IV. Implementation

IV-A Method 1: Direct Implementation

以 STFT 為例

$$X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Converting into the Discrete Form

$t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_t$

$$X(n\Delta_t, m\Delta_f)=\sum_{p=-\infty}^{\infty} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

Suppose that $w(t) \cong 0$ for $|t| > B$, $B/\Delta_t = Q$

$\Delta_t\Delta_f = \frac{1}{N}$

$$X(\underset{T}{n\Delta_t}, \underset{F}{m\Delta_f})=\sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x(p\Delta_t)e^{-j2\pi pm\Delta_t\Delta_f}\Delta_t$$

$w((n-p)\Delta t) \cong 0$

for $|n-p| > \frac{B}{\Delta t}$

$|p-n| > Q$

Therefore, only when

$-Q \leq p-n \leq Q$

$w((n-p)\Delta t)$ is

nonzero

Problem：對 scaled Gabor transform 而言，$Q = ?$

$w(t) = e^{-\pi\sigma t^2}$ $B = \frac{1.9143}{\sqrt{\sigma}}$, $Q = \frac{1.9143}{\sqrt{\sigma}\,\Delta t}$

• **Constraint for Δ_t** (The only constraint for the direct implementation method)

To avoid the aliasing effect,

$\Delta_t < 1/2\Omega$, Ω is the bandwidth of ? $x(\tau)w(t-\tau)$

suppose that
bandwidth of x : Ω_x
bandwidth of w : Ω_w

$\Omega = \Omega_x + \Omega_w$

$w(\tau) \xrightarrow{\mathcal{F}} W(f)$
$w(-\tau) \to W(-f)$
$w(t-\tau) \to e^{-j2\pi tf} W(-f)$

There is no constraint for Δ_f when using the direct implementation method.

Four Implementation Methods

(1) Direct implementation

Complexity: $TF(2Q+1) \cong 2TFQ$ $\quad O(TFQ)$

假設 t-axis 有 T 個 sampling points, f-axis 有 F 個 sampling points

(2) FFT-based method

unbalanced form

Complexity: $O(TN\log N)$ $\quad O(\frac{T}{S}N\log N)$

(3) FFT-based method with recursive formula

Complexity: $O(TF)$

(4) Chirp-Z transform method

Complexity: $O(TN\log N)$

4 (A) Direct Implementation

Advantage： simple, flexible   the only constraint: $\Delta t < \frac{1}{2(\Omega_x + \Omega_w)}$

Disadvantage ： higher complexity

2 (B) DFT-Based Method

Advantage ： lower complexity

Disadvantage ： with some constraints

$\frac{1}{\Delta t \Delta f} = N$ must be an integer

$N \geq 2Q+1 = 2\frac{B}{\Delta t}+1$

$\Delta t < \frac{1}{2(\Omega_x + \Omega_w)}$

1 (C) Recursive Method

Advantage ： least complexity

Disadvantage ： (i) only suitable for the rectangular window

(ii) accumulated error

3 (D) Chirp Z Transform

Advantage ： no constraint (except for $\Delta t < \frac{1}{2(\Omega_x + \Omega_w)}$)

Disadvantage ： complexity is higher than DFT-~~based~~ method

IV-B Method 2: FFT-Based Method

Constraints : (i) $\Delta_t \Delta_f = 1/N$,

(ii) $N = 1/(\Delta_t \Delta_f) \geqq 2Q + 1$: ($\Delta_t \Delta_f$是整數的倒數)

$$\boxed{X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)\, x(p\Delta_t)\, e^{-j\frac{2\pi pm}{N}} \Delta_t}$$

$w((n-q-n+Q)\Delta_t)x((q+n-Q)\Delta_t)$

Note that the input of the FFT has less than N points (others are set to zero).

Standard form of the DFT $\quad Y[m] = \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi mn}{N}}$

$O(N^2) \rightarrow O(N\log_2 N)$

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}}, \quad q = p-(n-Q) \rightarrow p = (n-Q)+q$$

$n-Q \le p \le n+Q$

$0 \le q \le 2Q$

window $\times$ $x(n)$ $(n-Q \le n \le n+Q)$

$w((q-Q)\Delta_t)$ $\quad N-1 \ge 2Q$

where $\quad x_1(q) = w((Q-q)\Delta_t)\, x((n-Q+q)\Delta_t) \quad$ for $0 \le q \le 2Q$,

$x_1(q) = 0 \quad$ for $2Q < q < N$.

