VII. Other Time Frequency Distributions (II)

The trend of time-frequency analysis in recent years:

(1) S transform and its generalization

(2) Time-frequency reassignment and the synchosqueezing transform

(3) Time-variant signal expansion (Compressive sensing)

(4) Improvement for the Hilbert-Huang transform

VII-A S Transform

(Modification from the Gabor transform)

比較 page 101

$\sigma = f^2$

$$S_x(t,f) = |f| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 f^2\right] \exp(-j2\pi f \tau)\, d\tau$$

no cross term

closely related to the wavelet transform

advantages and disadvantages

f	time resolution	frequency resolution
↓	↓	↑
↑	↑	↓

[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

S transform 和 Gabor transform 相似。

但是 Gaussian window 的寬度會隨著 f 而改變

$$w(t) = \exp\left[-\pi t^2\right] \qquad\qquad w(t) = |f|\exp\left[-\pi t^2 f^2\right]$$

music

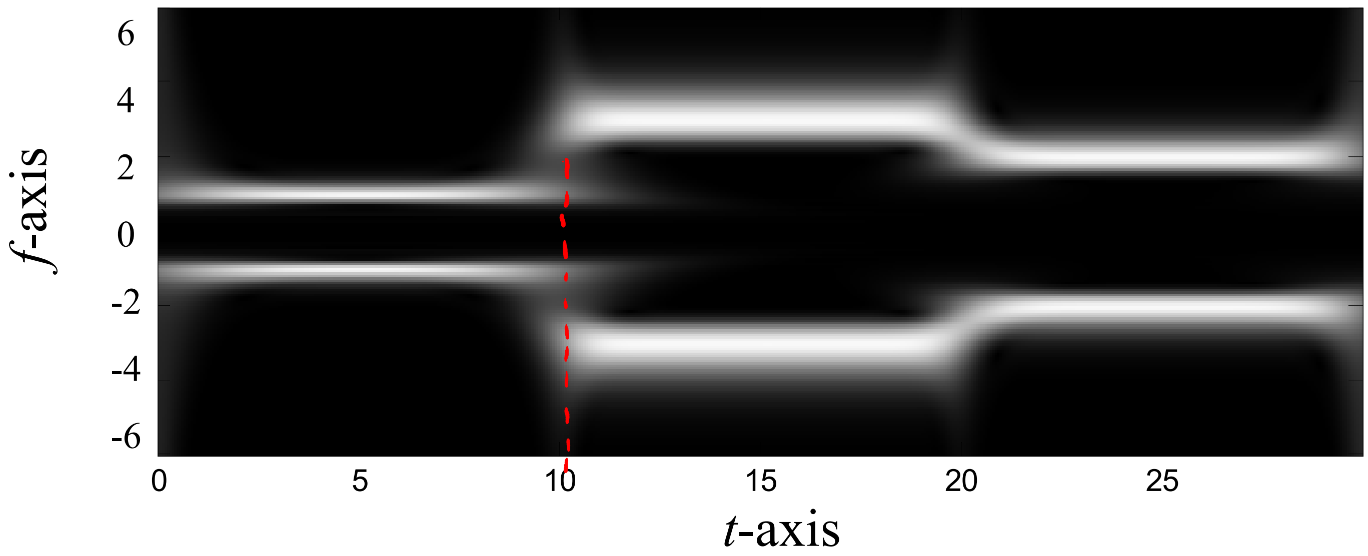
$261.63 \times 2^{\frac{k}{12}}$

Do: 261.63Hz

低頻： worse time resolution, better frequency resolution

高頻： better time resolution, worse frequency resolution

The result of the S transform (compared with page 94)



- General form

$$S_x(t,f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 s^2(f)\right] \exp(-j2\pi f\tau)\, d\tau$$

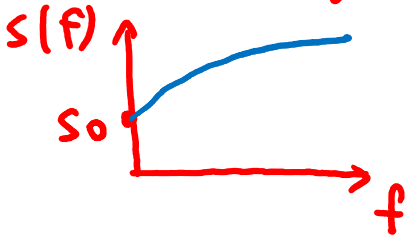
$s(f)$ increases with f original: $s(f) = f$, $\sigma = f^{-1}$

modified: $s(f)$ grows slowlier than f

$s(0) \neq 0$

ex: $s(f) = S_0 + f^{\alpha}$ $(\alpha < 1)$

$S_0 \neq 0$



C. R. Pinnegar and L. Mansinha, “The S-transform with windows of arbitrary and varying shape,” *Geophysics*, vol. 68, pp. 381-385, 2003.

Fast algorithm of the S transform

When f is fixed, the S transform can be expressed as a convolution form:

$$S_x(t,f) = |s(f)| \int_{-\infty}^{\infty} x(\tau)\exp\left[-\pi(t-\tau)^2 s^2(f)\right]\exp(-j2\pi f\tau)\,d\tau$$

$$\Downarrow$$

$$S_x(t,f) = |s(f)| \left(x(t)\exp(-j2\pi ft) \underset{\substack{convolution \\ along\ t-axis}}{*} \exp\left[-\pi t^2 s^2(f)\right] \right)$$

(for every fixed f)

Remember: $g(t) * h(t) = \int g(\tau)h(t-\tau)d\tau$

Q: Can we use the FFT-based method on page 119 to implement the S transform?

VII-B Generalized Spectrogram

[Ref] P. Boggiatto, G. De Donno, and A. Oliaro, “Two window spectrogram and their integrals," *Advances and Applications*, vol. 205, pp. 251-268, 2009.

Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^*_{x,w_2}(t,f)$

$$G_{x,w_1}(t,f) = \int_{-\infty}^{\infty} w_1(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

$$G_{x,w_2}(t,f) = \int_{-\infty}^{\infty} w_2(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau$$

Original spectrogram: $w_1(t) = w_2(t)$

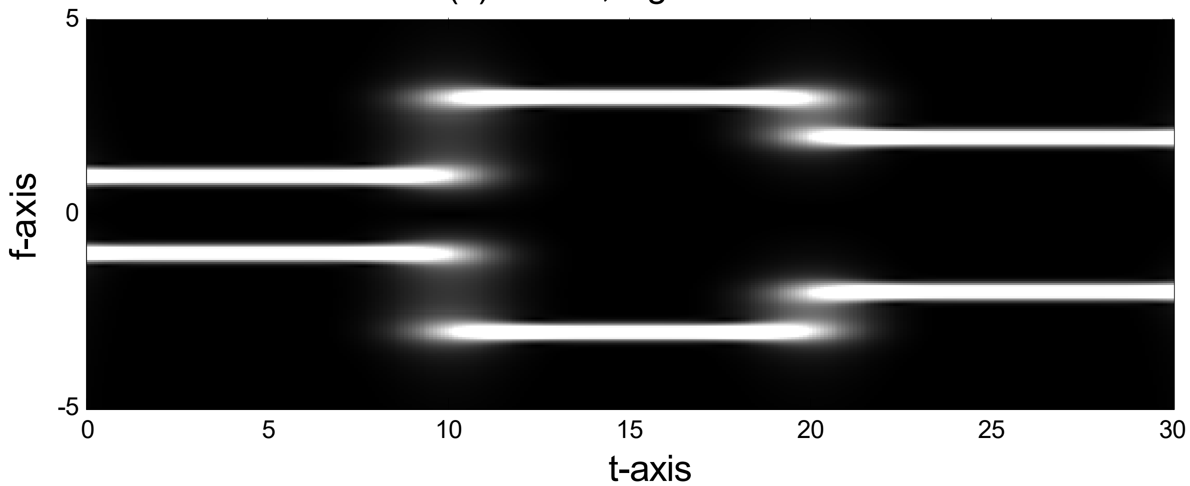
To achieve better clarity, $w_1(t)$ can be chosen as a wider window,
$w_2(t)$ can be chosen as a narrower window.

$x(t) = \cos(2\pi t)$ when $t < 10$,

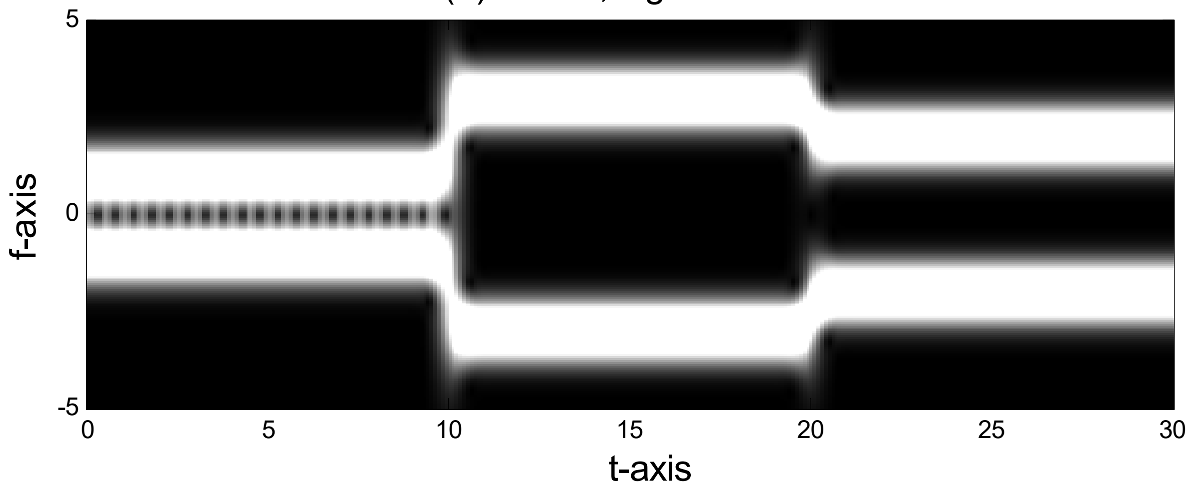
$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$

