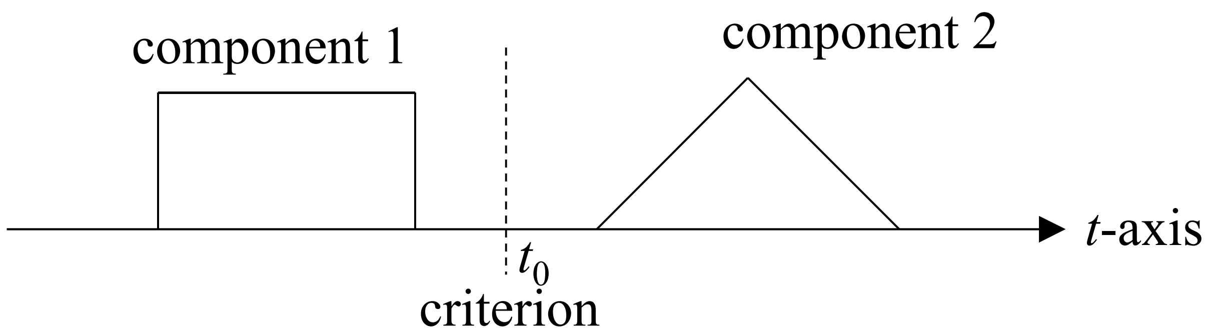
IX. Applications of Time-Frequency Analysis for Filter Design

9-1 Signal Decomposition and Filter Design

Signal Decomposition: Decompose a signal into several components.

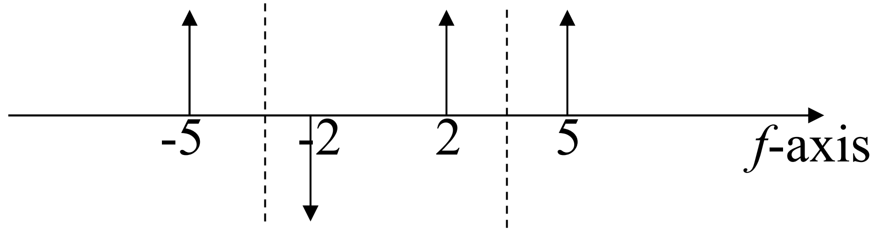
Filter: Remove the undesired component of a signal

(1) Decomposing in the time domain



(2) Decomposing in the frequency domain

$$x(t) = \sin(4\pi t) + \cos(10\pi t)$$



- Sometimes, signal and noise are separable in the time domain $\rightarrow$

  (1) without any transform

- Sometimes, signal and noise are separable in the frequency domain $\rightarrow$

  (2) using the FT (conventional filter)

$$x_o(t) = IFT\left[FT(x_i(t))H(f)\right]$$

$H(f)=1$ for $|f|<f_0$

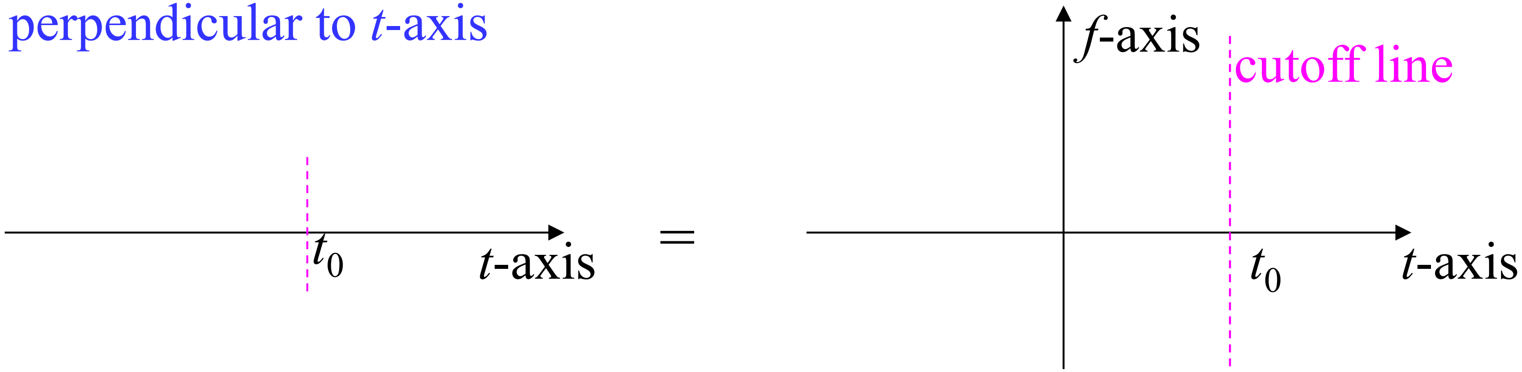
$2<f_0<5$

$H(f)=0$ for $|f|>f_0$

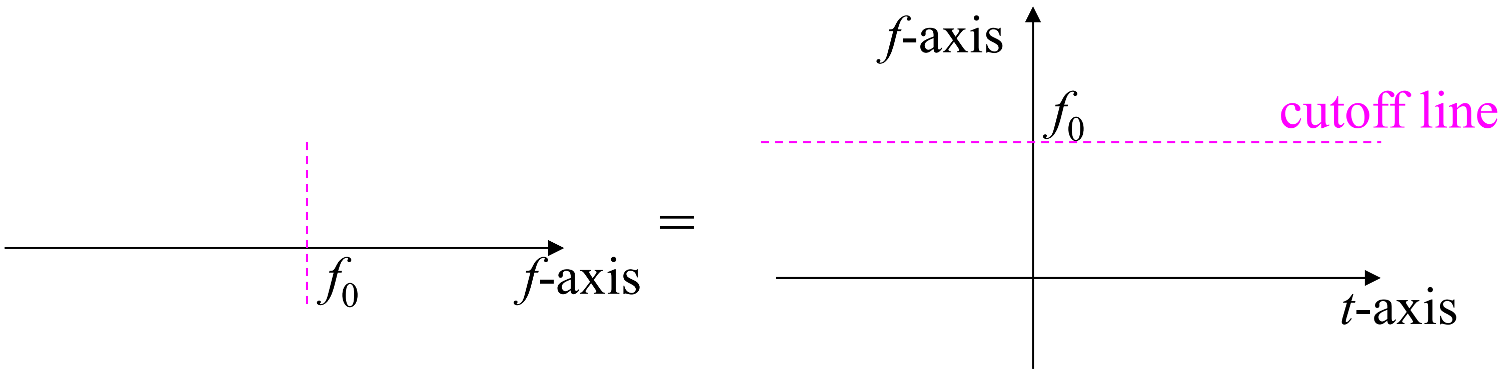
- If signal and noise are not separable in both the time and the frequency domains $\rightarrow$

  (3) Using the fractional Fourier transform and time-frequency analysis

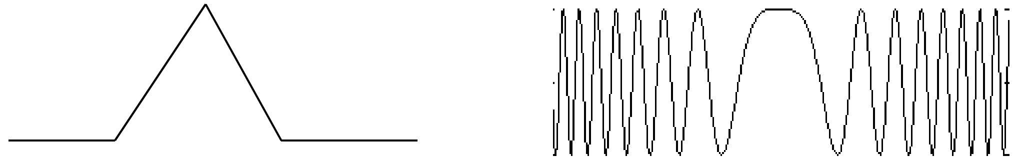
以時頻分析的觀點，criterion in the time domain 相當於 cutoff line perpendicular to t-axis

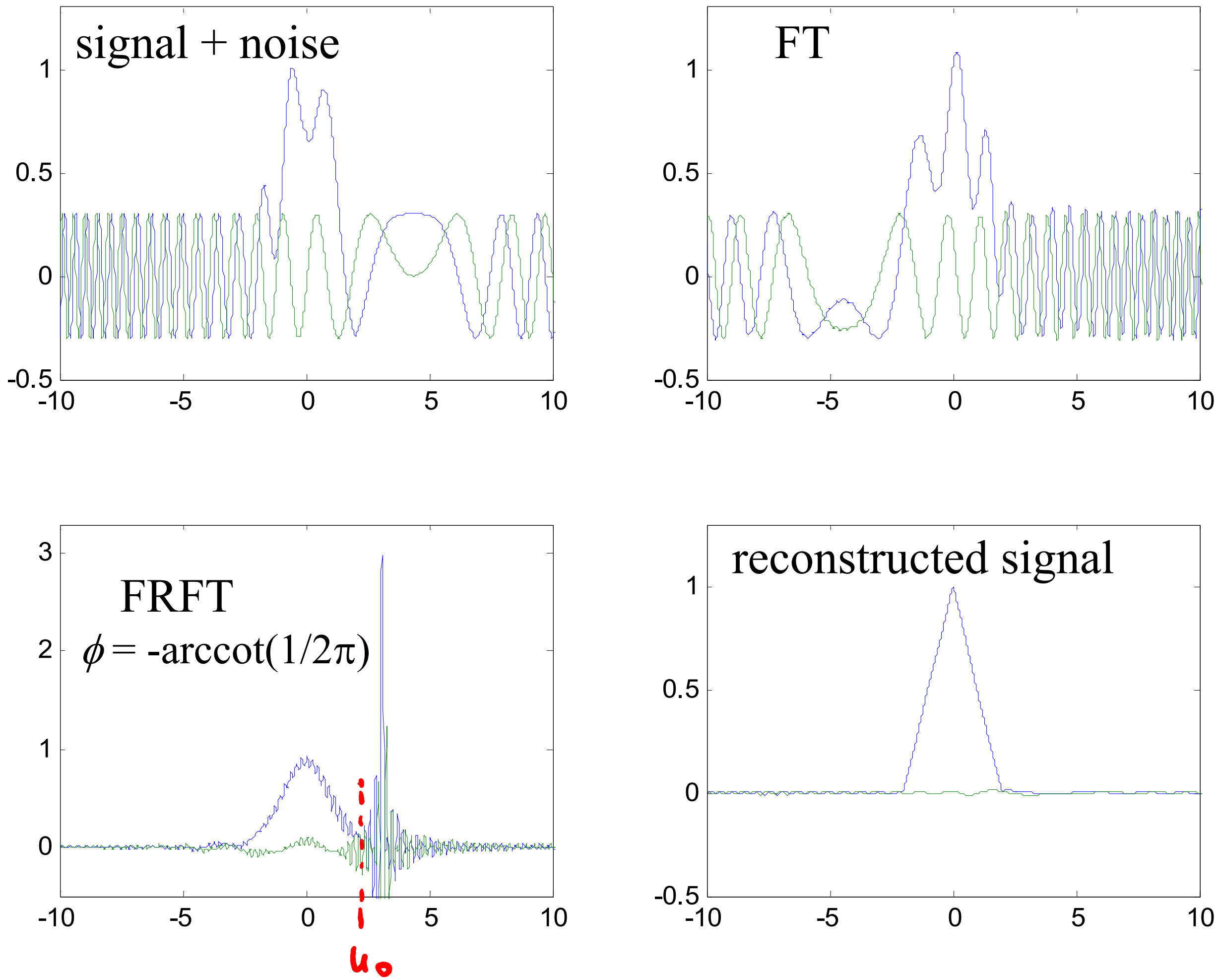


以時頻分析的觀點，criterion in the frequency domain 相當於 cutoff line perpendicular to f-axis



$x(t)$ = triangular signal + chirp noise $0.3\exp[j\,0.5(t-4.4)^2]$





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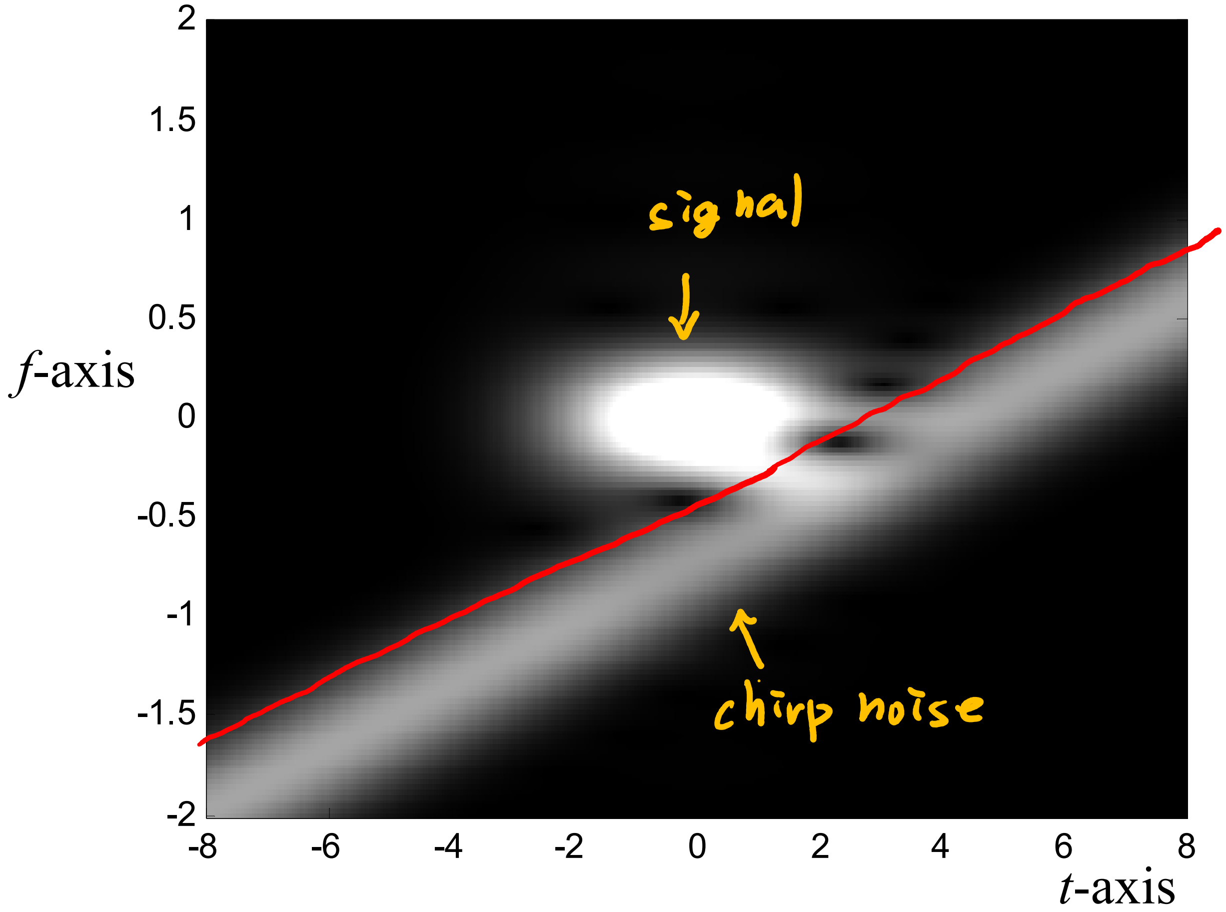
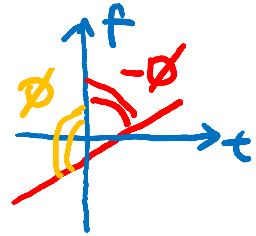




Decomposing in the time-frequency distribution

If $x(t) = 0$ for $t < T_1$ and $t > T_2$

$W_x\left(t, f\right) = 0$ for $t < T_1$ and $t > T_2$ (cutoff lines perpendicular to t-axis)

If $X(f) = FT[x(t)] = 0$ for $f < F_1$ and $f > F_2$

$W_x\left(t, f\right) = 0$ for $f < F_1$ and $f > F_2$ (cutoff lines parallel to t-axis)

What are the cutoff lines with other directions?

with the aid of the **FRFT**, the LCT, or the Fresnel transform

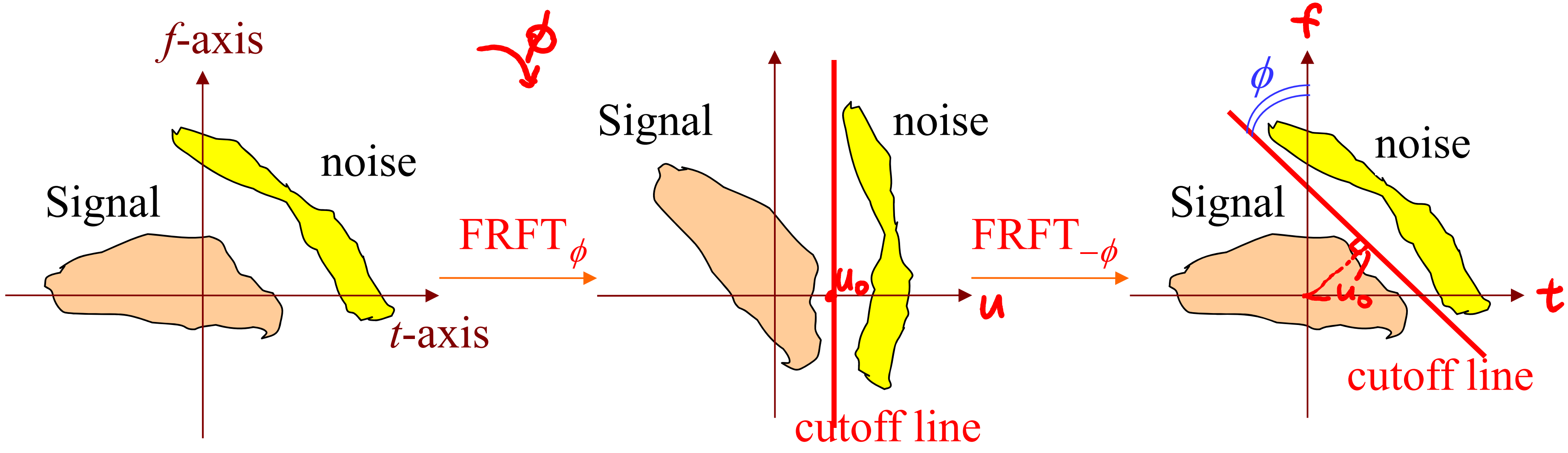
- Filter designed by the fractional Fourier transform

$$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]H(u)\right\}$$

比較： $x_o(t) = IFT\left[FT(x_i(t))H(f)\right]$

O_F^{ϕ} means the fractional Fourier transform:

$$O_F^{\phi}\left(x(t)\right) = \sqrt{1 - j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2}\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)\,dt$$



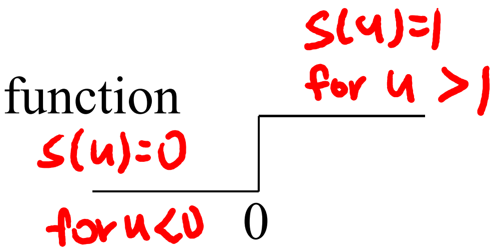
$$H(u) = \begin{cases} 1 & u < u_0 \\ 0 & u > u_0 \end{cases}$$

$$x_o(t) = O_F^{-\phi}\left\{O_F^{\phi}\left[x_i(t)\right]H(u)\right\}$$

If $\quad H(u) = S(-u+u_0) \qquad H(u) = \begin{cases} 1 & u < u_0 \\ 0 & u > u_0 \end{cases}$

If $\quad H(u) = S(u-u_0) \qquad H(u) = \begin{cases} 1 & u > u_0 \\ 0 & u < u_0 \end{cases}$

$S(u)$: Step function



(1) ϕ 由 cutoff line 和 f-axis 的夾角決定

(2) u_0 等於 cutoff line 距離原點的距離

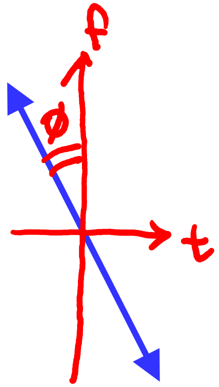
(注意正負號)

- Effect of the filter designed by the fractional Fourier transform (FRFT):

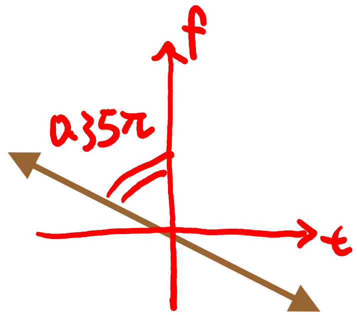
  Placing a cutoff line in the direction of $(-\sin\phi, \cos\phi)$

$\phi = 0$ (time domain)

$\phi = 0.15\pi$

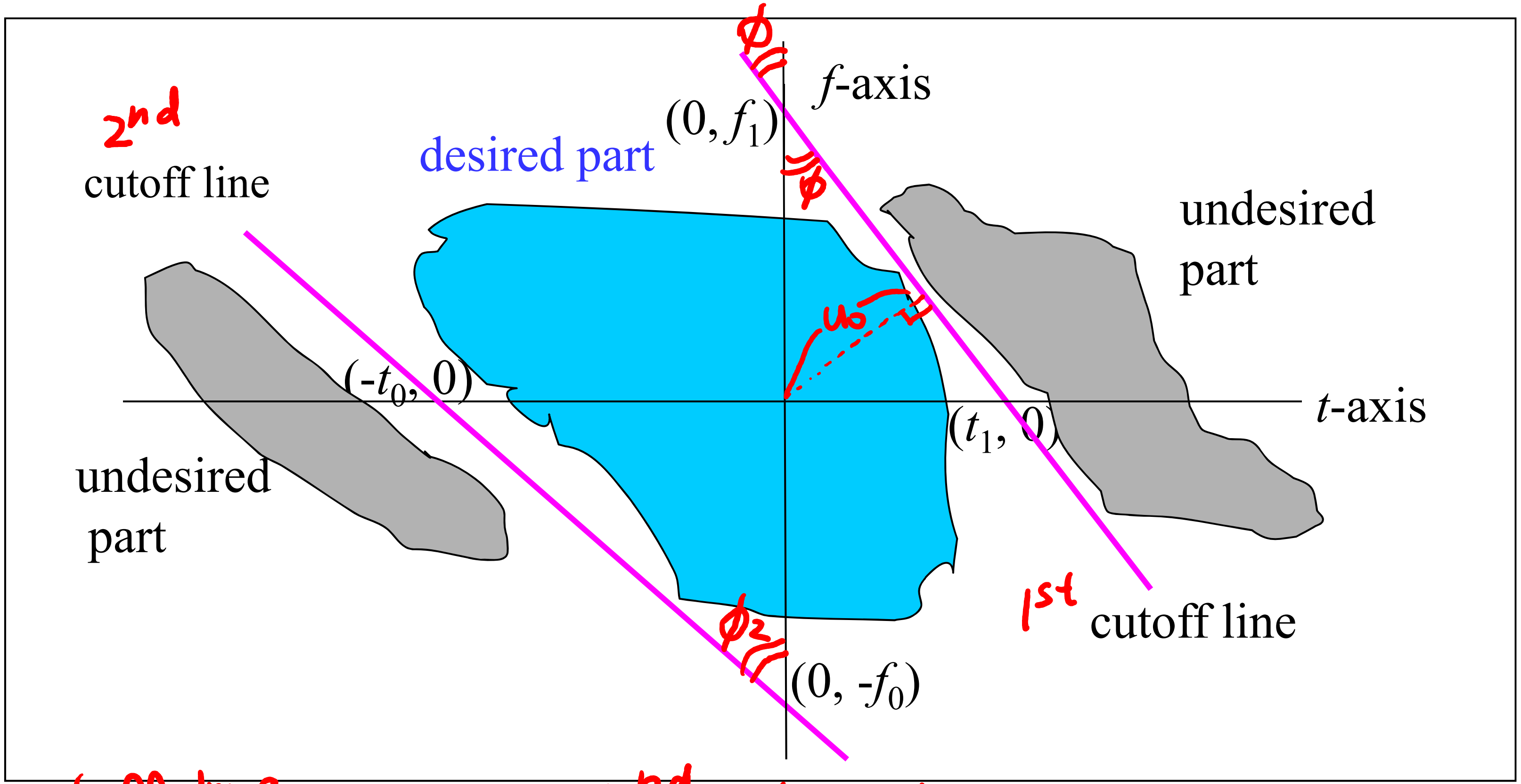


$\phi = 0.35\pi$



$\phi = 0.5\pi$ (FT)

2 times of FRFT filters are required



$\phi = ?$ $u_0 = ?$

1st cutoff line

$\phi = \arctan\left(\frac{t_1}{f_1}\right)$

$\frac{t_1 f_1}{2} = \frac{\sqrt{t_1^2 + f_1^2}\, u_0}{2}$

$u_0 = \frac{t_1 f_1}{\sqrt{t_1^2 + f_1^2}}$

2nd cutoff line

$x_2(t) = O_F^{-\phi_2}\left(O_F^{\phi_2}(x_1(t))\, H_2(u)\right)$

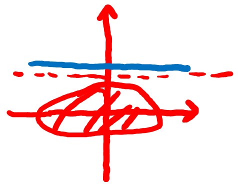
$H_2(u) = 1$ for $u > u_2$, $H_2(u) = 0$ for $u < u_2$

$\phi_2 = \arctan\left(\frac{t_0}{f_0}\right)$

$u_2 = ?$

- The Fourier transform is suitable to filter out the noise that is a combination of

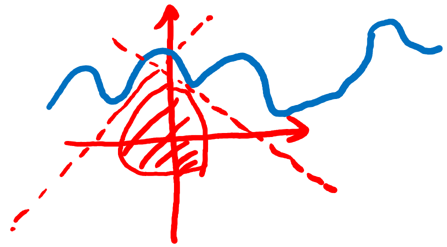
  sinusoid functions $\exp(jn_1 t)$.



