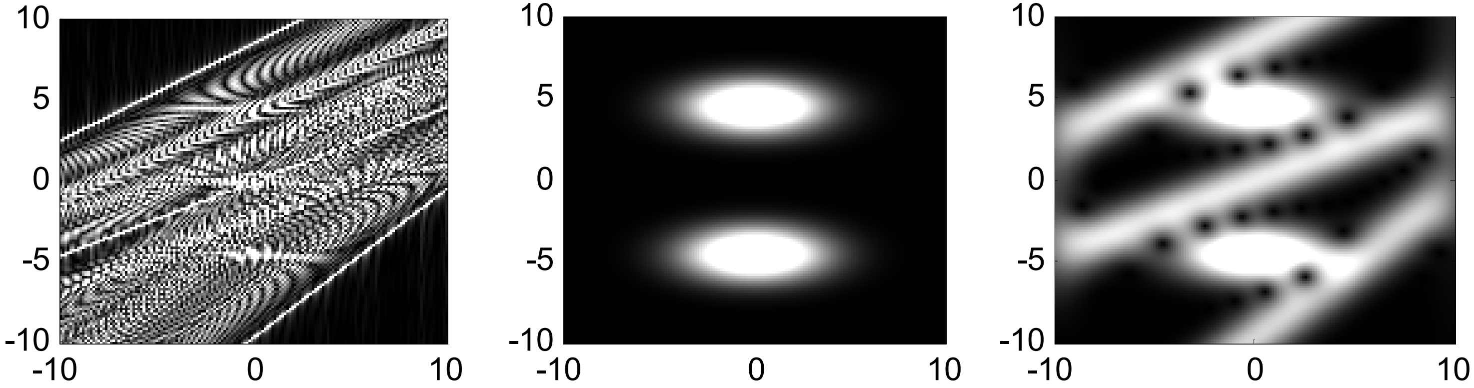
Example (I)



