XII. Wavelet Transform

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Main References

- [1] R. C. Gonzalez and R. E. Woods, *Digital Image Processing*, Chap. 7, 4th edition, Prentice Hall, New Jersey, 2017. (適合初學者閱讀)

- [2] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 3rd edition, 2009. (適合想深入研究的人閱讀)
  (若對時頻分析已經有足夠的概念，可以由這本書 Chapter 4 開始閱讀)

Other References

[3] I. Daubechies, “Orthonormal bases of compactly supported wavelets,” *Comm. Pure Appl. Math.*, vol. 4, pp. 909-996, Nov. 1988.

[4] S. Mallat, “Multiresolution approximations and wavelet orthonormal bases of L2(R),” *Trans. Amer. Math. Soc.*, vol. 315, pp. 69-87, Sept. 1989.

[5] C. Heil and D. Walnut, “Continuous and discrete wavelet transforms,” *SIAM Rev.*, vol. 31, pp. 628-666, 1989.

[6] I. Daubechies, “The wavelet transform, time-frequency localization and signal analysis,” *IEEE Trans. Information Theory*, pp. 961-1005, Sept. 1990.

[7] R. K. Young, *Wavelet Theory and Its Applications*, Kluwer Academic Pub., Boston, 1995.

- [8] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chapter 4, Prentice-Hall, New Jersey, 1996.

[9] L. Debnath, *Wavelet Transforms and Time-Frequency Signal Analysis*, Birkhäuser, Boston, 2001.

[10] B. E. Usevitch, “A Tutorial on Modern Lossy Wavelet Image Compression: Foundations of JPEG 2000,” *IEEE Signal Processing Magazine*, vol. 18, pp. 22-35, Sept. 2001.

[11] A. Kirsanov, “Wavelets: A mathematical microscope,” 影片：https://www.youtube.com/watch?v=jnxqHcObNK4

(1) Conventional method for signal analysis

- Fourier transform : $X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$
- Cosine and Sine transforms: if $x(t)$ is even and odd
- Orthogonal Polynomial Expansion

傳統方法共通的問題：

(2) Time frequency analysis

For example， STFT

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f \tau}d\tau$$

Time frequency analysis 共通的問題：

12-A Haar Transform

1910

一種最簡單又可以反應 time-variant spectrum 的 signal representation

low frequency 1
high frequency -1

8-point Haar transform

$$H[m,n]=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

low frequency

width=8

width=4

width=2

high frequency with different scales and different locations

8-point Haar transform

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

y_1: low frequency component $y_2 \sim y_8$: high frequency component

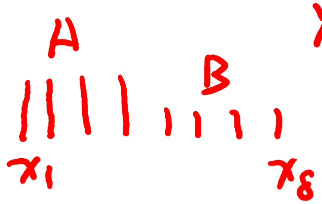
$$y_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

$$y_2 = x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8$$

$$y_3 = x_1 + x_2 - x_3 - x_4$$

$$y_4 = x_5 + x_6 - x_7 - x_8$$

$$y_5 = x_1 - x_2$$



$y_2 = 4A - 4B$

$y_3 = 0$

$y_4 = 0$

$H_2 = F_2$ (2-point DFT)

$F(m,n) = e^{-j\frac{2\pi}{N}mn}$

$N = 2$

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

when $N=2$
$m = 0, 1$
$n = 0, 1$

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$N = 4$

$$\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$N = 8$

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

H_{16}
1st row: all 1's
2nd row: 16 points are nonzero
3rd ~ 4th rows: 8 points are nonzero
5th ~ 8th rows: 4 points
9th ~ 16th rows: 2 points

H_{16} 6th row: [0 0 0 0 1 1 −1 −1 0 0 0 0 0 0 0 0]

H_{16} 15th row:

General way to generate the Haar transform:

$$\mathbf{H}_{2\mathbf{N}} = \begin{bmatrix} \mathbf{H}_{\mathbf{N}} \otimes [1,1] \\ \mathbf{I}_{\mathbf{N}} \otimes [1,-1] \end{bmatrix}$$

where $\otimes$ means the Kronecker product

$$\mathbf{I}_{\mathbf{N}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \cdots & a_{1,N}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \cdots & a_{2,N}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1}\mathbf{B} & a_{M,2}\mathbf{B} & \cdots & a_{M,N}\mathbf{B} \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} \end{bmatrix}$$

$N = 2^k$ 時

$$\mathbf{H} = \begin{bmatrix} \phi \\ h_{0,1} \\ h_{1,1} \\ h_{1,2} \\ \vdots \\ \vdots \\ h_{k-1,1} \\ h_{k-1,2} \\ \vdots \\ h_{k-1,2^{k-1}} \end{bmatrix}$$

H 除了第 1 個row 為 $\underbrace{\phi = [1 \quad 1 \quad 1 \quad \cdots \quad 1]}_{N \text{ 個 } 1}$ 以外

第 $2^p + q$ 個row 為 $h_{p,q}[n]$

$$p = 0, 1, \ldots, k-1, \quad q = 1, 2, \ldots, 2^p$$

$$k = \log_2 N$$

$$h_{p,q}[n] = 1 \qquad \text{when } (q\text{-}1)2^{k-p} < n \le (q\text{-}1/2)2^{k-p}$$

$$h_{p,q}[n] = -1 \qquad \text{when } (q\text{-}1/2)2^{k-p} < n \le q2^{k-p}$$

- Inverse 2^k-point Haar Transform

$$\mathbf{H}^{-1} = \mathbf{H}^{\mathrm{T}}\mathbf{D}$$

$D[m, n] = 0$ if $m \neq n$

$D[1, 1] = 2^{-k}$, $D[2, 2] = 2^{-k}$,

$D[n, n] = 2^{-k+p}$ if $2^p < n \leq 2^{p+1}$

When $k = 3$,

$$\mathbf{D} = \begin{bmatrix} 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

12-B Characteristics of Haar Transform

(1) No multiplications

(2) Input 和 Output 點數相同

(3) 頻率只分兩種：低頻 (全為 1) 和高頻 (一半為 1，一半為 −1)

(4) 可以分析一個信號的 localized feature

(5) Very fast, but not accurate

Example:

$$\mathbf{H}\begin{bmatrix}1.2\\1.2\\1.8\\0.8\\2\\2\\1.9\\2.1\end{bmatrix}=\begin{bmatrix}13\\-3\\-0.2\\0\\0\\1\\0\\-0.2\end{bmatrix}$$

Transforms	Running Time	terms required for NRMSE < 10^{-5}
DFT	9.5 sec 10^{-6}	43
Haar Transform	0.3 sec $\frac{10^{-6}}{32}$	128

References

- A. Haar, "Zur theorie der orthogonalen funktionensysteme ," *Math. Annal.*, vol. 69, pp. 331-371, 1910.
- H. F. Harmuth, *Transmission of Information by Orthogonal Functions*, Springer-Verlag, New York, 1972.

The Haar Transform is closely related to the Wavelet transform (especially the discrete wavelet transform).

12-C History of the Wavelet Transform

- 1910, Haar families.
- 1981, Morlet, wavelet concept.
- 1984, Morlet and Grossman, "wavelet".
- 1985, Meyer, "orthogonal wavelet".
- 1987, International conference in France.
- 1988, Mallat and Meyer, multiresolution.
- 1988, Daubechies, compact support orthogonal wavelet.
- 1989, Mallat, fast wavelet transform.
- 1990s, Discrete wavelet transforms
- 1999, Directional wavelet transform
- 2000, JPEG 2000

12-D Three Types of Wavelets

Wavelet 以 continuous / discrete 來分，有 3 種

	Input	Output	Name
Type 1	Continuous	Continuous	Continuous Wavelet Transform
Type 2	Continuous	Discrete	有時被稱為 discrete wavelet transform，但其實是continuous wavelet transform with discrete coefficients
Type 3	Discrete	Discrete	Discrete Wavelet Transform

比較：Fourier transform 有四種

C: continuous
D: discrete

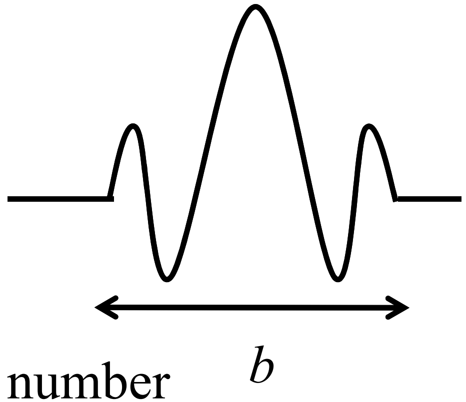
Input	Output	
C	C	continuous FT
C	D	Fourier series
D	C	discrete-time FT
D	D	discrete FT (DFT)

12-E Continuous Wavelet Transform (WT)

psy /saɪ/

Definition: $$X_w(a,b)=\frac{1}{\sqrt{b}}\int_{-\infty}^{\infty}x(t)\psi\left(\frac{t-a}{b}\right)dt$$

refer to page 34



$x(t)$: input, $\psi(t)$: mother wavelet → high-frequency function

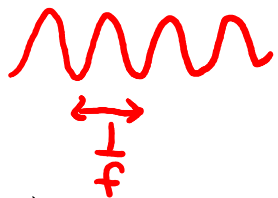
a: location, b: scaling

a is any real number, b is any positive real number

$a \in (-\infty, \infty)$, $b \in (0, \infty)$.

$e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$

Compare with time-frequency analysis:



location + modulation

Gabor Transform $$G_x(t,f)=\int_{-\infty}^{\infty}e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}x(\tau)d\tau$$

$t \leftrightarrow a$

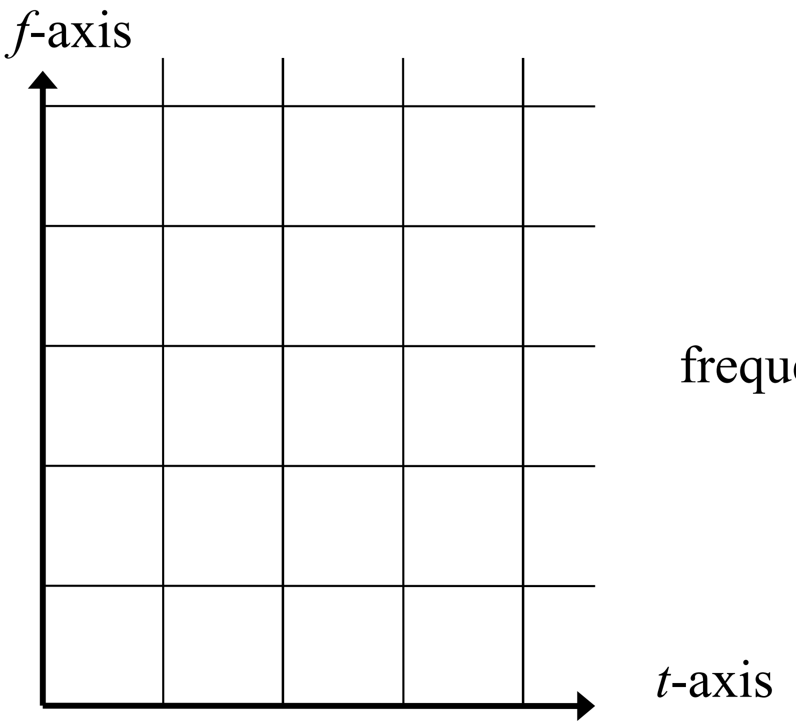
$f \leftrightarrow \frac{1}{b}$

small b → fast variation → large f

large b → small f

width of the kernel

Gabor

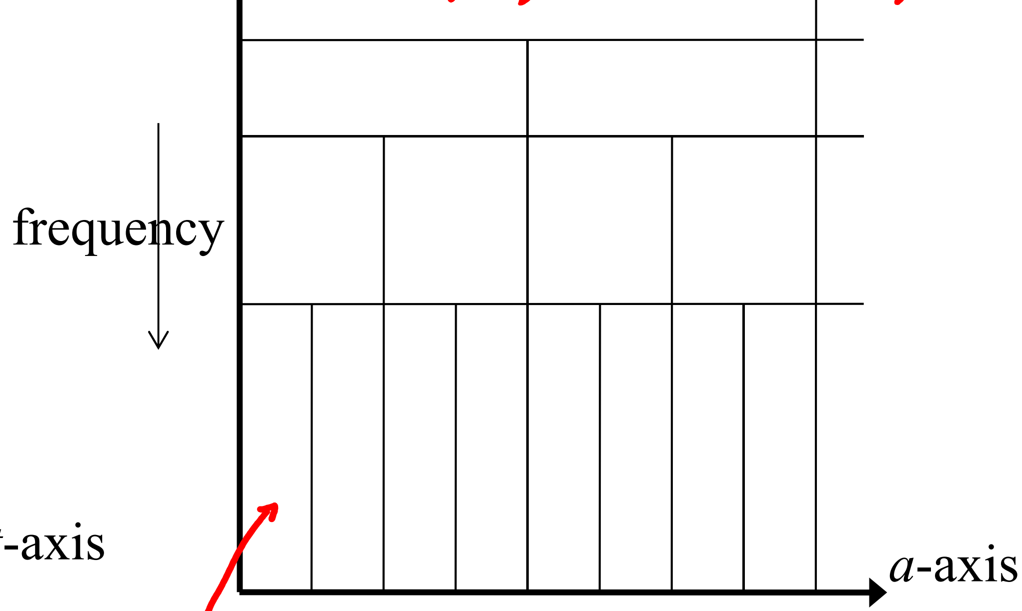


$e^{-\pi(\tau-t)^2}$

Wavelet transform

large b
(small f)