(a) Gabor, sigma = 0.1



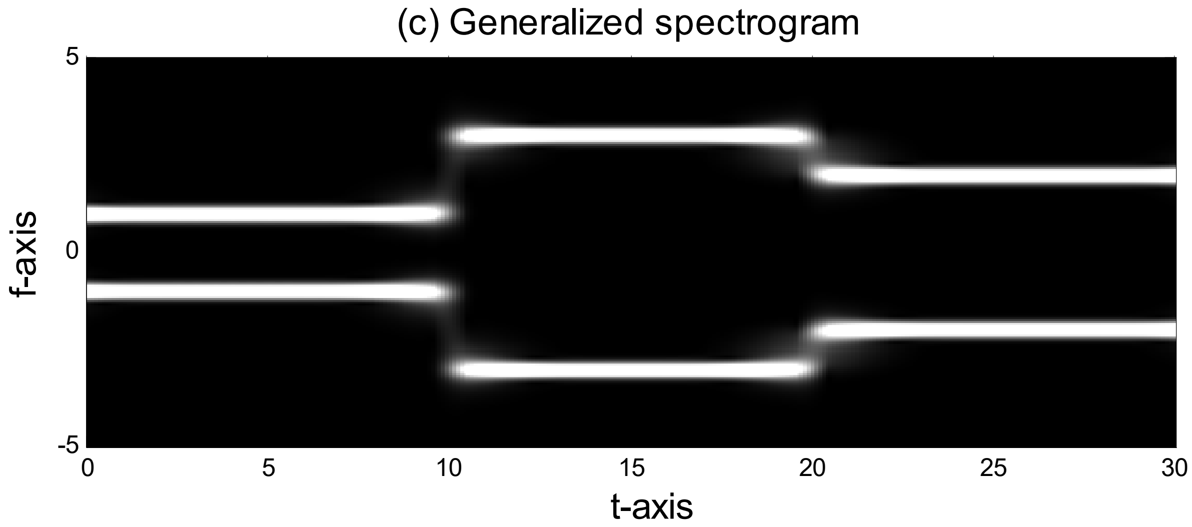
(b) Gabor, sigmal = 1.6

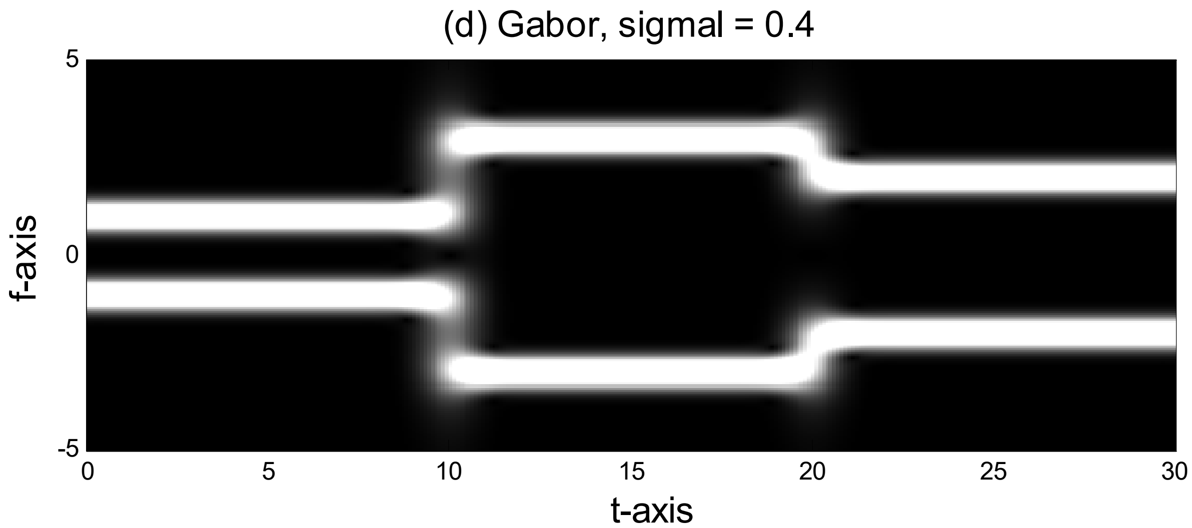


$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$





Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^*_{x,w_2}(t,f)$

Further Generalization for the spectrogram:

$$\boxed{SP_{x,w_1,w_2}(t,f) = G^{\alpha}_{x,w_1}(t,f)\,\overline{G^{\beta}_{x,w_2}(t,f)}}$$

or

$$\boxed{SP_{x,w_1,w_2}(t,f) = \left|G_{x,w_1}(t,f)\right|^{\alpha}\left|G_{x,w_2}(t,f)\right|^{\beta}}$$

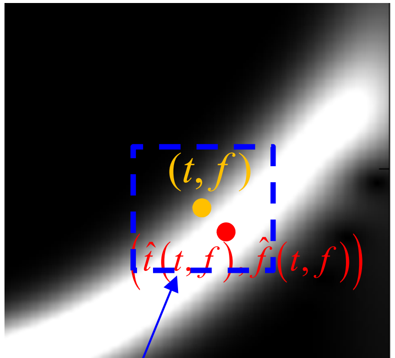
VII-C Reassignment Method

時頻再分配 (post-processing)

After computing the time-frequency distribution, we can use the following way to make the energy even more concentrated.

(1) First, estimate the offset.

E(t)



$\varphi(u-t, v-f)$

$$\hat{t}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}$$

E(f)

$$\hat{f}(t,f) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v \cdot \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(u-t, v-f) \cdot X(u,v)\,du\,dv}$$

$X(t, f)$: time-frequency analysis (STFT, WDF…) of $x(t)$,

$\varphi(u, v) = 1$ when $|u|, |v| < B$

$\varphi(u, v) = 0$ otherwise

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}(t,f), \hat{f}(t,f)\right)$

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}(t,f), \hat{f}(t,f)\right)$

$$\hat{X}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1, f_1)\,\delta\left(t - \hat{t}(t_1, f_1)\right)\delta\left(f - \hat{f}(t_1, f_1)\right)dt_1 df_1$$

(同步擠壓) synchrosqueezing: $\hat{X}(t,f) = \int_{-\infty}^{\infty} X(t_1, f_1)\,\delta(f - \hat{f}(t_1, f_1))\,dt_1 df_1$

similar to time-frequency reassignment

but only move along f

References

[1] F. Auger and P. Flandrin, "Improving the readability of time-frequency and time-scale representations by the reassignment method," *IEEE Trans. Signal Processing*, vol. 43, issue 5, pp. 1068-1089, May 1995.

[2] F. Auger, P. Flandrin, Y.T. Lin, S. McLaughlin, S. Meignen, T. Oberlin, and H.T. Wu, "Time-frequency reassignment and synchrosqueezing: An overview," *IEEE Signal Processing Magazine*, vol. 30, issue 6, pp. 32-41, 2013.

PS: 感謝 2017 年修課的盧德晏同學

VII-D Basis Expansion Time-Frequency Analysis

就如同

- Fourier series: $\varphi_m(t) = \exp(j2\pi f_m t)$, $x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$

$$a_m = \frac{\langle x(t), \varphi_m(t) \rangle}{\langle \varphi_m(t), \varphi_m(t) \rangle} = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi f_m t)dt \qquad f_m = m/T$$

部分的 Time-Frequency Analysis 也是意圖要將 signal 表示成如下的型態

$$x(t) \approx \sum_{m=1}^{M} a_m \varphi_m(t)$$

並且要求在 M 固定的情形下，

$$\text{approximation error} = \int_{-\infty}^{\infty} \left| x(t) - \sum_{m=1}^{M} a_m \varphi_m(t) \right|^2 dt$$ 為最小

將 $\varphi_m(t)$ 一般化，不同的 basis 之間不只是有 frequency 的差異

(1) Three Parameter Atoms

$$x(t) \approx \sum a_{t_0,f_0,\sigma}\varphi_{t_0,f_0,\sigma}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}}\exp(j2\pi f_0 t)\exp(-\frac{\pi(t-t_0)^2}{\sigma^2})$$

Gaussian amplitude

3 parameters: t_0 controls the central time
f_0 controls the frequency
σ controls the scaling factor

(i) too many choices of basis
(ii) not orthogonal
⇒ very large computation time

[Ref] S. G. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397-3415, Dec. 1993.

Since $\varphi_{t_0,f_0,\sigma}(t)$ are not orthogonal, $a_{t_0,f_0,\sigma}$ should be determined by a matching pursuit process.

(2) Four Parameter Atoms (Chirplet)

$$x(t) \approx \sum a_{t_0,f_0,\sigma,\eta}\varphi_{t_0,f_0,\sigma,\eta}(t)$$

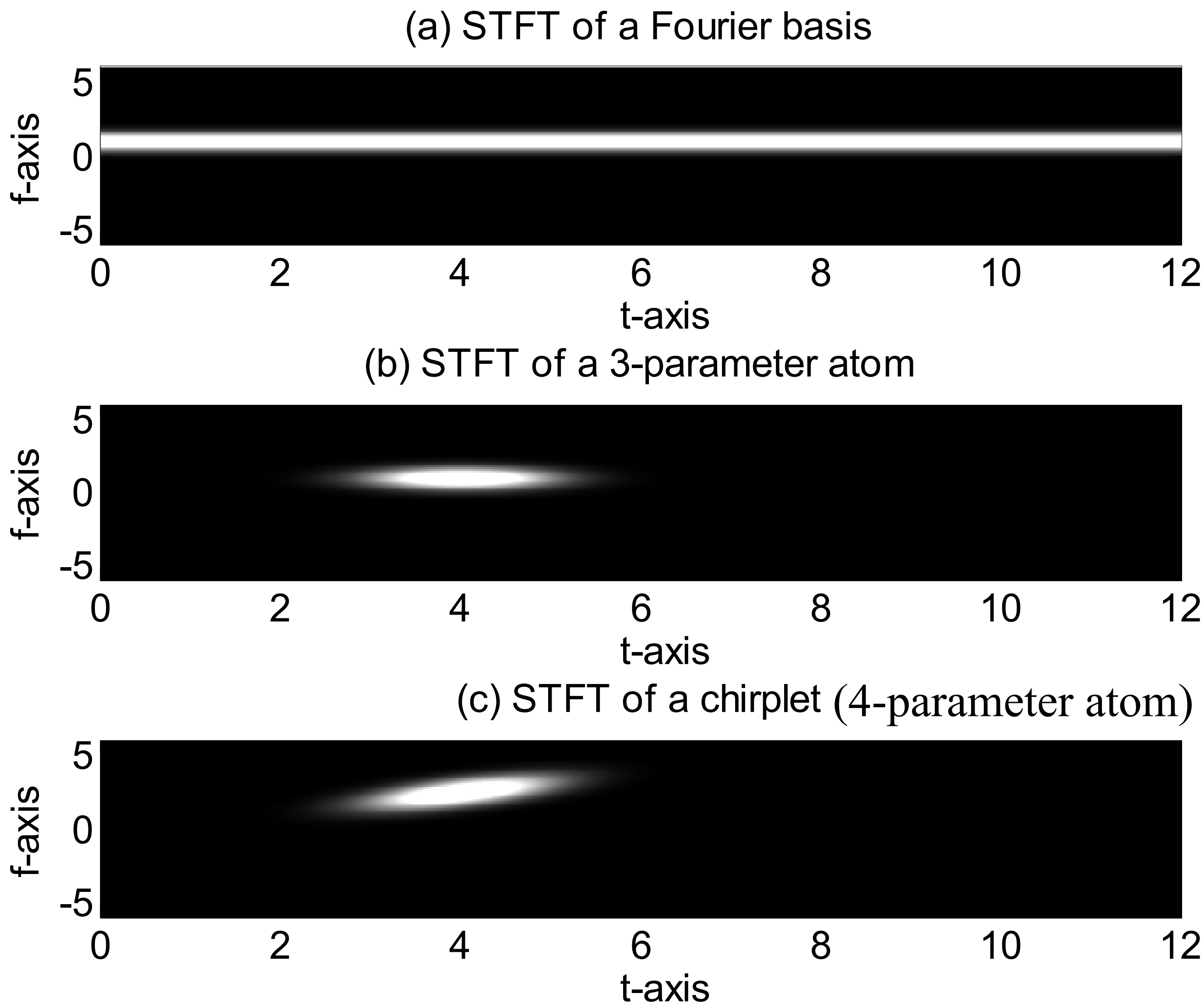
chirp

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}}\exp(j2\pi(f_0 t + \frac{\eta}{2}t^2) - \frac{\pi(t-t_0)^2}{\sigma^2})$$

4 parameters: t_0 controls the central time
f_0 controls the initial frequency
σ controls the scaling factor
η controls the chirp rate

[Ref] A. Bultan, "A four-parameter atomic decomposition of chirplets," *IEEE Trans. Signal Processing*, vol. 47, no. 3, pp. 731–745, Mar. 1999.