(a) Signal $s(t)$ (b) $f(t) = s(t)$ + noise (c) WDF of $s(t)$

$$s(t) = 2\cos(5t)\exp(-t^2/10)$$

$$n(t) = 0.5e^{j0.23t^2} + 0.5e^{j0.3t^2 + j8.5t} + 0.5e^{j0.46t^2 - j9.6t}$$

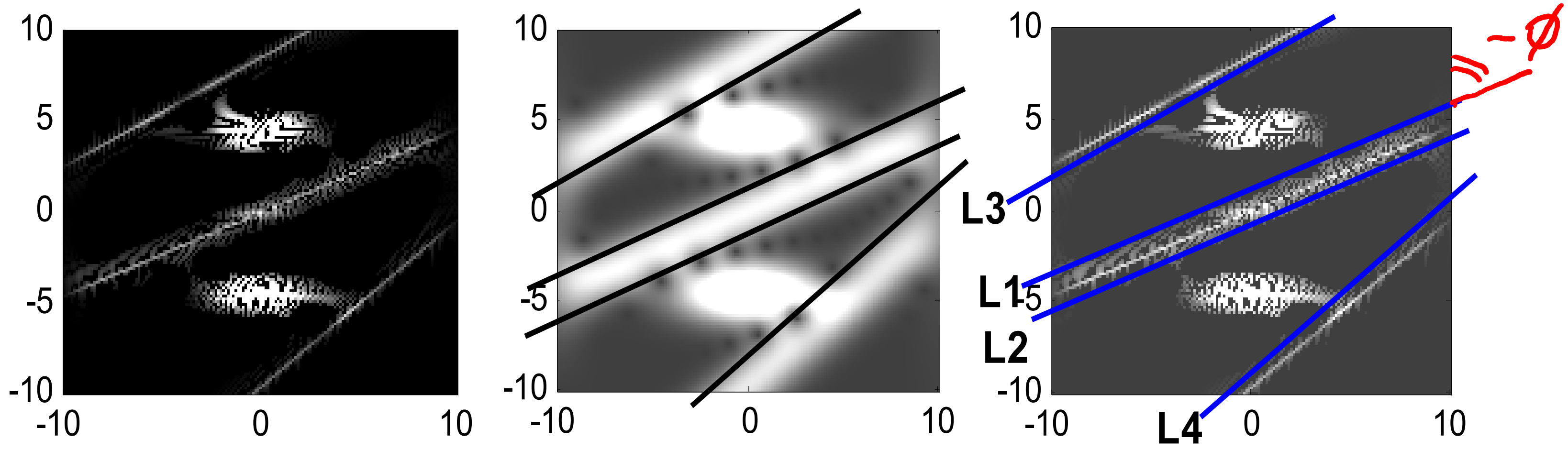


(d) WDF of $f(t)$ (e) GT of $s(t)$ (f) GT of $f(t)$

GT: Gabor transform, WDF: Wigner distribution function

horizontal: t-axis, vertical: ω-axis

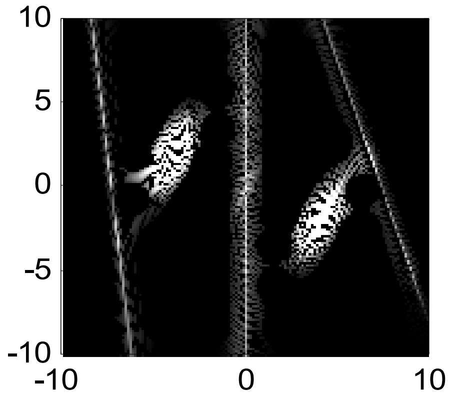
GWT: Gabor-Wigner transform



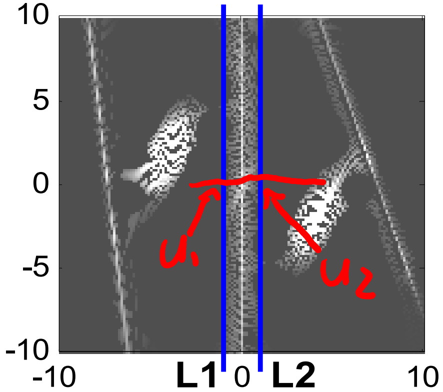
(g) GWT of $f(t)$ (h) Cutoff lines on GT (i) Cutoff lines on GWT