poor time resolution
better frequency resolution



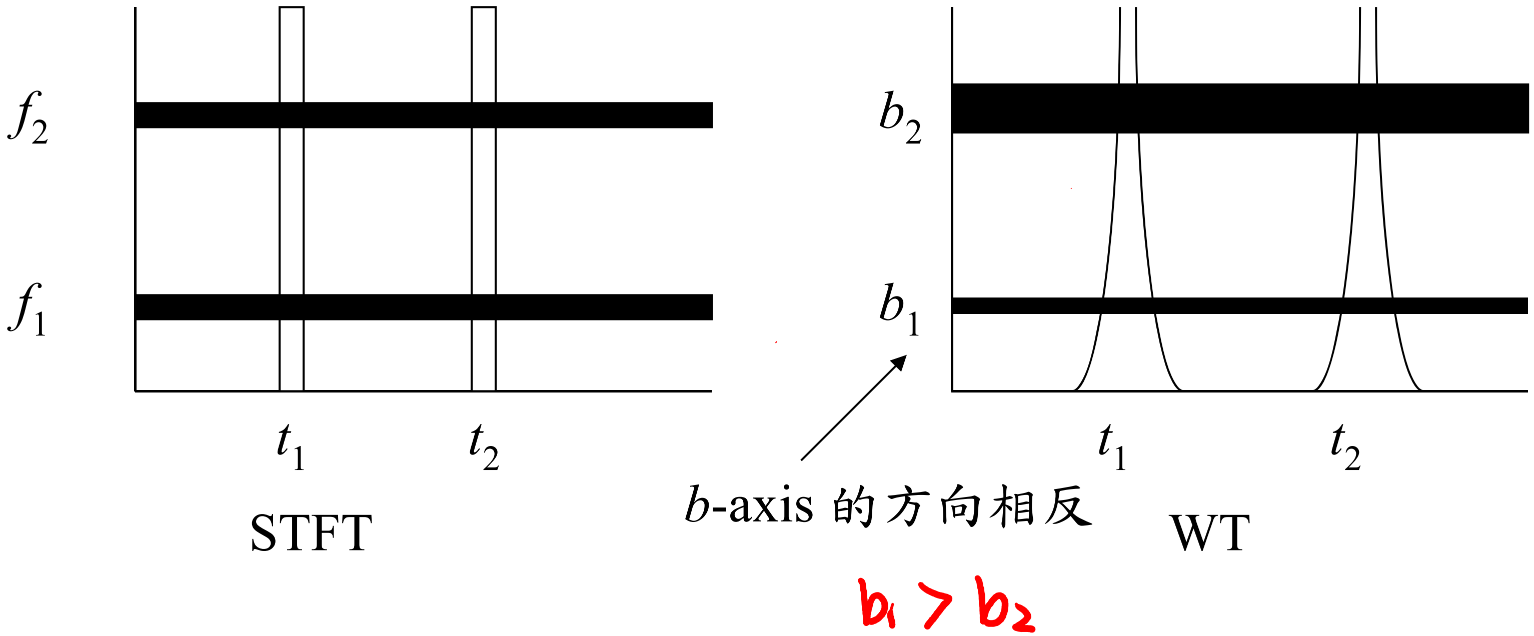
$\psi\left(\frac{t-a}{b}\right)$

small b
(large f)
better time resolution
poor frequency resolution

$$X_w(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\,\psi\left(\frac{t-a}{b}\right) dt$$ *a*: location, *b*: scaling

• The resolution of the wavelet transform is invariant along *a* (location-axis) but variant along *b* (scaling axis).

If $x(t) = \delta(t - t_1) + \delta(t - t_2) + \exp(j2\pi f_1 t) + \exp(j2\pi f_2 t)$,



12-F Mother Wavelet

There are many ways to choose the mother wavelet. For example,

odd symmetry

- Haar basis

1   -1   t=-B   t=0   t=B

vanish moment = 1

- Mexican hat function $\psi(t)=\dfrac{2^{5/4}}{\sqrt{3}}\left(1-2\pi t^2\right)e^{-\pi t^2}$

$\left(\frac{d^2}{dt^2}e^{-\pi t^2}\right)\times\frac{2^{5/4}}{-2\pi\sqrt{3}}$

LoG (Laplacian of Gaussian)

even symmetry   vanish moment = 2

$m_1=\int_{-\infty}^{\infty} t\,\psi(t)\,dt$ odd   even

$\because t\psi(t)$ is odd, $m_1=0$

In fact, the Mexican hat function is the 2nd order derivation of the Gaussian function.

$m_2=\int t^2\psi(t)\,dt=\int_{-\infty}^{\infty} C\,t^2\frac{d^2}{dt^2}e^{-\pi t^2}\,dt$

$m_0=\int_{-\infty}^{\infty}\psi(t)\,dt=\int_{-\infty}^{\infty}C\frac{d^2}{dt^2}e^{-\pi t^2}\,dt$

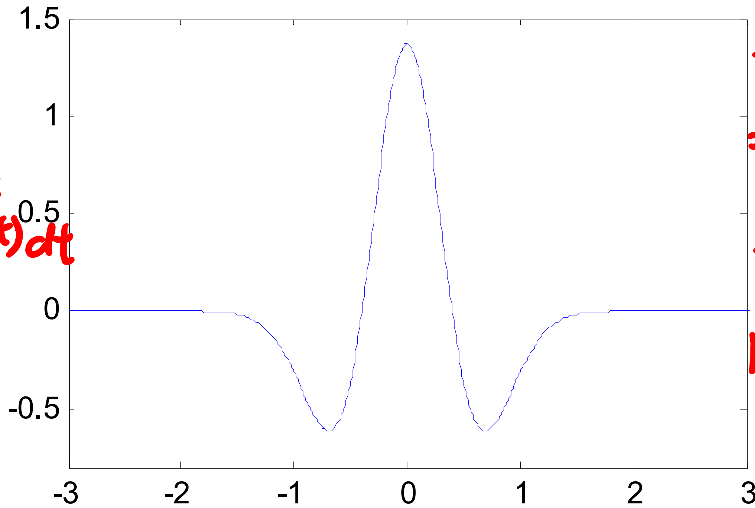
$C=\frac{2^{5/4}}{-2\pi\sqrt{3}}$

If $X(f)=FT(x(t))=\int_{-\infty}^{\infty}e^{-j2\pi ft}x(t)\,dt$

$X(0)=\int_{-\infty}^{\infty}x(t)\,dt$

$FT\left(\frac{d^2}{dt^2}e^{-\pi t^2}\right)=(j2\pi f)^2e^{-\pi f^2}$

$m_0=C(j2\pi f)^2e^{-\pi f^2}\Big|_{f=0}=0$



$FT\left(t^2\frac{d^2}{dt^2}e^{-\pi t^2}\right)$

$=\frac{1}{(-j2\pi)^2}\frac{d^2}{df^2}\left(FT\left(\frac{d^2}{dt^2}e^{-\pi t^2}\right)\right)$

$=\frac{-1}{4\pi^2}\frac{d^2}{df^2}\left((j2\pi f)^2e^{-\pi f^2}\right)$

$m_2=\frac{-1}{4\pi^2}\frac{d^2}{df^2}\left((j2\pi f)^2e^{-\pi f^2}\right)\Big|_{f=0}$

$\neq 0$

Constraints for the mother wavelet:

(1) Compact Support

support: the region where a function is not equal to zero

compact support: the width of the support is not infinite

$dt = -dt_1$

$t = 2a - t_1$

since $\int_{-\infty}^{a} x(t)\psi\left(\frac{t-a}{b}\right)dt$

$= -\int_{\infty}^{a} x(2a-t_1)\psi\left(\frac{a-t_1}{b}\right)(-dt_1)$

(2) Real

(3) Even Symmetric or Odd Symmetric

$x(t) = x(-t)$ (Even Symmetric) $\quad x(t) = -x(-t)$ (Odd Symmetric)

If $\psi(t) = \psi(-t)$

$\psi\left(\frac{t-a}{b}\right) = \psi\left(\frac{a-t}{b}\right)$

$\int_{-\infty}^{\infty} x(t)\psi\left(\frac{t-a}{b}\right)dt$

$= \int_{a}^{\infty} \left(x(t) + x(2a-t)\right)\psi\left(\frac{t-a}{b}\right)dt$

(4) Vanishing Moments

large vanish moment → high frequency
small vanish moment → low frequency

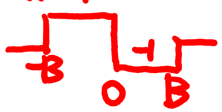
k^{th} moment: $m_k = \int_{-\infty}^{\infty} t^k \psi(t)\,dt$

but $m_p \neq 0$

$\int \psi(t) \sum_{n=0}^{p-1} c^n t^n dt = 0$

If $m_0 = m_1 = m_2 = \ldots\ldots = m_{p-1} = 0$, we say $\psi(t)$ has p vanishing moments.

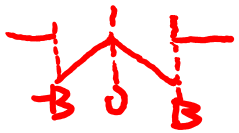
Haar



$\int \psi(t) dt = 0$

$m_0 = 0$

$m_1 = \int t\,\psi(t) dt \neq 0$



$p = 1$

In practice, vanish moment = 4, 5 is enough.