[Ref] C. Capus, and K. Brown. "Short-time fractional Fourier methods for the time-frequency representation of chirp signals," *J. Acoust. Soc. Am*. vol. 113, issue 6, pp. 3253-3263, 2003.


(a) STFT of a Fourier basis
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(b) STFT of a 3-parameter atom
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(c) STFT of a chirplet (4-parameter atom)
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis

(3) Prolate Spheroidal Wave Function (PSWF)

扁平圓波

$$x(t) \cong \sum_{n,T,\Omega,t_0,f_0} a_{n,T,\Omega,t_0,f_0} \psi_{n,T,\Omega}(t-t_0)\exp(j2\pi f_0 t)$$

where $\psi_{n,T,\Omega}(t)$ is the prolate spheroidal wave function

(i) No signal has finite supports in both the time and the frequency domain.

[Ref] D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty-I," *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, 1961.

(ii) PSWF is the time limited functions

$(x(t)=0$ for $|t|>T)$

(iii) However, among all the functions satisfy

$x(t)=0$ for $|t|>T$

PSWF can maximize $\dfrac{\int_{-\Omega}^{\Omega}|X(f)|^2 df}{\int_{-\infty}^{\infty}|X(f)|^2 df}$

Concept of the prolate spheroidal wave function (PSWF):

- FT: $X(f) = \int_{-\infty}^{\infty} \exp(-j2\pi f t) x(t) dt$, $x, f \in (-\infty, \infty)$.

energy preservation property (Parseval's property)

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- finite Fourier transform (fi-FT):

$$X_{fi}(f) = \int_{-T}^{T} \exp(-j2\pi f t) x(t) dt$$

space interval: $t \in [-T, T]$, frequency interval: $f \in [-\Omega, \Omega]$

$$0 < \text{energy preservation ratio} = \frac{\int_{-\Omega}^{\Omega} |X_{fi}(f)|^2 df}{\int_{-T}^{T} |x(t)|^2 dt} < 1$$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega} |X_{fi}(f)|^2 df}{\int_{-T}^{T} |x(t)|^2 dt}$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$

$\psi_{1,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$ and$\psi_{1,T,\Omega}$

$\psi_{2,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

and so on.

- Prolate spheroidal wave functions (PSWFs) are the continuous functions that satisfy: $\int_{-T}^{T} K_{F,\Omega}(t_1,t)\psi_{n,T,\Omega}(t)\,dt = \lambda_{n,T,\Omega}\psi_{n,T,\Omega}(t_1)$,

  where $K_{F,\Omega}(t_1,t) = \dfrac{\sin[2\pi\Omega(t_1-t)]}{\pi(t_1-t)}$ $= 2\,\mathrm{sinc}(2\Omega(t_1-t))$

PSWFs are orthonormal and can be sorted according to the values of $\lambda_{n,T,\Omega}$'s:

$$\int_{-T}^{T} \psi_{m,T,\Omega}(t)\psi_{n,T,\Omega}(t)\,dt = \delta_{m,n}$$

$1 > \lambda_{0,T,\Omega} > \lambda_{1,T,\Omega} > \lambda_{2,T,\Omega} > \ldots\ldots\ldots\ldots > 0.$ (All of $\lambda_{n,T,\Omega}$'s are real)

附錄十： Compressive Sensing and Matching Pursuit 的觀念

Different from orthogonal basis expansion, which applies a complete and orthogonal basis set, compressive sensing is to use an over-complete and non-orthogonal basis set to expand a signal.

Example:

Fourier series expansion is an orthogonal basis expansion method:

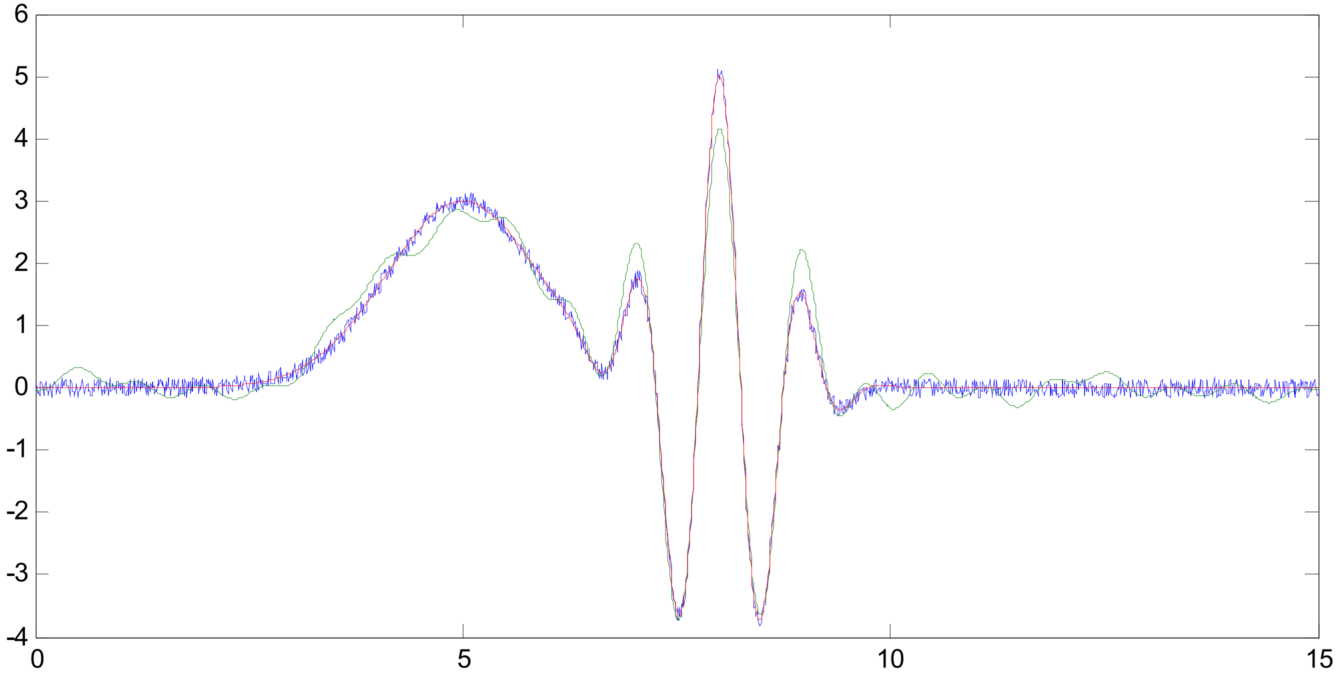
$$x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$$

$$\int \exp(j2\pi f_m t)\overline{\exp(j2\pi f_n t)}\, dt = 0 \qquad \text{if } f_m \neq f_n$$

Three-parameter atom expansion, Four-parameter atom (chirplet) expansion, and PSWF expansion are over-complete and non-orthogonal basis expansion methods.

$$x(t) \approx \sum a_{t_0, f_0, \sigma} \varphi_{t_0, f_0, \sigma}(t)$$

$\varphi_{t_0,f_0,\sigma}(t)$ do not form a complete and orthogonal set.



For example, in the above figure, the blue line is the original signal

- When using three-parameter atoms, the expansion result is the red line

$$x(t) = 3e^{-0.2\pi(t-5)^2} + 2.5e^{-0.4\pi(t-8)^2+j2\pi t} + 2.5e^{-0.4\pi(t-8)^2-j2\pi t}$$

Only 3 terms are used and the normalized root square error is 0.39%

- When using Fourier series, if 31 terms are used, the expansion result is the green line and the normalized root square error is 3.22%

(NRSE)

$$NRSE = \sqrt{\frac{\int |y(t) - x(t)|^2 dt}{\int |x(t)|^2 dt}}$$

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ ………….. form an over-complete and non-orthogonal basis set.

(matching pursuit problems)

(Problem 1) We want to minimize $||c||_0$ ($||\ ||_0$ 是 L_0 norm，$||c||_0$ 意指 c_m 的值不為 0 的個數） such that

Sparse: smaller L_0 norm

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_0$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_0$ is fixed to M, we want to minimize

$$\int \left(x(t) - \sum_{m=1}^{M} c_m b_m(t) \right)^2 dt$$

Question: How do we solve the optimization problems on page 229?

Method 1: Matching Pursuit (Greedy Algorithm) (not optimal)

atoms: $b_m(t)$, $m = 1, 2, 3, \ldots$.

$$\left|\int b_m(t) b_m^*(t)\,dt\right| = 1$$

(normalized)

Input: $x(t)$

Initial: $n = 0$, $x_r(t) = x(t)$

residue

↓

Find m such that $\left|\int x_r(t) b_m^*(t)\,dt\right|$ is maximal

↓

Set $\phi_n(t) = b_m(t)$, $\mu_n = \int x(t) b_m^*(t)\,dt$

↓

$x_r(t) = x_r(t) - \mu_n \phi_n(t)$

↓

$n = n+1$

↓

For Problem 1: Whether $x_r(t) = 0$

For Problem 2: Whether $\int |x_r|^2(t)\,dt < threshold$

For Problem 3: Whether $n = M$

No

Yes

$$x(t) \cong \sum_m \mu_m \phi_m(t)$$

Method 2: Basis Pursuit

Change the L_0 norm into the L_1 norm

$$||c||_1 = |c_0| + |c_1| + |c_2| + \ldots\ldots$$

(Problem 1) We want to minimize $||c||_1$ such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_1$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_1 \le M$, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

比較: for matching pursuit
the number of m such that $c_m \neq 0$ should be as small as possible

L_0-norm problem
hard to optimize

$x^0 = 1$ if $x \neq 0$
$x^0 = 0$ if $x = 0$

Norm (L_α norm): $\|x[n]\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$

$\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha = K$ where K is the number of points such that $x[n] \neq 0$

(Physical meaning: The number of nonzero points)

ex: L_0 norm of $[1, 0, 2, 0, 3, 0]$ is 3

L_1 norm: $\|x[n]\|_1 = \sum_{n=0}^{N-1} |x[n]|$

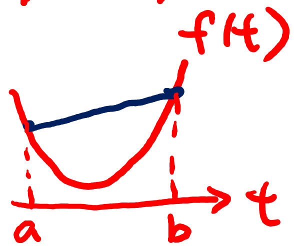
(Physical meaning: Sum of Amplitudes)

L_2 norm: $\|x[n]\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$

(Physical meaning: Distance)