根據斜率來決定 FrFT 的 order

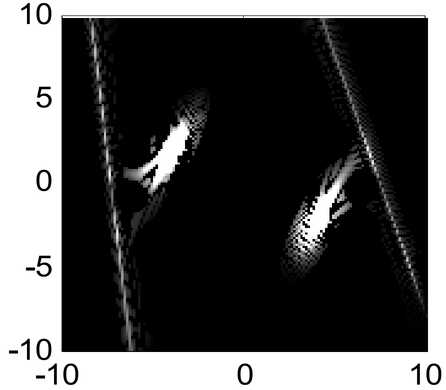
3 times FrFT filters



(j) performing the FRFT and calculate the GWT



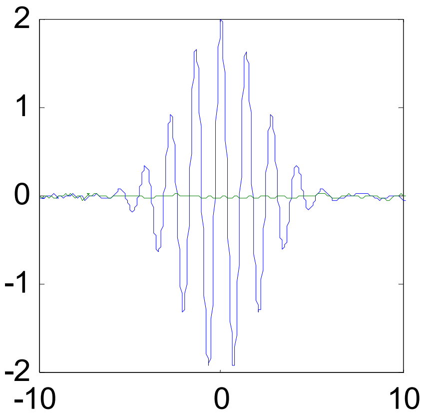
(k) High pass filter



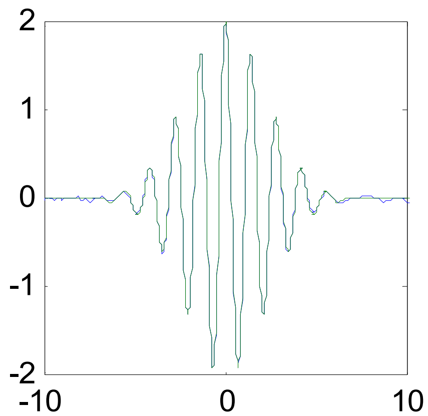
(l) GWT after filter

$$O_F^{-\phi}\left(O_F^{\phi}(x(t))\, H(u) \right)$$

$$H(u) = \begin{cases} 0 & u_1 < u < u_2 \\ 1 & \text{otherwise} \end{cases}$$



(m) recovered signal

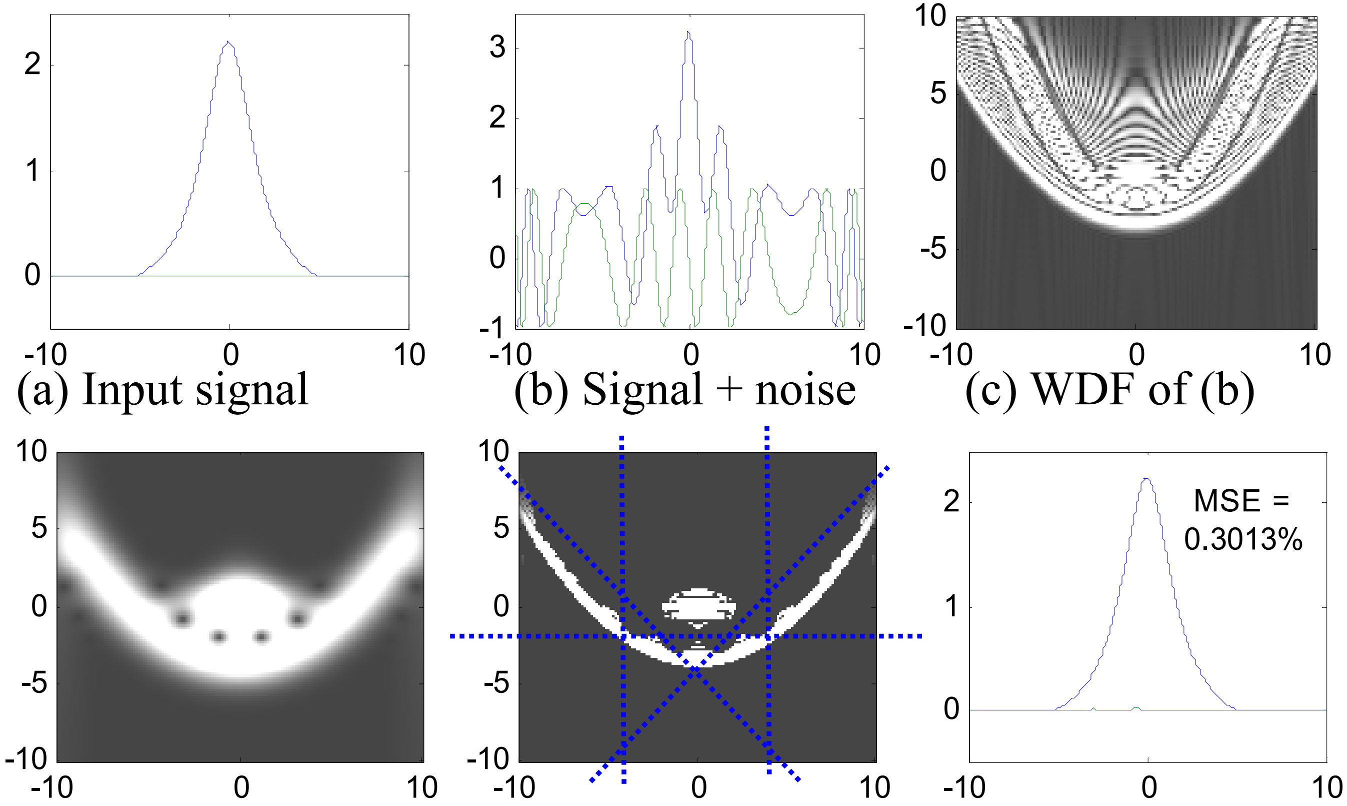


(n) recovered signal (green) and the original signal (blue)

mean square error (MSE) = 0.1128%

Example (II)

Signal + $0.7\exp(j0.032t^3 - j3.4t)$



(a) Input signal (b) Signal + noise (c) WDF of (b)

(d) Gabor transform of (b) (e) GWT of (b) (f) Recovered signal

How many times of FrFT filters are required

[Important Theory]:

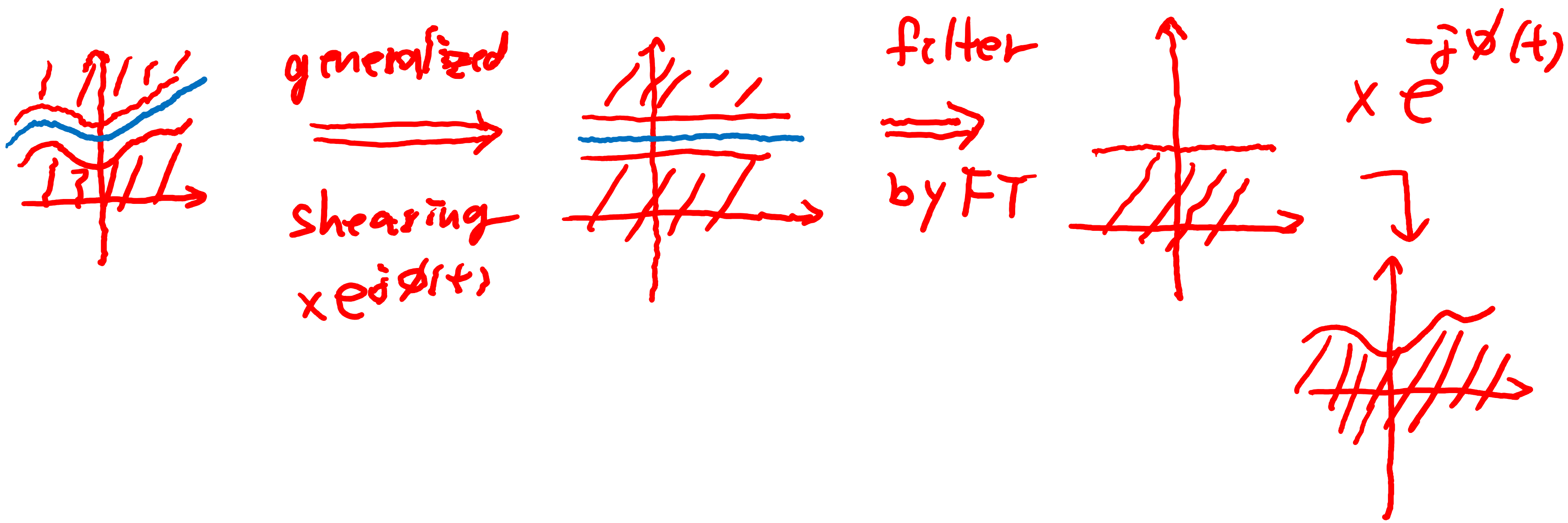
Using the **FT** can only filter the noises that do not overlap with the signals **in the frequency domain** (1-D)

In contrast, using the **FRFT** can filter the noises that do not overlap with the signals **on the time-frequency plane** (2-D)

[思考]

Q1: 哪些 time-frequency distribution 比較適合處理 filter 或 signal decomposition 的問題？

Q2: Cutoff lines 有可能是非直線的嗎？



[Ref] Z. Zalevsky and D. Mendlovic, “Fractional Wiener filter,” *Appl. Opt.*, vol. 35, no. 20, pp. 3930-3936, July 1996.

[Ref] M. A. Kutay, H. M. Ozaktas, O. Arikan, and L. Onural, “Optimal filter in fractional Fourier domains,” *IEEE Trans. Signal Processing*, vol. 45, no. 5, pp. 1129-1143, May 1997.

[Ref] B. Barshan, M. A. Kutay, H. M. Ozaktas, “Optimal filters with linear canonical transformations,” *Opt. Commun.*, vol. 135, pp. 32-36, 1997.

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] S. C. Pei and J. J. Ding, “Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing,” *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

9-2 TF analysis and Random Process

For a random process $x(t)$, we cannot find the explicit value of $x(t)$.
The value of $x(t)$ is expressed as a probability function.

original definition $\tilde{R}_x(t,\tau) = E\left(\left(x(t)-E(x(t))\right)\overline{\left(x(\tau)-E(x(\tau))\right)}\right)$
$= E\left(x(t)\overline{x(\tau)}\right)$

- Auto-covariance function $R_x(t,\tau)$

In usual, we suppose that $E[x(t)] = 0$ for any t

expected value

$$R_x(t,\tau) = E\left[x(t+\tau/2)x^*(t-\tau/2)\right]$$

$R_x(t,\tau) = \tilde{R}_x\left(t+\frac{\tau}{2},\, t-\frac{\tau}{2}\right)$

$$E\left[x(t+\tau/2)x^*(t-\tau/2)\right]$$
$$= \int\int x(t+\tau/2,\zeta_1)x^*(t-\tau/2,\zeta_2)P(\zeta_1,\zeta_2)\,d\zeta_1 d\zeta_2$$

(alternative definition of the auto-covariance function:

$$\hat{R}_x(t,\tau) = E\left[x(t)x^*(t-\tau)\right]$$

- Power spectral density (PSD) $S_x(t,f)$

$$S_x(t,f) = \int_{-\infty}^{\infty} R_x(t,\tau)e^{-j2\pi f\tau}d\tau = E\left(W_x(t,f)\right)$$

- Relation between the WDF and the random process

$$E\left[W_x(t,f)\right] = \int_{-\infty}^{\infty} E\left[x(t+\tau/2)x^*(t-\tau/2)\right]\cdot e^{-j2\pi f\tau}\cdot d\tau$$
$$= \int_{-\infty}^{\infty} R_x(t,\tau)\cdot e^{-j2\pi f\tau}\cdot d\tau$$
$$= \int_{-\infty}^{\infty} R_x(t,\tau)\cdot e^{-j2\pi f\tau}\cdot d\tau$$
$$= S_x(t,f)$$

- Relation between the ambiguity function and the random process

$$E\left[A_x(\eta,\tau)\right] = \int_{-\infty}^{\infty} E\left[x(t+\tau/2)x^*(t-\tau/2)\right]e^{-j2\pi t\eta}\,dt = \int_{-\infty}^{\infty} R_x(t,\tau)e^{-j2\pi t\eta}\,dt$$

- Stationary random process:

the statistical properties do not change with t.