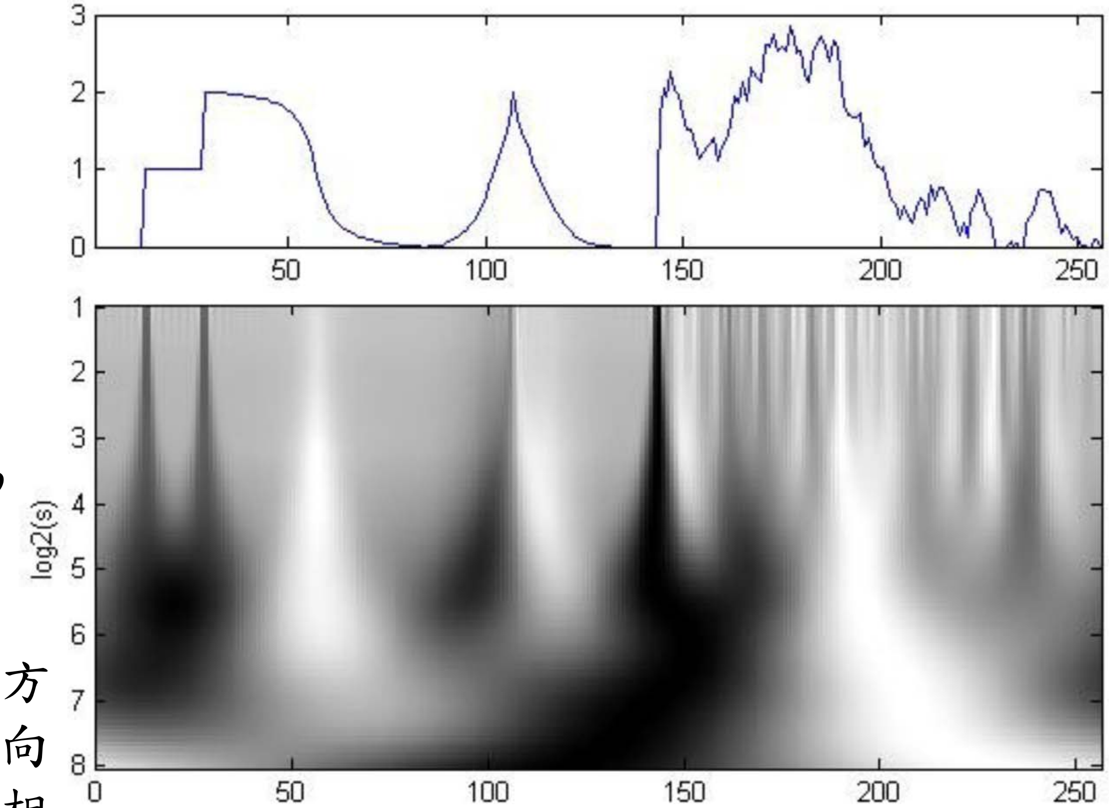
Vanishing moment 越高，經過內積後被濾掉的低頻成分越多

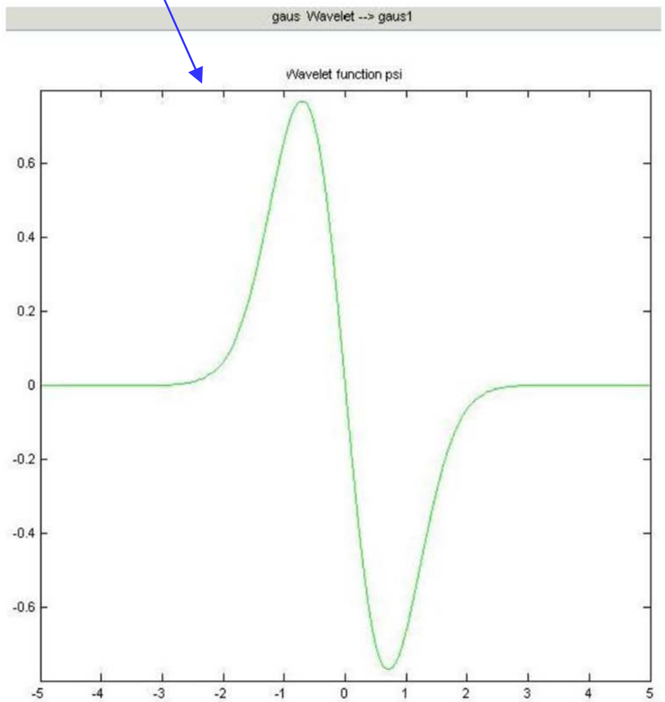
Question：為什麼要求 $\int_{-\infty}^{\infty} \psi(t)\,dt = 0$?

註：感謝2006年修課的張育思同學

vanishing moment = 1

the 1st order derivation of the Gaussian function





b

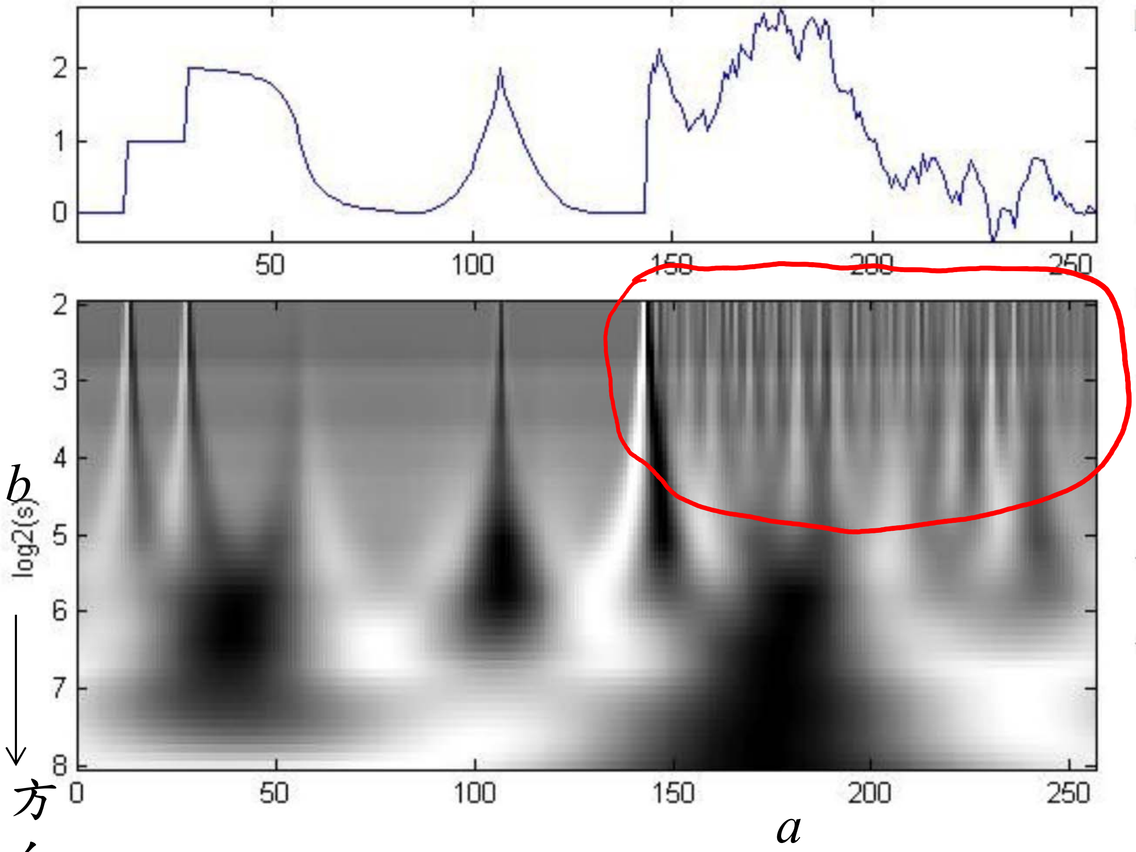
方向相反

a

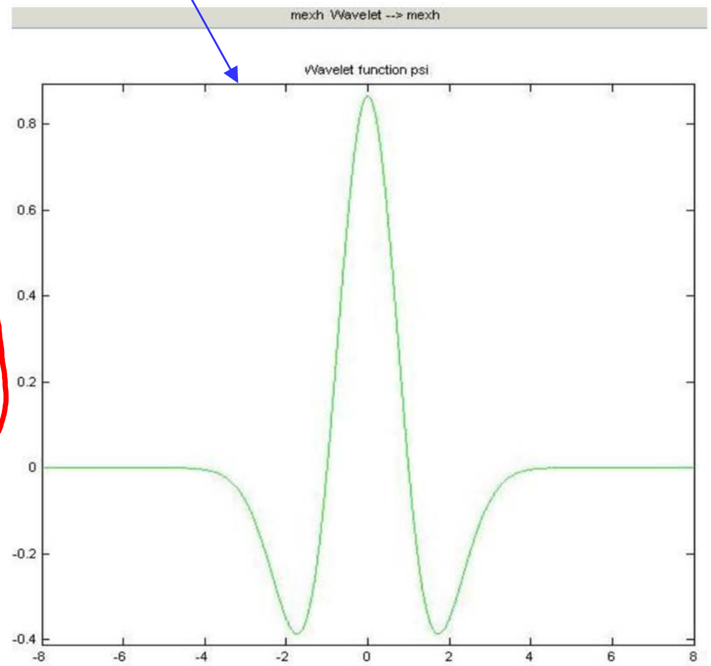
[Ref] S. Mallat, *A Wavelet Tour of Signal Processing*, 2nd Ed., Academic Press, San Diego, 1999.

vanishing moment = 2

the 2nd order derivation of the Gaussian function



方向相反



Similarly, when

$$\psi(t) = \frac{d^p}{dt^p} e^{-\pi t^2}$$

the vanishing moment is p

$$m_k = \int_{-\infty}^{\infty} t^k \frac{d^p}{dt^p} e^{-\pi t^2} dt$$

$$= FT\left(t^k \frac{d^p}{dt^p} e^{-\pi t^2}\right)\Big|_{f=0}$$

$$= FT\left(\frac{1}{(-j2\pi)^k} \frac{d^k}{df^k} (j2\pi f)^p e^{-\pi f^2}\right)\Big|_{f=0}$$

If $k < p$, $\frac{d^k}{df^k}(j2\pi f)^p = (j2\pi)^p \frac{p!}{(p-k)!} f^{p-k} +$ remained terms ($f^{p-k+1}, f^{p-k+2}, \cdots f^p$)

$m_k = 0$

If $k \geq p$, $m_k \neq 0$ (since the constant term appears)

$\therefore m_0, m_1, m_2, \cdots m_{p-1} = 0$

vanish moment $= p$

(5) Admissibility Criterion

$$C_\psi = \int_0^\infty \frac{|\Psi(f)|^2}{|f|} df < \infty$$,where $\Psi(f)$ is the Fourier transform of $\psi(t)$

For reversible

[Ref] A. Grossman and J. Morlet, "Decomposition of hardy functions into square integrable wavelets of constant shape," *SIAM J. Appl. Math.*, vol. 15, pp. 723-736, 1984.

12-G Inverse Wavelet Transform

$$x(t)=\frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db$$

where $C_\psi=\int_0^\infty\frac{|\Psi(f)|^2}{|f|}df<\infty$

$\psi(\frac{t-a}{b})\neq 0$

for $-t_0<\frac{t-a}{b}<t_0$

$t-bt_0<a<t+bt_0$

simplified $x(t)\simeq\frac{1}{C_\psi}\int_0^\infty\int_{t-bt_0}^{t+bt_0}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db$ if $\psi(t)\cong 0$ for $|t|>t_0$

(Proof): Since $X_w(a,b)=x(a)*\frac{1}{\sqrt{b}}\psi\left(\frac{-a}{b}\right)$

if $y(t)=\frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db$

then $y(t)=\frac{1}{C_\psi}\int_0^\infty x(t)*\psi\left(\frac{-t}{b}\right)*\psi\left(\frac{t}{b}\right)\frac{db}{b^3}$

$$y(t)=\frac{1}{C_\psi}\int_0^\infty x(t)*\psi\left(\frac{-t}{b}\right)*\psi\left(\frac{t}{b}\right)\frac{db}{b^3}$$

$$Y(f)=\frac{1}{C_\psi}\int_0^\infty X(f)\Psi(-bf)\Psi(bf)\frac{db}{b}$$

where

$$Y(f)=FT[y(t)]$$
$$X(f)=FT[x(t)]$$
$$\Psi(f)=FT[\psi(t)]$$

If $\psi(t)$ is real, $\Psi(-f)=\Psi^*(f)$, $\Psi(-bf)\,\Psi(bf)=\Psi^*(bf)\,\Psi(bf)=|\Psi(bf)|^2$

$$Y(f)=X(f)\frac{1}{C_\psi}\int_0^\infty\left|\Psi(bf)\right|^2\frac{db}{b}$$

$$=X(f)\frac{1}{C_\psi}\int_0^\infty\left|\Psi(f_1)\right|^2\frac{df_1}{bf}\qquad (f_1=bf,\ df_1=fdb)$$

$$=X(f)\frac{1}{C_\psi}\int_0^\infty\left|\Psi(f_1)\right|^2\frac{df_1}{f_1}$$

$$=X(f)$$

Therefore, $y(t)=x(t)$.