- The fractional Fourier transform (FRFT) is suitable to filter out the noise that is a combination of higher order exponential functions

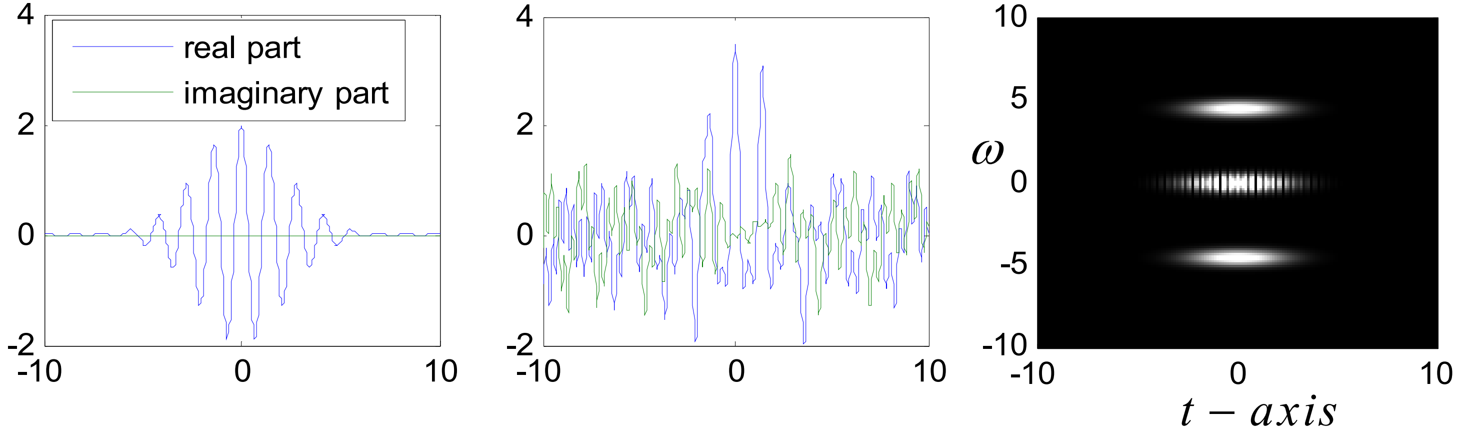
  $\exp[j(n_k t^k + n_{k-1} t^{k-1} + n_{k-2} t^{k-2} + \ldots\ldots + n_2 t^2 + n_1 t)]$

  For example: chirp function $\exp(jn_2 t^2)$



- With the FRFT, many noises that cannot be removed by the FT will be filtered out successfully.

Example (I)

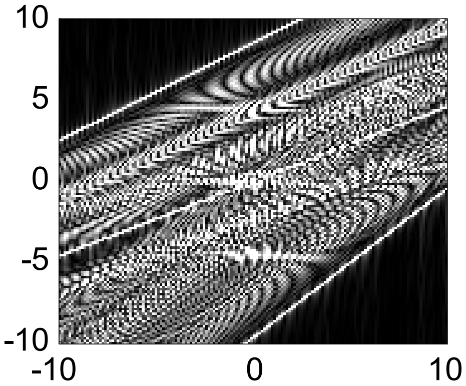


(a) Signal $s(t)$ (b) $f(t) = s(t)$ + noise (c) WDF of $s(t)$

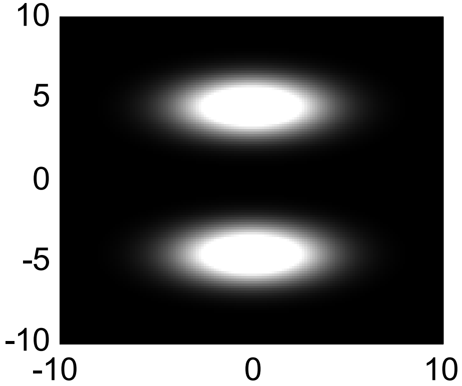
$$s(t) = 2\cos(5t)\exp(-t^2/10)$$

$$n(t) = 0.5e^{j0.23t^2} + 0.5e^{j0.3t^2 + j8.5t} + 0.5e^{j0.46t^2 - j9.6t}$$

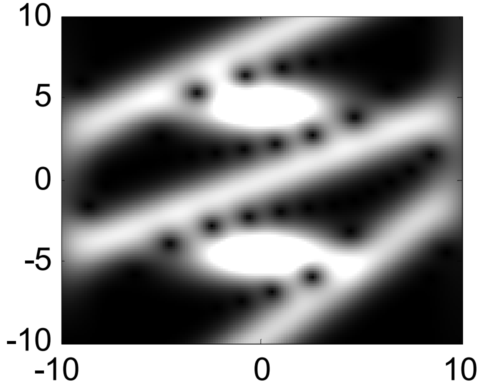
$f(t) = s(t) + n(t)$



(d) WDF of *f*(*t*)



(e) GT of *s*(*t*)

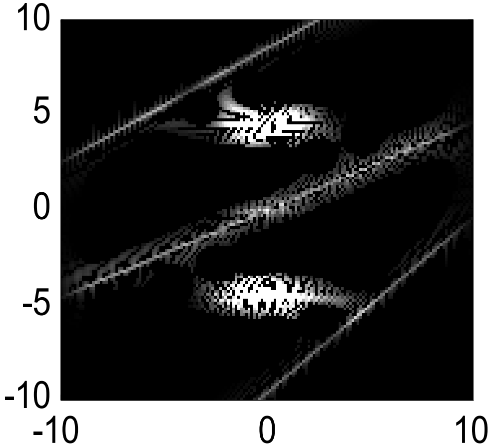


(f) GT of *f*(*t*)

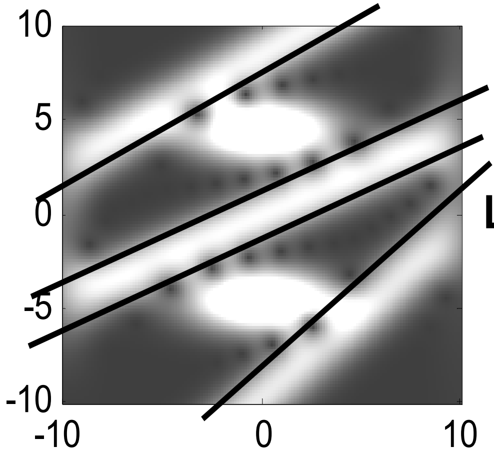
GT: Gabor transform,   WDF: Wigner distribution function

horizontal: *t*-axis,   vertical: ω-axis

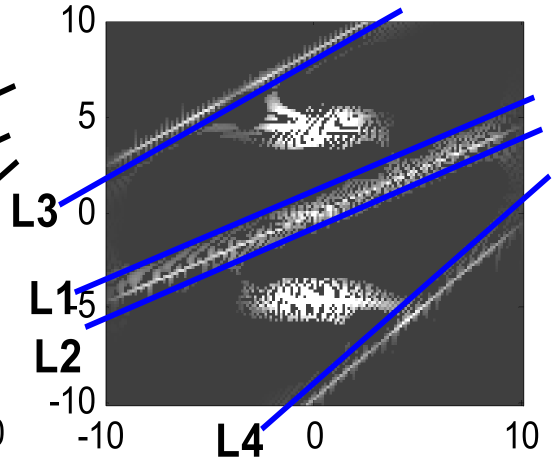
GWT: Gabor-Wigner transform



(g) GWT of $f(t)$



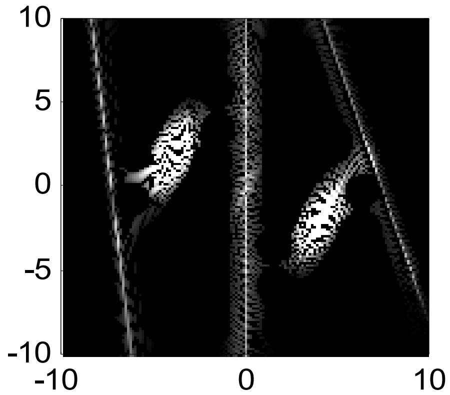
(h) Cutoff lines on GT



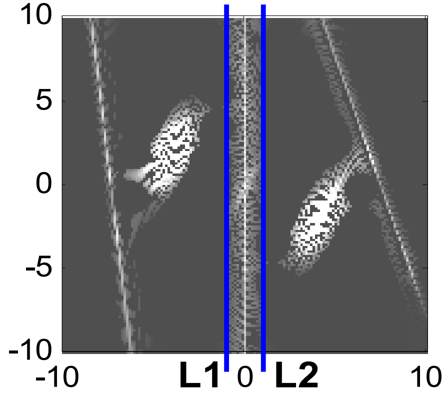
(i) Cutoff lines on GWT

3 times of FRFT filters

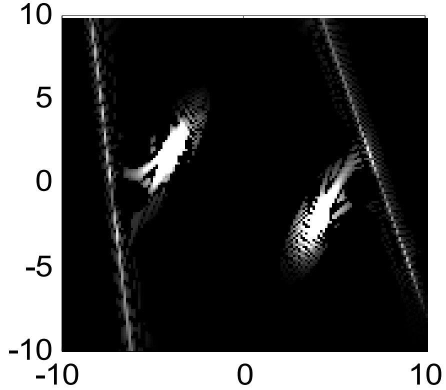
根據斜率來決定 FrFT 的 order



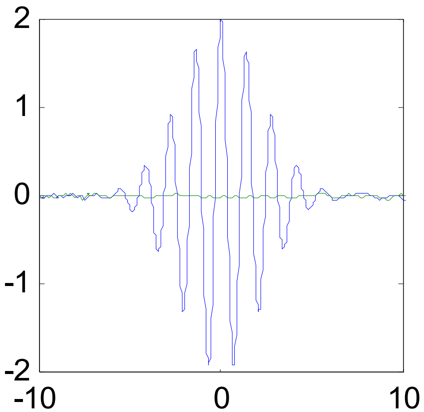
(j) performing the FRFT and calculate the GWT



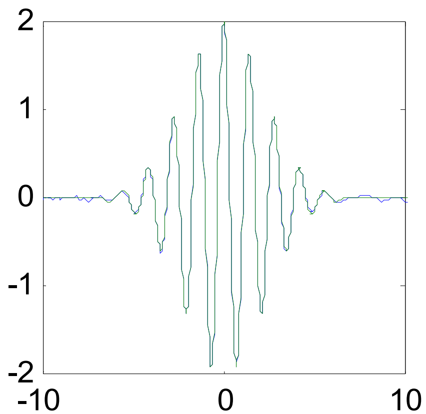
(k) High pass filter



(l) GWT after filter



(m) recovered signal



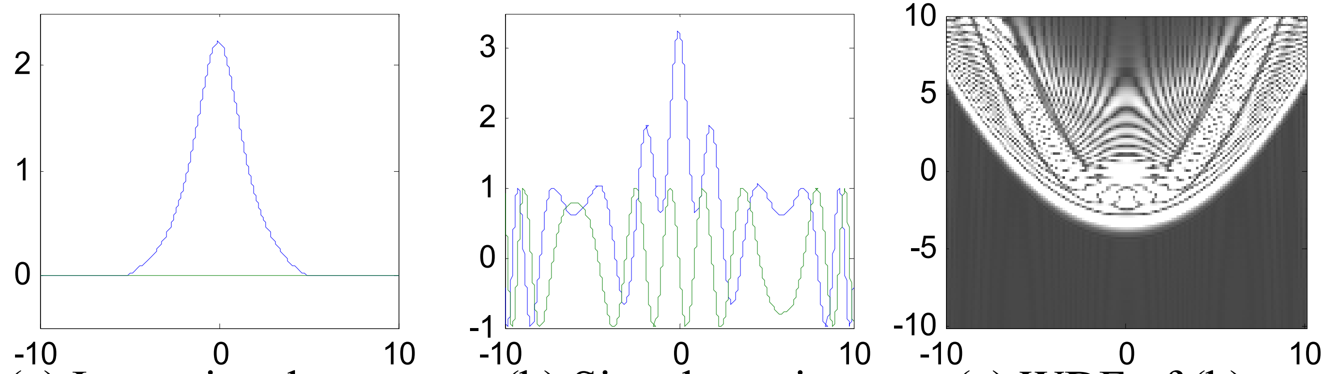
(n) recovered signal (green) and the original signal (blue)

mean square error (MSE) = 0.1128%

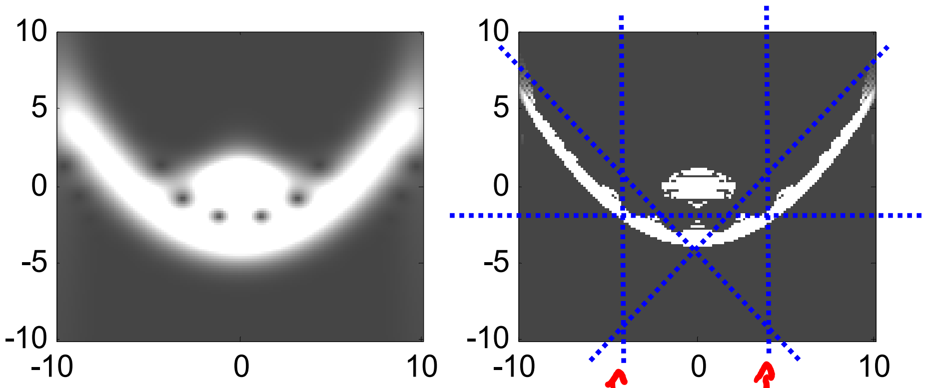
Example (II)

deterministic

Signal + $0.7\exp(j0.032t^3 - j3.4t)$



(a) Input signal (b) Signal + noise (c) WDF of (b)



(d) Gabor transform of (b) (e) GWT of (b) (f) Recovered signal

time domain

5 cutoff lines
2 of them can be performed in the time domain
3 FRFT filters

[Important Theory]:

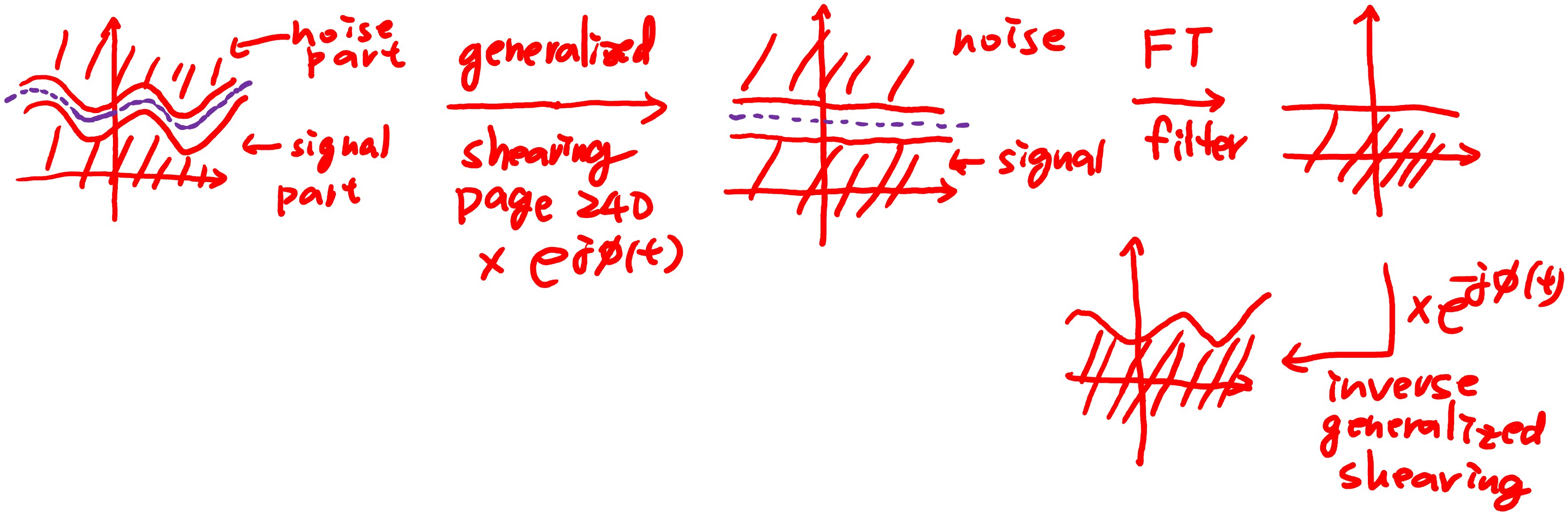
Using the **FT** can only filter the noises that do not overlap with the signals **in the** **frequency domain** (1-D)

In contrast, using the **FRFT** can filter the noises that do not overlap with the signals **on the** **time-frequency plane** (2-D)

[思考]

Q1: 哪些 time-frequency distribution 比較適合處理 filter 或 signal decomposition 的問題？

Q2: Cutoff lines 有可能是非直線的嗎？



[Ref] Z. Zalevsky and D. Mendlovic, "Fractional Wiener filter," *Appl. Opt.*, vol. 35, no. 20, pp. 3930-3936, July 1996.

[Ref] M. A. Kutay, H. M. Ozaktas, O. Arikan, and L. Onural, "Optimal filter in fractional Fourier domains," *IEEE Trans. Signal Processing*, vol. 45, no. 5, pp. 1129-1143, May 1997.

[Ref] B. Barshan, M. A. Kutay, H. M. Ozaktas, "Optimal filters with linear canonical transformations," *Opt. Commun.*, vol. 135, pp. 32-36, 1997.

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

9-2 TF analysis and Random Process

For a random process $x(t)$, we cannot find the explicit value of $x(t)$.
The value of $x(t)$ is expressed as a probability function.

- Auto-covariance function $R_x(t,\tau)$

$$R_x(t,\tau) = E\left[x(t+\tau/2)x^*(t-\tau/2)\right]$$

In usual, we suppose that $E[x(t)] = 0$ for any t

$$E\left[x(t+\tau/2)x^*(t-\tau/2)\right]$$
$$= \int\int x(t+\tau/2,\zeta_1)x^*(t-\tau/2,\zeta_2)P(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$$

(alternative definition of the auto-covariance function:

$$\hat{R}_x(t,\tau) = E\left[x(t)x^*(t-\tau)\right]$$

$R_x(t,\tau) = \hat{R}_x(t+\frac{\tau}{2},\tau)$

- Power spectral density (PSD) $S_x(t,f)$

$$S_x(t,f) = \int_{-\infty}^{\infty} R_x(t,\tau)e^{-j2\pi f\tau}d\tau$$

- Relation between the WDF and the random process

$$E\left[W_x(t,f)\right]=\int_{-\infty}^{\infty}E\left[x(t+\tau/2)x^*(t-\tau/2)\right]\cdot e^{-j2\pi f\tau}\cdot d\tau$$
$$=\int_{-\infty}^{\infty}R_x(t,\tau)\cdot e^{-j2\pi f\tau}\cdot d\tau$$
$$=\int_{-\infty}^{\infty}R_x(t,\tau)\cdot e^{-j2\pi f\tau}\cdot d\tau$$
$$=S_x(t,f)$$

- Relation between the ambiguity function and the random process

$$E\left[A_x(\eta,\tau)\right]=\int_{-\infty}^{\infty}E\left[x(t+\tau/2)x^*(t-\tau/2)\right]e^{-j2\pi t\eta}\,dt=\int_{-\infty}^{\infty}R_x(t,\tau)e^{-j2\pi t\eta}\,dt$$

$$E(X(t,f))=\int_{-\infty}^{\infty}E\left(x(\tau)\right)w(t-\tau)\,e^{-j2\pi f\tau}d\tau=0$$

- Stationary random process:

the statistical properties do not change with *t*.