$E(x(t_1+\frac{\tau}{2})x^*(t_1-\frac{\tau}{2}))$

$E(x(t_2+\frac{\tau}{2})x^*(t_2-\frac{\tau}{2}))$

Auto-covariance function $R_x(t_1,\tau)=R_x(t_2,\tau)=R_x(\tau)$ for any t,

$$R_x(\tau)=E\left[x(\tau/2)x^*(-\tau/2)\right]$$
$$=\int\int x(\tau/2,\zeta_1)x^*(-\tau/2,\zeta_2)P(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$$

PSD: $S_x(f)=\int_{-\infty}^{\infty}R_x(\tau)e^{-j2\pi f\tau}d\tau$

White noise: $S_x(f)=\sigma$ where σ is some constant.

$R_x(\tau)=\sigma\delta(\tau)$

- When $x(t)$ is stationary,

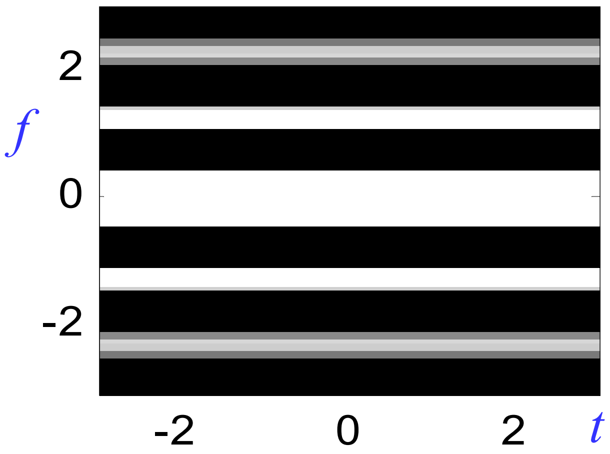
$$E\left[W_x(t,f)\right] = S_x(f)$$

(invariant with t)

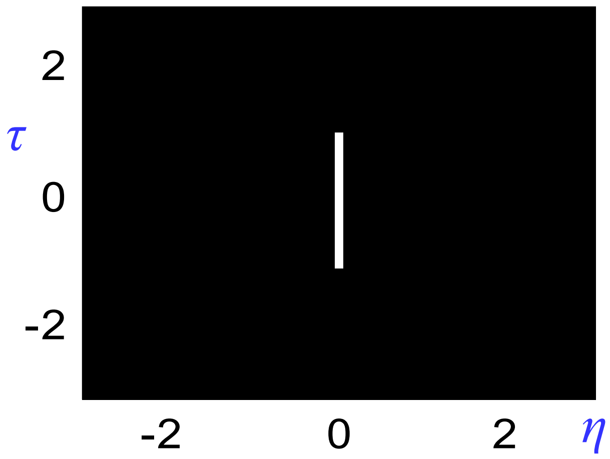
$$E\left[A_x(\eta,\tau)\right] = \int_{-\infty}^{\infty} R_x(\tau)\cdot e^{-j2\pi t\eta}\cdot dt = R_x(\tau)\int_{-\infty}^{\infty}\cdot e^{-j2\pi t\eta}\cdot dt = R_x(\tau)\delta(\eta)$$

(nonzero only when $\eta = 0$)

a typical $E[W_x(t, f)]$ for stationary random process



a typical $E[A_x(\eta, \tau)]$ for stationary random process



If $x(t)$ is stationary

✓(i) $x(t/a)$

✓(ii) $e^{j2\pi f_0 t} x(t-t_0)$

(iii) $FT(x(t))$

(iv) $e^{j\pi a t^2} x(t)$

(v) $e^{j\pi \frac{t^2}{a}} * x(t)$

(i)(ii)(v) are stationary

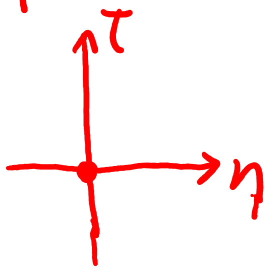
(iii)(iv) may be nonstationary

If $x(t)$ is white

(i)(ii)(iii)(iv)(v) are white

- For white noise,

$$E\left[W_x(t,f)\right]=\sigma$$



$$E\left[A_x(\eta,\tau)\right]=\sigma\delta(\tau)\delta(\eta)$$

[Ref 1] W. Martin, "Time-frequency analysis of random signals", *ICASSP'82*, pp. 1325-1328, 1982.