12-H Scaling Function (low frequency)

It can simplify the calculation of the Inverse wavelet transform.

定義 scaling function 為

$$\phi(t)=\int_{-\infty}^{\infty}\Phi(f)e^{j2\pi ft}df$$

where $$\left|\Phi(f)\right|^2=\int_f^{\infty}\frac{|\Psi(f_1)|^2}{|f_1|}df_1$$ for $f>0$, $\Phi(-f)=\Phi^*(f)$

$$\left|\Phi(f)e^{j\theta(f)}\right|^2=\left|\Phi(f)\right|^2$$

$\phi(t)$ is usually a lowpass filter (Why?)

$f\uparrow$ $\int_f^{\infty}\frac{|\Psi(f)|^2}{|f|}df\downarrow$

$|\Phi(f)|^2$ decays with f

修正型的 Wavelet transform

(1) $$X_w(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\,\psi\left(\frac{t-a}{b}\right) dt$$

a is any real number, $0 < b < b_0$

scaling function

(2) $$LX_w(a,b_0) = \frac{1}{\sqrt{b_0}} \int_{-\infty}^{\infty} x(t)\,\phi\left(\frac{t-a}{b_0}\right) dt$$

reconstruction:

$$x(t) = \frac{1}{C_\psi}\left[\int_0^{b_0}\int_{-\infty}^{\infty} \frac{1}{b^{5/2}} X_w(a,b)\,\psi\left(\frac{t-a}{b}\right) da\,db + \int_{-\infty}^{\infty} \frac{1}{b_0^{3/2}} LX_w(a,b_0)\,\phi\left(\frac{t-a}{b_0}\right) da\right]$$

由 b_0 至 ∞ 的積分被第二項取代

If $\psi(t) \cong 0$ for $|t| > t_0$, $\phi(t) \cong 0$ for $|t| > t_1$

$\int_{b_0}^{\infty}$ is replaced by this term

$$x(t) \cong \frac{1}{C_\psi}\left[\int_0^{b_0}\int_{t-bt_0}^{t+bt_0} \frac{1}{b^{5/2}} X_w(a,b)\,\psi\left(\frac{t-a}{b}\right) da\,db + \int_{t-b_0t_1}^{t+b_0t_1} \frac{1}{b_0^{3/2}} LX_w(a,b_0)\,\phi\left(\frac{t-a}{b_0}\right) da\right]$$

(Proof): If $y_1(t) = \frac{1}{C_\psi}\int_0^{b_0}\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db$

$$y_2(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da$$

$$Y_1(f) = X(f)\frac{1}{C_\psi}\int_0^{b_0}\left|\Psi(bf)\right|^2\frac{db}{b}$$

(from the similar process on pages 392 and 393)

$$= X(f)\frac{1}{C_\psi}\int_0^{b_0 f}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1}$$

$$y_2(t) = \frac{1}{b_0^2 C_\psi}x(t)*\phi\left(\frac{-t}{b_0}\right)*\phi\left(\frac{t}{b_0}\right)$$

$$Y_2(f) = X(f)\frac{1}{C_\psi}\Phi(-b_0 f)\Phi(b_0 f) = X(f)\frac{1}{C_\psi}\Phi^*(b_0 f)\Phi(b_0 f)$$

$$= X(f)\frac{1}{C_\psi}\left|\Phi(b_0 f)\right|^2$$

Key process

$$= X(f)\frac{1}{C_\psi}\int_{b_0 f}^{\infty}\frac{|\Psi(f_1)|^2}{|f_1|}df_1$$

Therefore, if $y(t) = y_1(t) + y_2(t)$,

$$\begin{aligned} Y(f) &= Y_1(f) + Y_2(f) \\ &= X(f)\frac{1}{C_\psi}\int_0^{b_0 f}\left|\Psi(f_1)\right|^2 \frac{df_1}{f_1} + X(f)\frac{1}{C_\psi}\int_{b_0 f}^{\infty}\left|\Psi(f_1)\right|^2 \frac{df_1}{f_1} \\ &= X(f)\frac{1}{C_\psi}\int_0^{\infty}\left|\Psi(f_1)\right|^2 \frac{df_1}{f_1} \\ &= X(f) \end{aligned}$$

$$y(t) = x(t)$$

12-I Property

(1) real input $\longrightarrow$ real output

(2) If $x(t) \longrightarrow X_w(a, b)$, then $x(t-\tau) \longrightarrow X_w(a-\tau, b)$,

(3) If $x(t) \longrightarrow X_w(a, b)$, then $x(t/\sigma) \longrightarrow \sqrt{\sigma} X_w(a/\sigma, b/\sigma)$

(4) Parseval's Theory:

$$\int |x(t)|^2 \, dt = \frac{1}{C} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{b^2} |X_w(a,b)|^2 \, da \, db$$

12-J Scalogram

Scalogram 即 Wavelet transform 的絕對值平方

比較 spectrogram

$$Sc_x(a,b)=\left|X_w(a,b)\right|^2=\frac{1}{|b|}\left|\int_{-\infty}^{\infty}x(t)\psi\left(\frac{t-a}{b}\right)dt\right|^2$$

有時，會將 Scalogram 定義成

$$Sc_x(a,\zeta)=\left|X_w\left(a,\frac{\eta}{\zeta}\right)\right|^2$$

$$\eta=\frac{\int_0^{\infty}f\left|\Psi(f)\right|^2df}{\int_0^{\infty}\left|\Psi(f)\right|^2df}$$

↑ zeta

$$\zeta=\frac{\eta}{b}$$

正比於 frequency

$$\Psi(f)=\int_{-\infty}^{\infty}\psi(t)e^{-j2\pi ft}dt$$

12-K Problems

Problems of the continuous WT

(1) hard to implement

(2) hard to find $\phi(t)$

Continuous WT is good in mathematics.

In practical, the discrete WT and the continuous WT with discrete coefficients are more useful.

附錄十二 電機+資訊領域的中研院院士

王兆振 (電子物理學家，1968年當選院士)
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朱經武 (超導體，1987年當選院士)
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崔琦 (量子霍爾效應，1992年當選院士，1998年諾貝爾物理獎)
王佑曾 (資料庫管理理論先驅，1992年當選院士)
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孟懷縈 (無線通信與信號處理，2010年當選院士)
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馬佐平 (微電子，2012年當選院士)
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註：歷年中研院院士當中，屬於電機+資訊相關領域的有44人，佔了全部的 8 %

其中和通信、信號處理、影像處理相關的有10位，大多是2004年以後當選院士

XIII. Continuous WT with Discrete Coefficients

13-A Definition

The parameters *a* and *b* are not chosen arbitrarily.

For example, $a = nb$ $\quad \psi\left(\frac{t-a}{b}\right) = \psi\left(\frac{t}{b} - n\right) = \psi(2^m t - n)$

$a = n2^{-m}$ and $b = 2^{-m}$.

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt \qquad \begin{array}{ll} n \in Z, & n \in (-\infty,\infty) \\ m \in Z, & m \in (-\infty,\infty) \end{array}$$

註：某些文獻把這個式子稱作是 discrete wavelet transform，實際上仍然是 continuous wavelet transform 的特例

- Main reason for constrain *a* and *b* to be $n2^{-m}$ and 2^{-m} :

Easy to implementation

$X_w(n, m)$ can be computed from $X_w(n, m-1)$ by digital convolution.

If A is orthogonal

$AA^T = D$

$A(A^T D^{-1}) = I$

$A^{-1} = A^T D^{-1}$

13-B Inverse Wavelet Transform

$$x(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m/2}\psi\left(2^m t-n\right)X_w\left(n,m\right)$$

$$x(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m/2}\psi\left(2^m t-n\right)2^{m/2}\int_{-\infty}^{\infty}x(t_1)\psi(2^m t_1-n)dt_1$$

$$=\int_{-\infty}^{\infty}\{\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m}\psi\left(2^m t-n\right)\psi(2^m t_1-n)\}x(t_1)dt_1$$

since $x(t)=\int_{-\infty}^{\infty}\delta\left(t-t_1\right)x(t_1)dt_1$

Constraint: $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m}\psi\left(2^m t-n\right)\psi\left(2^m t_1-n\right)=\delta\left(t-t_1\right)$

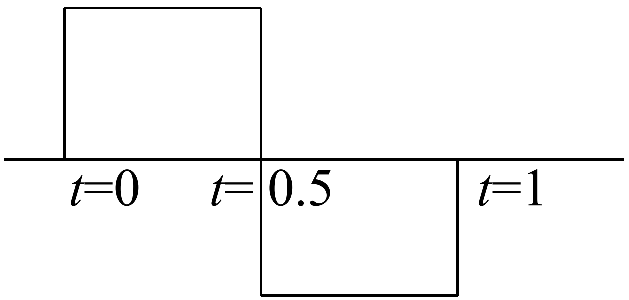
duality ⇓

i.e., $\int_{-\infty}^{\infty}2^{m}\psi\left(2^{m_1} t-n_1\right)\psi\left(2^m t-n\right)dt=\delta\left(m-m_1\right)\delta\left(n-n_1\right)$ ← constraint

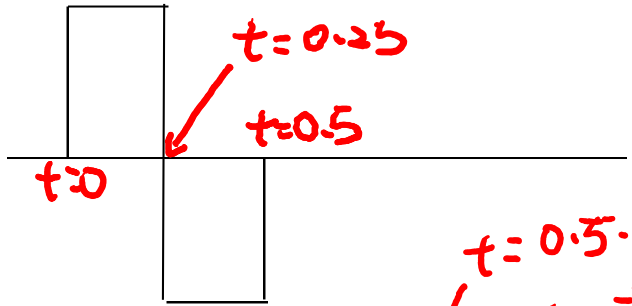
should be satisfied.

13-C Haar Wavelet

$\psi(t)$ mother wavelet
(wavelet function)

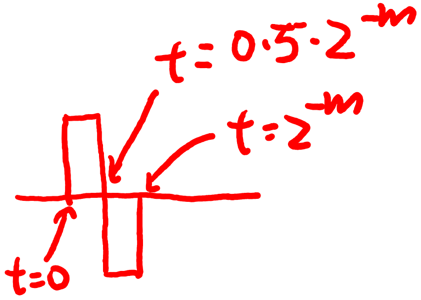


$\psi(2t)$

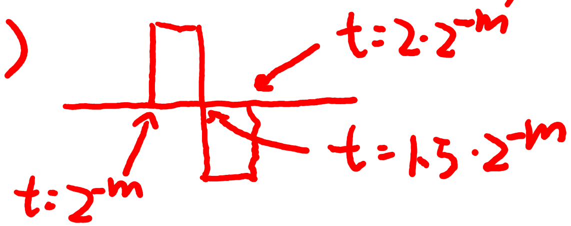


$\psi(2^m t - n) = \psi(2^m(t - n 2^{-m}))$

$\psi(2^m t)$

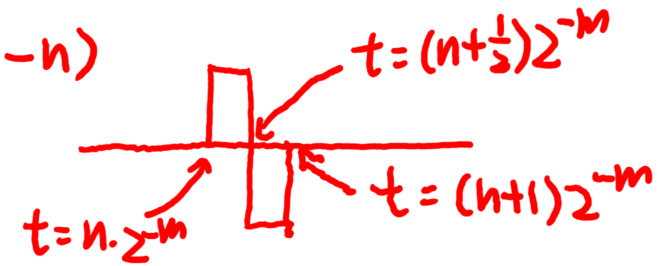


$\psi(2^m t - 1)$



- $\int \psi(t)\,\psi(2^m t)\,dt$ $m > 0$
  $= \int_{-\infty}^{\infty} 1 \cdot \psi(2^m t)\,dt = 0$