$R_x(t,\tau) = R_x(0,\tau)$
$= E(x(\frac{\tau}{2})x^*(-\frac{\tau}{2}))$

Auto-covariance function $R_x(t_1,\tau) = R_x(t_2,\tau) = R_x(\tau)$ for any *t*,

$$R_x(\tau) = E\left[x(\tau/2)x^*(-\tau/2)\right]$$

$$= \int\int x(\tau/2,\zeta_1)x^*(-\tau/2,\zeta_2)P(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2$$

PSD: $S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j2\pi f\tau}d\tau$

White noise: $S_x(f) = \sigma$ where σ is some constant.

$$R_x(\tau) = \sigma\delta(\tau)$$

$R_x(t,\tau) = E(x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2}))$
if $\tau \neq 0$ and $x(t+\frac{\tau}{2})$ is independent of $x(t-\frac{\tau}{2})$
$R_x(t,\tau) = E(x(t+\frac{\tau}{2}))E(x^*(t-\frac{\tau}{2}))$
$= 0 \cdot 0 = 0$

• When $x(t)$ is stationary,

$$E\left[W_x(t,f)\right] = S_x(f)$$

(invariant with t)

$$E\left[A_x(\eta,\tau)\right] = \int_{-\infty}^{\infty} R_x(\tau)\cdot e^{-j2\pi t\eta}\cdot dt = R_x(\tau)\int_{-\infty}^{\infty}\cdot e^{-j2\pi t\eta}\cdot dt = R_x(\tau)\delta(\eta)$$

(nonzero only when $\eta = 0$)

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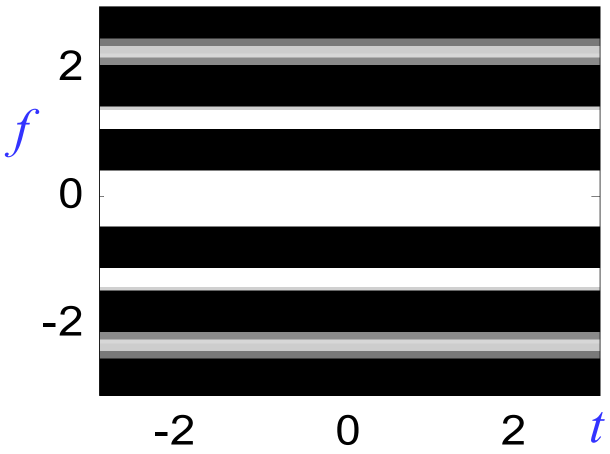
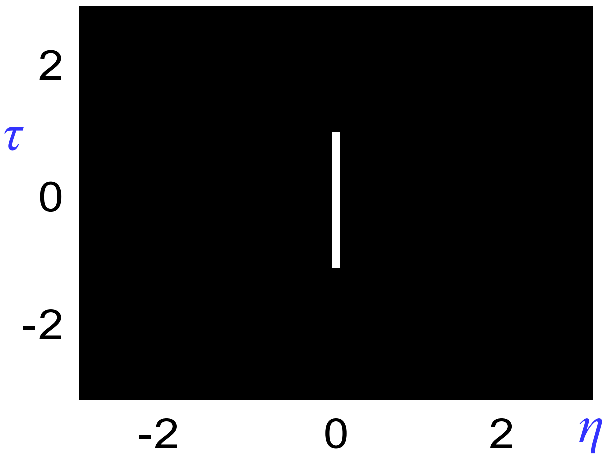
a typical $E[W_x(t, f)]$ for stationary random process



a typical $E[A_x(\eta, \tau)]$ for stationary random process



$\Pi(f)$

-0.5 0.5

(7) $\text{sinc}(t) * x(t) = IFT(X(f)\,\Pi(f))$

Suppose that $x(t)$ is stationary but not white which of the following signals are stationary?

(1) $x(t-3)$ stationary

(2) $x(2t)$ stationary

(3) $FT(x(t))$ not stationary

(4) $e^{j\pi t^2} x(t)$ non-stationary

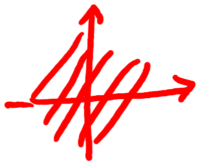
(5) $e^{j\pi t^2} * x(t)$ stationary

(6) $e^{-|t|} x(t)$ not stationary

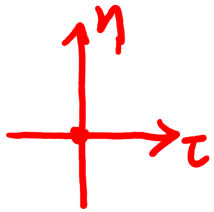
If $x(t)$ is white

- For white noise,

$$E\left[W_x(t,f)\right]=\sigma$$



$$E\left[A_x(\eta,\tau)\right]=\sigma\delta(\tau)\delta(\eta)$$



[Ref 1] W. Martin, "Time-frequency analysis of random signals", *ICASSP'82*, pp. 1325-1328, 1982.

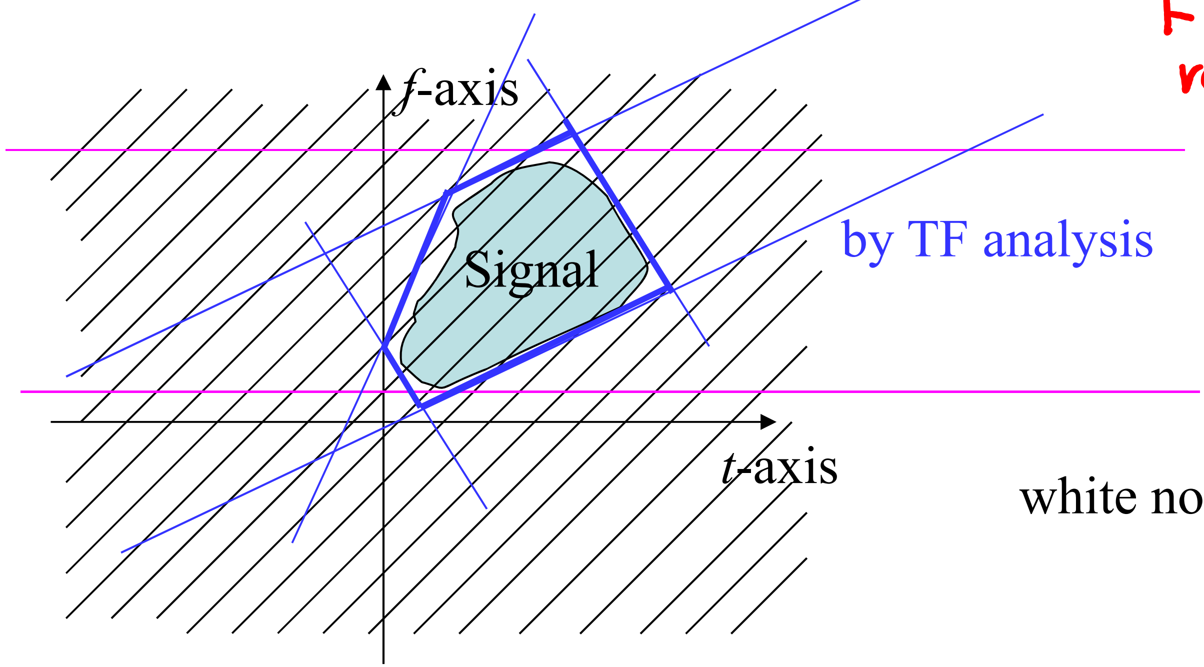
[Ref 2] W. Martin and P. Flandrin, "Wigner-Ville spectrum analysis of nonstationary processed", *IEEE Trans. ASSP*, vol. 33, no. 6, pp. 1461-1470, Dec. 1983.

[Ref 3] P. Flandrin, "A time-frequency formulation of optimum detection" , *IEEE Trans. ASSP*, vol. 36, pp. 1377-1384, 1988.

[Ref 4] S. C. Pei and J. J. Ding, "Fractional Fourier transform, Wigner distribution, and filter design for stationary and nonstationary random processes," *IEEE Trans. Signal Processing*, vol. 58, no. 8, pp. 4079-4092, Aug. 2010.

How many times of FRFT filters are required?

Filter Design for White noise



white noise everywhere

$$SNR \approx 10\log_{10} \frac{E_{signal}}{\int\int_{\substack{(t,f)\in \\ \text{signal part}}} W_{noise}(t,f)\,dtdf}$$

E_{signal}: energy of the signal

A: area of the time frequency distribution of the signal

$$SNR \approx 10\log_{10} \frac{E_{signal}}{\sigma A}$$

The PSD of the white noise is $S_{noise}(f) = \sigma$

- If $E\left[W_x\left(t,f\right)\right]$ varies with t and $E\left[A_x\left(\eta,\tau\right)\right]$ is nonzero when $\eta \neq 0$, then $x(t)$ is a non-stationary random process.

- If ① $h\left(t\right)=x_1\left(t\right)+x_2\left(t\right)+x_3\left(t\right)+\cdots\cdots+x_k\left(t\right)$

  ② $x_n(t)$'s have zero mean for all t's

  ③ $x_n(t)$'s are mutually independent for all t's and τ's

$$E\left[x_m(t+\tau/2)x_n^*(t-\tau/2)\right]=E\left[x_m(t+\tau/2)\right]E\left[x_n^*(t-\tau/2)\right]=0$$

if $m \neq n$, then

$$E\left[W_h\left(t,f\right)\right]=\sum_{n=1}^{k}E\left[W_{x_n}\left(t,f\right)\right], \quad E\left[A_h\left(\eta,\tau\right)\right]=\sum_{n=1}^{k}E[A_{x_n}\left(\eta,\tau\right)]$$

(1) Random process for the STFT

$E[x(t)] \neq 0$ should be satisfied.

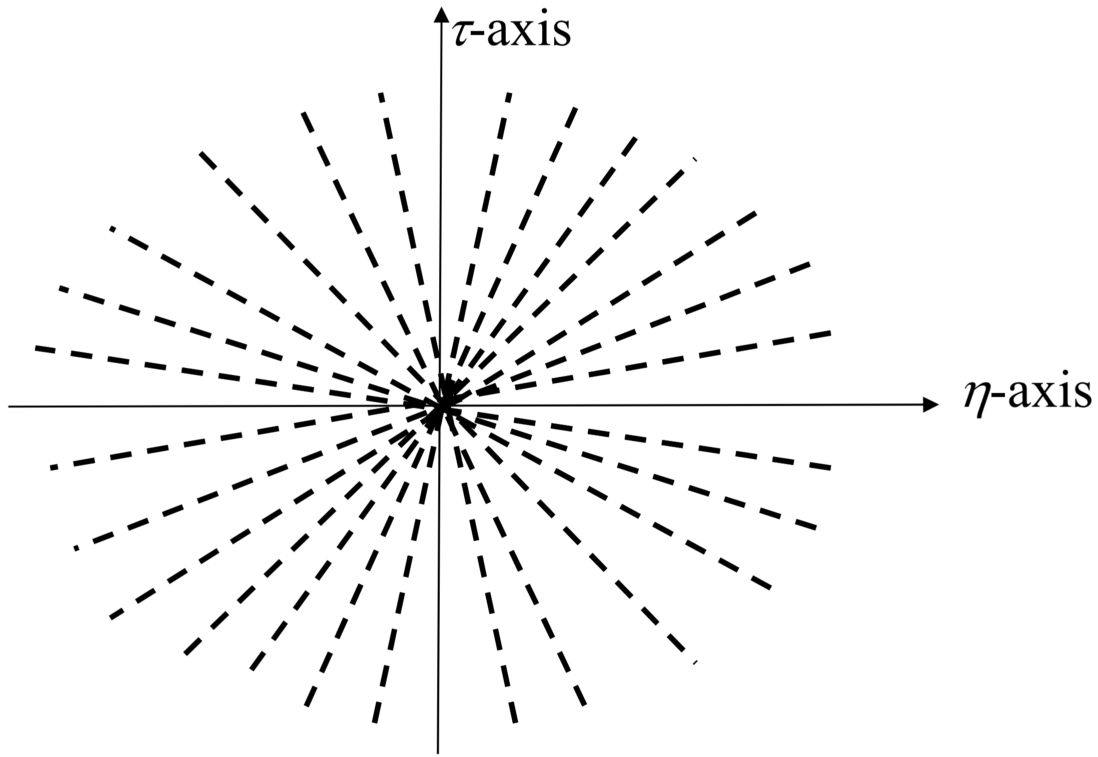
Otherwise,

$$E[X(t,f)] = E[\int_{t-B}^{t+B} x(\tau)w(t-\tau)e^{-j2\pi f\tau}d\tau] = \int_{t-B}^{t+B} E[x(\tau)]w(t-\tau)e^{-j2\pi f\tau}d\tau$$

for zero-mean random process, $E[X(t,f)] = 0$

(2) Decompose by the AF and the FRFT

Any non-stationary random process can be expressed as a summation of the fractional Fourier transform (or chirp multiplication) of stationary random process.



An ambiguity function plane can be viewed as a combination of infinite number of radial lines.

Each radial line can be viewed as the fractional Fourier transform of a stationary random process.

信號處理小常識

$S(f) = \sigma$ white noise

$S(f) = \dfrac{\sigma}{|f|}$ $\alpha<0$ pink noise

$S(f) = \sigma |f|$ $\alpha>1$ purple noise

$S(f) = \sigma |f|^{\alpha}$ $\alpha \neq 0$ color noise

$\alpha=0 \rightarrow$ white

X. Other Applications of Time-Frequency Analysis

Applications

(1) Finding Instantaneous Frequency

(2) Signal Decomposition

(3) Filter Design

(4) Sampling Theory

(5) Modulation and Multiplexing

(6) Electromagnetic Wave Propagation

(7) Optics

(8) Radar System Analysis

(9) Random Process Analysis

(10) Music Signal Analysis

(11) Biomedical Engineering 生醫 ECG

(12) Accelerometer Signal Analysis 加速規

(13) Acoustics

(14) Data Compression

(15) Spread Spectrum Analysis

(16) System Modeling

(17) Economic Data Analysis

(18) Signal Representation

(19) Seismology 地震波

(20) Geology 地質學

(21) Astronomy 天文學

(22) Oceanography 海洋學

(23) Satellite Signal Analysis 衛星

(24) Image Processing??

10-1 Sampling Theory

Number of sampling points == Sum of areas of time frequency distributions
+ the number of extra parameters

- How to make the area of time-frequency smaller?

(1) Divide into several components.

(2) Use chirp multiplications, chirp convolutions, fractional Fourier transforms, or linear canonical transforms to reduce the area.

[Ref] X. G. Xia, "On bandlimited signals with fractional Fourier transform," *IEEE Signal Processing Letters*, vol. 3, no. 3, pp. 72-74, March 1996.

[Ref] J. J. Ding, S. C. Pei, and T. Y. Ko, "Higher order modulation and the efficient sampling algorithm for time variant signal," *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

Analytic Signal Conversion

$$x(t) \rightarrow x_a(t) = x(t) + jx_H(t)$$

If $x(t)$ is real
$X(f) = X^*(-f)$
$X(t,f) = X^*(t,-f)$

$x_H(t) = IFT(FT(x(t))\,H(f))$
Hilbert transform

$$H(f) = \begin{cases} -j & f>0 \\ j & f<0 \end{cases}$$

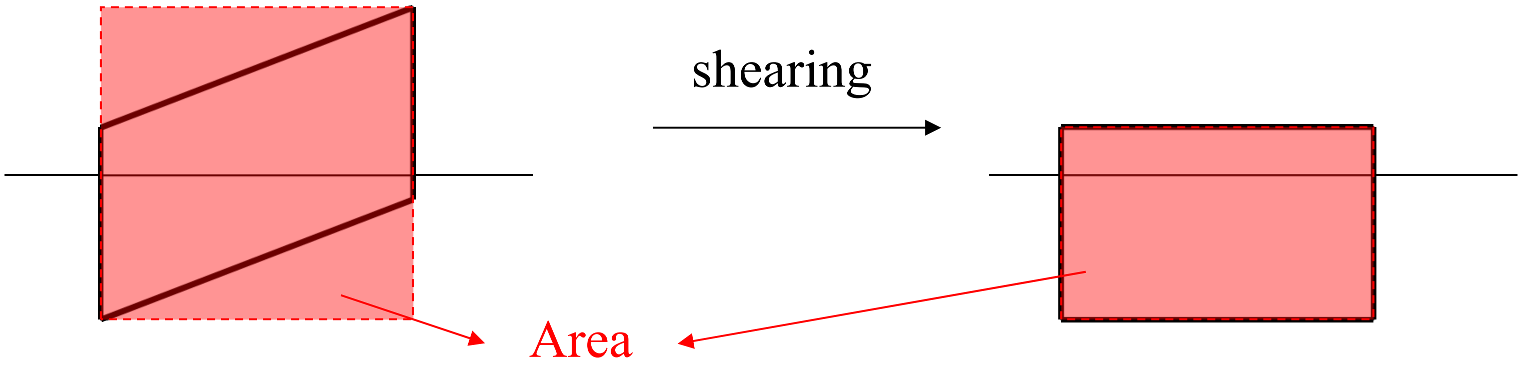
Since $H(f) = H^*(-f)$
$\therefore x_H(t)$ is real

$X_a(f) = X(f) + jH(f)X(f)$
$= (1+jH(f))X(f)$

$$X_a(f) = \begin{cases} 2X(f) & f>0 \\ 0 & f<0 \end{cases}$$

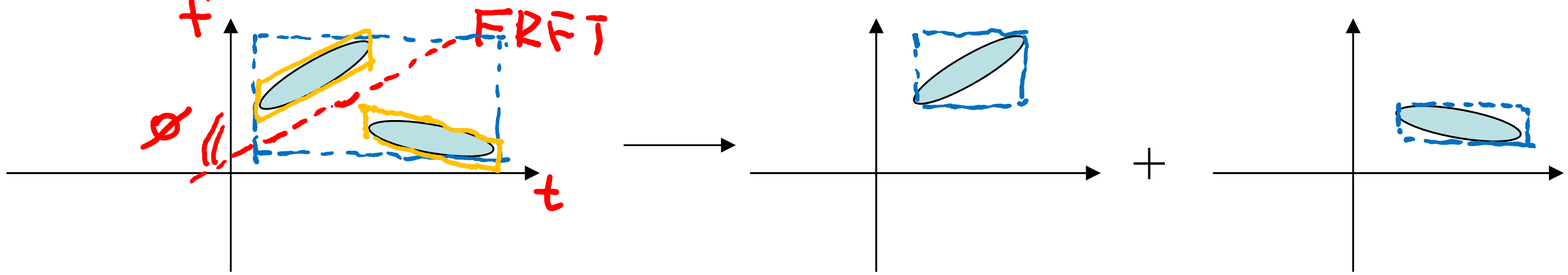
single sided band conversion

Shearing

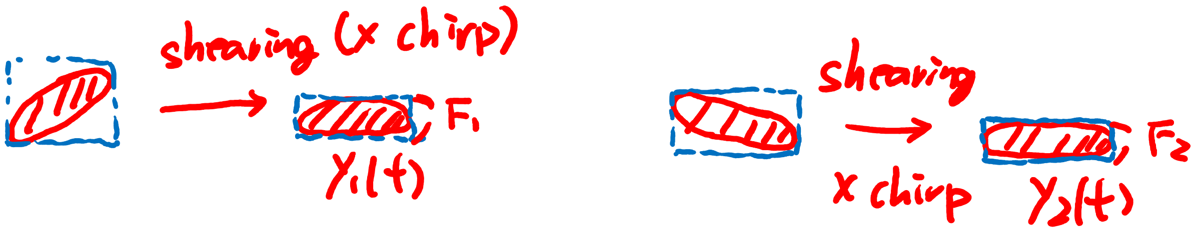


Step 1 Analytic Signal Conversion

Step 2 Separate the components (a) (b)



Step 3 Use shearing or rotation to minimize the "area" to each component



Step 4 Use the conventional sampling theory to sample each components

傳統的取樣方式

$$x_d[n] = x(n\Delta_t) \qquad \Delta_t < 1/F$$

重建： $$x(t) = \sum_n x_d[n]\operatorname{sinc}\left(\frac{t}{\Delta_t} - n\right)$$

新的取樣方式 $x_H(t)$: Hilbert transform of $x(t)$

(1) $x(t) \to x_a(t) = x(t) + jx_H(t)$ $x(t) = \mathrm{Re}(x_a(t))$

(2) $x_a(t) \to x_a(t) = x_1(t) + x_2(t) + \cdots + x_K(t)$

(3) $y_k(t) = \exp\left(j2\pi a_k t^2\right)x_k(t)$ $\qquad k = 1, 2, \ldots, K$

(4) $x_{d,k}[n] = y_k(n\Delta_{t,k})$ $\qquad k = 1, 2, \ldots, K$

$$= \exp\left(j2\pi a_k n^2 \Delta_{t,k}^2\right)x_k(n\Delta_{t,k})$$

$\Delta_{t,k} < \dfrac{1}{F_k}$, F_k: width of $y_k(t)$ along f-axis

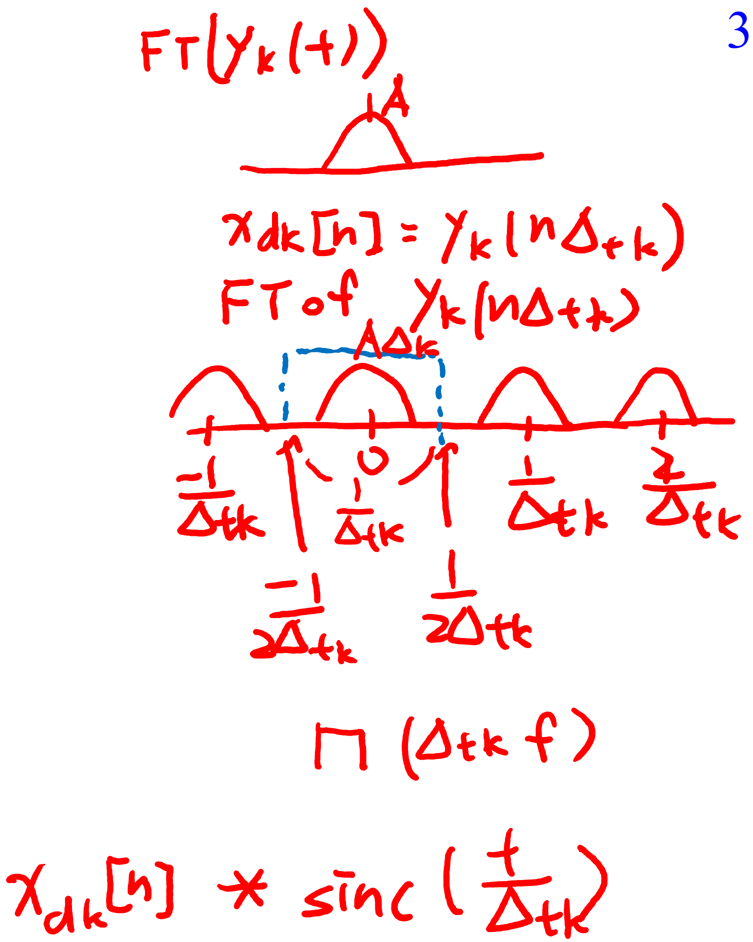
重建：

(1) $y_k(t) = \sum_n x_{d,k}[n] \operatorname{sinc}\left(\frac{t}{\Delta_{t,k}} - n\right)$