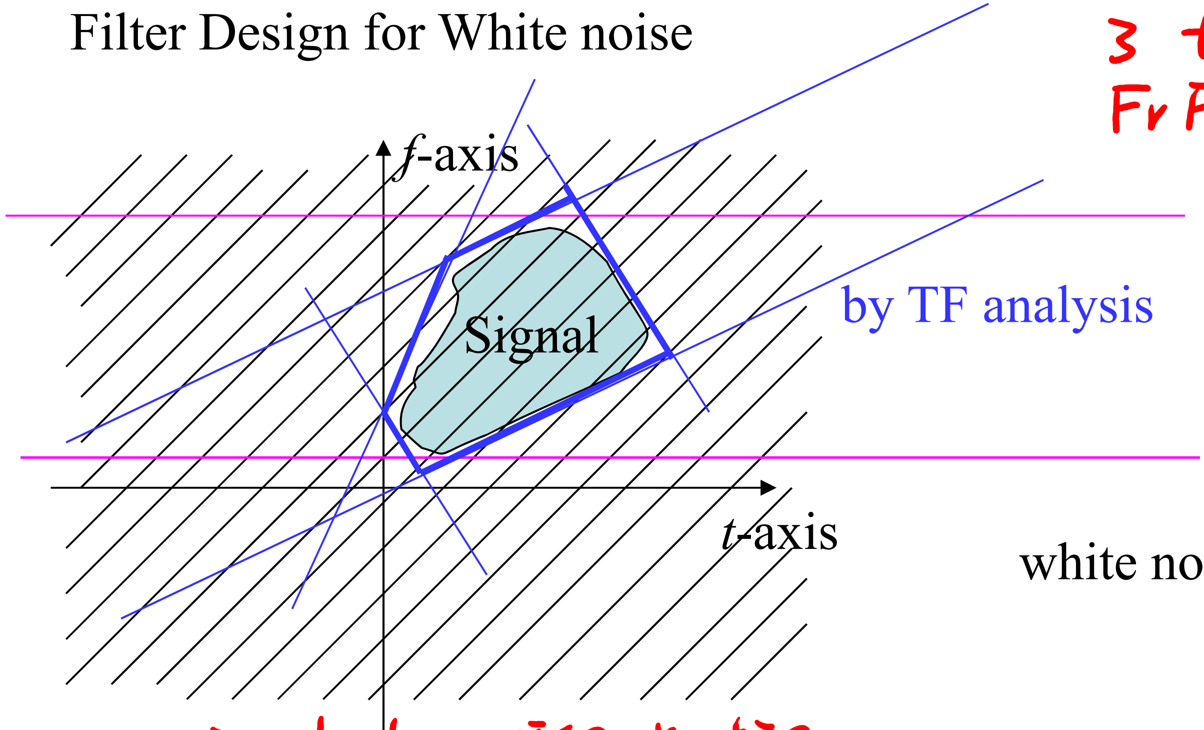
[Ref 2] W. Martin and P. Flandrin, "Wigner-Ville spectrum analysis of nonstationary processed", *IEEE Trans. ASSP*, vol. 33, no. 6, pp. 1461-1470, Dec. 1983.

[Ref 3] P. Flandrin, "A time-frequency formulation of optimum detection" , *IEEE Trans. ASSP*, vol. 36, pp. 1377-1384, 1988.

[Ref 4] S. C. Pei and J. J. Ding, "Fractional Fourier transform, Wigner distribution, and filter design for stationary and nonstationary random processes," *IEEE Trans. Signal Processing*, vol. 58, no. 8, pp. 4079-4092, Aug. 2010.

Filter Design for White noise

3 times of the FrFT filters



conventional filter

white noise everywhere

signal to noise ratio

$$SNR \approx 10\log_{10}\frac{E_{signal}}{\int\int_{\substack{(t,f)\in\\ \text{signal part}}} W_{noise}(t,f)\,dtdf}$$

E_{signal}: energy of the signal

A: area of the time frequency distribution of the signal

$$SNR \approx 10\log_{10}\frac{E_{signal}}{\sigma A}$$

The PSD of the white noise is $S_{noise}(f) = \sigma$

- If $E\left[W_x(t,f)\right]$ varies with t and $E\left[A_x(\eta,\tau)\right]$ is nonzero when $\eta \neq 0$, then $x(t)$ is a non-stationary random process.