$\psi(2^m t - n)$



- $\int \psi(2^m t)\,\psi(2^m t - n)\,dt = 0$ for $n \neq 0$

when $m \geq 1$
$\psi(t)$ is a constant within
$n 2^{-m} < t < (n+1) 2^{-m}$

$\int \psi(t)\,\psi(2^m t - n)\,dt = 0$

The Haar wavelet satisfies

$$2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

Without the loss of generalization, suppose that $m_1 \geq m$. Set

$$t_1 = 2^m t - n \qquad dt_1 = 2^m dt$$

$$2^{m_1} t - n_1 = 2^{m_1 - m} t_1 + 2^{m_1 - m} n - n_1$$

$$2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \int_{-\infty}^{\infty} \psi\left(2^{m_1 - m} t_1 + 2^{m_1 - m} n - n_1\right) \psi\left(t_1\right) dt_1$$

$n_{new} = n_1 - 2^{m_1 - m} n$

if $m_1 = m$

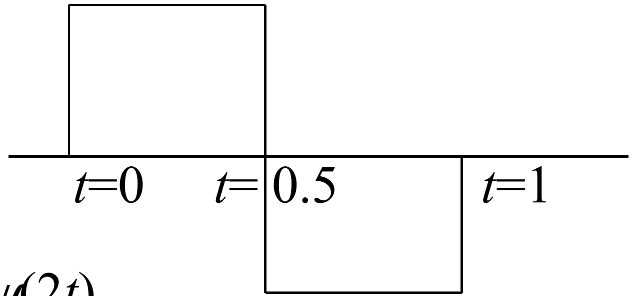
$n_{new} = n_1 - n$

Therefore, we only have to prove that

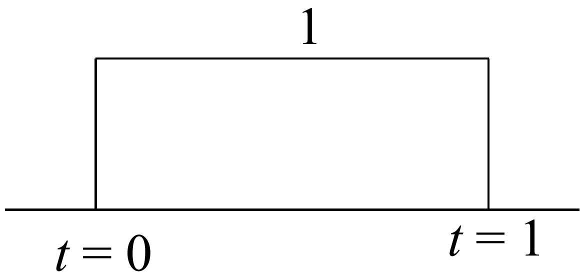
$$\int_{-\infty}^{\infty} \psi\left(2^m t - n\right) \psi\left(t\right) dt = \delta\left(m\right) \delta\left(n\right)$$

for $m \geq 0$.

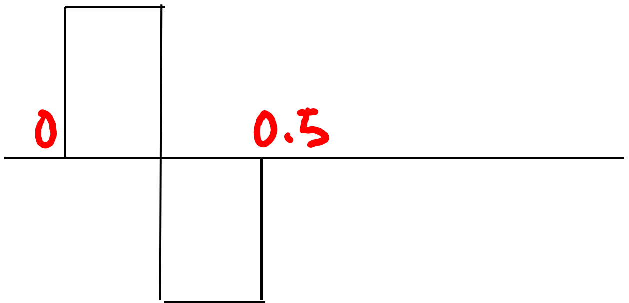
$\psi(t)$ mother wavelet
(wavelet function)



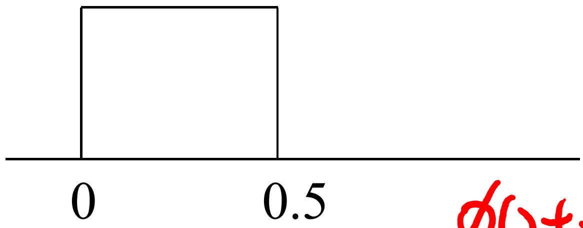
$\phi(t)$ scaling function



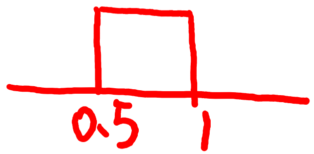
$\psi(2t)$



$\phi(2t)$ $2\phi(2t) = \phi(t) + \psi(t)$



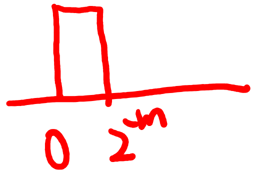
$\phi(2t-1)$

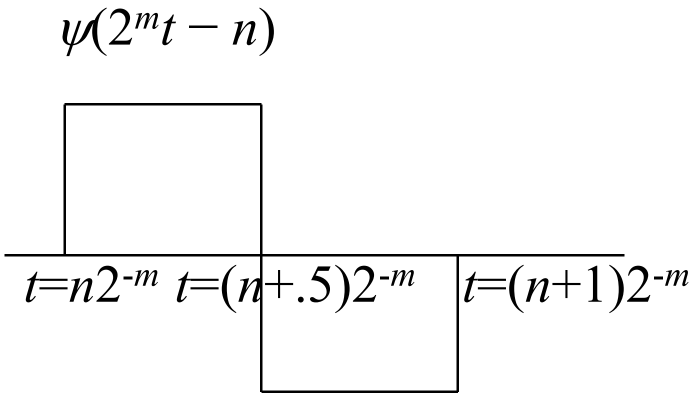


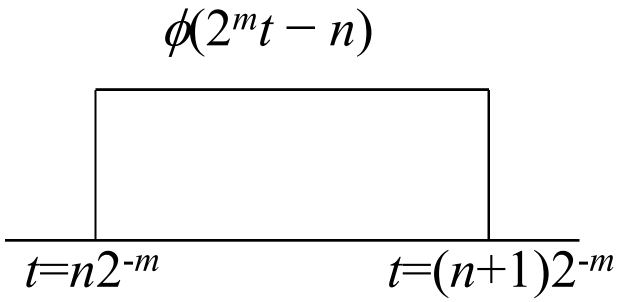
recursive

$$\begin{cases} \phi(t) = \phi(2t) + \phi(2t-1) \\ \psi(t) = \phi(2t) - \phi(2t-1) \end{cases}$$

$\phi(2^m t)$







- Advantages of Haar wavelet

  (1) Simple

  (2) Fast algorithm

  (3) Orthogonal →reversible

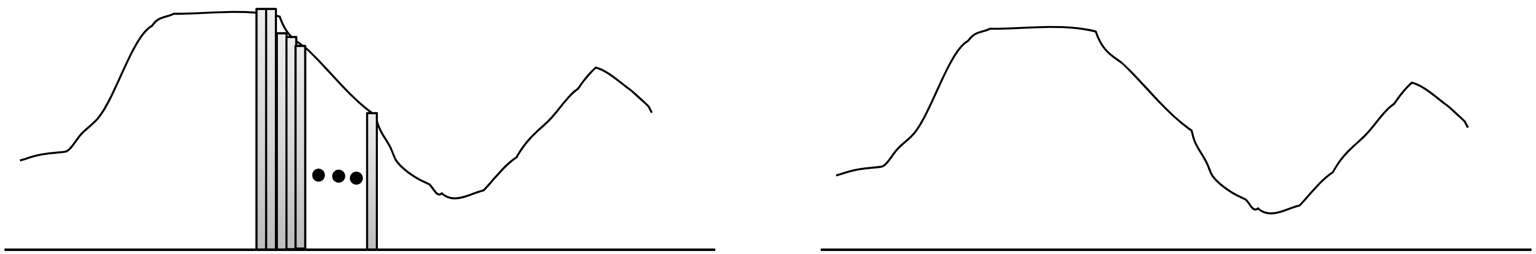
  (4) Compact, real, odd

- Disadvantages of Haar wavelet

  vanishing moment =

Properties

(1) Any function can be expressed by a linear combination of $\phi(t)$, $\phi(2t)$, $\phi(4t)$, $\phi(8t)$, $\phi(16t)$, ……….., and their shifting.



(2) 任何平均為 0 的function 都可以由 $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ………..
所組成

換句話說……. 任何 function 都可以由 constant, $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ……….. 所組成

(4) 不同寬度 (也就是不同 m) 的 wavelet / scaling functions 之間會有一個關係

$$\boxed{\phi(t) = \phi(2t) + \phi(2t-1)}$$

$$\phi(t-n) = \phi(2t-2n) + \phi(2t-2n-1)$$

$$\phi(2^m t-n) = \phi(2^{m+1} t-2n) + \phi(2^{m+1} t-2n-1)$$

$$\boxed{\psi(t) = \phi(2t) - \phi(2t-1)}$$

$$\psi(t-n) = \phi(2t-2n) - \phi(2t-2n-1)$$

$$\psi(2^m t-n) = \phi(2^{m+1} t-2n) - \phi(2^{m+1} t-2n-1)$$

(5) 可以用 m+1 的 coefficients 來算 m 的 coefficients

/kaI/

若 $\chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^m t - n)dt$

$\phi(t) = \phi(2t) + \phi(2t+1)$

$$\chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n)dt + 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}}\left(\chi_w(2n, m+1) + \chi_w(2n+1, m+1)\right)$$

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

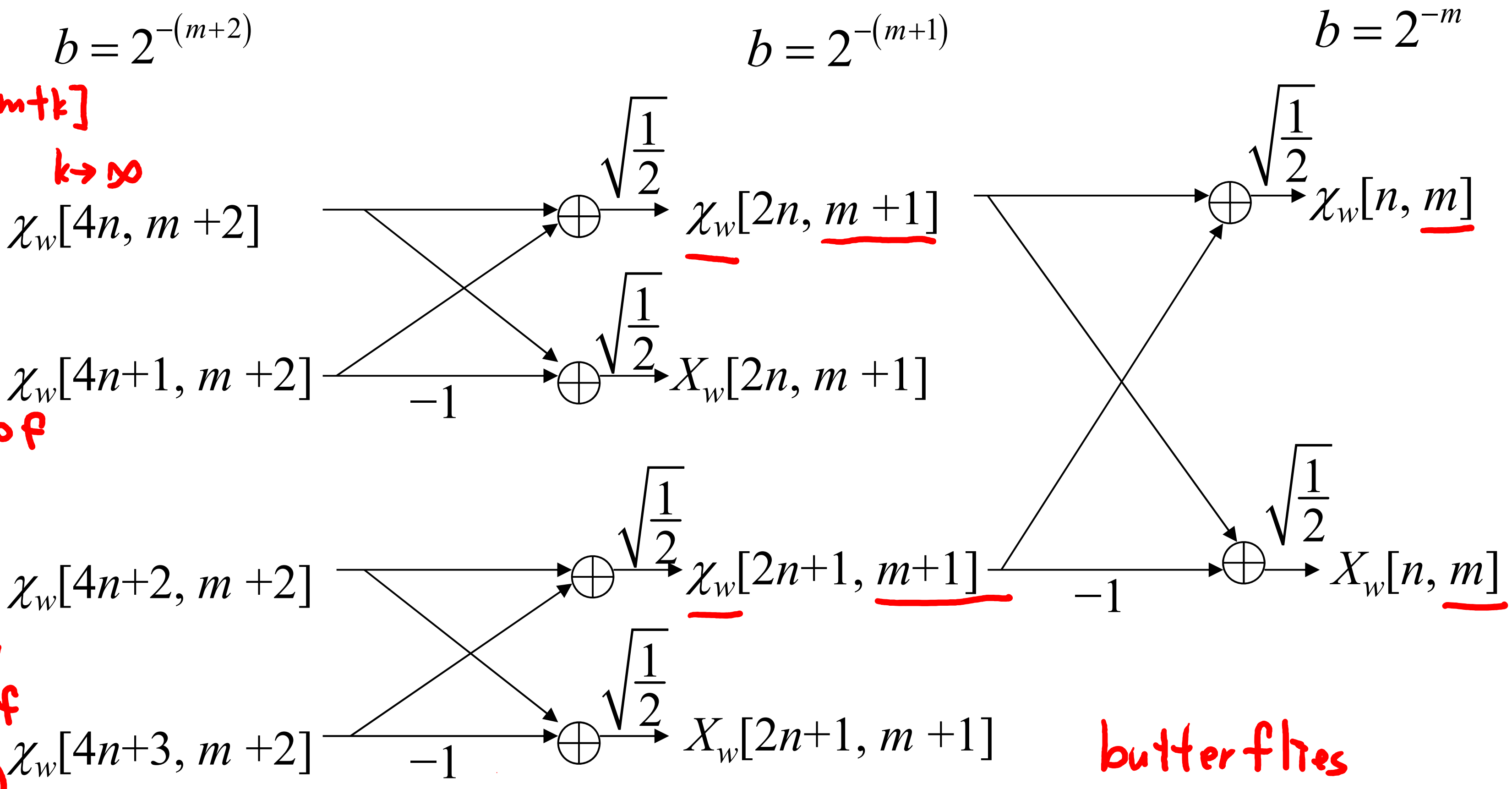
← cont. wavelet transform with disc. coefficients