$f(a) + \frac{f(b)-f(a)}{b-a}(t-a)$
$> f(t)$ for $a < t < b$

Matching Pursuit: Zero order norm $\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha$



convex

Basis Pursuit: First order norm L_1 norm

The L_α norm is convex if $\alpha \geq 1$. $\alpha < 1$ not convex

凸函數
only one local minimum

[Compressive Sensing 參考文獻]

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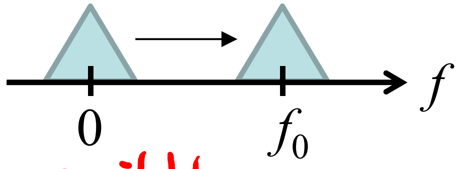
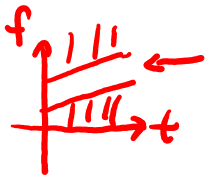
VIII. Motions on the Time-Frequency Distribution

Fourier spectrum 為 1-D form，只有二種可能的運動或變形：

Modulation $e^{j2\pi f_0 t}x(t) \xrightarrow{FT} X(f-f_0)$



Scaling $x(t/a) \xrightarrow{FT} |a|X(af)$



available band

TF distribution for the signal

Time-frequency analysis 為 2-D，在 2-D 平面上有多種可能的運動或變形

(1) Horizontal shifting (2) Vertical shifting

(3) Dilation (scaling) (4) Shearing 剪羊毛

(5) Generalized Shearing (6) Rotation

(7) Twisting

belt

扭曲

8-1 Basic Motions

(1) Horizontal Shifting

$$x(t-t_0) \rightarrow S_x(t-t_0, f)\,e^{-j2\pi f t_0} \quad \text{,STFT, Gabor}$$

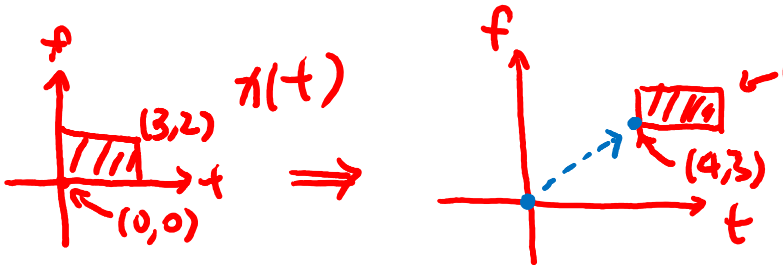
$$\rightarrow W_x(t-t_0, f) \quad \text{,Wigner}$$

(2) Vertical Shifting

instantaneous frequency = f_0

$$e^{j2\pi f_0 t}x(t) \rightarrow S_x(t, f-f_0) \quad \text{,STFT,Gabor}$$

$$\rightarrow W_x(t, f-f_0) \quad \text{,Wigner}$$



$t_0 = 4$
$f_0 = 3$

$e^{j6\pi t}x(t-4)$

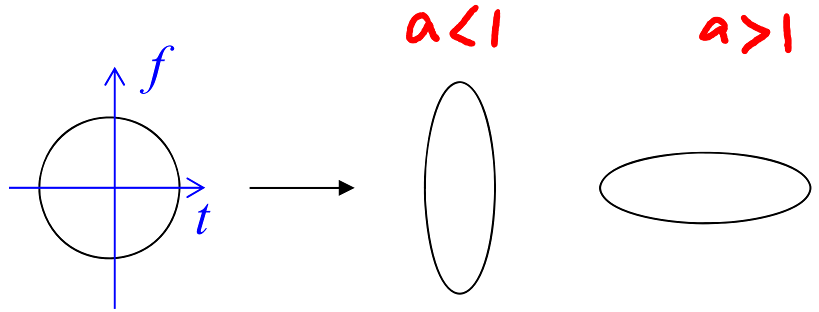
(3) Dilation (scaling)

$$\frac{1}{\sqrt{|a|}}x\left(\frac{t}{a}\right) \to \approx S_x\left(\frac{t}{a}, af\right) \text{ ,STFT,Gabor}$$

$$\to W_x\left(\frac{t}{a}, af\right) \text{ ,WDF}$$

$a>1$
$\Rightarrow$ wider

$a<1 \Rightarrow$ narrower



(4) Shearing

instantaneous frequency = at

lens (page 268)

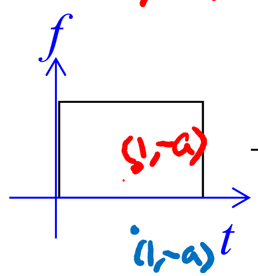
$a = \frac{-1}{\lambda f}$

① $x(t) = e^{j\pi a t^2}\, y(t)$

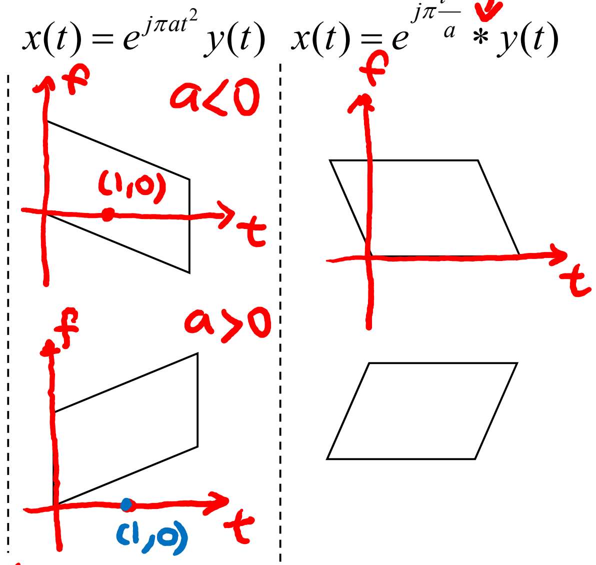
$S_x(t,f) \approx S_y(t, f-at)$,STFT,Gabor

$W_x(t,f) = W_y(t, f-at)$,WDF

$S_x(1,0) \approx S_y(1,-a)$



convolution



② $x(t) = e^{j\pi \frac{t^2}{a}} * y(t)$ (* means convolution)

Fresnel transform (page 267)

$a = \lambda z$

$S_x(t,f) \approx S_y(t-af, f)$, STFT,Gabor

$W_x(t,f) = W_y(t-af, f)$, WDF

$S_x(0,1) \approx S_y(-af, f)$

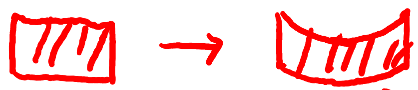
$X(f) = FT(e^{j\pi \frac{t^2}{a}})\, Y(f)$

$= \sqrt{ja}\, e^{-j\pi a f^2}\, Y(f)$

$e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

$e^{-\pi \sigma t^2} \rightarrow \frac{1}{\sqrt{\sigma}} e^{-\pi \frac{f^2}{\sigma}}$

$\sigma = \frac{-j}{a}$



$x(t) = e^{j\phi(t)}\, y(t)$

$\phi(t)$: 3rd order polynomial

(Proof): When $x(t) = e^{j\pi at^2} y(t)$,

$$W_x\left(t,f\right) = \int_{-\infty}^{\infty} x\left(t+\tau/2\right)x^*\left(t-\tau/2\right)e^{-j2\pi\tau f}\cdot d\tau$$

$$= \int_{-\infty}^{\infty} e^{j\pi a\left(t+\tau/2\right)^2} e^{-j\pi a\left(t-\tau/2\right)^2} y\left(t+\tau/2\right)y^*\left(t-\tau/2\right)e^{-j2\pi\tau f}d\tau$$

$$= \int_{-\infty}^{\infty} e^{j2\pi at\tau} y\left(t+\tau/2\right)y^*\left(t-\tau/2\right)e^{-j2\pi\tau f}d\tau$$

$$= \int_{-\infty}^{\infty} y\left(t+\tau/2\right)y^*\left(t-\tau/2\right)e^{-j2\pi\tau(f-at)}d\tau$$

$$= W_y\left(t, f-at\right)$$

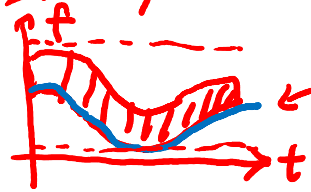
(5) Generalized Shearing

$$x(t) = e^{j\phi(t)} y(t) \quad \text{的影響?}$$

instantaneou frequency $= \sum_{k=1}^{n} \frac{a_k k}{2\pi} t^{k-1}$

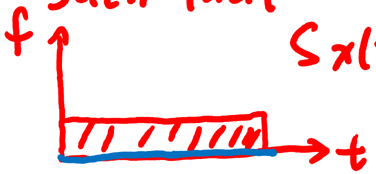
$$\phi(t) = \sum_{k=0}^{n} a_k t^k$$

If $S_y(t,f)$



$p(t)$

We want to obtain $x(t)$ such that



$S_x(t,f)$

$$S_x(t,f) \cong S_y\Big(t, f - \sum_{k=1}^{n} \frac{a_k k}{2\pi} t^{k-1}\Big) \text{ ,STFT,Gabor}$$

$$W_x(t,f) \cong W_y\Big(t, f - \sum_{k=1}^{n} \frac{a_k k}{2\pi} t^{k-1}\Big) \text{ ,WDF}$$

then

$x(t) = e^{j\phi(t)} y(t)$

$\phi(t) = 2\pi \int -p(t)\, dt$

J. J. Ding, S. C. Pei, and T. Y. Ko, "Higher order modulation and the efficient sampling algorithm for time variant signal," *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

J. J. Ding and C. H. Lee, "Noise removing for time-variant vocal signal by generalized modulation," *APSIPA ASC*, pp. 1-10, Kaohsiung, Taiwan, Oct. 2013

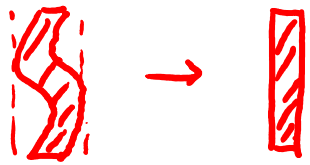
Q:

If $x(t) = h(t) * y(t)$ where $h(t) = IFT\left(\exp\left(j\sum_{k=0}^{n} a_k f^k\right)\right)$

then

$$S_x(t,f) \cong S_y(t + \frac{1}{2\pi}\sum_{k=1}^{n} k a_k f^{k-1}, \quad f) \text{ ,STFT,Gabor}$$

$$W_x(t,f) \cong W_y(t + \frac{1}{2\pi}\sum_{k=1}^{n} k a_k f^{k-1}, \quad f) \text{ ,WDF}$$



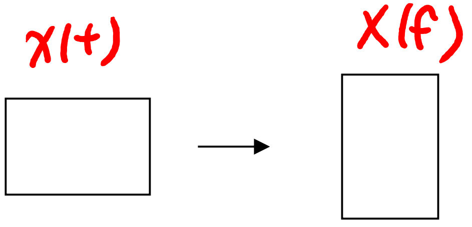
8-2 Rotation by $\pi/2$: Fourier Transform

$X(f) = FT(x(t))$

$|S_X(t,f)| \approx |S_x(-f,t)|$,STFT

$G_X(t,f) = G_x(-f,t)e^{-j2\pi ft}$,Gabor

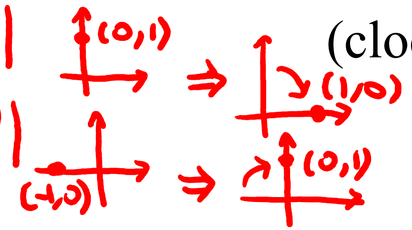
$W_X(t,f) = W_x(-f,t)$,WDF



(clockwise rotation by 90°)

$|S_X(1,0)| \approx |S_x(0,1)|$

$|S_X(0,1)| \approx |S_x(-1,0)|$



$FT(FT(FT(x(t)))) = X(-f)$ rotated by 270°

$FT(FT(x(t))) = x(-t)$ rotated by π (180°)

$FT(FT(FT(FT(x(t))))) = x(t)$ rotated by 2π (360°)

Strictly speaking, the rec-STFT have no rotation property.