(2) $x_k(t) = \exp\left(-j2\pi a_k t^2\right) y_k(t)$

(3) $x_a(t) = x_1(t) + x_2(t) + \cdots + x_K(t)$

(4) $x(t) = \mathcal{Re}\{x_a(t)\}$



嚴格來說，沒有一個信號的 時頻分佈的「面積」是有限的。

Theorem:

If $x(t)$ is time limited ($x(t) = 0$ for $t < t_1$ and $t > t_2$)

then it is impossible to be frequency limited

If $x(t)$ is frequency limited ($X(f) = 0$ for $f < f_1$ and $f > f_2$)

then it is impossible to be time limited

但是我們可以選一個 “threshold” Δ

時頻分析 $|X(t, f)| > \Delta$ 或 的區域的面積是有限的

實際上，以「面積」來討論取樣點數，是犧牲了一些精確度。

只取 $t \in [t_1, t_2]$ and $f \in [f_1, f_2]$ 犧牲的能量所佔的比例

$$err = \frac{\int_{-\infty}^{t_1} |x(t)|^2 dt + \int_{t_2}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{f_1} |X_1(f)|^2 df + \int_{f_2}^{\infty} |X_1(f)|^2 df}{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

$X_1(f) = FT[x_1(t)]$, $x_1(t) = x(t)$ for $t \in [t_1, t_2]$, $x_1(t) =$ 0 otherwise

- For the Wigner distribution function (WDF)

$$|x(t)|^2 = \int_{-\infty}^{\infty} W_x(t, f)\, df, \qquad |X(f)|^2 = \int_{-\infty}^{\infty} W_x(t, f)\, dt$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t, f)\, df dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad = \quad \text{energy of } x(t).$$

$$|x(t)|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,df \qquad |X(f)|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,dt$$

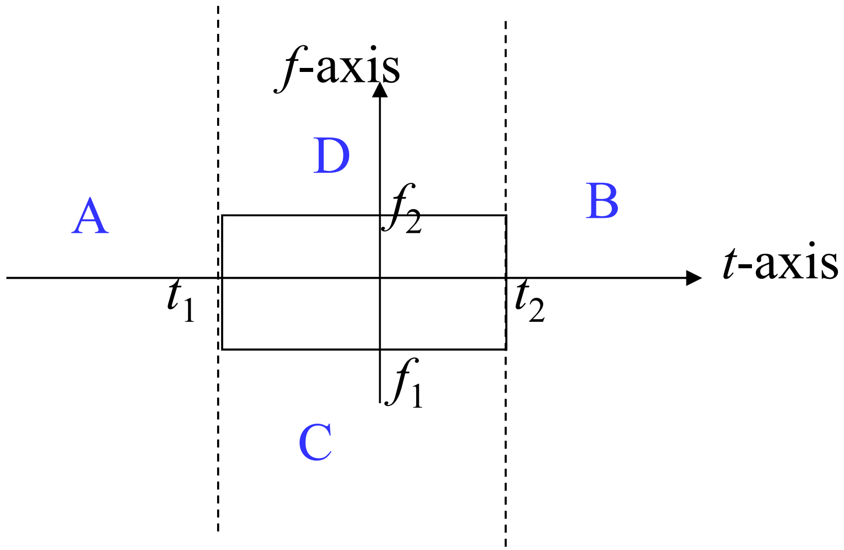
$$\int_{-\infty}^{t_1} |x(t)|^2\,dt + \int_{t_2}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{f_1} |X_1(f)|^2\,df + \int_{f_2}^{\infty} |X_1(f)|^2\,df$$

$$= \int_{-\infty}^{t_1}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{t_2}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{-\infty}^{\infty}\int_{-\infty}^{f_1} W_{x_1}(t,f)\,dfdt + \int_{-\infty}^{\infty}\int_{f_2}^{\infty} W_{x_1}(t,f)\,dfdt$$

$$= \int_{-\infty}^{t_1}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{t_2}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{t_1}^{t_2}\int_{-\infty}^{f_1} W_{x_1}(t,f)\,dfdt + \int_{t_1}^{t_2}\int_{f_2}^{\infty} W_{x_1}(t,f)\,dfdt$$

$$\cong \underbrace{\int_{-\infty}^{t_1}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt}_{A} + \underbrace{\int_{t_2}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt}_{B} + \underbrace{\int_{t_1}^{t_2}\int_{-\infty}^{f_1} W_x(t,f)\,dfdt}_{C} + \underbrace{\int_{t_1}^{t_2}\int_{f_2}^{\infty} W_x(t,f)\,dfdt}_{D}$$

$$err \cong 1 - \frac{\int_{t_1}^{t_2}\int_{f_1}^{f_2} W_x(t,f)\,dfdt}{\int_{-\infty}^{\infty} |x(t)|^2\,dt}$$



10-2 Modulation and Multiplexing

With the aid of

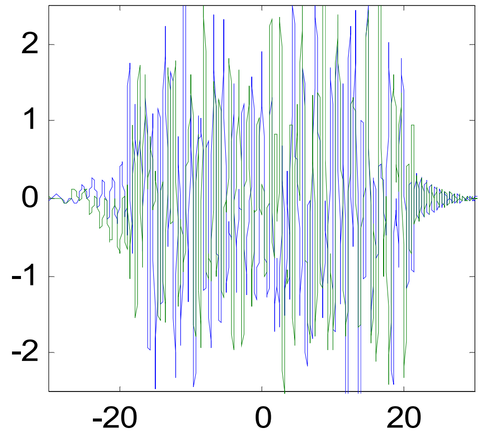
(1) the Gabor transform (or the Gabor-Wigner transform)

(2) horizontal and vertical shifting, dilation, shearing, generalized shearing, and rotation.

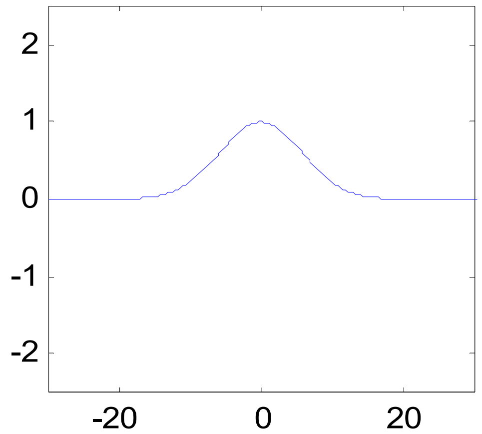
[Ref] C. Mendlovic and A. W. Lohmann, "Space-bandwidth product adaptation and its application to superresolution: fundamentals," *J. Opt. Soc. Am. A*, vol. 14, pp. 558-562, Mar. 1997.

[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," vol. 55, issue 10, pp. 4839-4850, *IEEE Trans. Signal Processing*, 2007.

Example

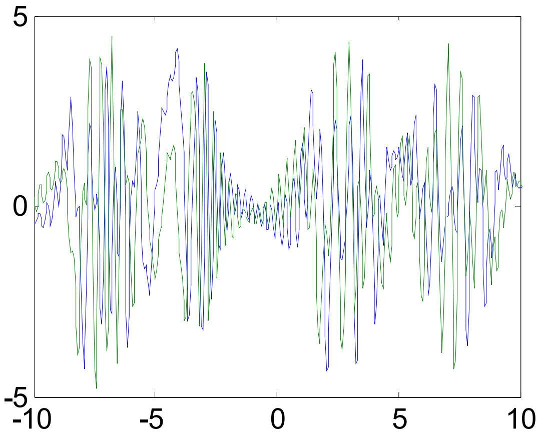


(a) G(u), consisted of 7 components

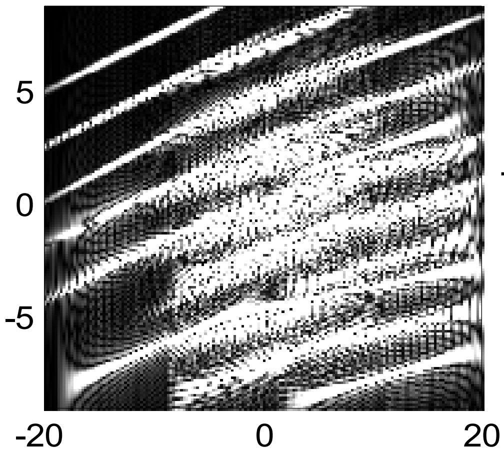


(b) f(t), the signal to be modulated

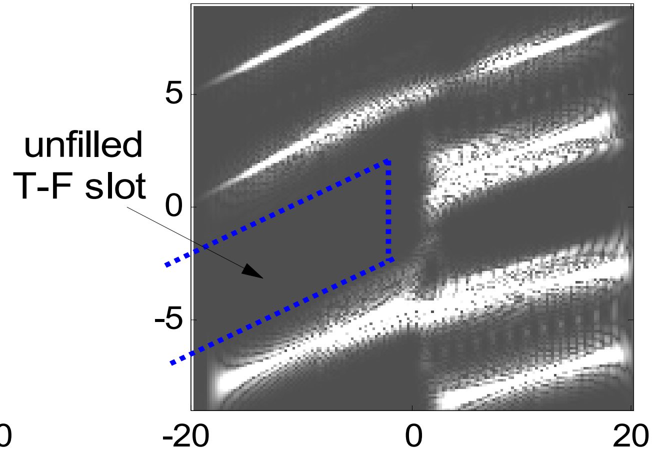
FT ↓ We want to add $f(t)$ into $G(u)$



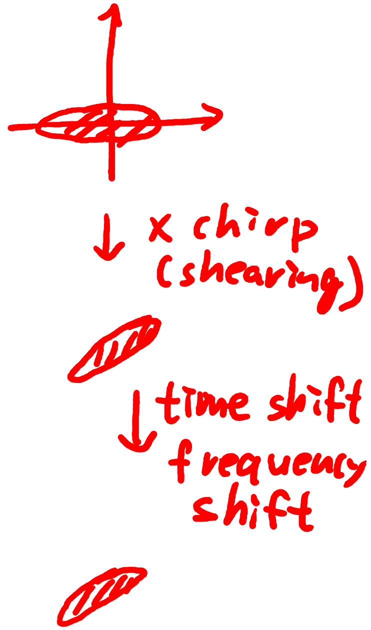
(no empty band)

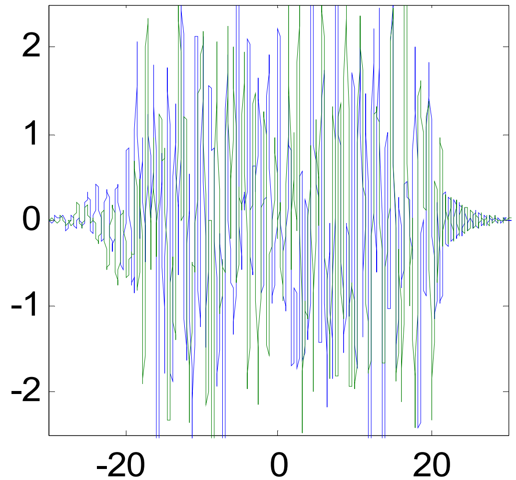


(c) WDF of G(u)

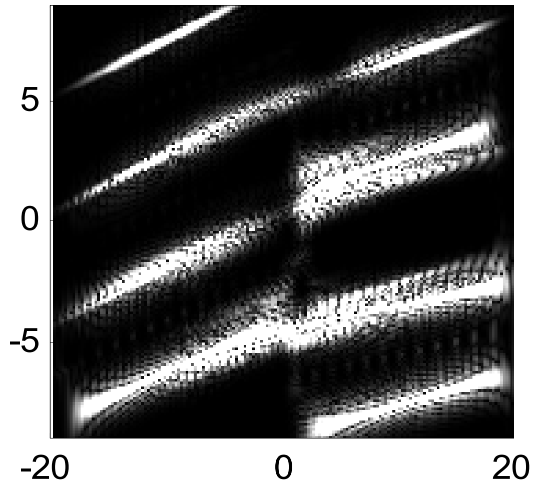


(d) GWT of G(u)





(e) multiplexing f(t) into G(u)



(f) GWT of (e)

◎ Conventional Modulation Theory

The signals $x_1(t)$, $x_2(t)$, $x_3(t)$, ……., $x_K(t)$ can be transmitted successfully if

$$\text{Allowed Bandwidth} \geq \sum_{k=1}^{K} B_k$$

B_k: the bandwidth (including the negative frequency part) of $x_k(t)$

◎ Modulation Theory Based on Time-Frequency Analysis

The signals $x_1(t)$, $x_2(t)$, $x_3(t)$, ……., $x_K(t)$ can be transmitted successfully if

$$\text{Allowed Time duration} \times \text{Allowed Bandwidth} \geq \sum_{k=1}^{K} A_k$$

A_k: the area of the time-frequency distribution of $x_k(t)$

Suppose that time duration = 60

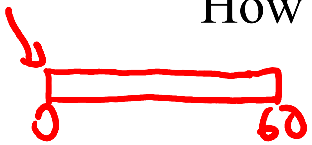
freq. width = 10^6

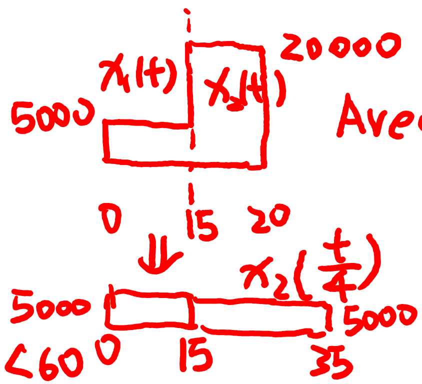
$60 \times 10^6 = 6 \times 10^7$

- The interference is inevitable.

How to estimate the interference?

$\frac{5000}{60} \times 35$





Area: $15 \times 5000 + 5 \times 20000 = 175000$

$x_3(t) = [x_1(t), x_2(\frac{t}{4})]$

$x_4(t) = x_3(\frac{t}{60/35})$ $\quad 0 < t < 60$

10-3 Electromagnetic Wave Propagation

Time-Frequency analysis can be used for

Wireless Communication

Optical system analysis

Laser

Radar system analysis

Propagation through the free space (Fresnel transform): chirp convolution

Propagation through the lens or the radar disk: chirp multiplication

Fresnel Transform ：描述電磁波在空氣中的傳播 (See pages 267-271)

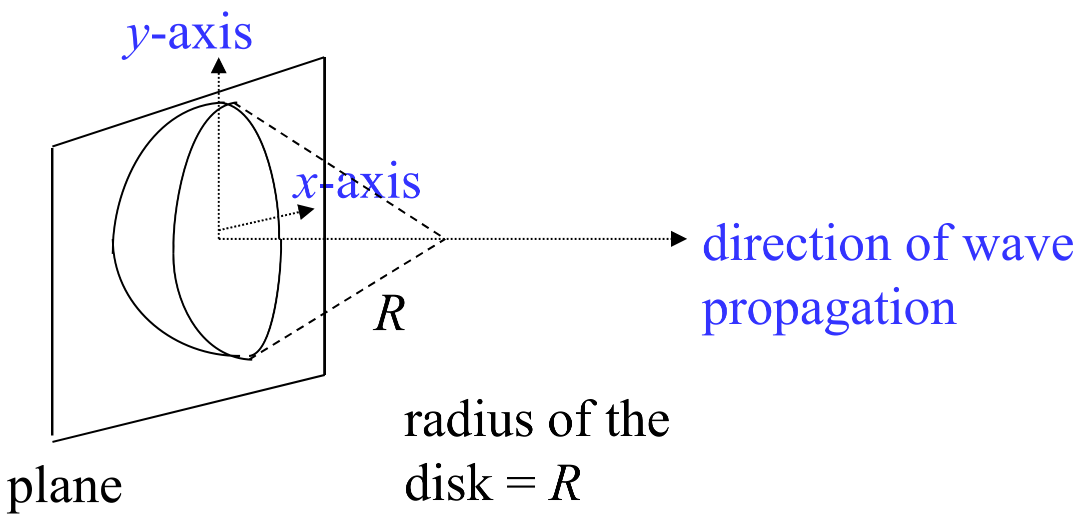
電磁波包括光波、雷達波、紅外線、紫外線………

Fresnel transform == LCT with parameters $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

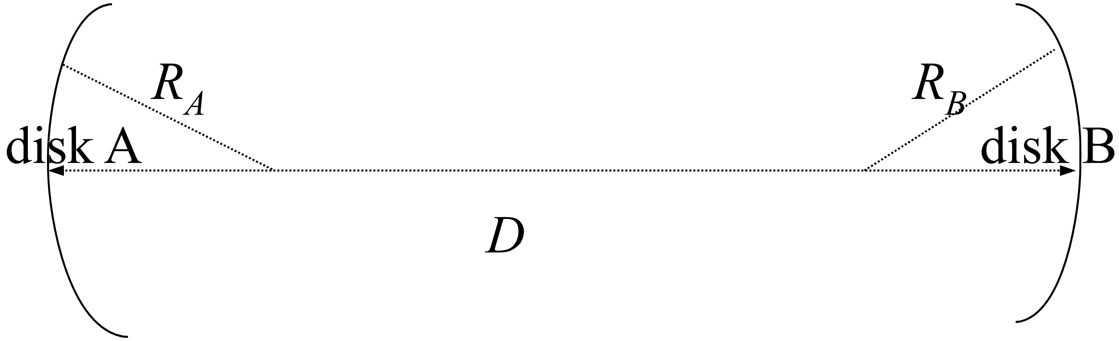
思考： (1) STFT 或 WDF 哪一個比較適合用在電磁波傳播的分析？

(2) 為何波長越短的電磁波，在空氣中散射的情形越少？

(4) Spherical Disk



Disk 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\lambda R & 1 \end{bmatrix}$ 的情形



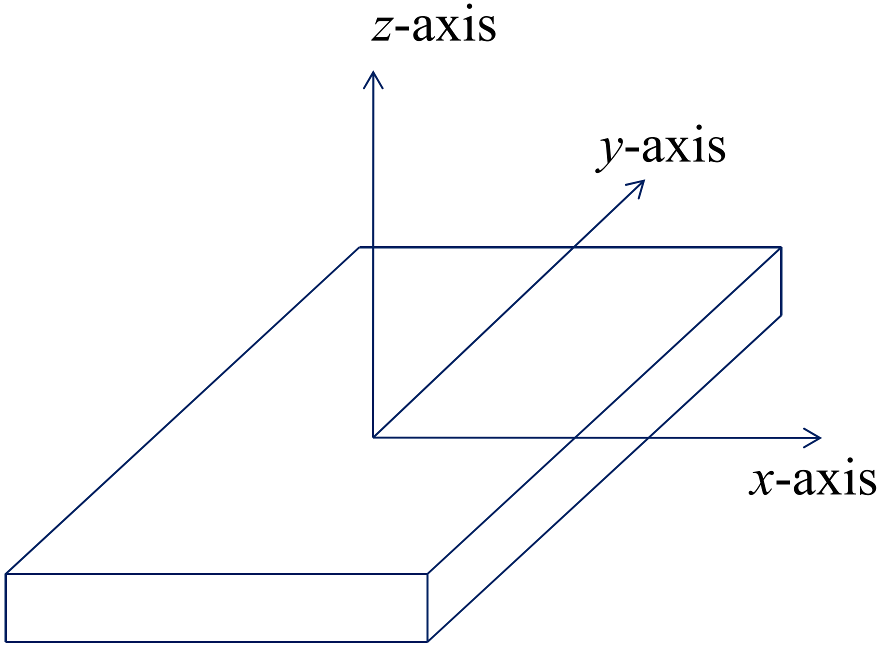
相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\lambda R_B & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda R_A & 1 \end{bmatrix}$

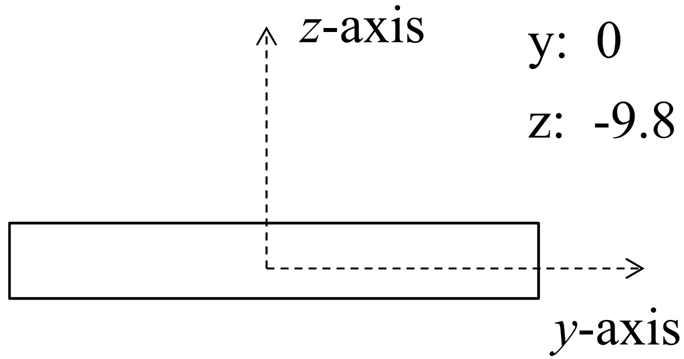
$$= \begin{bmatrix} 1 - D/R_A & -\lambda D \\ -\frac{1}{\lambda}\left(R_A^{-1} - R_B^{-1} + R_A^{-1} R_B^{-1} D\right) & 1 + D/R_B \end{bmatrix}$$

的情形

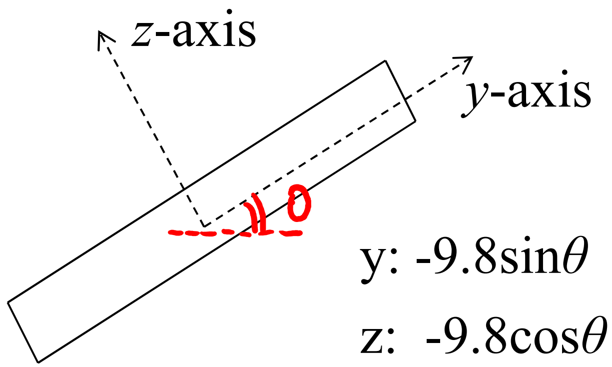
10-3 Accelerometer Signal Analysis

The 3-D Accelerometer (三軸加速規) can be used for identifying the activity of a person.





tilted by θ

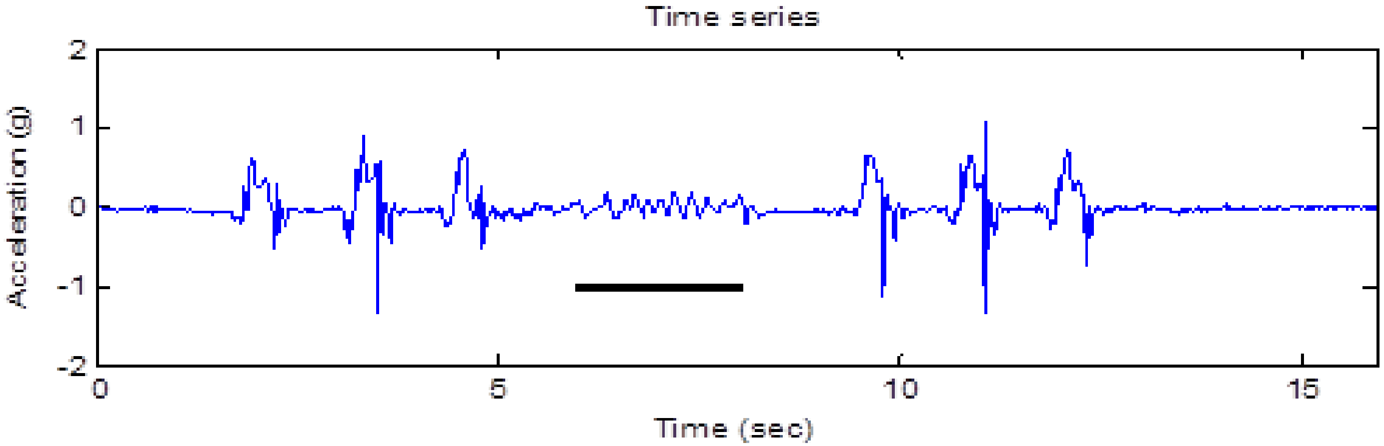


Using the 3D accelerometer + time-frequency analysis, one can analyze the activity of a person.