$\psi(t) = \phi(2t) - \phi(2t-1)$

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n)dt - 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}}\left(\chi_w(2n, m+1) - \chi_w(2n+1, m+1)\right)$$

layer:



structure of multiresolution analysis (MRA)

13-D General Methods to Define the Mother Wavelet and the Scaling Function

Constraints:
- (a) nearly compact support
- (b) fast algorithm
- (c) real
- (d) vanishing moment
- (e) orthogonal

8-point Daubechies wavelet
or 10-point

和 continuous wavelet transform 比較：

(1) compact support 放寬為 "nearly compact support"

(2) 沒有 even, odd symmetric 的限制

(3) 由於只要是 complete and orthogonal, 必定可以 reconstruction

所以不需要 admissibility criterion 的限制

(4) 多了對 fast algorithm 的要求

13-E Fast Algorithm Constraints

Higher and lower resolutions 的 recursive relation 的一般化

$$\phi(t) = 2\sum_k g_k \phi(2t-k)$$ 稱作 dilation equation

$$\psi(t) = 2\sum_k h_k \phi(2t-k)$$

compared to page 411

$k=0, 1$

$g_0 = g_1 = 1/2$

$h_0 = 1/2$, $h_1 = -1/2$

$\psi(t)$: mother wavelet, $\phi(t)$: scaling function

這些關係式成立，才有fast algorithms

g_k : low-frequency sequence

h_k : high-frequency sequence

$$\phi(t) = 2\sum_k g_k \phi(2t-k) \qquad \psi(t) = 2\sum_k h_k \phi(2t-k)$$

If $\quad \chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^m t - n)dt$

then $\quad \chi_w(n,m) = \sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty} x(t) g_k \phi(2^{m+1}t - 2n - k)dt$

$$= 2^{\frac{1}{2}} \sum_k g_k \chi_w(2n+k, m+1)$$

If $\quad X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$

then $\quad X_w(n,m) = \sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty} x(t) h_k \phi(2^{m+1}t - 2n - k)dt$

$$= 2^{\frac{1}{2}} \sum_k h_k \chi_w(2n+k, m+1)$$

(Step 1) convolution

$\tilde{x}_w(n) = 2^{\frac{1}{2}}(\tilde{g}_n * x_w(n, m+1))$

$h[n]*x[n] = \sum_k h[k]x[n-k]$

$$\tilde{\chi}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{g}_k \chi_w(n-k, m+1) \qquad \tilde{g}_k = g_{-k}$$

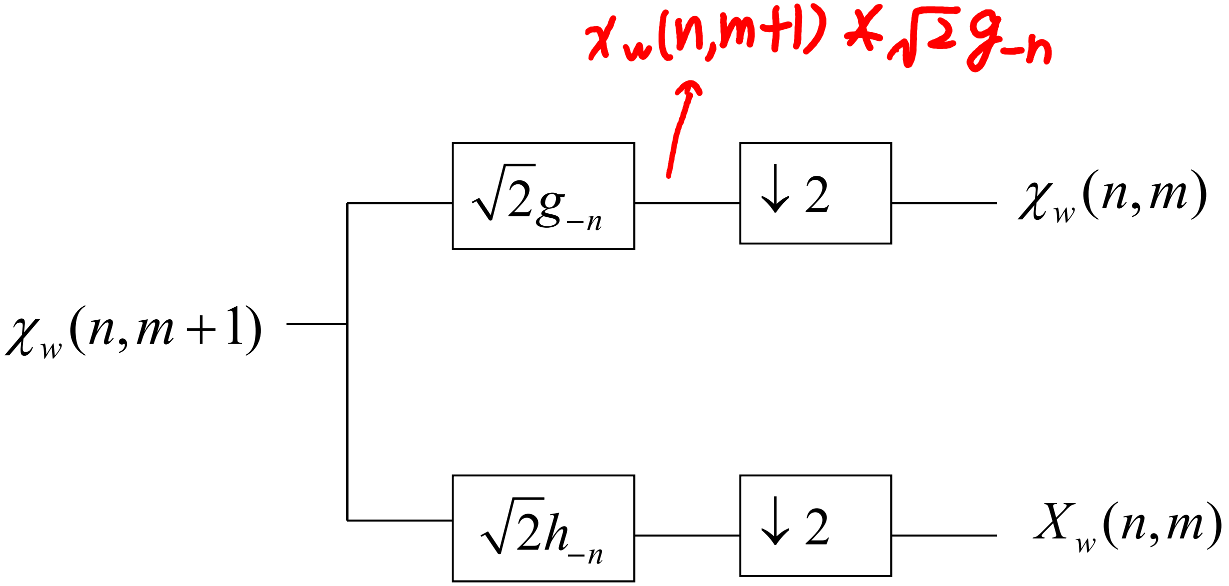
$$\tilde{X}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{h}_k \chi_w(n-k, m+1) \qquad \tilde{h}_k = h_{-k}$$

(Step 2) down sampling

$$\chi_w(n,m) = \tilde{\chi}_w(2n) = 2^{\frac{1}{2}} \sum_k \tilde{g}_k \chi_w[2n-k, m+1] = 2^{\frac{1}{2}} \sum_k g_{-k} \chi_w[2n-k, m+1]$$

$$= 2^{\frac{1}{2}} \sum_k g_k \chi_w(2n+k, m+1)$$

$$X_w(n,m) = \tilde{X}_w(2n)$$


$\chi_w(n,m+1) * \sqrt{2}g_{-n}$
$\sqrt{2}g_{-n}$
$\downarrow 2$
$\chi_w(n,m)$
$\chi_w(n,m+1)$
$\sqrt{2}h_{-n}$
$\downarrow 2$
$X_w(n,m)$

m 越大，越屬於細節

- To satisfy $\phi(t) = 2\sum_k g_k \phi(2t-k)$,

$$\phi(t/2) = 2\sum_k g_k \phi(t-k) = 2\sum_k g_k \delta(t-k) * \phi(t)$$

FT ↓ FT ↓

$$2\Phi(2f) = 2G(f)\Phi(f)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

where $\Phi(f) = FT\left[\phi(t)\right] = \int_{-\infty}^{\infty} \phi(t) e^{-j2\pi f t} dt$

$$G(f) = FT\left[\sum_k g_k \delta(t-k)\right]$$

$$= \sum_k g_k \int_{-\infty}^{\infty} \delta(t-k) e^{-j2\pi f t} dt$$

$$= \sum_k g_k e^{-j2\pi f k}$$

$G(0) = \sum_k g_k$

$\Phi(f)$ 是 $\phi(t)$ 的 continuous Fourier transform

$G(f)$ 是 $\{g_k\}$ 的 discrete time Fourier transform

$$\Phi(f)=G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right) \qquad \Phi\left(\frac{f}{2}\right)=G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right)$$

$$\Phi(f)=G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right)=G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)G\left(\frac{f}{8}\right)\Phi\left(\frac{f}{8}\right)=\cdots\cdots$$

$$\Phi(f)=\Phi\left(\frac{f}{2^\infty}\right)\prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)=\Phi(0)\prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)$$

↑ 連乘

$$\Phi(0)=\int_{-\infty}^{\infty}\phi(t)\,dt$$ (可以藉由 normalization, 讓 $\Phi(0) = 1$)

$$\boxed{\Phi(f)=\prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)}$$

constraint 1

若 $G(f)$ 決定了，則 $\Phi(f)$ 可以被算出來

FT of $\phi(t)$

discrete-time Fourier transform of g_k

$G(f)$: 被稱作 generating function

- 同理

$$\psi(t) = 2\sum_k h_k \phi(2t-k)$$

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi f t} dt$$

$$\psi(t/2) = 2\sum_k h_k \phi(t-k)$$

$$\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$H(f) = \sum_k h_k e^{-j2\pi f k}$$

$$\boxed{\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)}$$

constraint 2

- 另外，由於

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$\Phi(0) = G(0)\Phi(0) \qquad (f = 0 \text{ 代入})$$

$$\boxed{G(0) = 1}$$ 必需滿足

constraint 3

13-F Real Coefficient Constraints

Since $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

If $G(f) = G^*(-f)$ $H(f) = H^*(-f)$ are satisfied,

constraint 4 constraint 5

then $\Phi(f) = \Phi^*(-f)$, $\Psi(f) = \Psi^*(-f)$, and $\phi(t)$, $\psi(t)$ are real.

Note: If these constraints are satisfied, g_k, h_k on page 415 are also real.

13-G Vanishing Moment Constraint

If $\psi(t)$ has p vanishing moments,

$$\int_{-\infty}^{\infty} t^k \psi(t)\, dt = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$$

Since $FT\left[t^k \psi(t)\right] = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi(f)$

$$\int_{-\infty}^{\infty} x(t)\, dt = X(0) \quad \text{if} \quad X(f) = FT(x(t))$$

$$\int_{-\infty}^{\infty} t^k \psi(t)\, dt = 0 \Longrightarrow FT\left[t^k \psi(t)\right]\Big|_{f=0} = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi(f)\Big|_{f=0} = 0$$

Therefore, $\frac{d^k}{df^k}\Psi(f)\Big|_{f=0} = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$

$$\left.\frac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$$

Since $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

$$\frac{d^k}{df^k}\Psi(f) = \sum_{n=0}^{k}\binom{k}{n}\frac{d^n}{df^n}H\left(\frac{f}{2}\right)\frac{d^{k-n}}{df^{k-n}}\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

$$= \sum_{n=0}^{k}\binom{k}{n}\frac{1}{2^n}\frac{d^n}{df^n}H(f)\frac{d^{k-n}}{df^{k-n}}\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

if $\left.\frac{d^k}{df^k}H(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$ is satisfied,

constraint 6

then $\left.\frac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$ are satisfied

and the wavelet function has p vanishing moments.

13-H Orthogonality Constraints

- orthogonality constraint:

$$\int_{-\infty}^{\infty} 2^{m} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^{m} t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

$\psi(t)$: wavelet function

If the above equality is satisfied,

forward wavelet transform:

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

inverse wavelet transform:

$$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w\left(m,n\right)$$

(much easier for inverse) C = mean of $x(t)$

(證明於後頁)

If $$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w(m,n)$$

and $$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta(m - m_1)\delta(n - n_1),$$

then $$2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} \left[C + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \psi\left(2^{m_1} t - n_1\right) X_w(m_1, n_1) \right] \psi(2^m t - n) dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} C \psi(2^m t - n) dt + 2^{m/2} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi(2^m t - n) dt X_w(m_1, n_1)$$

$$= 0 + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \delta(m_1 - m)\delta(n_1 - n) X_w(m_1, n_1)$$

$$= X_w(m, n)$$

due to $\int_{-\infty}^{\infty} \psi(t) dt = 0$

Therefore, $2^{m/2}\int_{-\infty}^{\infty}x(t)\psi(2^m t-n)dt$ is the inverse operation of

$$C+\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m/2}\psi\left(2^m t-n\right)X_w\left(m,n\right) \qquad \#$$

※ 要滿足

$$\int_{-\infty}^{\infty} 2^{m} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^{m} t - n\right) dt = \delta(m - m_1) \delta(n - n_1)$$

之前，需要滿足以下三個條件

(1) $\int_{-\infty}^{\infty} \psi(t - n_1) \psi(t - n) dt = \delta(n_1 - n)$ for mother wavelet

這個條件若滿足， $\int_{-\infty}^{\infty} 2^{m} \psi\left(2^{m} t - n_1\right) \psi\left(2^{m} t - n\right) dt = \delta(n - n_1)$

對所有的 m 皆成立

(2) $\int_{-\infty}^{\infty} \phi(t - n_1) \phi(t - n) dt = \delta(n_1 - n)$ for scaling function