For Gabor transforms, if

$$\boxed{G_x(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau} \;,$$

$$G_X(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} X(\tau)\,d\tau \qquad X(f)=FT[x(t)]=\int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

then $\boxed{G_X(t,f)=G_x(-f,t)e^{-j2\pi t f}}$

(clockwise rotation by 90° for amplitude)

(Proof): $G_X(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau}\int_{-\infty}^{\infty} x(u)e^{-j2\pi\tau u}du\,d\tau$

$$=\int_{-\infty}^{\infty} x(u)e^{-\pi(\tau-t)^2}\left(\int_{-\infty}^{\infty} e^{-j2\pi\tau(f+u)}d\tau\right)du$$

$$=\int_{-\infty}^{\infty} x(u)\left(\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi\tau(f+u)}d\tau\right)du=\int_{-\infty}^{\infty} x(u)\left(FT\left(e^{-\pi(\tau-t)^2}\right)\Big|_{f\to f+u}\right)du$$

Since $\quad FT\left(e^{-\pi\tau^2}\right)=e^{-\pi f^2}, \quad FT\left(e^{-\pi(\tau-t)^2}\right)=e^{-j2\pi tf}e^{-\pi f^2}$

$$G_X(t,f)=\int_{-\infty}^{\infty} x(u)e^{-j2\pi t(f+u)}e^{-\pi(f+u)^2}du$$

$$=e^{-j2\pi tf}\int_{-\infty}^{\infty} x(u)e^{-j2\pi tu}e^{-\pi(u-(-f))^2}du\;=G_x(-f,t)e^{-j2\pi t f}$$

If we define the Gabor transform as

$$\boxed{G_x(t,f) = e^{j\pi f t}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f \tau} x(\tau)\, d\tau}\ ,$$

and $G_X(t,f) = e^{j\pi f t}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f \tau} X(\tau)\, d\tau$

then $\boxed{G_X(t,f) = G_x(-f,t)}$

If $W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$ is the WDF of $x(t)$,

$W_X(t,f)=\int_{-\infty}^{\infty} X(t+\tau/2)\cdot X^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$ is the WDF of $X(f)$,

then $W_X(t,f)=W_x(-f,t)$

(clockwise rotation by 90°)

還有哪些 time-frequency distribution 也有類似性質？

- If $X(f) = IFT\left[x(t)\right] = \int_{-\infty}^{\infty} x(t) e^{j2\pi f t} dt$, then

$$W_X(t,f) = W_x(f,-t), \qquad G_X(t,f) = G_x(f,-t)e^{j2\pi t f}$$

(counterclockwise rotation by 90°).

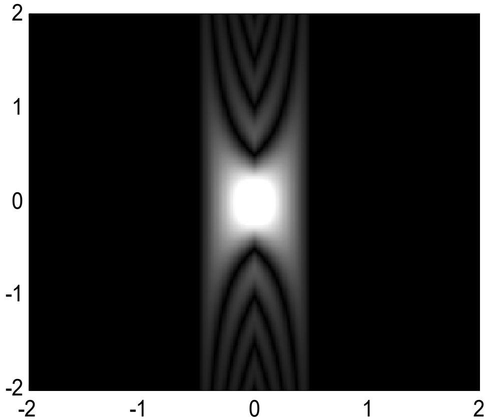
- If $X(f) = x(-t)$, then

$$W_X(t,f) = W_x(-t,-f) \quad , \qquad G_X(t,f) = G_x(-t,-f).$$

(rotation by 180°).

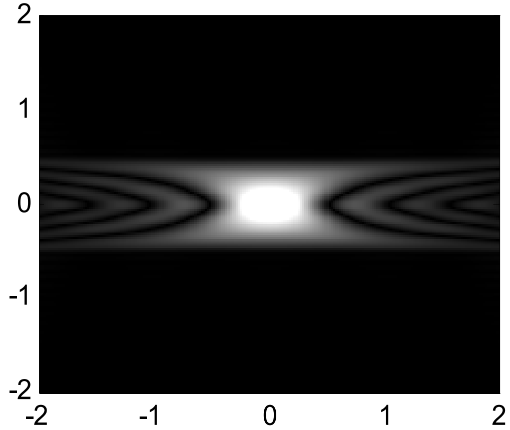
Examples: $x(t) = \Pi(t)$, $X(f) = FT[x(t)] = \text{sinc}(f)$.

WDF of $\Pi(t)$

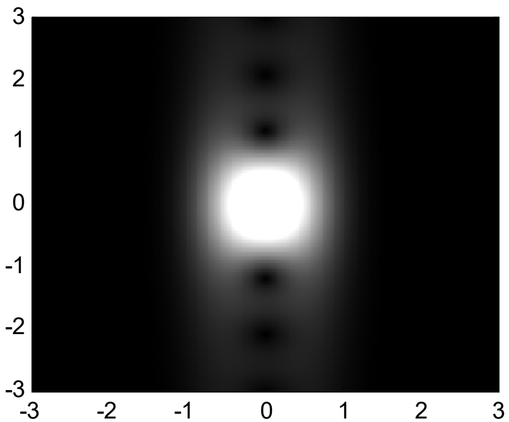


$\pi/2$

WDF of sinc(t)

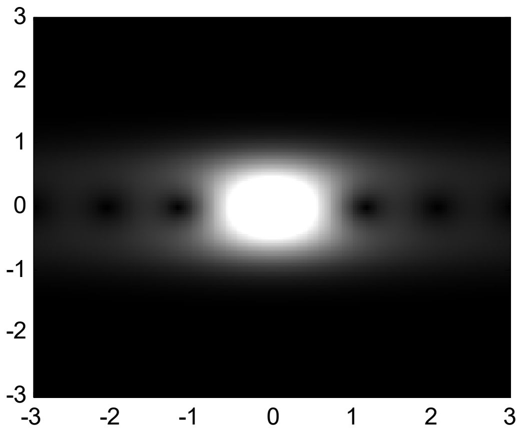


Gabor transform of $\Pi(t)$



$\pi/2$

Gabor transform of sinc(t)



If a function is an eigenfunction of the Fourier transform,

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} x(t)\,dt = \lambda x(f) \qquad \lambda = 1, -j, -1, j$$

then its WDF and Gabor transform have the property of

$$W_x(t,f) = W_x(f,-t) \qquad \left|G_x(t,f)\right| = \left|G_x(f,-t)\right|$$

(轉了 90°之後，和原來還是一樣)

Example: Gaussian function

$$\exp\left(-\pi t^2\right)$$

Hermite-Gaussian function

$$\phi_m(t) = \exp\left(-\pi t^2\right) H_m(t)$$

Hermite polynomials: $H_m(t) = C_m e^{2\pi t^2} \dfrac{d^m}{dt^m} e^{-2\pi t^2}$, C_m is some constant,

$$H_0(t) = 1 \qquad H_1(t) = t \qquad H_2(t) = 4\pi t^2 - 1$$

$$H_3(t) = 4\pi t^3 - 3t \qquad H_4(t) = 16\pi^2 t^4 - 24\pi t^2 + 3$$

$$\int_{-\infty}^{\infty} e^{-2\pi t^2} H_m(t) H_n(t) = D_m \delta_{m,n}$$, D_m is some constant,

$\delta_{m,n} = 1$ when $m = n$, $\delta_{m,n} = 0$ otherwise.

[Ref] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 1990.

Hermite-Gaussian functions are eigenfunctions of the Fourier transform

$$\int_{-\infty}^{\infty} \phi_m(t) e^{-j2\pi f t} dt = (-j)^m \phi_m(f)$$

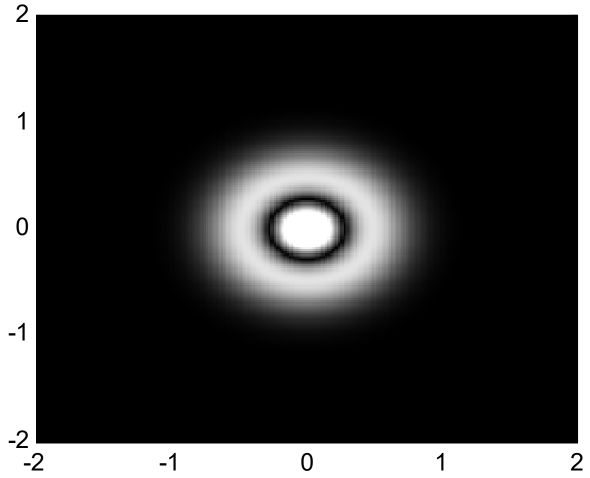
Any eigenfunction of the Fourier transform can be expressed as the form of

$$k(t) = \sum_{q=0}^{\infty} a_{4q+r} \phi_{4q+r}(t)$$

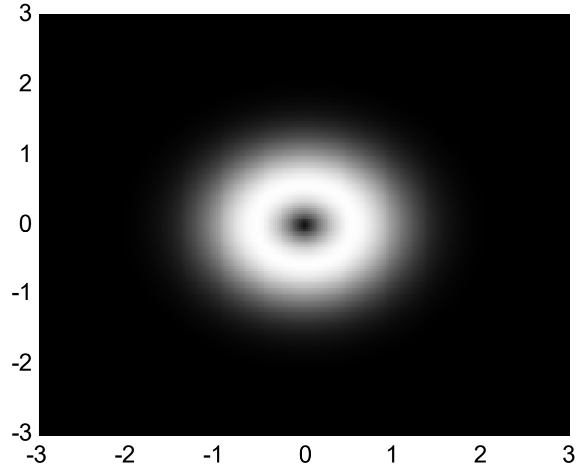
where $r = 0, 1, 2,$ or 3,
a_{4q+r} are some constants