- If ① $h(t) = x_1(t) + x_2(t) + x_3(t) + \cdots\cdots + x_k(t)$

  ② $x_n(t)$'s have zero mean for all t's

  ③ $x_n(t)$'s are mutually independent for all t's and τ's

  $$E\left[x_m(t+\tau/2)x_n^*(t-\tau/2)\right] = E\left[x_m(t+\tau/2)\right]E\left[x_n^*(t-\tau/2)\right] = 0$$

  if $m \neq n$, then

$$E\left[W_h(t,f)\right] = \sum_{n=1}^{k} E\left[W_{x_n}(t,f)\right], \quad E\left[A_h(\eta,\tau)\right] = \sum_{n=1}^{k} E[A_{x_n}(\eta,\tau)]$$

(1) Random process for the STFT

$E[x(t)] \neq 0$ should be satisfied.

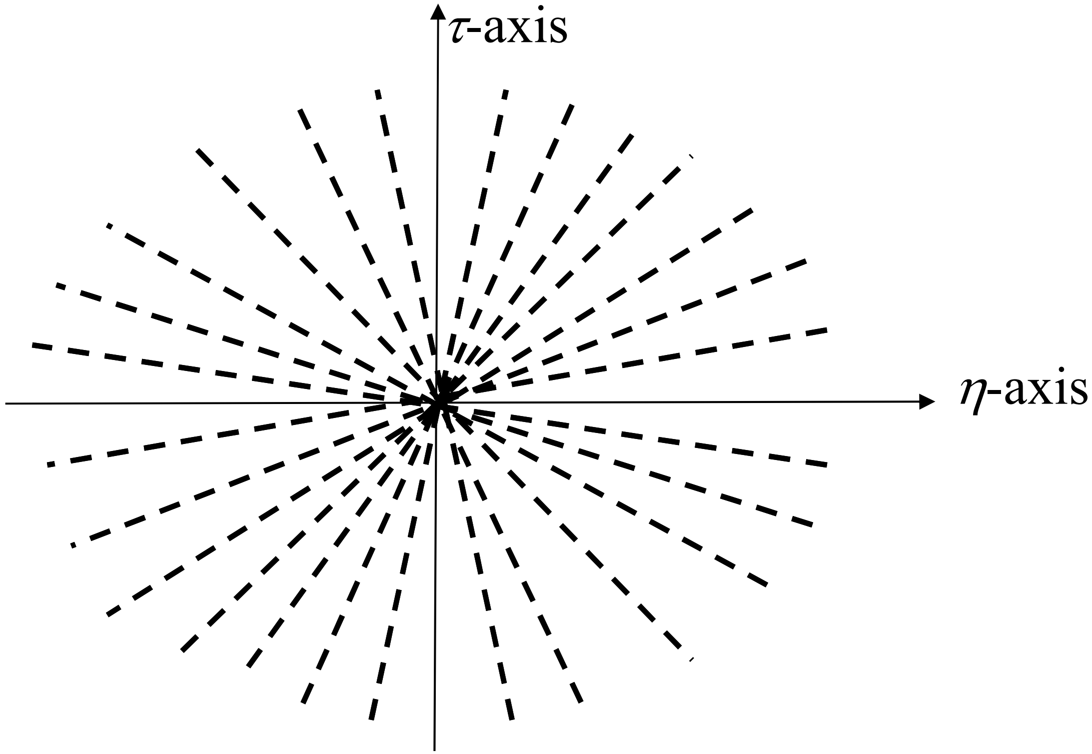
Otherwise,

$$E[X(t,f)] = E[\int_{t-B}^{t+B} x(\tau)w(t-\tau)e^{-j2\pi f\tau}d\tau] = \int_{t-B}^{t+B} E[x(\tau)]w(t-\tau)e^{-j2\pi f\tau}d\tau$$

for zero-mean random process, $E[X(t,f)] = 0$

(2) Decompose by the AF and the FRFT

Any non-stationary random process can be expressed as a summation of the fractional Fourier transform (or chirp multiplication) of stationary random process.



An ambiguity function plane can be viewed as a combination of infinite number of radial lines.

Each radial line can be viewed as the fractional Fourier transform of a stationary random process.