嚴格來說，這並不是必要條件，但是可以簡化 第 (3) 個條件的計算

(3) $\int_{-\infty}^{\infty} \psi(t-n_1)\psi(2^{-k}t-n)dt = 0$ for any n, n_1 if $k > 0$

若 (1) 和 (3) 的條件滿足，則

$$\boxed{\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t-n_1)\psi(2^m t-n)dt = \delta(m-m_1)\delta(n-n_1)}$$

也將滿足

(Proof): Set $t_1 = 2^m t$, $dt_1 = 2^m dt$

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t-n_1)\psi(2^m t-n)dt = \int_{-\infty}^{\infty} \psi(2^{m_1-m}t_1-n_1)\psi(t_1-n)dt_1$$

If (3) is satisfied,

$$\int_{-\infty}^{\infty} 2^m \psi(2^{m_1}t-n_1)\psi(2^m t-n)dt = 0 \quad \text{when } m \neq m_1$$

In the case where $m = m_1$, if (1) is satisfied, then

$$\int_{-\infty}^{\infty} 2^m \psi(2^m t-n_1)\psi(2^m t-n)dt = \int_{-\infty}^{\infty} \psi(t_1-n_1)\psi(t_1-n)dt_1 = \delta(n_1-n)$$

#

- 由 Page 428 的條件 (1)

$$\int_{-\infty}^{\infty}\psi(t-n_1)\psi(t-n)dt$$

$$=\int_{-\infty}^{\infty}e^{-j2\pi n_1 f}\Psi(f)e^{j2\pi n f}\Psi^*(f)df$$

Parseval's theorem

$$\int_{-\infty}^{\infty}x(t)y^*(t)dt=\int_{-\infty}^{\infty}X(f)Y^*(f)df$$

$$=\int_{-\infty}^{\infty}e^{j2\pi(n-n_1)f}\Psi(f)\Psi^*(f)df$$

$$=\sum_{p=-\infty}^{\infty}\int_0^1 e^{j2\pi(n-n_1)(f'+p)}\Psi(f'+p)\Psi^*(f'+p)df'$$

$$=\int_0^1 e^{j2\pi(n-n_1)f'}\sum_{p=-\infty}^{\infty}|\Psi(f'+p)|^2 df'=\delta(n-n_1)$$

$$e^{j2\pi(n-n_1)(f'+p)}=e^{j2\pi(n-n_1)f'}$$

if p is an integer

Therefore,

$$\int_0^1 e^{-j2\pi n_2 f'}\sum_{p=-\infty}^{\infty}|\Psi(f'+p)|^2 df=\delta(-n_2)=\delta(n_2)$$

$$\boxed{\sum_{p=-\infty}^{\infty}|\Psi(f'+p)|^2=1}$$

for all f' should be satisfied

- 同理，由 Page 428 的條件 (2)

$$\int_{-\infty}^{\infty} \phi(t-n_1)\phi(t-n)\,dt = \delta(n_1 - n)$$ for scaling function

↓ 推導過程類似 page 430

$$\boxed{\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1}$$ for all f should be satisfied

衍生的條件：將 $\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty} |\Psi(f+p)|^2 = 1$

(page 430)

$$\sum_{p=-\infty}^{\infty} \left| H\left(\frac{f}{2}+\frac{p}{2}\right)\Phi\left(\frac{f}{2}+\frac{p}{2}\right)\right|^2 = 1$$

$$\sum_{q=-\infty}^{\infty} \left| H\left(\frac{f}{2}+q\right)\Phi\left(\frac{f}{2}+q\right)\right|^2 + \sum_{q=-\infty}^{\infty} \left| H\left(\frac{f}{2}+q+\frac{1}{2}\right)\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 = 1$$

因為 h_k 是 discrete sequence, $H(f)$ 是 h_k 的 discrete-time Fourier transform

$$H(f) = H(f+1) = H(f+2) = \cdots\cdots$$

$$\left| H\left(\frac{f}{2}\right)\right|^2 \sum_{q=-\infty}^{\infty} \left|\Phi\left(\frac{f}{2}+q\right)\right|^2 + \left| H\left(\frac{f}{2}+\frac{1}{2}\right)\right|^2 \sum_{q=-\infty}^{\infty} \left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 = 1$$

$$|H\left(\frac{f}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2}+q\right)|^2 + |H\left(\frac{f}{2}+\frac{1}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)|^2 = 1$$

因為 $\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$ for all f

(page 430 的條件)

$$|H\left(\frac{f}{2}\right)|^2 + |H\left(\frac{f}{2}+\frac{1}{2}\right)|^2 = 1$$

$$|H(f)|^2 + |H\left(f+\frac{1}{2}\right)|^2 = 1$$

constraint 7

同理，將 $\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$

(page 430)

經過運算可得

$$|G(f)|^2 + \left|G\left(f+\frac{1}{2}\right)\right|^2 = 1$$

constraint 8

- Page 429 條件 (3) 的處理

由於

$\psi\left(2^{-k}t-n\right)$ 是 $\phi\left(2^{-k+1}t-n_1\right)$ 的 linear combination

$\phi\left(2^{-k+1}t-n_1\right)$ 是 $\phi\left(2^{-k+2}t-n_2\right)$ 的 linear combination

$\phi\left(2^{-k+2}t-n_2\right)$ 是 $\phi\left(2^{-k+3}t-n_3\right)$ 的 linear combination

$$\psi(t)=2\sum_k h_k\phi(2t-k)$$

$$\phi(t)=2\sum_k g_k\phi(2t-k)$$

⋮

⋮

$\phi\left(2^{-1}t-n_{k-1}\right)$ 是 $\phi\left(t-n_k\right)$ 的 linear combination

所以

$\psi\left(2^{-k}t-n\right)$ 必定可以表示成 $\phi\left(t-n_k\right)$ 的 linear combination

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k} b_{n_k}\phi\left(t-n_k\right)$$

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k} b_{n_k}\phi\left(t-n_k\right)$$

所以，若 $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\phi\left(t-n_k\right)dt=0$ for any n_1, n_k 可以滿足

則 $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\psi\left(2^{-k}t-n\right)dt=0$ for any n_1, n_k 必定能夠成立

Page 429 條件 (3) 可改寫成

$$\boxed{\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\phi\left(t-n_k\right)dt=0}$$

$$\int_{-\infty}^{\infty}\psi\left(t\right)\phi\left(t-\tau\right)dt=0$$ (將 $t-n_1$ 變成 t, $\tau= n_k-n_1$)

$$\int_{-\infty}^{\infty}\Psi\left(f\right)\Phi^*\left(f\right)e^{j2\pi\tau f}\,df=0$$ (from Parseval's theorem)

$$\int_{-\infty}^{\infty} \Psi(f)\Phi^*(f)e^{j2\pi\tau f}\,df = 0$$

Since $\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ $\quad \Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$

$$\int_{-\infty}^{\infty} H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\left|\Phi\left(\frac{f}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

$$\sum_{p=-\infty}^{\infty}\int_0^1 H\left(\frac{f+p}{2}\right)G^*\left(\frac{f+p}{2}\right)\left|\Phi\left(\frac{f+p}{2}\right)\right|^2 e^{j2\pi\tau(f+p)}\,df = 0$$

$e^{j2\pi\tau(f+p)} = e^{j2\pi\tau f}$ (since from page 436, τ is an integer)

$$\sum_{q=-\infty}^{\infty}\int_0^1 H\left(\frac{f}{2}+q\right)G^*\left(\frac{f}{2}+q\right)\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\,df$$

$$+\sum_{q=-\infty}^{\infty}\int_0^1 H\left(\frac{f}{2}+q+\frac{1}{2}\right)G^*\left(\frac{f}{2}+q+\frac{1}{2}\right)\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

Since $H(f)=H(f+1)=H(f+2)=\cdots\cdots$

$$G(f)=G(f+1)=G(f+2)=\cdots\cdots$$

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\int_0^1\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\,df$$

$$+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)\int_0^1\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\,df=0$$

Since $\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2=1$ for all f (page 430)

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)=0$$

$$H(f)G^*(f)+H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right)=0$$

constraint 9

13-I Nine Constraints

整理： 設計 mother wavelet 和 scaling function 的九大條件
(皆由 page 414 的 constraints 衍生而來)

(1) $\Phi(f)=\prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)$ for fast algorithm , page 420

(2) $\Psi(f)=H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty}G\left(\frac{f}{2^q}\right)$ for fast algorithm , page 421

(3) $G(0)=1$ for fast algorithm , page 421

(4) $H(f)=H^*(-f)$ for real , page 422

(5) $G(f)=G^*(-f)$ for real , page 422

(6) $\left.\frac{d^k}{df^k}H(f)\right|_{f=0}=0$ for p vanishing moments , page 424

for $k = 0, 1, \ldots, p$-1

(7) $|H(f)|^2 + \left|H\left(f+\frac{1}{2}\right)\right|^2 = 1$ for orthogonal , page 433

(8) $|G(f)|^2 + \left|G\left(f+\frac{1}{2}\right)\right|^2 = 1$ for orthogonal , page 434

(9) $H(f)G^*(f) + H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right) = 0$ for orthogonal , page 438

$G(f)$, $H(f)$ are the discrete-time Fourier transform of $\{g_k\}$, $\{h_k\}$ on page 415.

- Simplification

Let

$$\left|H(f)\right| = \left|G(f+1/2)\right|$$

$$G(f) = \sum_k g_k e^{-j2\pi fk}, \quad H(f) = \sum_k h_k e^{-j2\pi fk}$$

$$G(f) = G(f+1), \quad H(f) = H(f+1)$$

Low frequency: around $f = 0$

High frequency: around $f = \pm 1/2$

Specially, if we set that

$$h_k = (-1)^k g_{1-k} \qquad\qquad H(f) = -e^{-j2\pi f} G^*(f+1/2)$$

when the following constraints are satisfied:

$$|G(f)|^2 + \left|G\left(f+\frac{1}{2}\right)\right|^2 = 1$$

$$G(f) = G^*(-f) \qquad\qquad \text{(條件 (5), (8) 滿足)}$$

then

$$|H(f)|^2 + \left|H\left(f+\frac{1}{2}\right)\right|^2 = \left|G\left(f+\frac{1}{2}\right)\right|^2 + |G(f)|^2 = 1$$

$$H(f)G^*(f) + H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right)$$

$$= -e^{-j2\pi f} G^*\left(f+\frac{1}{2}\right)G^*(f) - e^{-j2\pi(f+\frac{1}{2})} G^*(f)G^*\left(f+\frac{1}{2}\right)$$

$$= -e^{-j2\pi f} G^*\left(f+\frac{1}{2}\right)G^*(f) + e^{-j2\pi f} G^*(f)G^*\left(f+\frac{1}{2}\right) = 0$$

$$H^*(-f) = -e^{-j2\pi f} G(-f+1/2) = -e^{-j2\pi f} G^*(f-1/2) = H(f)$$

條件 (4), (7), (9) 也將滿足

整理：設計 mother wavelet 和 scaling function 的幾個要求 (簡化版)

(1) $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm

(2) $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm

(3) $G(0) = 1$ for fast algorithm

(4) $G(f) = G^*(-f)$ for real

(5) $\left.\frac{d^k}{df^k} H(f)\right|_{f=0} = 0$ for p vanishing moments

for $k = 0, 1, \ldots, p\text{-}1$

$G(f+\frac{1}{2}) = G(f-\frac{1}{2})$
$= G^*(\frac{1}{2}-f)$

$f \in (0, \frac{1}{4})$ $f+\frac{1}{2} \in (\frac{1}{2}, \frac{3}{4})$

if $f \in (0, \frac{1}{4})$, then $\frac{1}{2} - f \in (\frac{1}{4}, \frac{1}{2})$

(6) $|G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1$ for orthogonal

(7) $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$

key of designing the continuous wavelet transform with discrete coefficients: $G(f)$