$$\int_{-\infty}^{\infty} k(t) e^{-j2\pi f t} dt = (-j)^r k(f)$$

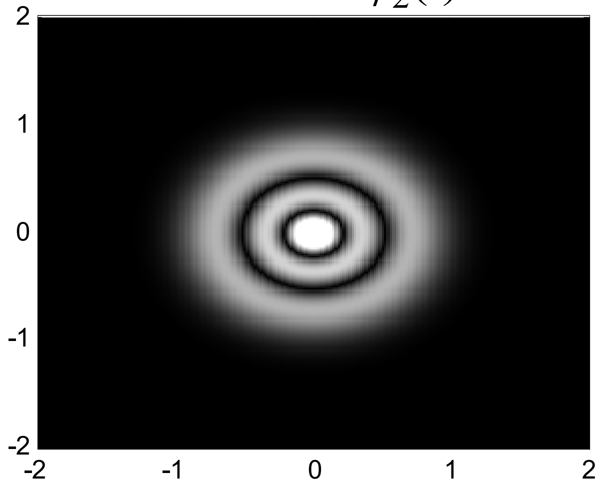
WDF for $\phi_1(t)$



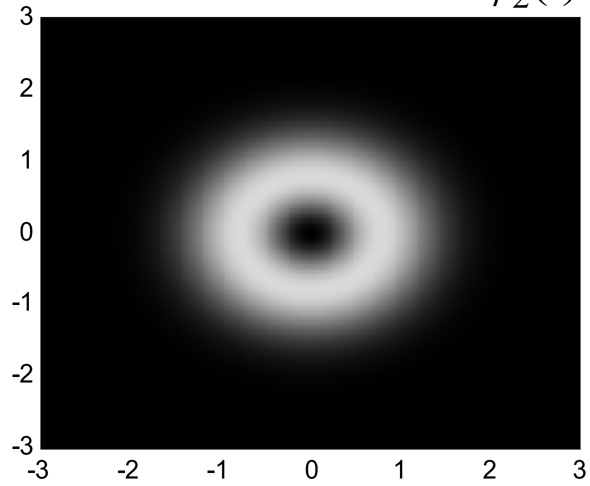
Gabor transform for $\phi_1(t)$



WDF for $\phi_2(t)$



Gabor transform for $\phi_2(t)$



Problem: How to rotate the time-frequency distribution by the angle other than $\pi/2$, π, and $3\pi/2$?

8-3 Rotation: Fractional Fourier Transforms (FRFTs)

x chirp   scaled FT   x chirp

$$X_\phi(u) = \sqrt{1 - j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2} \int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t} e^{j\pi\cdot\cot\phi\cdot t^2} x(t)\,dt\;, \qquad \phi = 0.5a\pi$$

When $\phi = 0.5\pi$, the FRFT becomes the FT.

$\csc(0.5\pi)=1$
$\cot(0.5\pi)=0$

Additivity property:

If we denote the FRFT as O_F^ϕ (i.e., $X_\phi(u) = O_F^\phi[x(t)]$)

then $O_F^\sigma\left\{O_F^\phi[x(t)]\right\} = O_F^{\phi+\sigma}[x(t)]$

Physical meaning: Performing the FT a times.

$a=\frac{1}{2}$ $\phi=\frac{\pi}{4}$
$a=\frac{2}{3}$ $\phi=\frac{\pi}{3}$

Another definition $$X_{\phi}(u)=\sqrt{\frac{1-j\cot\phi}{2\pi}}\,e^{j\frac{\cot\phi}{2}\cdot u^2}\int_{-\infty}^{\infty}e^{-j\csc\phi\cdot u\,t}\,e^{j\frac{\cot\phi}{2}\cdot t^2}\,x(t)\,dt$$

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] N. Wiener, "Hermitian polynomials and Fourier analysis," *Journal of Mathematics Physics MIT*, vol. 18, pp. 70-73, 1929.

[Ref] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

[Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

[Ref] S. C. Pei and J. J. Ding, "Closed form discrete fractional and affine Fourier transforms," *IEEE Trans. Signal Processing*, vol. 48, no. 5, pp. 1338-1353, May 2000.

$$FT[x(t)] = X(f)$$

$$FT\{FT[x(t)]\} = x(-t)$$

$$FT(FT\{FT[x(t)]\}) = X(-f) = IFT[f(t)]$$

$$FT[FT(FT\{FT[x(t)]\})] = x(t)$$

What happen if we do the FT non-integer times?

Physical Meaning:

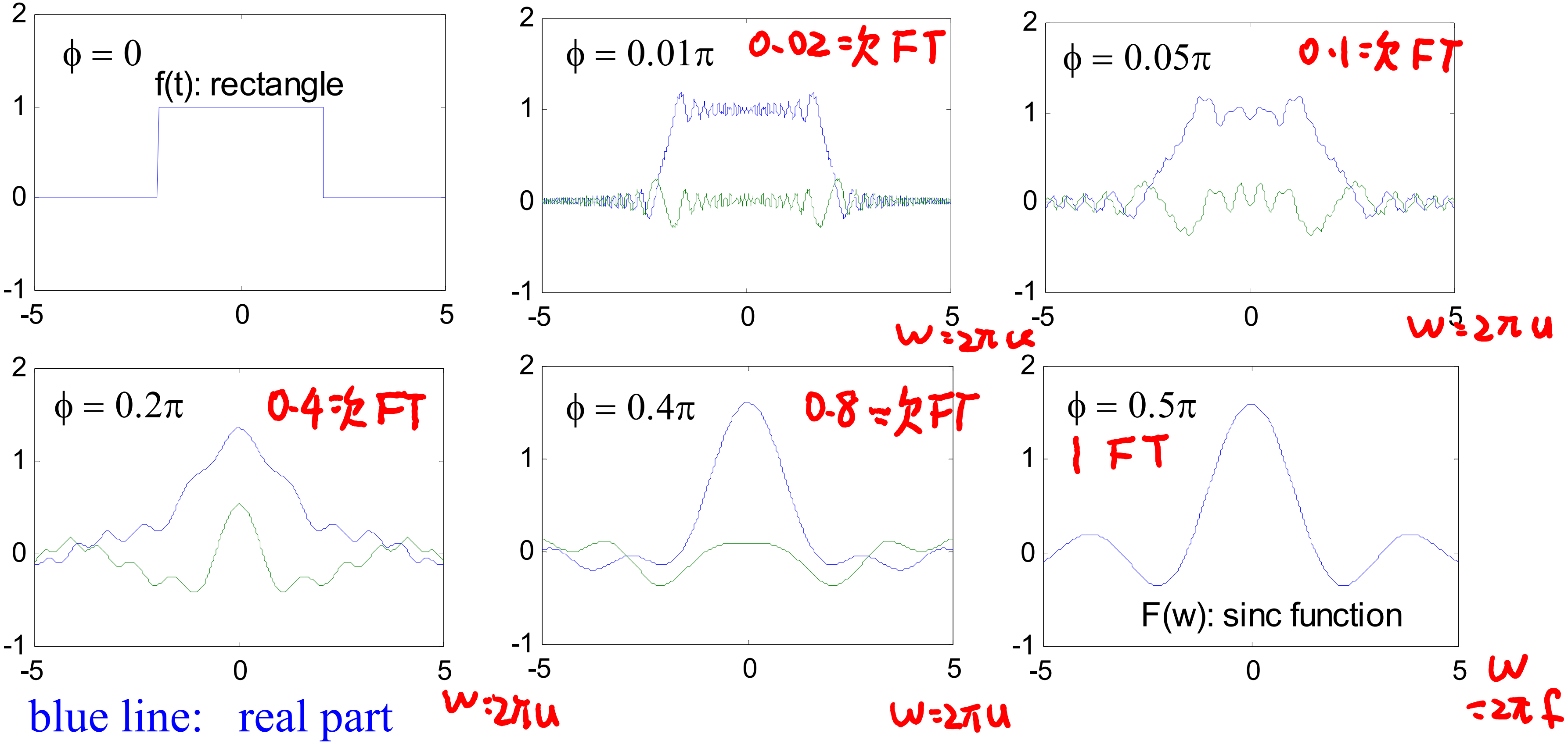
Fourier Transform: time domain $\rightarrow$ frequency domain

Fractional Fourier transform: time domain $\rightarrow$ fractional domain

Fractional domain: the domain between time and frequency

(partially like time and partially like frequency)

Experiment:



blue line: real part

green line: imaginary part

[Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

Time domain	Frequency domain	fractional domain
Modulation	Shifting	Modulation + Shifting
Shifting	Modulation	Modulation + Shifting
Differentiation	$\times j2\pi f$	Differentiation and $\times j2\pi f$
$\times -j2\pi f$	Differentiation	Differentiation and $\times -j2\pi f$

$$x(t-t_0) \xrightarrow{FT} \exp(-j2\pi f t_0)X(f)$$

$$x(t-t_0) \xrightarrow{fractional\ FT} \exp(j\varphi - j2\pi u t_0 \sin\phi)X(u - t_0\cos\phi)$$

$$\varphi = \pi t_0^2 \sin\phi\cos\phi$$

$$\frac{d}{dt}x(t) \xrightarrow{FT} j2\pi f\, X(f)$$

$$\frac{d}{dt}x(t) \xrightarrow{fractional\ FT} j2\pi u\, X(u)\sin\phi + \frac{d}{du}X(u)\cos\phi$$

[Theorem] The fractional Fourier transform (FRFT) with angle ϕ is equivalent to <u>the **clockwise rotation operation** with angle ϕ</u> for the Wigner distribution function (or for the Gabor transform)

FRFT with parameter ϕ = $\curvearrowright$ with angle ϕ

For the WDF

If $W_x(t, f)$ is the WDF of $x(t)$, and $W_{X\phi}(u, v)$ is the WDF of $X_\phi(u)$, ($X_\phi(u)$ is the FRFT of $x(t)$), then

$$\boxed{W_{X_\phi}(u,v) = W_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

For the Gabor transform (with standard definition)

If $G_x(t, f)$ is the Gabor transform of $x(t)$,
and $G_{X\phi}(u, v)$ is the Gabor transform of $X_\phi(u)$, then

$$\boxed{G_{X_\phi}(u,v) = e^{j[-2\pi uv\sin^2\phi + \pi(u^2-v^2)\sin(2\phi)/2]} G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

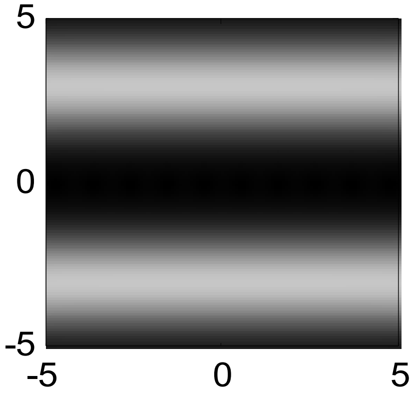
$$\boxed{\left|G_{X_\phi}(u,v)\right| = \left|G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)\right|}$$

For the Gabor transform (with another definition on page 244)

$$\boxed{G_{X_\phi}(u,v) = G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)}$$

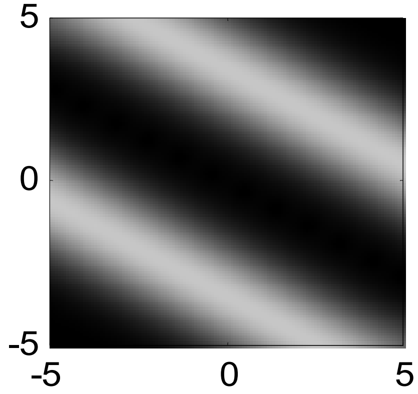
The Cohen's class distribution and the Gabor-Wigner transform also have the rotation property