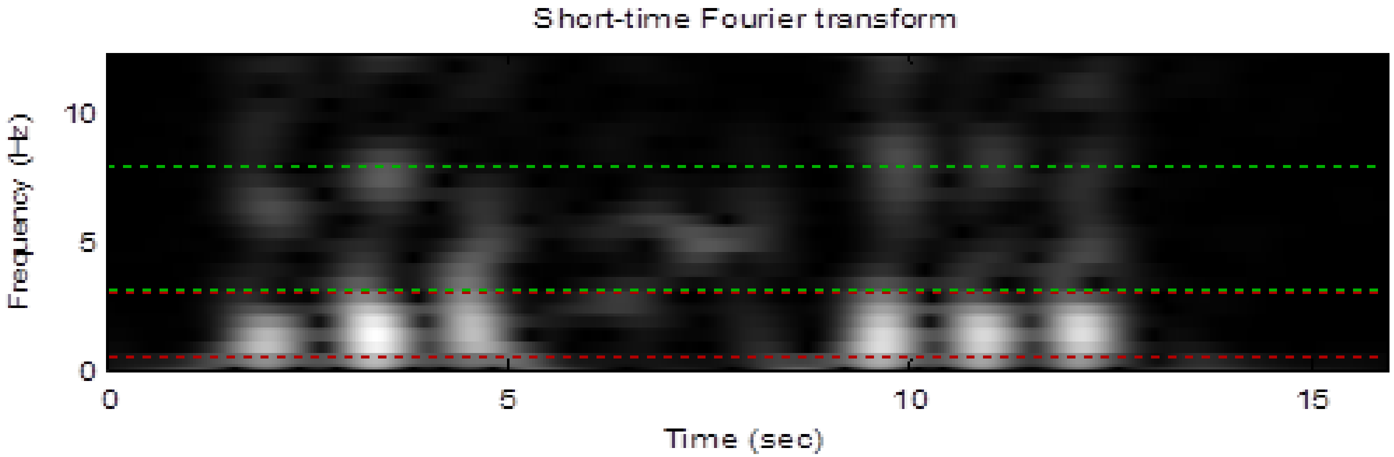
Walk, Run (Pedometer 計步器)

Healthcare for the person suffered from Parkinson’s disease

3D accelerometer signal for a person suffering from Parkinson's disease



The result of the short-time Fourier transform



Y. F. Chang, J. J. Ding, H. Hu, Wen-Chieh Yang, and K. H. Lin, "A real-time detection algorithm for freezing of gait in Parkinson's disease," *IEEE International Symposium on Circuits and Systems*, Melbourne, Australia, pp. 1312-1315, May 2014

10-5 Music and Acoustic Signal Analysis

Music Signal Analysis

Acoustic

Voiceprint (Speaker) Recognition

Speech Signal ：

(1) 不同的人說話聲音頻譜不同 (聲紋 voiceprint)

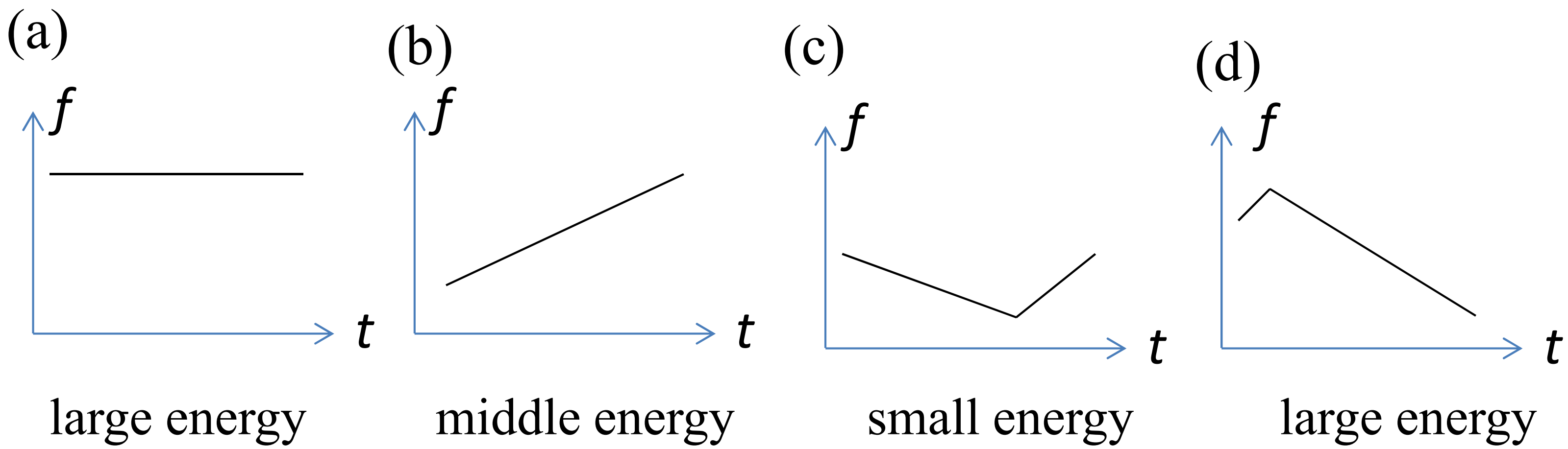
(2) 同一個人但不同的字音，頻譜不一樣

(3) 語調 (第一、二、三、四聲和輕聲) 不同，則頻譜變化的情形也不同

(4) 即使同一個字音，子音和母音的頻譜亦不相同

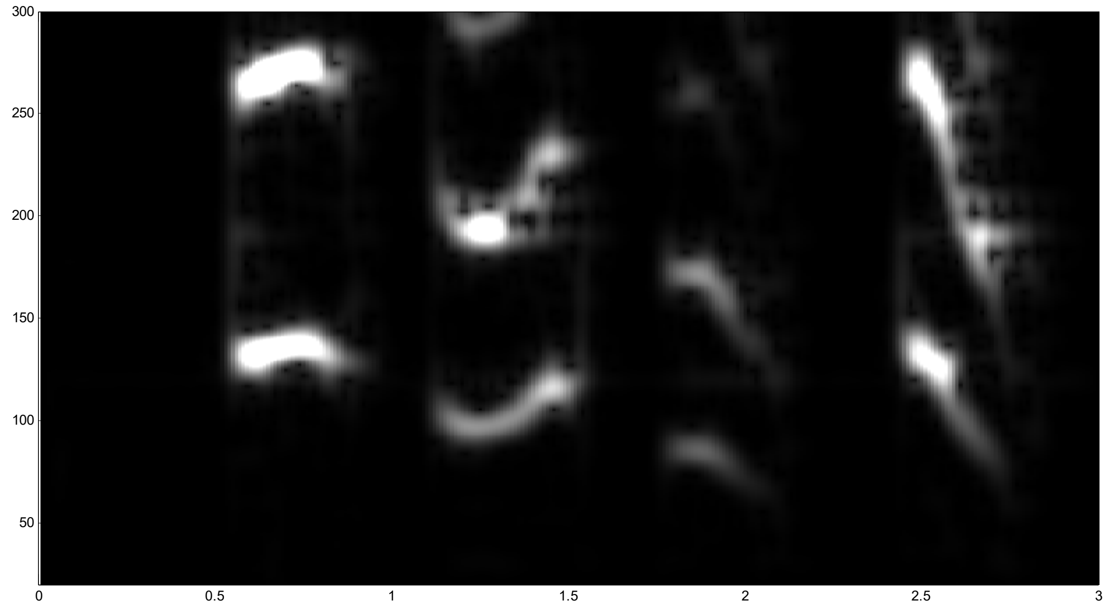
(5) 雙母音本身就會有頻譜的變化

• 王小川，“語音訊號處理”，第二章，全華出版，台北，民國94年。



Typical relations between time and the instantaneous frequencies for (a) the 1st tone, (b) the 2nd tone, (c) the 3rd tone, and (d) the 4th tone in Chinese.

X. X. Chen, C. N. Cai, P. Guo, and Y. Sun, "A hidden Markov model applied to Chinese four-tone recognition," *ICASSP*, vol. 12, pp. 797-800, 1987.



ㄚ1,　ㄚ2,　ㄚ3,　ㄚ4

ㄚ　ㄚˊ　ㄚˇ　ㄚˋ

10-6 Other Applications

時頻分析適用於頻譜會隨著時間而改變的信號

Biomedical Engineering (心電圖 (ECG), 肌電圖 (EMG), 腦電圖, ……)

Communication and Spread Spectrum Analysis

Economic Data Analysis

Seismology

Geology

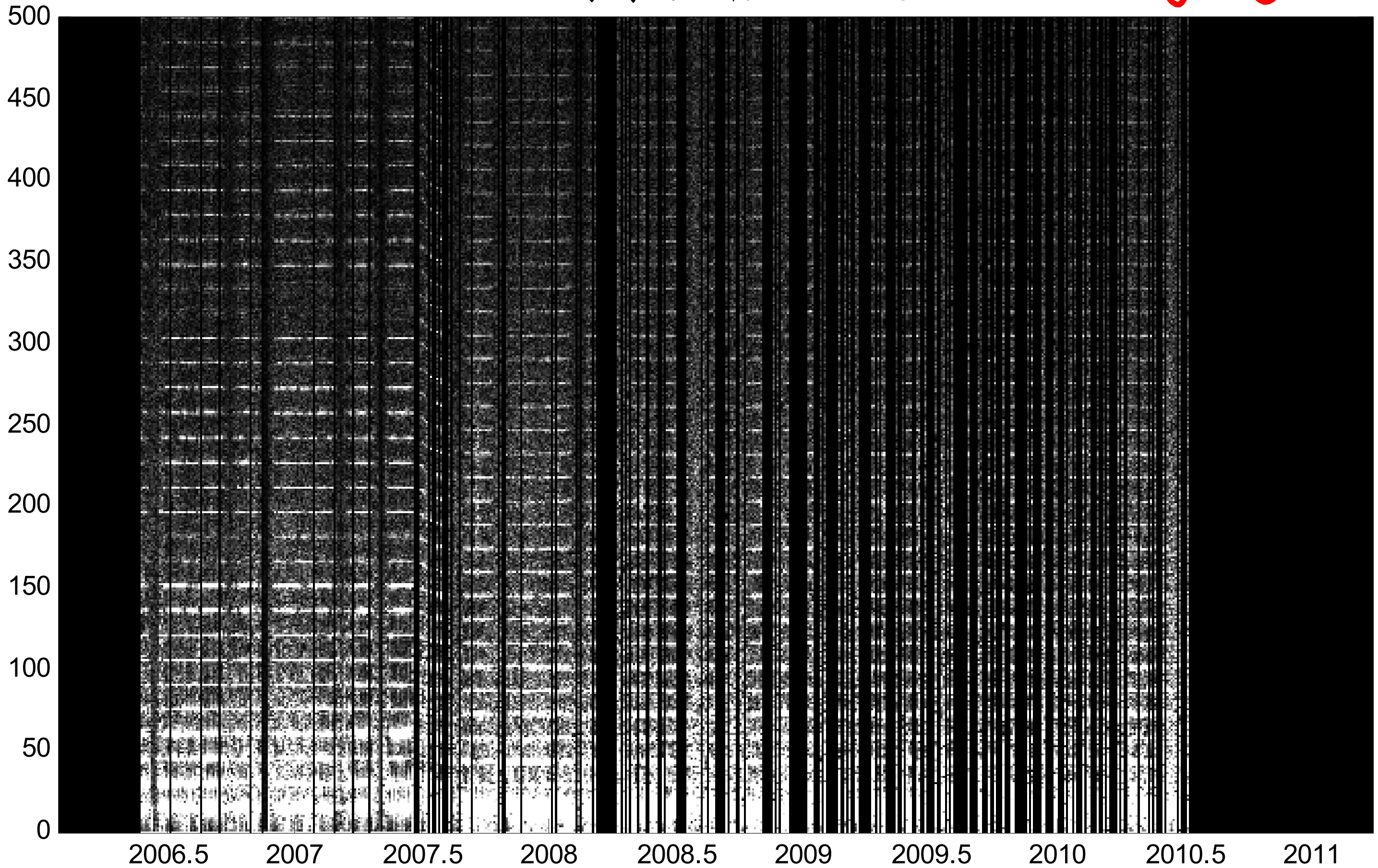
Astronomy

Oceanography

Satellite Signal

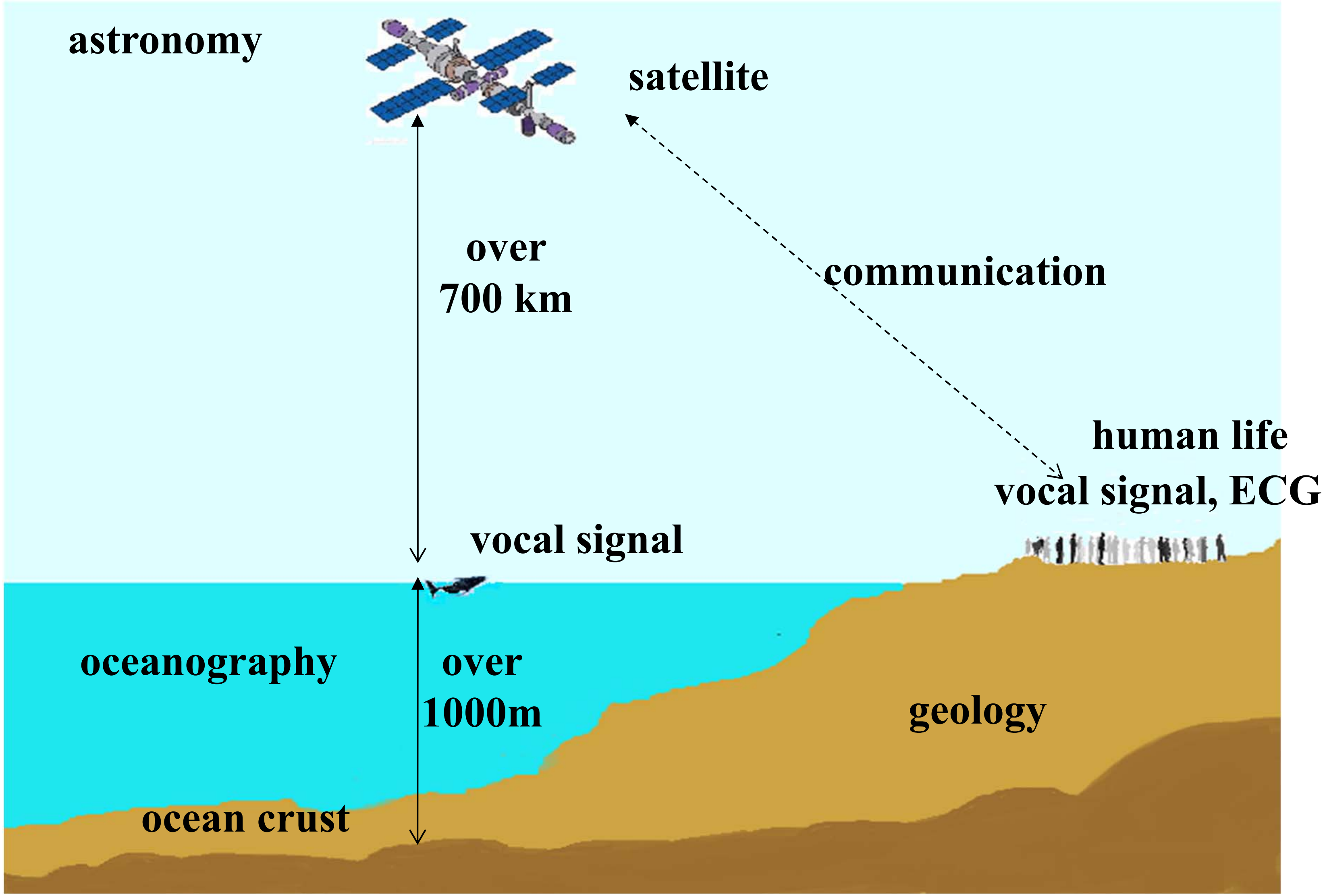
Short-time Fourier transform of the power signal from a satellite

福爾摩沙衛星三號



C. J. Fong, S. K. Yang, N. L. Yen, T. P. Lee, C. Y. Huang, H. F. Tsai, S. Wang, Y. Wang, and J. J. Ding, "Preliminary studies of the applications of HHT (Hilbert-Huang transform) on FORMOSAT-3/COSMIC GOX payload trending data," *6th FORMOSAT-3/COSMIC Data Users' Workshop*, Boulder, Colorado, USA, Oct. 2012

時頻分析的應用範圍


astronomy
satellite
over
700 km
communication
human life
vocal signal, ECG
vocal signal
oceanography
over
1000m
geology
ocean crust

附錄十二：幾個常見的資料蒐尋方法

(1) Google 學術搜尋

http://scholar.google.com.tw/

(太重要了，不可以不知道) 只要任何的書籍或論文，在網路上有電子版，都可以用這個功能查得到

輸入關鍵字，或期刊名，或作者

再按「搜尋」，就可找到想要的資料

(2) 尋找 IEEE 的論文

http://ieeexplore.ieee.org/Xplore/guesthome.jsp

(3) Wikipedia

(4) Github (搜尋 code)

(5) ChatGPT (萬事通)

(6) 數學的百科網站

http://eqworld.ipmnet.ru/index.htm

有多個 tables，以及對數學定理的介紹

(7) 傳統方法：去圖書館找資料

台大圖書館首頁 http://www.lib.ntu.edu.tw/

或者去 http://www.lib.ntu.edu.tw/tulips

(8) 查詢其他圖書館有沒有我要找的書

「台大圖書館首頁」———→「其他圖書館」

(9) 找尋電子書

「台大圖書館首頁」———→「電子書」或「免費電子書」

(10) 查詢一個期刊是否為 SCI

Step 1: 先去 http://scientific.thomson.com/mjl/

Step 2: 在 Search Terms 輸入期刊全名

Search Type 選擇 “Full Journal Title”，再按 “Search”

Step 3: 如果有找到這期刊，那就代表這個期刊的確被收錄在 SCI

(11) 想要對一個東西作入門但較深入的了解:

看 journal papers 會比看 conference papers 適宜

看書會比看 journal papers 適宜

(12) 如果實在沒有適合的書籍，可以看 “review”， “survey”，或 “tutorial” 性質的論文

有了相當基礎之後，再閱讀 journal papers

(以 Paper Title， Abstract， 以及其他 Papers 對這篇文章的描述，來判斷這篇 journal papers 應該詳讀或大略了解即可)

附錄十三　Time-Frequency Analysis 理論發展年表

AD 1785　The Laplace transform was invented

AD 1812　The Fourier transform was invented

AD 1822　The work of the Fourier transform was published

AD 1898　Schuster proposed the periodogram.

AD 1910　The Haar Transform was proposed

AD 1927　Heisenberg discovered the uncertainty principle

AD 1929　The fractional Fourier transform was invented by Wiener

AD 1932　The Wigner distribution function was proposed

AD 1946　The short-time Fourier transform and the Gabor transform was proposed.
In the same year, the computer was invented

註：沒列出發明者的，指的是 transform / distribution 的名稱和發明者的名字相同

AD 1961 Slepian and Pollak found the prolate spheroidal wave function

AD 1965 The Cooley-Tukey algorithm (FFT) was developed

AD 1966 Cohen's class distribution was invented

AD 1970s VLSI was developed

AD 1971 Moshinsky and Quesne proposed the linear canonical transform

AD 1980 The fractional Fourier transform was re-invented by Namias

AD 1981 Morlet proposed the wavelet transform

AD 1982 The relations between the random process and the Wigner distribution function was found by Martin and Flandrin

AD 1988 Mallat and Meyer proposed the multiresolution structure of the wavelet transform;
In the same year, Daubechies proposed the compact support orthogonal wavelet

註：沒列出發明者的，指的是 transform / distribution 的名稱和發明者的名字相同

AD 1989 The Choi-Williams distribution was proposed; In the same year, Mallat proposed the fast wavelet transform
Kontis proposed the chromagram. (我所知最早的 chromagram 論文)

AD 1990 The cone-Shape distribution was proposed by Zhao, Atlas, and Marks

AD 1990s The discrete wavelet transform was widely used in image processing

AD 1992 The generalized wavelet transform was proposed by Wilson et. al.

AD 1993 Mallat and Zhang proposed the matching pursuit;
In the same year, the rotation relation between the WDF and the fractional Fourier transform was found by Lohmann

AD 1994 The applications of the fractional Fourier transform in signal processing were found by Almeida, Ozaktas, Wolf, Lohmann, and Pei;
Boashash and O'Shea developed polynomial Wigner-Ville distributions

AD 1995 Auger and Flandrin proposed time-frequency reassignment

L. J. Stankovic, S. Stankovic, and Fakultet proposed the pseudo Wigner distribution

AD 1995 Katkovnik proposed the local polynomial Fourier transform and the local polynomial time-frequency transform

AD 1996 Stockwell, Mansinha, and Lowe proposed the S transform

Daubechies and Maes proposed the synchrosqueezing transform

AD 1998 N. E. Huang proposed the Hilbert-Huang transform

Chen, Donoho, and Saunders proposed the basis pursuit

AD 1999 Bultan proposed the four-parameter atom (i.e., the chirplet)

Wakefield applied the chromagram in music signal processing.