13-J Design Process

設計時，只要 $G(f)$ $(0 \le f \le 1/4)$ 決定了，mother wavelet 和scaling function 皆可決定

$G(f)$: 被稱作 generating function

Design Process (設計流程):

(Step 1): 給定 $G(f)$ $(0 \le f \le 1/4)$，滿足以下的條件

(a) $G(0) = 1$

(b) $\left.\dfrac{d^k}{df^k} G(f)\right|_{f=\frac{1}{2}} = 0$ for $k = 0, 1, 2, \ldots, p\text{-}1$

$G(f) = \sum_k g_k e^{-j2\pi fk}$

$G(f+1) = \sum_k g_k e^{-j2\pi(f+1)k} = \sum_k g_k e^{-j2\pi fk} = G(f)$

$f \in (-\infty, \infty)$

$\to f \in (-1/2, 1/2)$

$\to f \in (0, 1/2)$

$\to f(0, 1/4)$

(Step 2) 由 $G(f) = G^*(-f)$ 決定 $G(f)$ $(-1/4 \le f < 0)$

(Step 3) 由 $|G(f)|^2 + |G\left(f + \frac{1}{2}\right)|^2 = 1$ 決定 $G(f)$ $(1/4 < f < 1/2)$

$(-1/2 < f < -1/4)$

再根據 $G(f) = G(f+1)$，決定所有的 $G(f)$ 值

(Step 4) 由 $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$ 決定 $H(f)$

(Step 5) 由 $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$

$$\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$ 決定 $\Phi(f)$, $\Psi(f)$

註：(1) 當 Step 1 的兩個條件滿足，由於 $|G(f)|^2 + |G(f+1/2)|^2 = 1$

$$\left.\frac{d^k}{df^k}G(f)\right|_{f=1/2} = 0 \quad \text{for } k = 0, 1, 2, \ldots, p\text{-}1$$

又由於 $H(f) = -e^{-j2\pi f}G^*(f+1/2)$

equivalent

$$\left.\frac{d^k}{df^k}H(f)\right|_{f=0} = 0 \quad \text{for } k = 0, 1, 2, \ldots, p\text{-}1$$

(2) $|G(f)|^2 + |G(f+1/2)|^2 = 1 \qquad |G(f)|^2 = |G(-f)|^2$

所以當 $G(f)$ $(0 \le f \le 1/4)$ 給定，$|G(f)|$ 有唯一解

(3) 對於離散信號而言，$G(f) = G(f+1)$

有意義的頻率範圍為 $-1/2 < f < 1/2$

$$G(f) = \sum_k g_k e^{-j2\pi f k}$$

13-K Several Continuous Wavelets with Discrete Coefficients

(1) Haar Wavelet = Daubechies wavelet 2 points (p=1)

$g[0] = 1,\ g[1] = 1$ $\qquad G(f) = 1 + \exp(-j2\pi f)$

$h[0] = 1,\ h[1] = -1$ $\qquad H(f) = 1 - \exp(-j2\pi f)$

$G(1/2) = \sum_k g_k e^{-j\pi k}$

$G(f) = \sum_k g_k e^{-j2\pi k f}$

或

$G(1/2) = (1-1)/2 = 0$ $\qquad G'(f) = -j\pi f\, e^{-j2\pi f}$

$g[0] = 1/2,\ g[1] = 1/2$ $\qquad G(f) = [1 + \exp(-j2\pi f)]/2$

$G'(1/2) = j\frac{\pi}{2} \neq 0$

$h[0] = 1/2,\ h[1] = -1/2$ $\qquad H(f) = [1 - \exp(-j2\pi f)]/2$

vanishing moment = ? 1

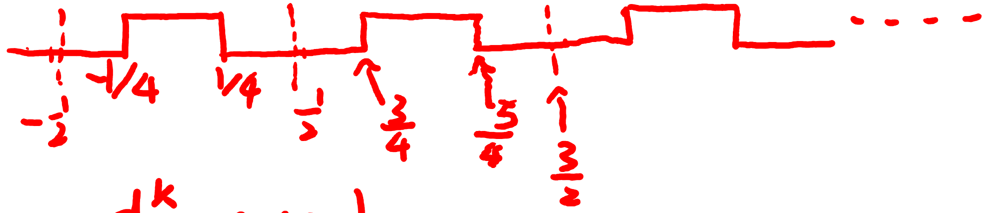
Daubechies with infinite points vanish moment → ∞

(2) Sinc Wavelet

$G(f) = 1 \quad \text{for } |f| \le 1/4$

$G(f) = 0 \quad \text{otherwise}$

$G(f) = G(f+1)$



$\frac{d^k}{df^k} G(f)\Big|_{f=\frac{1}{2}} = 0\Big|_{f=1/2} = 0$ for all k

vanishing moment = ?

(3) 4-point Daubechies Wavelet

多貝西

$$g_k : \left[\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8} \right]$$

$G(1/2) = \sum g_k (-1)^k$

$= g_0 - g_1 + g_2 - g_3 = 0$

$G'(f) = \sum_k g_k (-j2\pi k) e^{-j2\pi f k}$

$G'(1/2) = -j2\pi \sum_k g_k k (-1)^k$

$= -j2\pi(-g_1 + 2g_2 - 3g_3) = 0$

p=1 Haar ⟵ Daubechies 2p points, vanish moment = p ⟶ p → ∞ sinc

vanishing moment = ? 2

vanishing moment VS the number of coefficients

p 2p

Daubechies Wavelet:

It can be viewed as a generalization of the Haar wavelet.

(Haar wavelet = 2-point Daubechies wavelet).

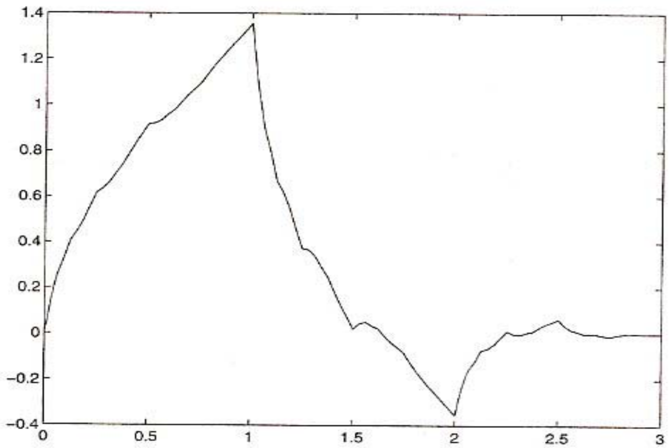
The $2p$-point Daubechies wavelet has the vanish moment of p.

[Ref]: Ingrid Daubechies: *Ten Lectures on Wavelets*, SIAM 1992.

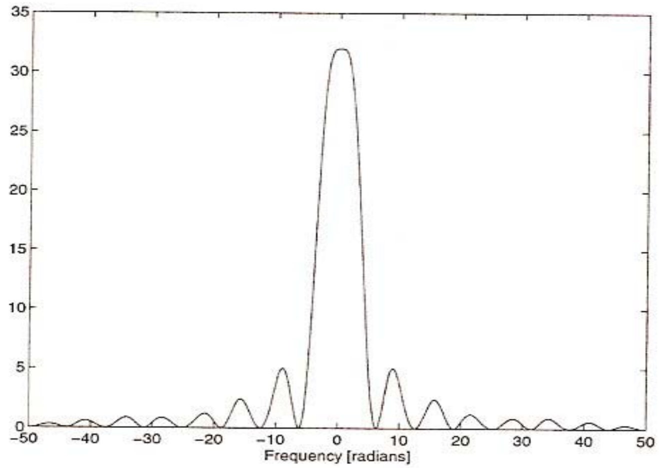
[Ref]: "Daubechies wavelets", Encyclopedia of Mathematics, EMS Press, 2001, https://encyclopediaofmath.org/index.php?title=Daubechies_wavelets.

Ingrid Daubechies
https://en.wikipedia.org/wiki/Ingrid_Daubechies

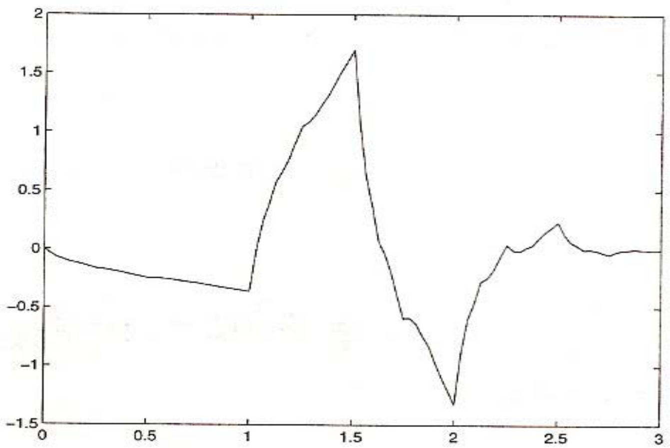
From: S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice Hall, N.J., 1996.



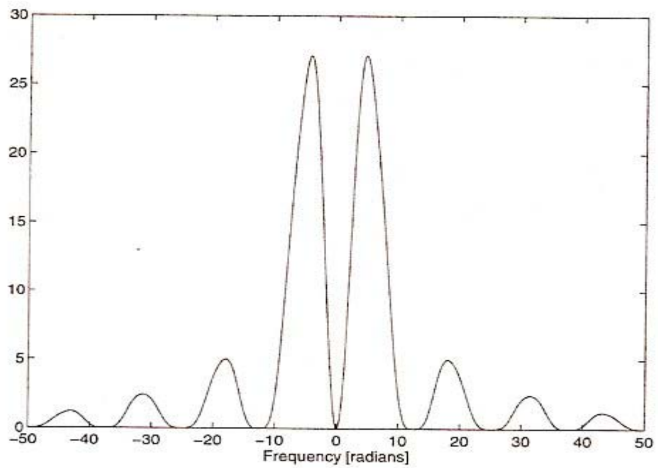
(a) Scaling function $\phi(t)$



(b) $|\Phi(\omega)|$



(c) Daubechies wavelet $\psi(t)$



(d) $|\Psi(\omega)|$

13-L Continuous Wavelet with Discrete Coefficients 優缺點

• Advantages:

(1) Fast algorithm for MRA

(2) Non-uniform frequency analysis

$$\psi\left(2^m t - n\right) \xrightarrow{\text{FT}} 2^{-m} e^{-j2\pi n 2^{-m} f} \Psi\left(2^{-m} f\right)$$

(3) Orthogonal

- Disadvantages:

(a) 無限多項連乘

(b) problem of initial

$\chi_w(n,m), \quad X_w(n,m)$ 皆由 $\chi_w(n,m+1)$ 算出

$\chi_w(n,m)\big|_{m\to\infty}$ 如何算

(c) 難以保證 compact support

(d) 仍然太複雜

附錄十三 幾種常見的影像壓縮格式

(1) JPEG: 使用 discrete cosine transform (DCT) 和 8×8 blocks
是當前最常用的壓縮格式 (副檔名為 *.jpg 的圖檔都是用 JPEG 來壓縮)

可將圖檔資料量壓縮至原來的 1/8 (對灰階影像而言) 或 1/16 (對彩色影像而言)

(2) JPEG2000: 使用 discrete wavelet transform (DWT)
壓縮率是 JPEG 的 5 倍左右

(3) JPEG-LS: 是一種 lossless compression
壓縮率較低，但是可以完全重建原來的影像

(4) JPEG2000-LS: 是 JPEF2000 的 lossless compression 版本

(5) JBIG: 針對 bi-level image (非黑即白的影像) 設計的壓縮格式

(6) GIF: 使用 LZW (Lempel–Ziv–Welch) algorithm (類似字典的建構)

適合卡通圖案和動畫製作，lossless

(7) PNG: 使用 LZ77 algorithm (類似字典的建構，並使用 sliding window)

lossless

(8) JPEG XR (又稱 HD Photo): 使用 Integer DCT，lossless

在 lossy compression 的情形下壓縮率可和 JPEG 2000 差不多

(9) TIFF: 使用標籤，最初是為圖形的印刷和掃描而設計的，lossless