The Gabor Transform for the FRFT of a cosine function



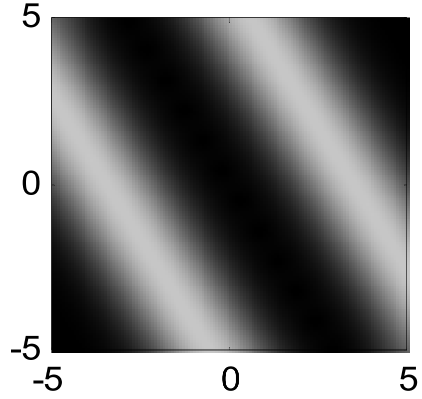
(a) $\phi = 0$

0次 FT



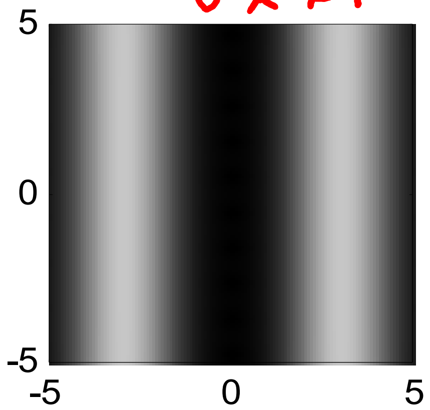
(b) $\phi = \pi/6$

$\frac{1}{3}$次 FT



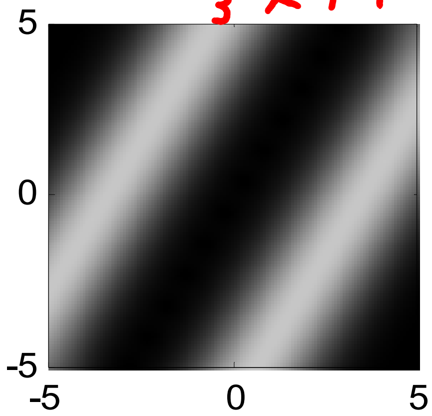
(c) $\phi = 2\pi/6$

$\frac{2}{3}$次 FT



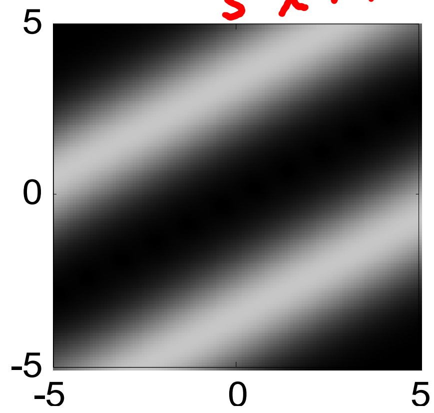
(d) $\phi = 3\pi/6$

1次 FT



(e) $\phi = 4\pi/6$

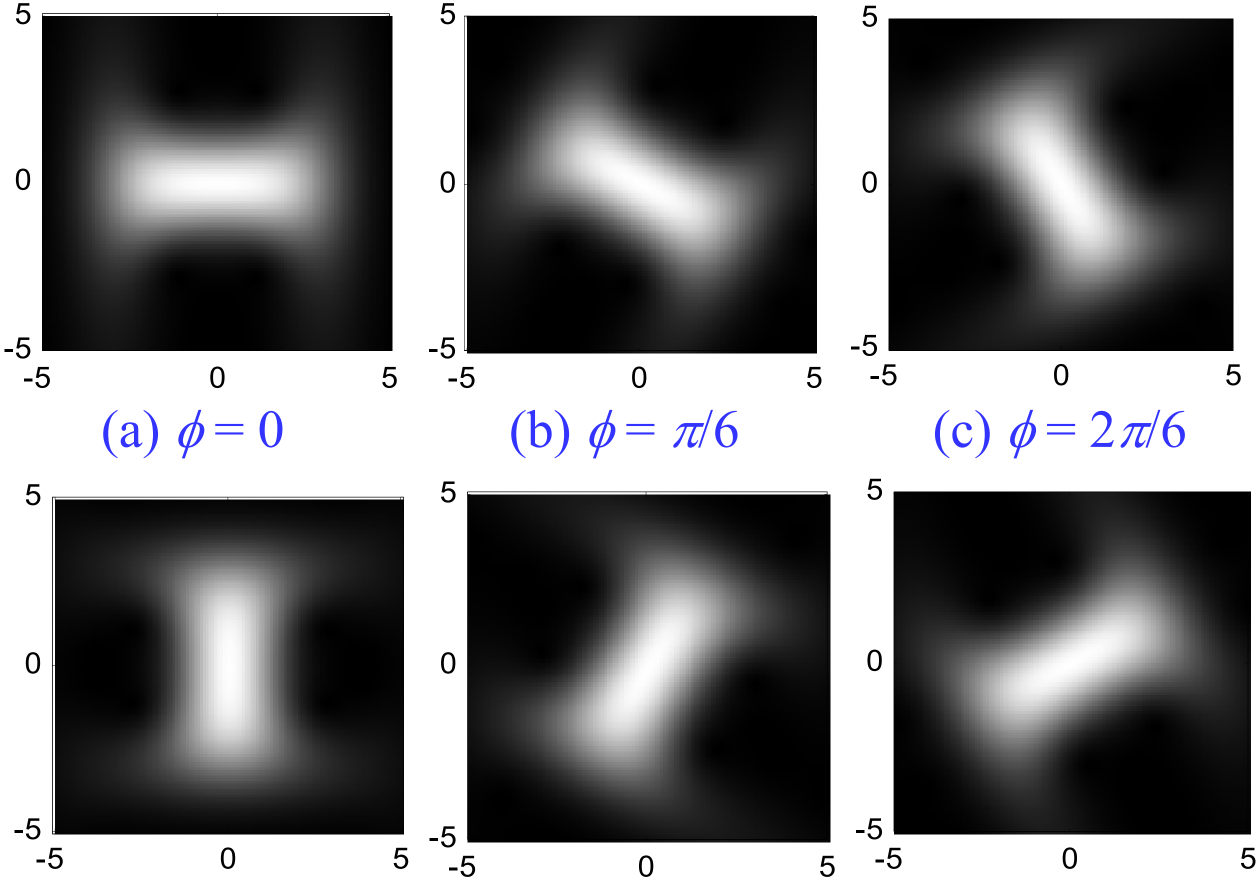
$\frac{4}{3}$次 FT



(f) $\phi = 5\pi/6$

$\frac{5}{3}$次 FT

The Gabor Transform for the FRFT of a rectangular function.



(a) $\phi = 0$ (b) $\phi = \pi/6$ (c) $\phi = 2\pi/6$

(d) $\phi = 3\pi/6$ (e) $\phi = 4\pi/6$ (f) $\phi = 5\pi/6$

8-4 Twisting: Linear Canonical Transform (LCT)

chirp 完整 scaled FT × chirp 比較 page 253

$$X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{jb}}\,e^{j\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-j2\pi\frac{1}{b}ut}\,e^{j\pi\frac{a}{b}t^2}\,x(t)\,dt \qquad \text{when } b\neq 0$$

$$X_{(a,0,c,d)}(u)=\sqrt{d}\cdot e^{j\pi cd u^2}\,x(du) \qquad \text{when } b=0$$

$ad-bc=1$ should be satisfied

Four parameters a, b, c, d

$c=\frac{ad-1}{b}$

$a=?$ $b=?$ $d=?$

The LCT is reduced to the FRFT with parameter ϕ ?

When $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

$X_{abcd}(u)=\sqrt{\frac{1}{j\lambda z}}\,e^{j\frac{\pi}{\lambda z}u^2}\int_{-\infty}^{\infty}e^{-j\frac{2\pi}{\lambda z}ut}\,e^{j\frac{\pi}{\lambda z}t^2}\,x(t)\,dt$

$=\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty}e^{j\frac{\pi}{\lambda z}(u-t)^2}\,x(t)\,dt=\sqrt{\frac{1}{j\lambda z}}\,e^{j\frac{\pi}{\lambda z}t^2}*x(t)$

When $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ \frac{-1}{\lambda f} & 1 \end{bmatrix}$

$X_{abcd}(u)=e^{-j\frac{\pi}{\lambda f}u^2}\,x(u)$

Additivity property of the WDF

If we denote the LCT by $O_F^{(a,b,c,d)}$, i.e., $X_{(a,b,c,d)}(u) = O_F^{(a,b,c,d)}[x(t)]$

then $$O_F^{(a_2,b_2,c_2,d_2)}\left\{O_F^{(a_1,b_1,c_1,d_1)}[x(t)]\right\} = O_F^{(a_3,b_3,c_3,d_3)}[x(t)]$$

where $$\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

[Ref] K. B. Wolf, "*Integral Transforms in Science and Engineering*," Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

If $W_{X_{(a,b,c,d)}}(u,v)$ is the WDF of $X_{(a,b,c,d)}(u)$, where $X_{(a,b,c,d)}(u)$ is the LCT of $x(t)$, then

$$W_{X_{(a,b,c,d)}}(u,v) = W_x(du - bv\ , -cu + av)$$

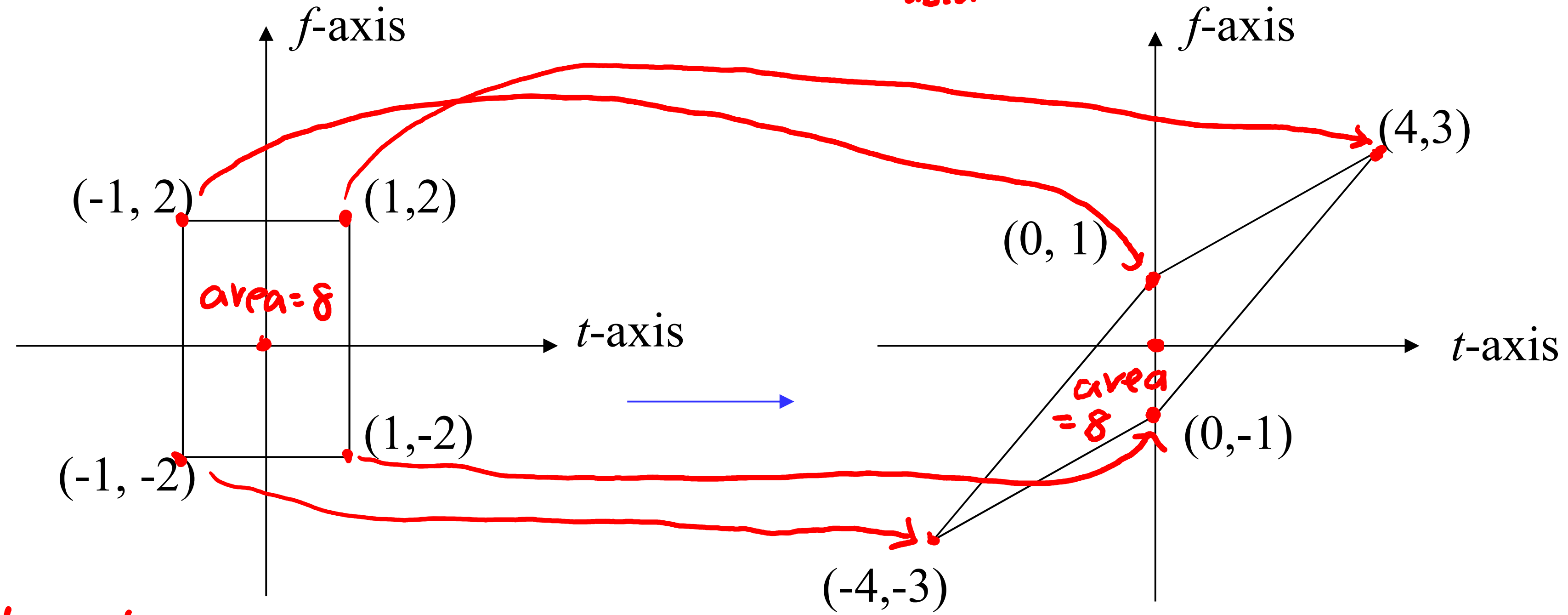
$$W_{X_{(a,b,c,d)}}(au + bv\ , cu + dv) = W_x(u,v)$$

LCT == twisting operation for the WDF

The Cohen's class distribution also has the twisting operation.