AD 2000 The standard of JPEG 2000 was published by ISO

Another wavelet-based compression algorithm, SPIHT, was proposed by Kim, Xiong, and Pearlman

The curvelet was developed by Donoho and Candes

AD 2000s The applications of the Hilbert Huang transform in signal processing, climate analysis, geology, economics, and speech were developed

AD 2002 The bandlet was developed by Mallet and Peyre;
Stankovic proposed the time frequency distribution with complex arguments

AD 2003 Pinnegar and Mansinha proposed the general form of the S transform

Liebling et al. proposed the Fresnelet.

AD 2005 The contourlet was developed by Do and Vetterli;

The shearlet was developed by Kutyniok and Labate

The generalized spectrogram was proposed by Boggiatto, et al.

AD 2006 Donoho proposed compressive sensing

AD 2006~ Accelerometer signal analysis becomes a new application.

AD 2007 The Gabor-Wigner transform was proposed by Pei and Ding

AD 2007 The multiscale STFT was proposed by Zhong and Zeng.

AD 2007~ Many theories about compressive sensing were developed by Donoho, Candes, Tao, Zhang ….

AD 2010~ Many applications about compressive sensing are found.

AD 2012 The generalized synchrosqueezing transform was proposed by Li and Liang

AD 2014 The variational mode decomposition was proposed by Dragomiretskiy and Zosso

AD 2015~ Time-frequency analysis was widely combined with the deep learning technique for signal identification

The second-order synchrosqueezing transform was proposed by Oberlin, Meignen, and Perrier.

AD 2017 The wavelet convolutional neural network was proposed by Kang et al.

The higher order synchrosqueezing transform was proposed by Pham and Meignen

AD 2018 Shen et al applied the Mel spectrogram in speech recognition.

AD 2018~ With the fast development of hardware and software, the time-frequency distribution of a 10^6-point data can be analyzed efficiently within 0.1 Second

時頻分析理論與應用未來的發展，還看各位同學們大顯身手

XI. Hilbert Huang Transform (HHT)

Proposed by 黃鍔院士 (AD. 1998)

黃鍔院士的生平可參考

http://djj.ee.ntu.edu.tw/%E9%BB%83%E9%8D%94%E9%99%A2%E5%A3%AB.pdf

References

[1] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N. C. Yen, C. C. Tung, and H. H. Liu, "The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis," *Proc. R. Soc. Lond. A*, vol. 454, pp. 903-995, 1998.

[2] N. E. Huang and S. Shen, *Hilbert-Huang Transform and Its Applications*, World Scientific, Singapore, 2005.

(PS: 謝謝 2007 年修課的趙逸群同學和王文阜同學)

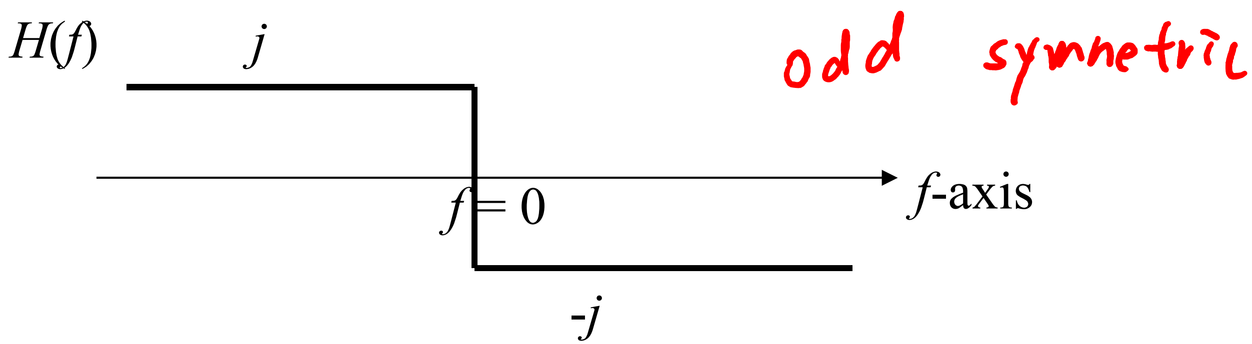
11-A The Origin of the Concept

Another instantaneous frequency analysis method： Hilbert transform

- Hilbert transform

$$x_H(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

or $$x_H(t) = IFT\{FT[x(t)]H(f)\}$$

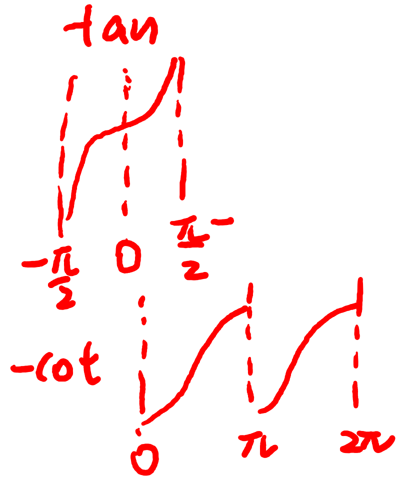


Applications of the Hilbert Transform

- analytic signal

$$x_a(t) = x(t) + jx_H(t)$$

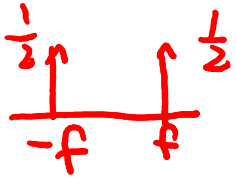
- edge detection
- another way to define the instantaneous frequency:

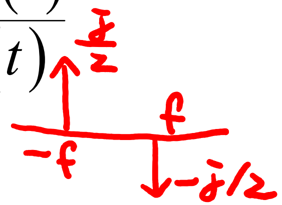


$$\text{instantaneous frequency} = \frac{1}{2\pi}\frac{d}{dt}\theta$$

$$\text{where} \quad \theta = \tan^{-1}\frac{x_H(t)}{x(t)}$$

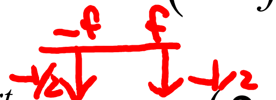
Example:





$\theta = \arctan\left(\frac{\sin(2\pi ft)}{\cos(2\pi ft)}\right) = 2\pi ft$

$$\cos(2\pi ft) \xrightarrow{Hilbert} \sin(2\pi ft) \qquad \theta = 2\pi ft \qquad \frac{1}{2\pi}\frac{d}{dt}\theta = \frac{2\pi f}{2\pi} = f$$



$$\sin(2\pi ft) \xrightarrow{Hilbert} -\cos(2\pi ft) \qquad \theta = 2\pi ft + \pi/2 \qquad \frac{1}{2\pi}\frac{d}{dt}\theta = f$$

$\theta = \arctan\left(\frac{-\cos(2\pi ft)}{\sin(2\pi ft)}\right) = \arctan(-\cot(2\pi ft)) = 2\pi ft + \frac{\pi}{2}$

$\sin(2\pi ft) = -\frac{j}{2}e^{j2\pi ft} + \frac{j}{2}e^{-j2\pi ft}$

Problem of using Hilbert transforms to determine the instantaneous frequency:

This method is only good for cosine and sine functions with single component.

Not suitable for (1) complex function

(2) non-sinusoid-like function

(3) multiple components

Moreover, (4) θ has multiple solutions.

$\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example: $\pm f_1,\ \pm f_2$

$$\cos(2\pi f_1 t)+\cos(2\pi f_2 t)\xrightarrow{\text{Hilbert}}\sin(2\pi f_1 t)+\sin(2\pi f_2 t)$$

$$\theta=\arctan\left(\frac{\sin(2\pi f_1 t)+\sin(2\pi f_2 t)}{\cos(2\pi f_1 t)+\cos(2\pi f_2 t)}\right)=\arctan\left(\frac{2\sin(\pi(f_1+f_2)t)\cos(\pi(f_1-f_2)t)}{2\cos(\pi(f_1+f_2)t)\cos(\pi(f_1-f_2)t)}\right)$$

$$=\pi(f_1+f_2)t$$

$$\frac{1}{2\pi}\frac{d\theta}{dt}=\frac{f_1+f_2}{2}$$

- Hilbert-Huang transform 的基本精神：

IMFs

先將一個信號分成多個 sinusoid-like components + trend

(和 Fourier analysis 不同的地方在於，這些 sinusoid-like components 的 period 和 amplitude 可以不是固定的)

再運用 Hilbert transform (或 STFT，number of zero crossings) 來分析每個 components 的 instantaneous frequency

完全不需用到 Fourier transform

11-B Intrinsic Mode Function (IMF)

= generalization of sinusoid functions

① Amplitude and ② frequency can vary with time.

但要滿足

local maximums & local minimums

$x(t) > x(t \pm \Delta)$ $x(t) < x(t \pm \Delta)$

(1) The number of extremes and the number of zero-crossings must either equal or differ at most by one. = local maximums > 0, local minimums < 0

(2) At any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is near to zero.

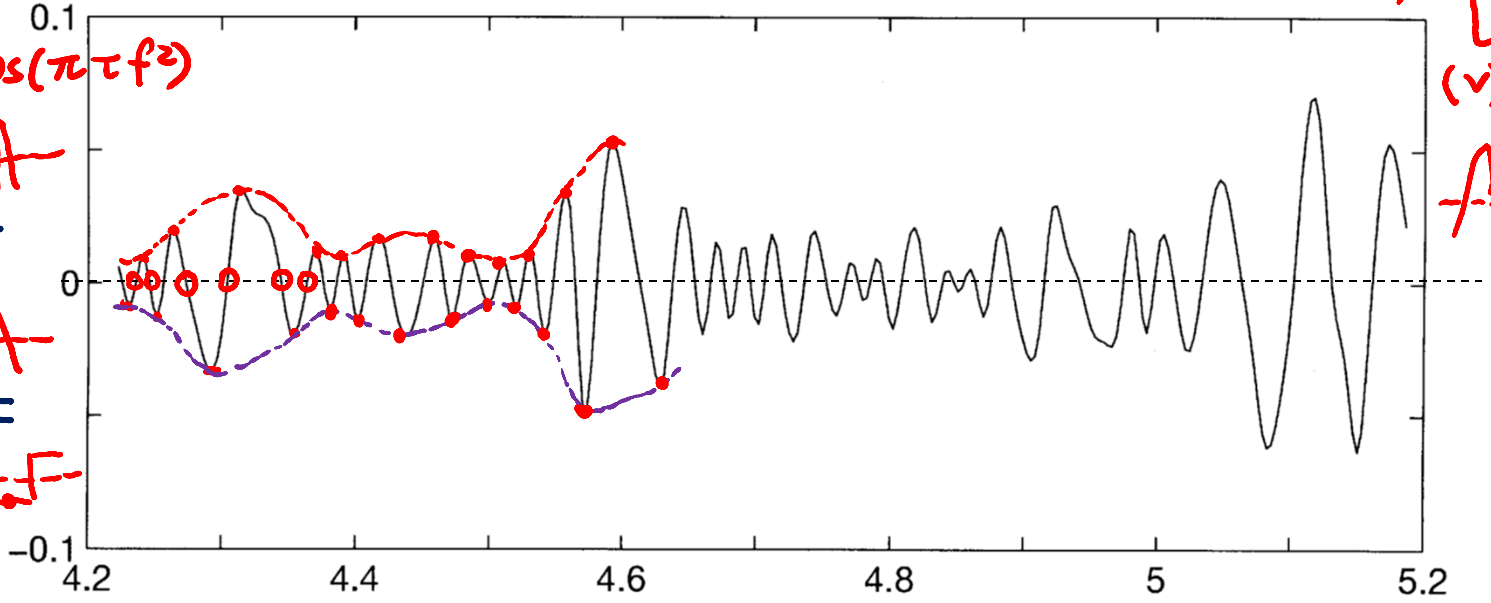
(i) IMF real part of chirp $\cos(\pi \tau f^2)$

(ii) IMF

(iii) IMF

(iv) IMF

(v) not IMF



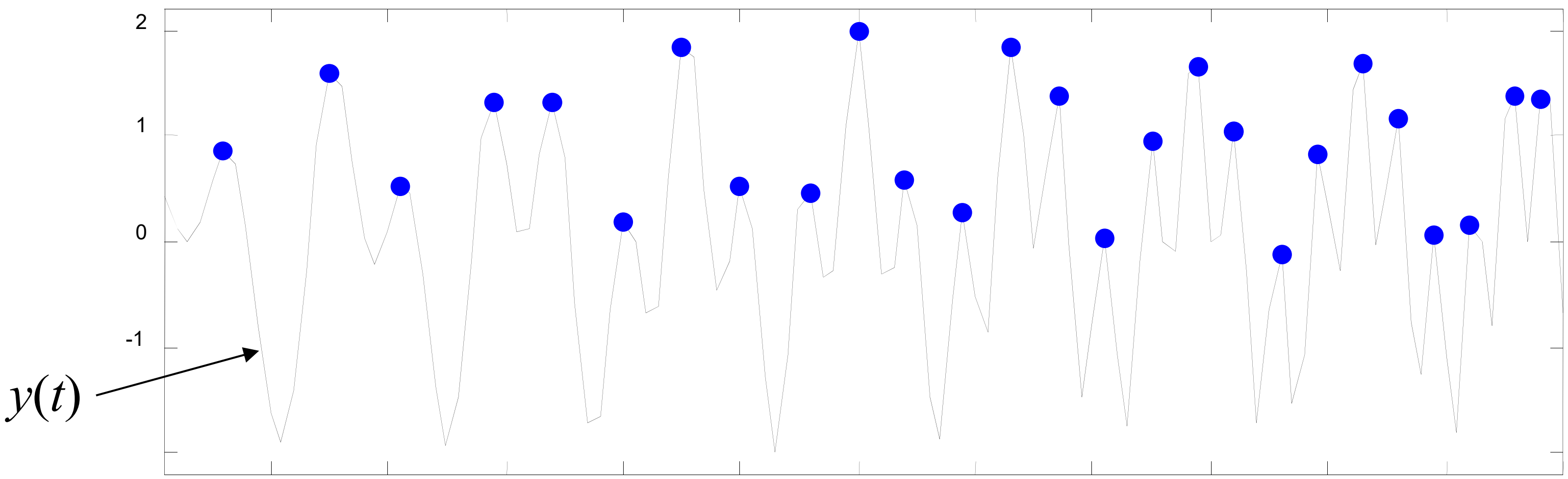
11-C Procedure of the Hilbert Huang Transform

Steps 1~8 are called Empirical Mode Decomposition (EMD)

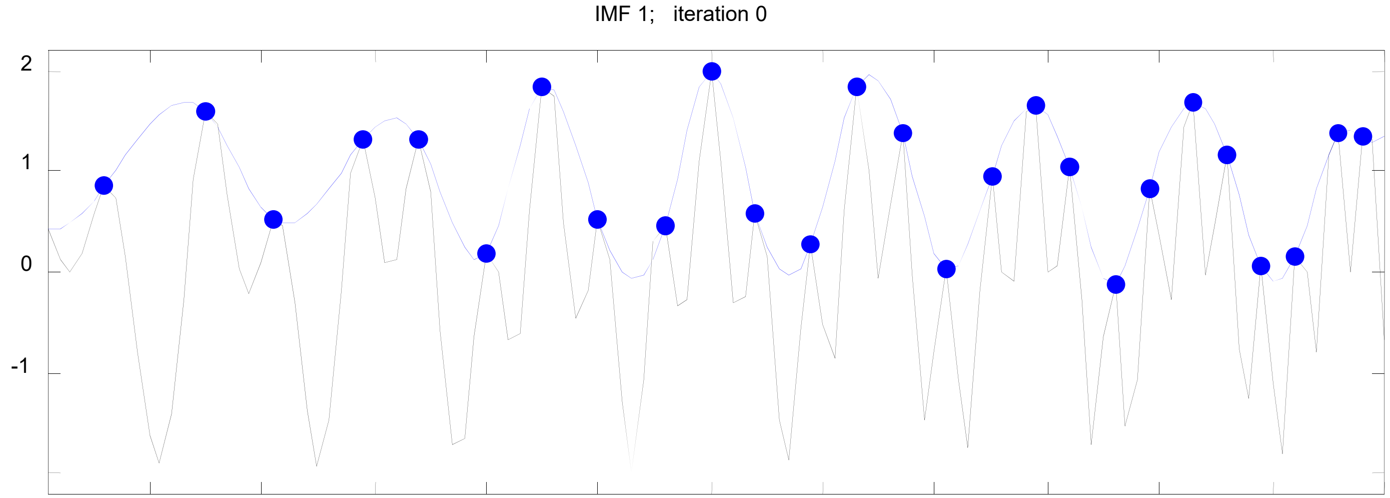
(Step 1) Initial: $y(t) = x(t)$, ($x(t)$ is the input) $n = 1$, $k = 1$

(Step 2) Find the local peaks

$y(t) > y(t+\Delta)$
$y(t) > y(t-\Delta)$



(Step 3) Connect local peaks



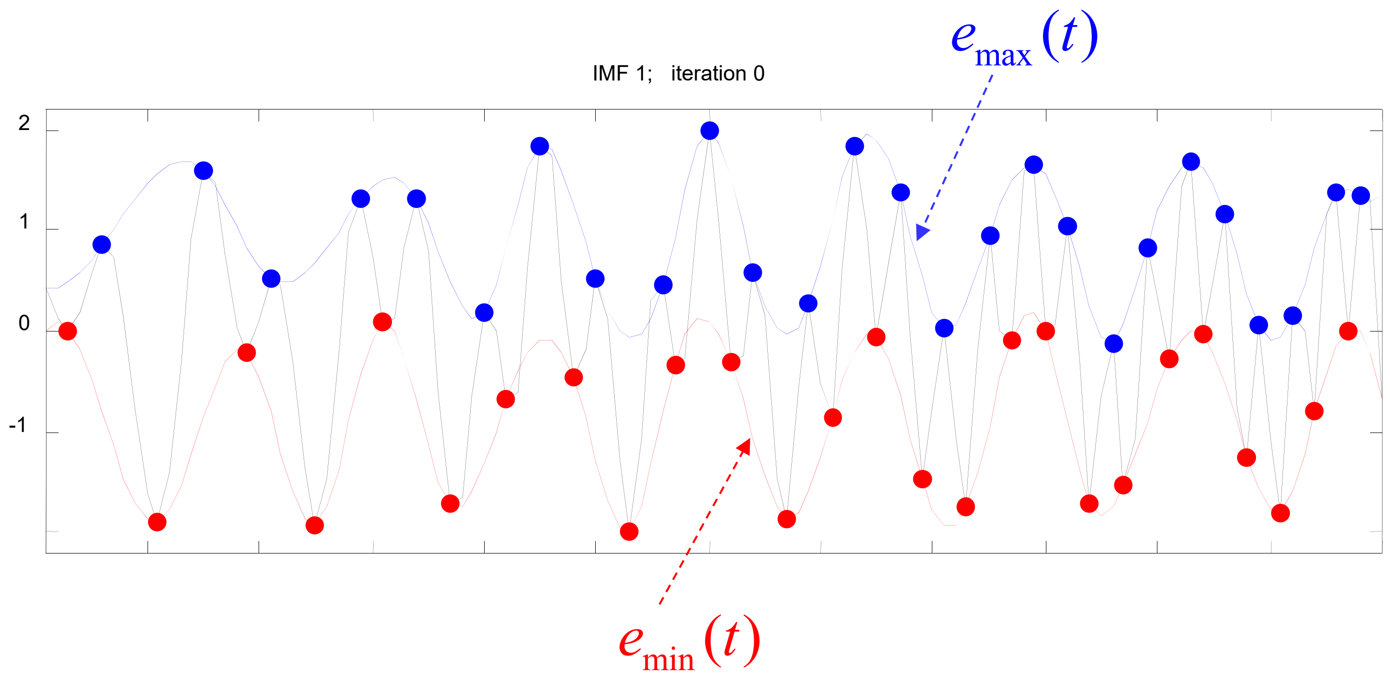
通常使用 B-spline，尤其是 cubic B-spline 來連接

(參考附錄十一) pages 365-368

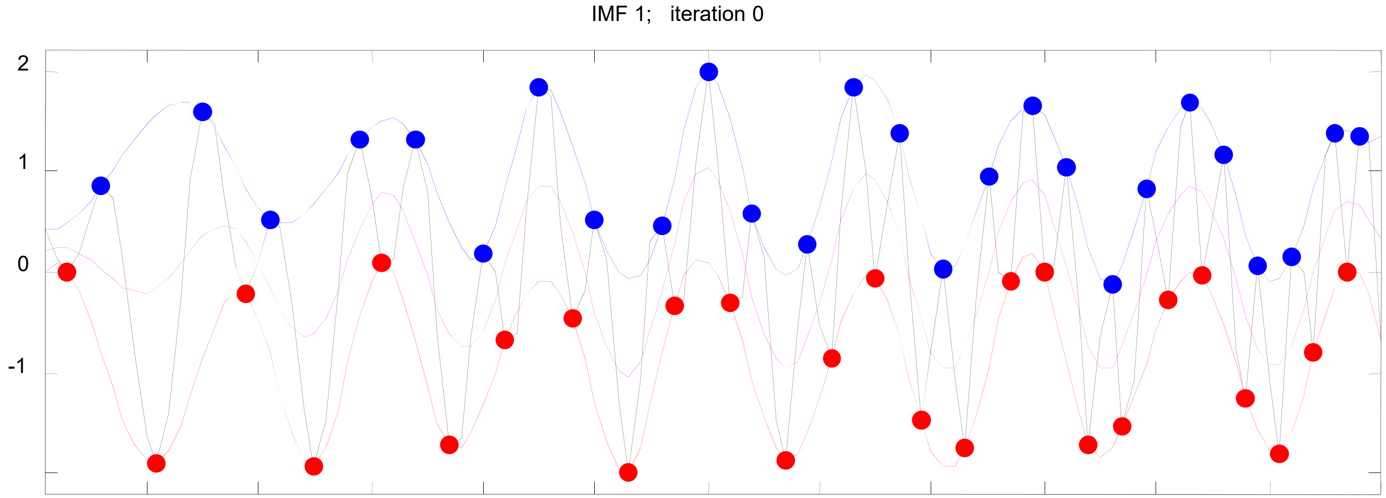
$x(t) < x(t+\Delta)$, $x(t) < x(t-\Delta)$

(Step 4) Find the local dips

(Step 5) Connect the local dips



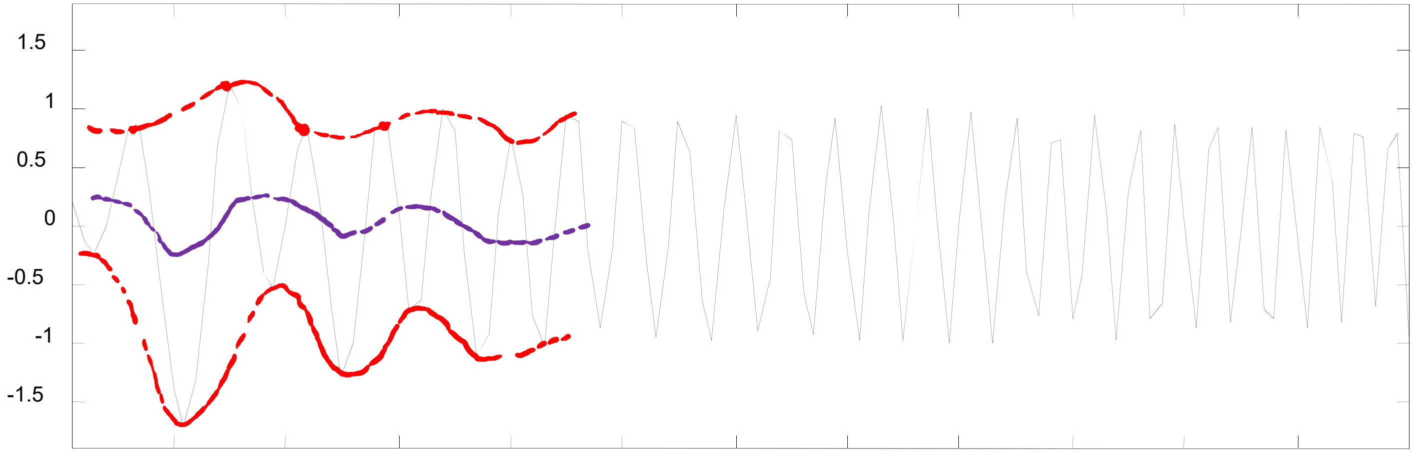
(Step 6-1) Compute the mean



$$z(t) = \frac{e_{\min}(t) + e_{\max}(t)}{2}$$

(pink line)

(Step 6-2) Compute the residue



$$h_k(t) = y(t) - z(t)$$

(Step 7) Check whether $h_k(t)$ is an intrinsic mode function (IMF)

(1) 檢查是否 local maximums 皆大於 0
local minimums 皆小於 0

(2) 上封包： $u_1(t)$， 下封包： $u_0(t)$

檢查是否 $\left|\frac{u_1(t)+u_0(t)}{2}\right| < threshold$ for all t

thr

If they are satisfied (or $k \geqq K$), set $c_n(t) = h_k(t)$ and continue to Step 8

$c_n(t)$ is the n^{th} IMF of $x(t)$.