信號處理小常識

$S(f)=\sigma$ white noise

$\alpha=-1$ $S(f)=\frac{\sigma}{|f|}$ pink noise

$\alpha=1$ $S(f)=\sigma|f|$ purple noise

$S(f)=\sigma|f|^{\alpha}$ $\alpha\neq 0$ color noise

附錄十二 Time-Frequency Analysis 理論發展年表

AD 1785 The Laplace transform was invented

AD 1812 The Fourier transform was invented

AD 1822 The work of the Fourier transform was published

AD 1898 Schuster proposed the periodogram.

AD 1910 The Haar Transform was proposed

AD 1927 Heisenberg discovered the uncertainty principle

AD 1929 The fractional Fourier transform was invented by Wiener

AD 1932 The Wigner distribution function was proposed

AD 1946 The short-time Fourier transform and the Gabor transform was proposed.
In the same year, the computer was invented

註：沒列出發明者的，指的是 transform / distribution 的名稱和發明者的名字相同

AD 1961 Slepian and Pollak found the prolate spheroidal wave function

AD 1965 The Cooley-Tukey algorithm (FFT) was developed

AD 1966 Cohen's class distribution was invented

AD 1970s VLSI was developed

AD 1971 Moshinsky and Quesne proposed the linear canonical transform

AD 1980 The fractional Fourier transform was re-invented by Namias

AD 1981 Morlet proposed the wavelet transform

AD 1982 The relations between the random process and the Wigner distribution function was found by Martin and Flandrin

AD 1988 Mallat and Meyer proposed the multiresolution structure of the wavelet transform;
In the same year, Daubechies proposed the compact support orthogonal wavelet

註：沒列出發明者的，指的是 transform / distribution 的名稱和發明者的名字相同

AD 1989 The Choi-Williams distribution was proposed; In the same year, Mallat proposed the fast wavelet transform

AD 1990 The cone-Shape distribution was proposed by Zhao, Atlas, and Marks

AD 1990s The discrete wavelet transform was widely used in image processing

AD 1992 The generalized wavelet transform was proposed by Wilson et. al.

AD 1993 Mallat and Zhang proposed the matching pursuit;
In the same year, the rotation relation between the WDF and the fractional Fourier transform was found by Lohmann

AD 1994 The applications of the fractional Fourier transform in signal processing were found by Almeida, Ozaktas, Wolf, Lohmann, and Pei;
Boashash and O'Shea developed polynomial Wigner-Ville distributions

AD 1995 Auger and Flandrin proposed time-frequency reassignment

L. J. Stankovic, S. Stankovic, and Fakultet proposed the pseudo Wigner distribution

AD 1996 Stockwell, Mansinha, and Lowe proposed the S transform

Daubechies and Maes proposed the synchrosqueezing transform

AD 1998 N. E. Huang proposed the Hilbert-Huang transform

Chen, Donoho, and Saunders proposed the basis pursuit

AD 1999 Bultan proposed the four-parameter atom (i.e., the chirplet)

AD 2000 The standard of JPEG 2000 was published by ISO

Another wavelet-based compression algorithm, SPIHT, was proposed by Kim, Xiong, and Pearlman

The curvelet was developed by Donoho and Candes

AD 2000s The applications of the Hilbert Huang transform in signal processing, climate analysis, geology, economics, and speech were developed

AD 2002 The bandlet was developed by Mallet and Peyre;
Stankovic proposed the time frequency distribution with complex arguments

AD 2003 Pinnegar and Mansinha proposed the general form of the S transform

Liebling et al. proposed the Fresnelet.

AD 2005 The contourlet was developed by Do and Vetterli;

The shearlet was developed by Kutyniok and Labate

The generalized spectrogram was proposed by Boggiatto, et al.

AD 2006 Donoho proposed compressive sensing

AD 2006~ Accelerometer signal analysis becomes a new application.

AD 2007 The Gabor-Wigner transform was proposed by Pei and Ding

AD 2007 The multiscale STFT was proposed by Zhong and Zeng.

AD 2007~ Many theories about compressive sensing were developed by Donoho, Candes, Tao, Zhang ….

AD 2010~ Many applications about compressive sensing are found.

AD 2012 The generalized synchrosqueezing transform was proposed by Li and Liang

AD 2015~ Time-frequency analysis was widely combined with the deep learning technique for signal identification

The second-order synchrosqueezing transform was proposed by Oberlin, Meignen, and Perrier.

AD 2017 The wavelet convolutional neural network was proposed by Kang et al.

The higher order synchrosqueezing transform was proposed by Pham and Meignen

AD 2018~ With the fast development of hardware and software, the time-frequency distribution of a 10^6-point data can be analyzed efficiently within 0.1 Second

時頻分析理論與應用未來的發展，還看各位同學們大顯身手