我們可以自由的用 LCT 將一個中心在 (0, 0) 的平行四邊形的區域，扭曲成另外一個面積一樣且中心也在 (0, 0) 的平行四邊形區域。

$W_x(u,v)$ $\qquad\qquad$ $W_{x_{abcd}}(au+bv, cu+dv) = W_x(u,v)$



$W_{x_{abcd}}(0,1) = W_x(-1,2)$

$-a+2b=0 \qquad -c+2d=1$

$W_{x_{abcd}}(4,3) = W_x(1,2)$

$a+2b=4 \qquad c+2d=3$

$\Downarrow \qquad\qquad \Downarrow$

$a=2,\ b=1 \qquad c=1,\ d=1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}}\, e^{j\pi \frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b} u t}\, e^{j\pi \frac{a}{b} t^2}\, x(t)\, dt \quad \text{when } b \neq 0$$

$$X_{(a,0,c,d)}(u) = \sqrt{d} \cdot e^{j\pi c d u^2}\, x(d\,u) \quad \text{when } b = 0$$

$ad - bc = 1$ should be satisfied

linear canonical transform

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ fractional Fourier transform
  - $\phi = \pi/2$ Fourier transform
  - $\phi = 0$ identity operation
  - $\phi = -\pi/2$ inverse Fourier transform
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$ Fresnel transform (convolution with a chirp)
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}$ chirp multiplication $X_{(a,0,c,d)}(u) = e^{j\pi \tau u^2}\, x(u)$
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/\sigma & 0 \\ 0 & \sigma \end{bmatrix}$ scaling

附錄十一　Linear Canonical Transform 和光學系統的關係

(1) Fresnel Transform (電磁波在空氣中的傳播)

$$U_o(x,y) = -\frac{i}{\lambda}\frac{e^{ikz}}{z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}\left[(x-x_i)^2+(y-y_i)^2\right]} U_i(x_i,y_i)\,dx_i dy_i$$

$k = 2\pi/\lambda$: wave number　　λ: wavelength　　z: distance of propagation

$U_o(x,y) = \frac{e^{jkz}}{j\lambda z}\, e^{j\frac{k}{2z}y^2} * e^{j\frac{k}{2z}x^2} * U_i(x,y)$

$$U_o(x,y) = e^{ikz}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(y-y_i)^2}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(x-x_i)^2} U_i(x_i,y_i)\,dx_i dy_i$$

(2 個 1-D 的 LCT)　　$\frac{k}{2z} = \frac{2\pi}{2\lambda z} = \frac{\pi}{\lambda z}$

$$e^{j\frac{k}{2z}x^2} * U_i(x,y_i) = e^{j\frac{\pi}{\lambda z}x^2} * U_i(x,y_i)$$

Fresnel transform 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

(or spherical disk)

$k = \frac{2\pi}{\lambda}$

(2) Spherical lens, refractive index = n

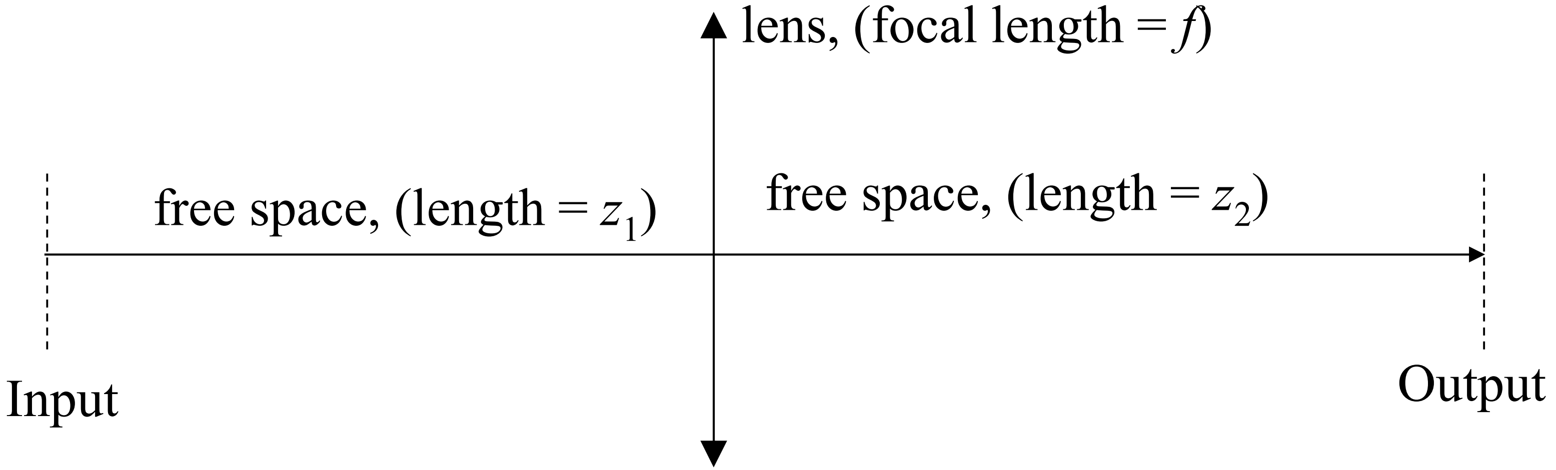
$$U_o(x,y) = e^{ikn\Delta} e^{-j\frac{k}{2f}\left[x^2+y^2\right]} U_i(x,y)$$

$e^{-j\frac{k}{2f}(x^2+y^2)} = e^{-j\frac{\pi}{\lambda f}(x^2+y^2)}$

f: focal length　　Δ: thickness of lens

經過 lens 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}$ 的情形

(3) Free spaces + Spherical lens



Input 和 output 之間的關係，可以用 LCT 表示

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_2}{f} & \lambda(z_1+z_2)-\dfrac{\lambda z_1 z_2}{f} \\ -\dfrac{1}{\lambda f} & 1-\dfrac{z_1}{f} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_2}{f} & \lambda(z_1+z_2)-\dfrac{\lambda z_1 z_2}{f} \\ -\dfrac{1}{\lambda f} & 1-\dfrac{z_1}{f} \end{bmatrix}$$

$z_1 = z_2 = 2f \rightarrow$ 即高中物理所學的「倒立成像」

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -\dfrac{1}{\lambda f} & -1 \end{bmatrix}$$

from page 262

$X_{abcd}(u) = \sqrt{-1}\, e^{j\frac{\pi}{\lambda f}u^2} x(-u)$

$|X_{abcd}(u)| = |x(-u)|$

$z_1 = z_2 = f \rightarrow$ Fourier Transform + Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & \lambda f \\ -\dfrac{1}{\lambda f} & 0 \end{bmatrix}$$

from page 262

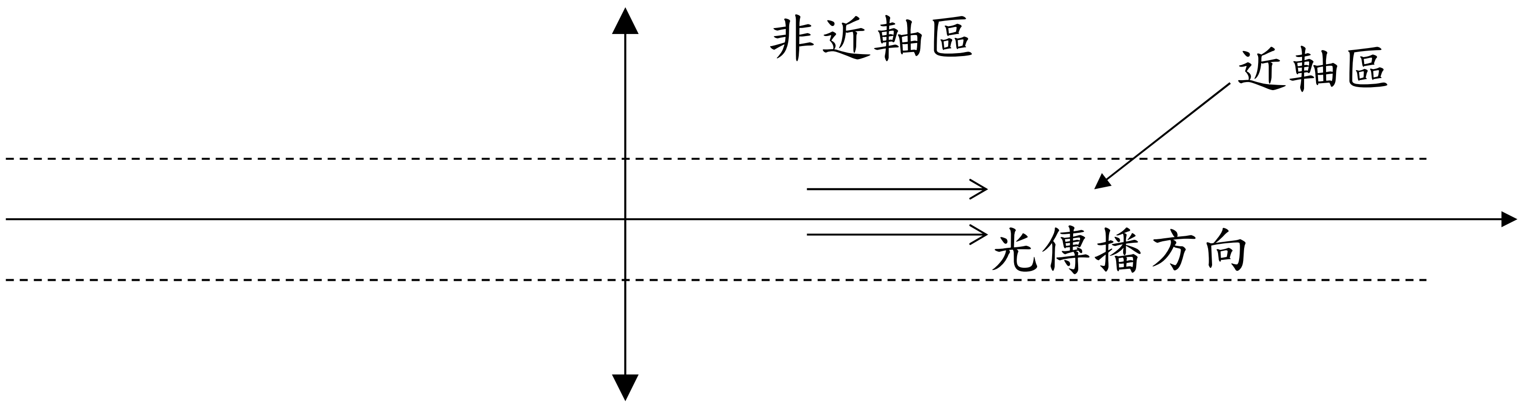
$X_{abcd}(u) = \sqrt{\frac{1}{j\lambda f}} \int_{-\infty}^{\infty} e^{-j\frac{2\pi}{\lambda f}ut} x(t)\,dt$

$z_1 = z_2 \rightarrow$ fractional Fourier Transform + Scaling

用 LCT 來分析光學系統的好處：

只需要用到 2×2 的矩陣運算，避免了複雜的物理理論和數學積分

但是 LCT 來分析光學系統的結果，只有在「近軸」的情形下才準確



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[1] H. M. Ozaktas and D. Mendlovic, “Fractional Fourier optics,” *J. Opt. Soc. Am. A*, vol.12, 743-751, 1995.

[2] L. M. Bernardo, “ABCD matrix formalism of fractional Fourier optics,” *Optical Eng.*, vol. 35, no. 3, pp. 732-740, March 1996.