If not, set $y(t) = h_k(t)$,
$k = k + 1$, and repeat Steps 2~6

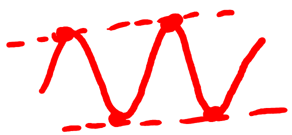
(為了避免無止盡的迴圈，可以定 k 的上限 K)

(Step 8) Calculate $x_0(t) = x(t) - \sum_{s=1}^{n} c_s(t)$

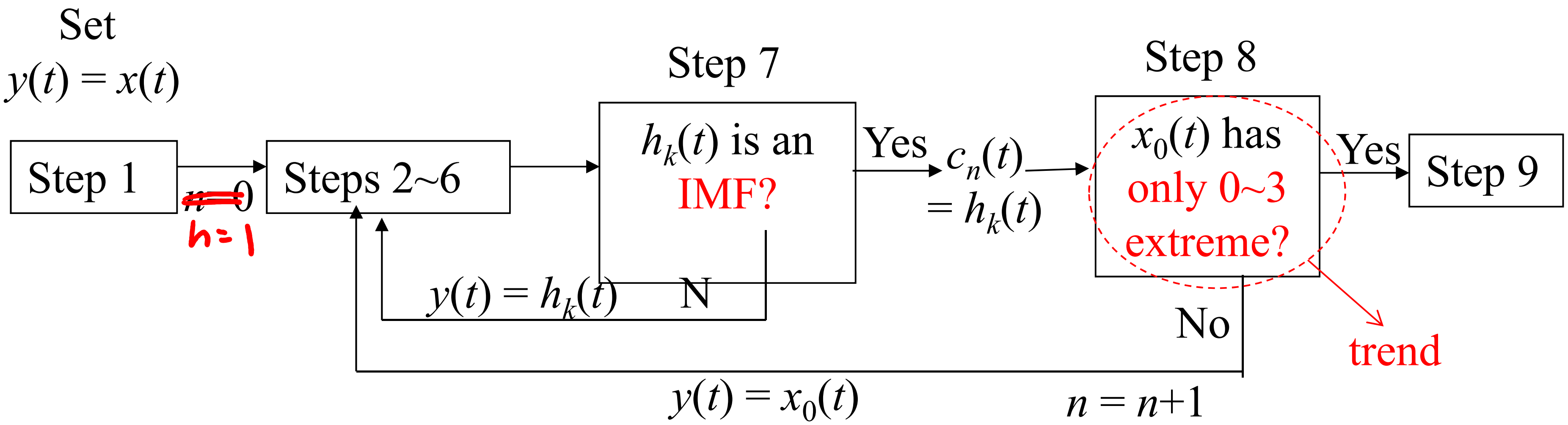
and check whether $x_0(t)$ is a function with no more than one extreme point.

If not, set $n = n+1$, $y(t) = x_0(t)$

and repeat Steps 2~7



If so, the empirical mode decomposition is completed.



$$x(t) = x_0(t) + \sum_{s=1}^{n} c_s(t)$$

(Step 9) Find the instantaneous frequency for each IMF $c_s(t)$ ($s = 1, 2, \ldots, n$).

Method 1: Using the Hilbert transform

Method 2: Calculating the STFT for $c_s(t)$.

Method 3: Furthermore, we can also calculate the instantaneous frequency from the number of zero-crossings directly.

1 period = 2 zero crossings

IMF

instantaneous frequency $F_s(t)$ of $c_s(t)$

$$= \frac{\text{the number of zero-crossings of } c_s(t) \text{ between } t-B \text{ and } t+B}{4B}$$

Technique Problems of the Hilbert Huang Transform

(A) 邊界處理的問題：

目前尚未有一致的方法，可行的方式有

(1) 只使用非邊界的 extreme points

(2) 將最左、最右的點當成是 extreme points

(3) 預測邊界之外的 extreme points 的位置和大小

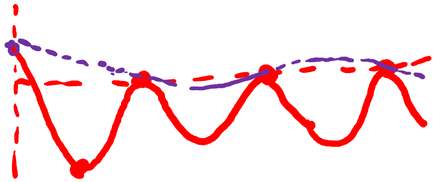
(4) 用邊界和最近的 extreme point 的距離來判斷是否邊界要當成 extreme points

d_1: 邊界和最近 extreme points 距離

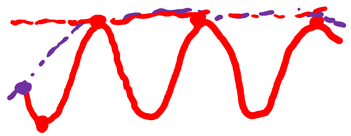
d_2: 非邊界 extreme points 平均間隔

$\frac{d_1}{d_2} < \frac{2}{3}$: not extreme

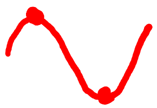
$\frac{d_1}{d_2} \geq \frac{2}{3}$: extreme



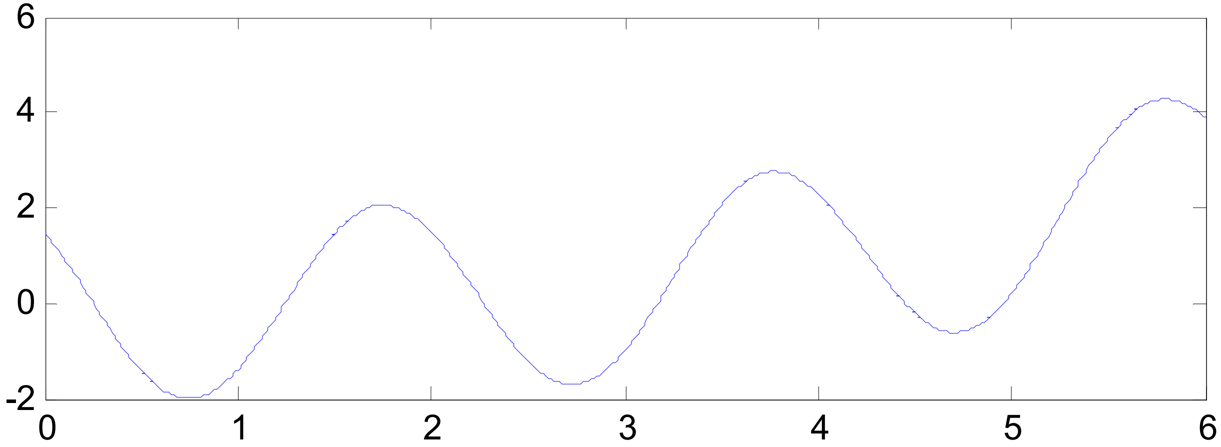
(B) Noise 的問題：

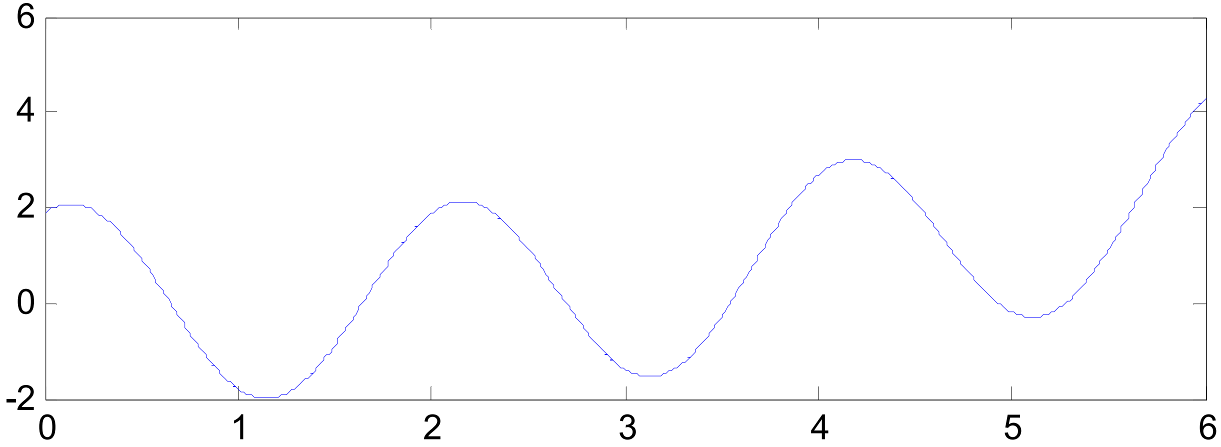


先用 pre-filter 來處理



最左、最右的點是否要當成是 extreme points

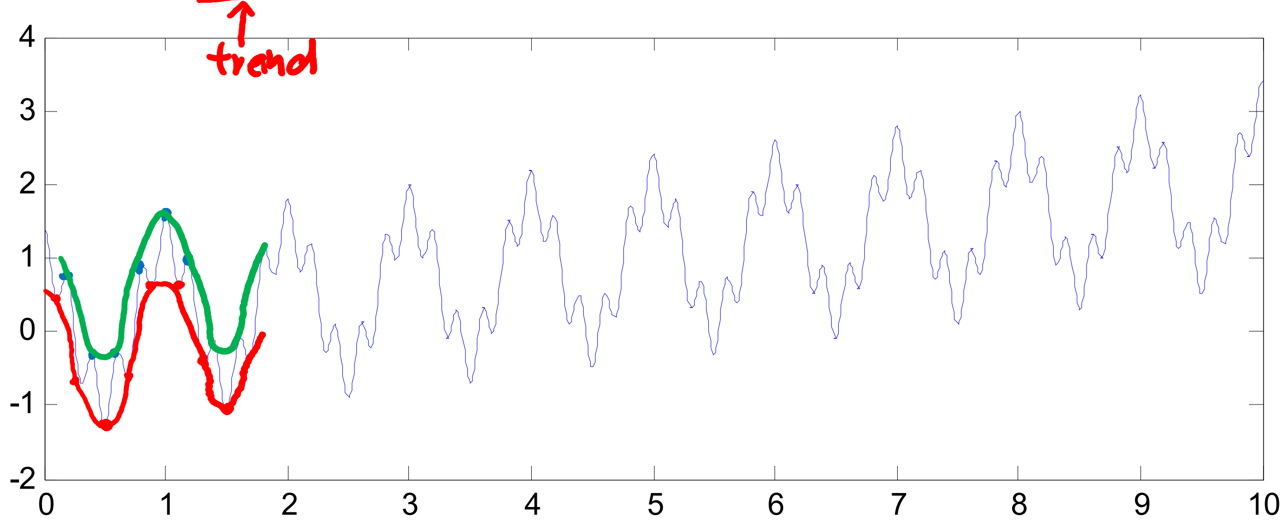




11-D Example

$FT(0.2t) = \frac{j0.2}{2\pi}\delta'(f)$

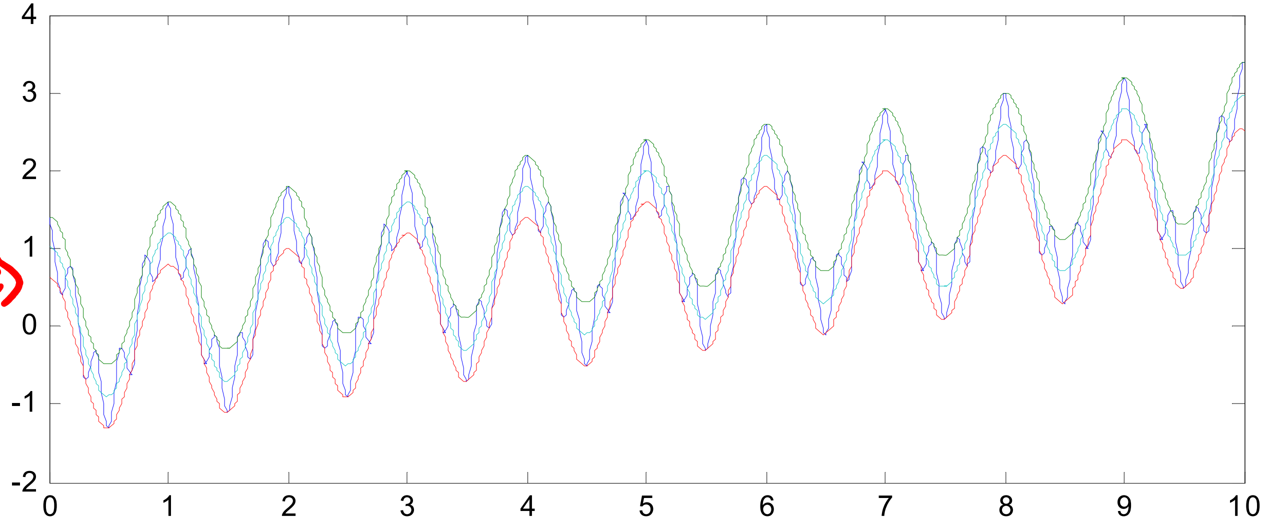
Example 1 $\quad x(t) = 0.2t + \cos(2\pi t) + 0.4\cos(10\pi t)$



After Step 6

上下封包平均

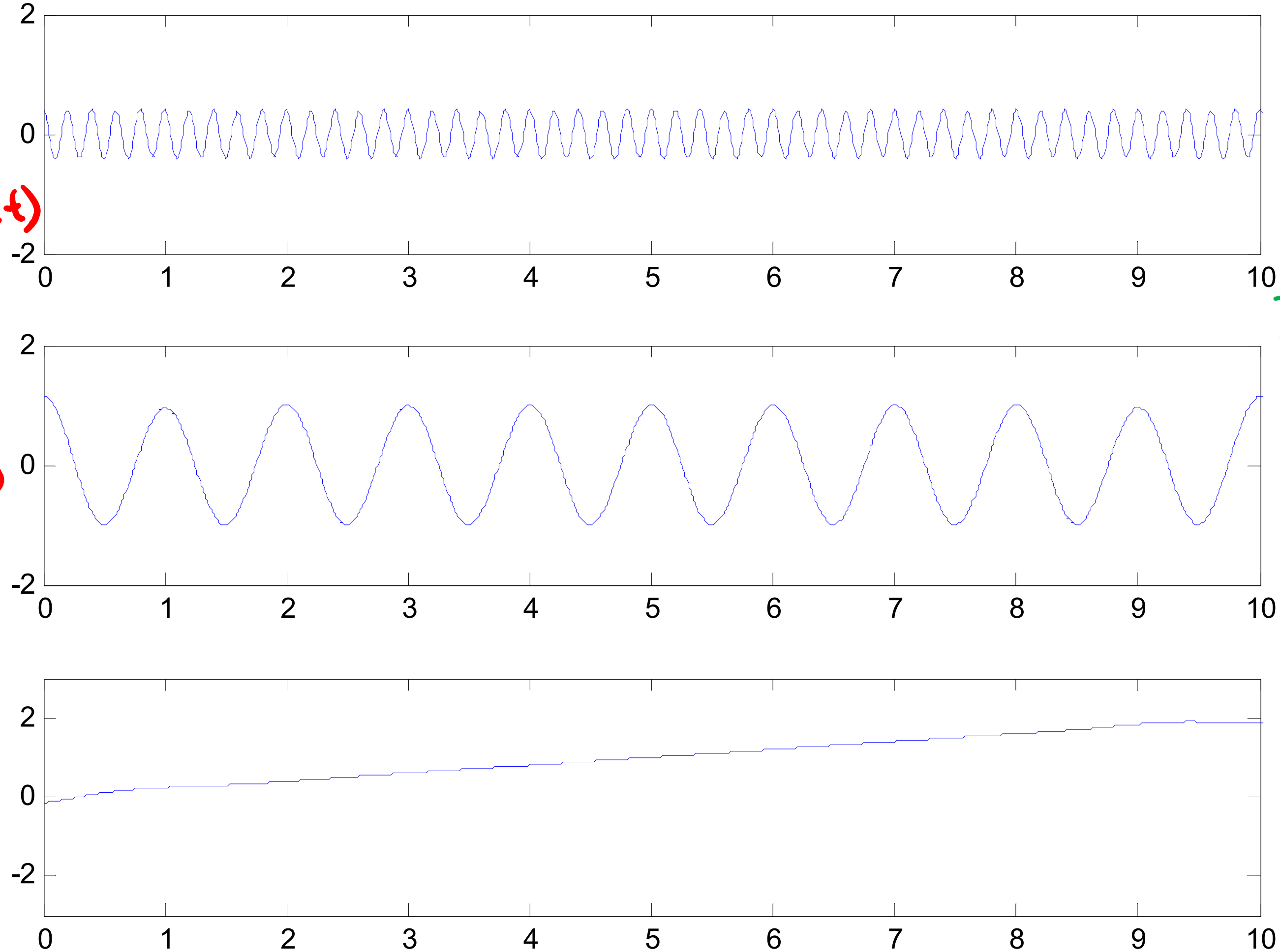
$\cong 0.2t + \cos(2\pi t)$



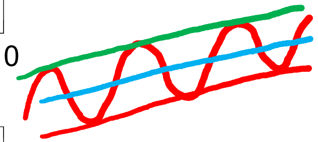
IMF1
$\cong 0.4\cos(10\pi t)$

IMF2
$\cong \cos(2\pi t)$

$x_0(t)$
$\cong 0.2t$

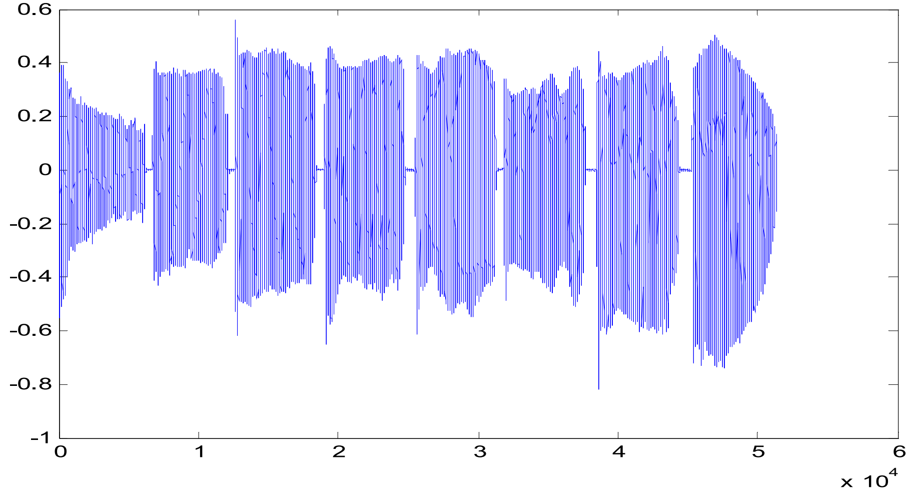


$x(t) - \text{IMF1}$

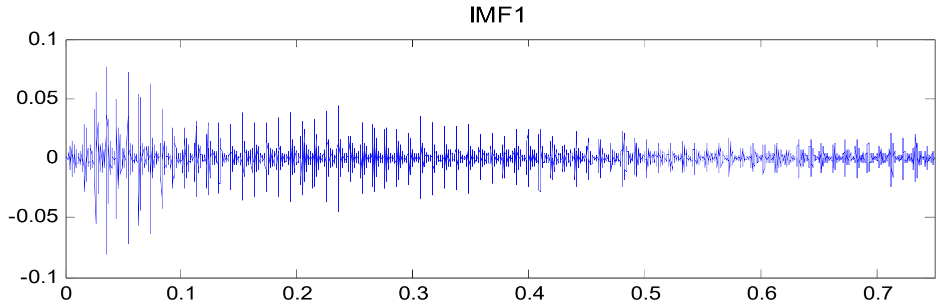


Example 2

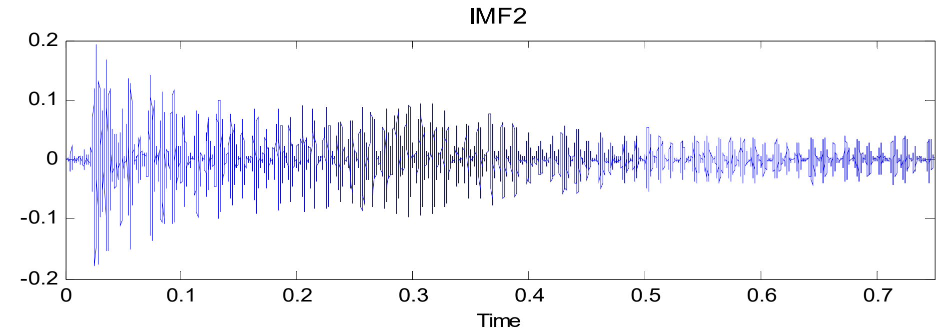
hum signal



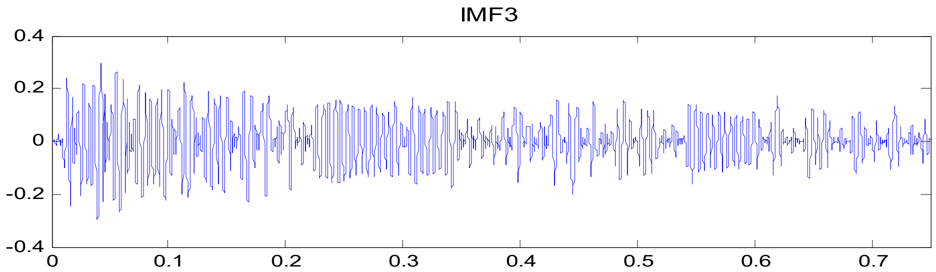
IMF1



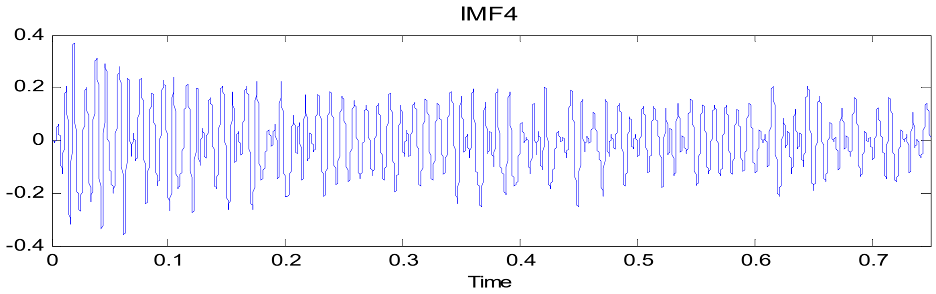
IMF2



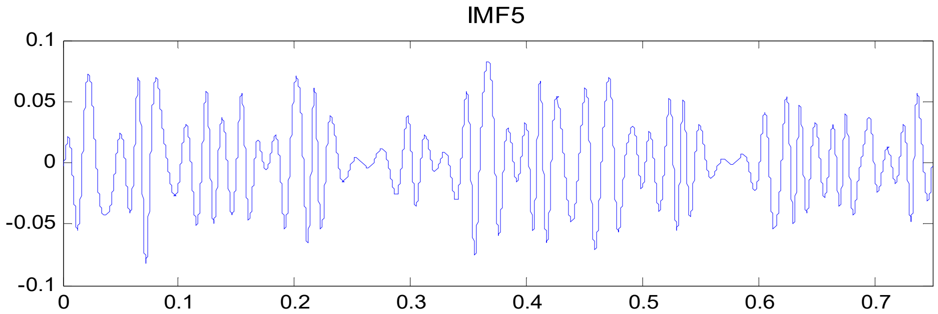
IMF3



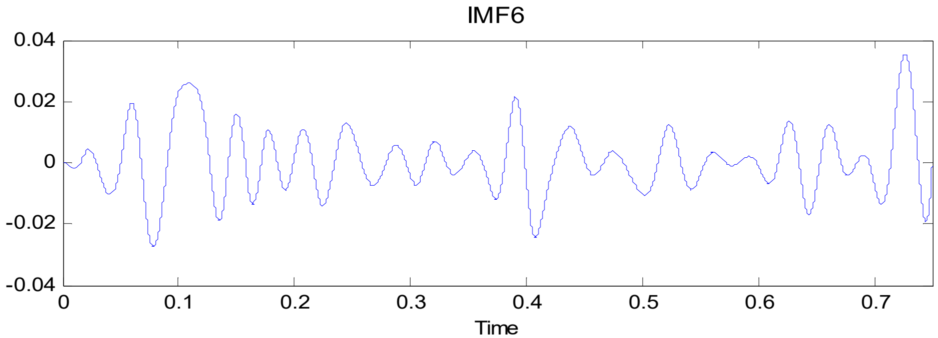
IMF4



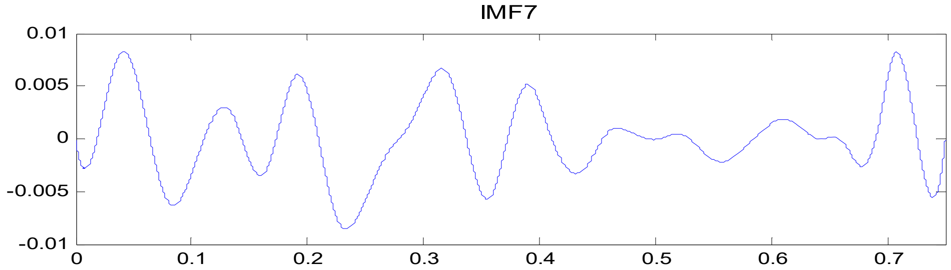
IMF5



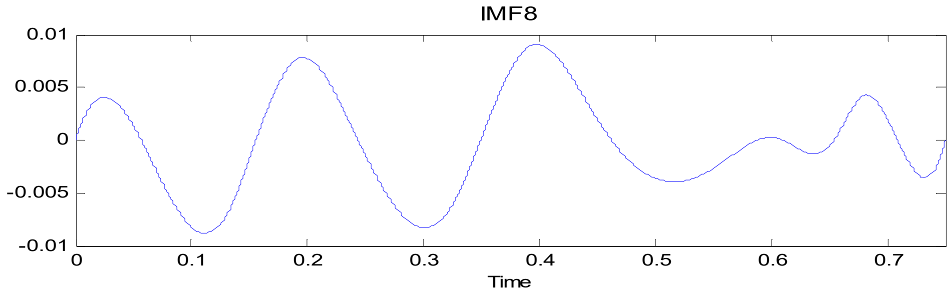
IMF6



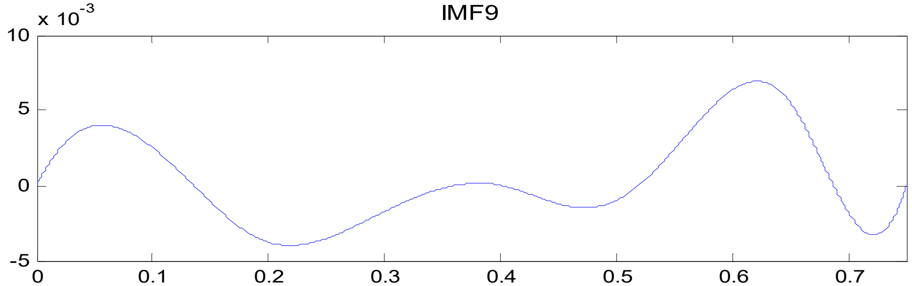
IMF7



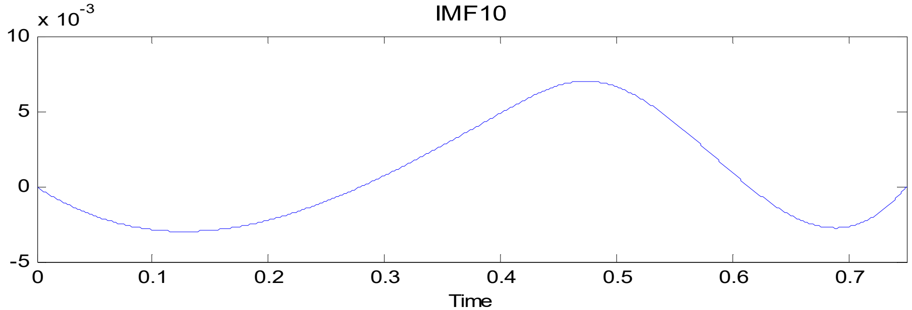
IMF8



IMF9

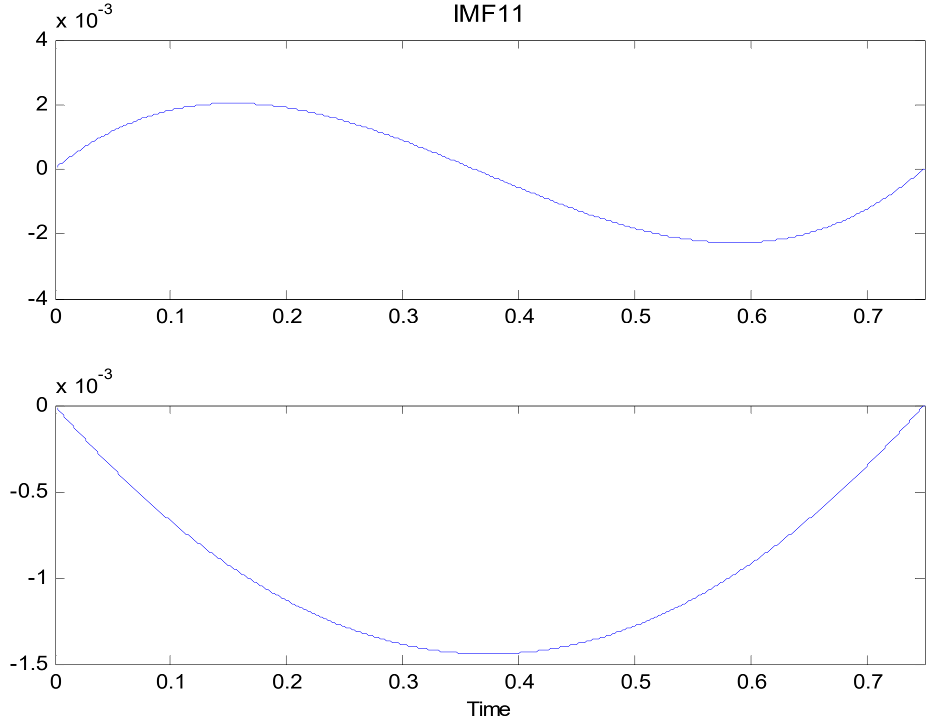


IMF10



IMF11

$x_0(t)$


IMF11
x 10^-3
4
2
0
-2
-4
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
x 10^-3
0
-0.5
-1
-1.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
Time

11-E Comparison

(1) 避免了複雜的數學理論分析

(2) 可以找到一個 function 的「趨勢」

(3) 和其他的時頻分析一樣，可以分析頻率會隨著時間而改變的信號

(4) 適合於
- Climate analysis
- Economical data
- Geology
- Acoustics
- Music signal

- Conclusion

當信號含有「趨勢」

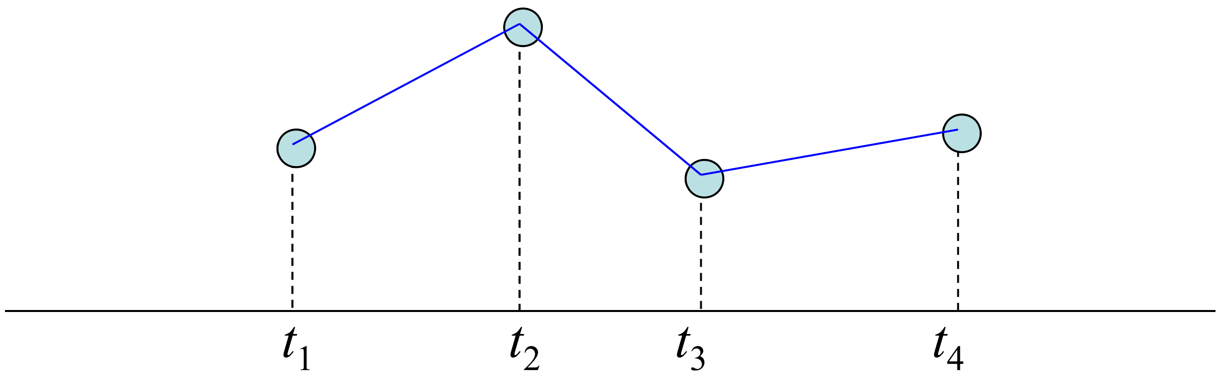
或是由少數幾個 sinusoid functions 所組合而成，而且這些sinusoid functions 的 amplitudes 相差懸殊時，可以用 HHT 來分析

附錄十一 Interpolation and the B-Spline

Suppose that the sampling points are t_1, t_2, t_3, …, t_N
and we have known the values of $x(t)$ at these sampling points.

There are several ways for interpolation.

(1) The simplest way: Using the straight lines (i.e., linear interpolation)



(2) Lagrange interpolation

$$x(t) = \sum_{n=1}^{N} \frac{\prod_{\substack{j=1 \\ j\neq n}}^{N} (t - t_j)}{\prod_{\substack{j=1 \\ j\neq n}}^{N} t_n - t_j} x(t_n)$$

$\prod$ 指的是連乘符號，

$$\prod_{j=1}^{N} h_j = h_1 h_2 h_3 \cdots\cdots h_N$$

Example: When $N = 4$,

$$x(t) = \frac{(t-t_2)(t-t_3)(t-t_4)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} x(t_1) + \frac{(t-t_1)(t-t_3)(t-t_4)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} x(t_2)$$

$$+ \frac{(t-t_1)(t-t_2)(t-t_4)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} x(t_3) + \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} x(t_4)$$

(3) Polynomial interpolation

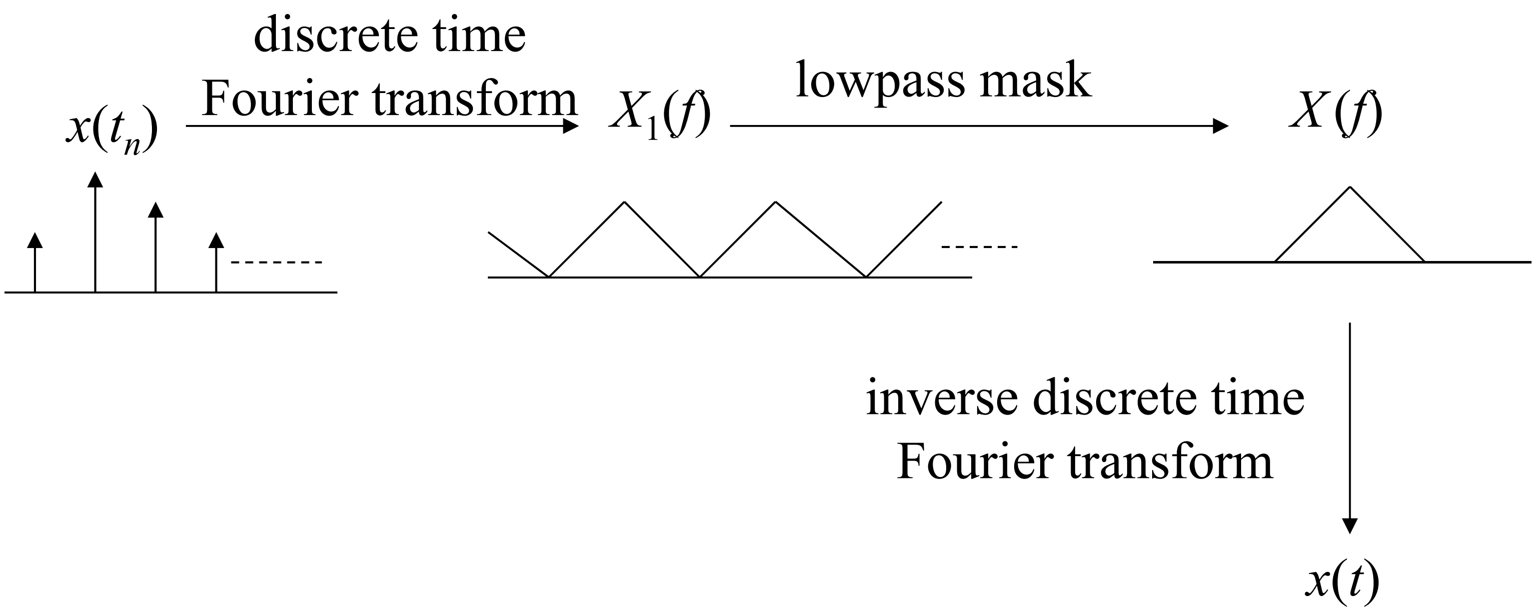
$$x(t) = \sum_{n=1}^{N} a_n t^{n-1}, \qquad \text{solve } a_1, a_2, a_3, \ldots\ldots, a_{N\text{-}1} \text{ from}$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{N-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{N-1} \\ 1 & t_3 & t_3^2 & \cdots & t_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^{N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ x(t_3) \\ \vdots \\ x(t_N) \end{bmatrix}$$

(4) Lowpass Filter Interpolation

適用於 sampling interval 為固定的情形 $t_{n+1} - t_n = \Delta_t$ for all n

$$x(t) = \sum_{n=1}^{N} x(t_n)\,\mathrm{sinc}\left(\frac{t - t_n}{\Delta_t}\right)$$



(5) B-Spline Interpolation

B-spline 簡稱為 spline

$$B_{n,0}(t) = 1 \quad \text{for } t_n < t < t_{n+1}$$

$$B_{n,0}(t) = 0 \quad \text{otherwise}$$

$$B_{n,m}(t) = \frac{t - t_n}{t_{n+m} - t_n} B_{n,m-1}(t) + \frac{t_{n+m+1} - t}{t_{n+m+1} - t_{n+1}} B_{n+1,m-1}(t)$$

for $t_n < t < t_{n+m+1}$

$$x(t) = \sum_{n=1}^{N} x(t_n) B_{n,m}(t)$$

$B_{n,m}(t)$: m^{th} order polynomial

$m = 1$: linear B-spline — $x(t)$ is continuous, $x'(t)$ is not continuous

$m = 2$: quadratic B-spline — $x(t)$, $x'(t)$ are constinuous, $x''(t)$ is not continuous

$m = 3$: cubic B-spline (通常使用) — $x(t)$, $x'(t)$, $x''(t)$ are continuous

$x'''(t)$ is not continuous

In **Matlab**，the command “spline” can be used for spline interpolation.

(Note： In the command, the cubic B-spline is used)

Cubic B-Spline Interpolation by Matlab:

Generating a sine-like spline curve and samples it over a finer mesh:

```
x = 0:1:10;         % original sampling points
y = sin(x);
xx = 0:0.1:10.2;    % new sampling points
yy = spline(x,y,xx);
plot(x,y,'o',xx,yy)
```

value at x

x: locations of local maximums
minimums

y: values at x

xx: 0:0.1:10

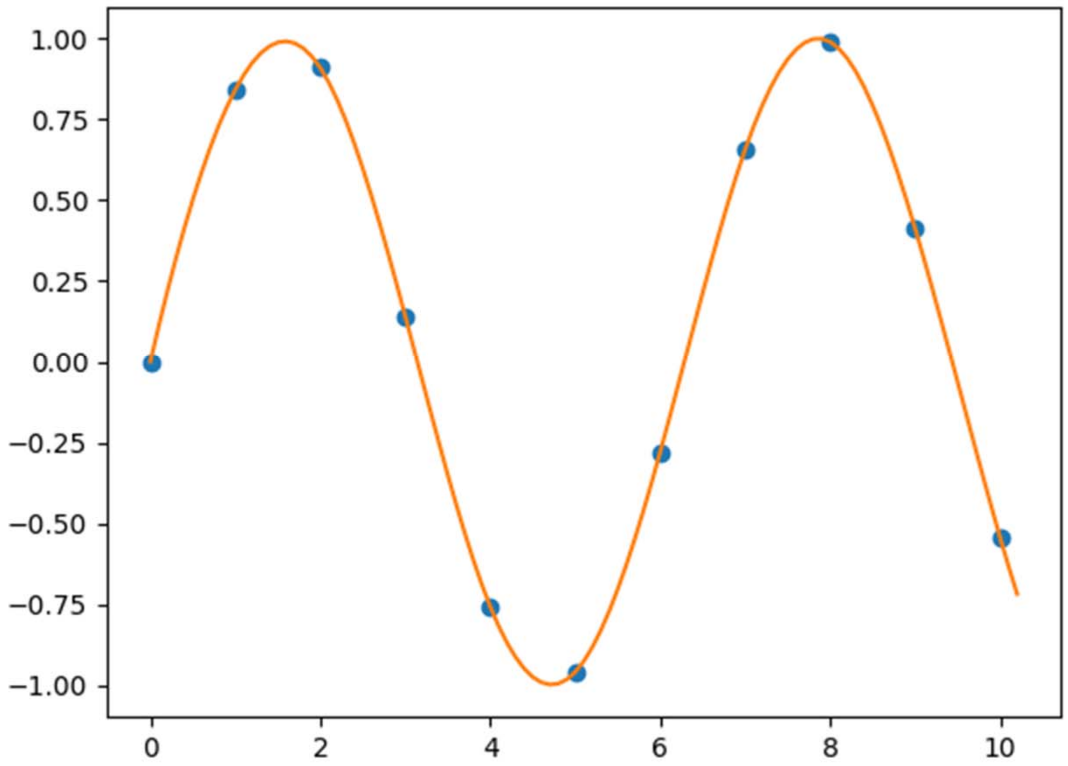
In **Python**, we can use the following way to perform cubic B-spline interpolation.

事前安裝模組

pip install numpy

pip install scipy

pip install matplotlib



Reference :
https://docs.scipy.org/doc/scipy/reference/reference/generated/scipy.interpolate.interp1d.html#scipy.interpolate.interp1d

Cubic B-Spline Interpolation by Python

```
import numpy as np
import scipy.interpolate as interpolate
import matplotlib.pyplot as plt
x = np.arange(0, 11)    # original sample points, [0, 1, 2, …, 9, 10]
y = np.sin(x)
t, c, k = interpolate.splrep(x, y, k=3)
x_new = np.arange(0, 10.3, 0.1)
# new sample points, [0, 0.1, 0.2, ….., 10, 10.1, 10.2]
f = interpolate.BSpline(t, c, k)
y_new = f(x_new)
plt.plot(x,y,'o',x_new, y_new)
plt.show()
```

x: locations of local maximums
minimus

y: the values at x

(0, 10.1, 0.1)