$-Q \le q-Q \le Q$

$0 \le q \le 2Q$

$n-Q \le n-Q+q \le n+Q$

n-$Q \le n$-Q+$q \le n$+Q

$Q \ge Q$ - $q \ge$ -Q

$$\boxed{X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}}$$

T F

where $x_1(q) = w(k\Delta_t)x((n+k)\Delta_t)$ for $0 \le q \le 2Q$, $k = q\text{-}Q$ $\quad -Q \le k \le Q$

$x_1(q) = 0$ for $2Q < q < N$.

(Suppose that $w(t) = w(-t)$)

注意：

$\Theta(N\log N)$

(1) 可以使用 Matlab 的 FFT 指令來計算 $\sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}} = X_1(m)$

If $m<0$, we can apply

$X_1(m+N) = X_1(m)$

$q=0,1,\cdots N-1$

$m=0,1,\cdots N-1$

$X_1(m+N) = \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi}{N}qm} \cancel{e^{-j\frac{2\pi}{N}qN}} = X_1(m)$

(2) 對每一個固定的 n，都要計算一次下方的式子

$$X\left(n\Delta_t, m\Delta_f\right) = \Delta_t e^{j\frac{2\pi(Q-n)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}$$

$T(F+N\log N)$

(fixed n) T

$F + N\log N$

$\doteq TN\log N$

假設 $t = n_0\Delta_t,\ (n_0+1)\,\Delta_t,\ (n_0+2)\,\Delta_t,\ \cdots\cdots,\ (n_0+T\text{-}1)\Delta_t$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F\text{-}1)\Delta_f$

$N = \frac{1}{\Delta_t\Delta_f}$

Step 1: Calculate n_0, m_0, T, F, N, Q

$Q = \frac{B}{\Delta_t}$

Step 2: $n = n_0$

loop

Step 3: Determine $x_1(q)$

Step 4: $X_1(m) = \text{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(n\Delta_t, m\Delta_f)$

page 115

$m = f/\Delta_f$

$m_1 = \text{mod}(m, N)+1$

Matlab 語法

$$X\left(n\Delta_t, m\Delta_f\right) = X_1(?)\times ?$$

$X_1(\text{mod}(m,N))\, e^{j\frac{2\pi}{N}(Q-n)m}\,\Delta_t$

ex: if $N=400$

$X_1[-100] = X_1[300]$

$\text{mod}(m,N) = m+N$

if $-N \le m < 0$

$$X_1[m] = \sum_{q=0}^{N-1} x_1(q)\, e^{-j\frac{2\pi qm}{N}}$$

$$X_1[m] = X_1[m+N]$$

Step 6: Set $n = n+1$ and return to Step 3 until $n = n_0+T\text{-}1$.

IV-C Method 3: Recursive Method

限制

- **A very fast way for implementing the rec-STFT**

$\Delta_t \Delta_f = \frac{1}{N}$, $w(t) = 1$ for $|t| \le B$

(n 和 $n-1$ 有recursive的關係)

$$X(n\Delta_t, m\Delta_f) = \sum_{p=n-Q}^{n+Q} x(p\Delta_t) e^{-j\frac{2\pi pm}{N}} \Delta_t$$

$$X((n-1)\Delta_t, m\Delta_f) = \sum_{p=n-1-Q}^{n-1+Q} x(p\Delta_t) e^{-j\frac{2\pi}{N}pm} \Delta_t$$

(1) Calculate $X(\min(n)\Delta_t, m\Delta_f)$ by the N-point FFT

$$X(n_0\Delta_t, m\Delta_f) = \Delta_t e^{j\frac{2\pi(Q-n_0)m}{N}} \sum_{q=0}^{N-1} x_1(q) e^{-j\frac{2\pi qm}{N}}, \qquad n_0 = \min(n),$$

$x_1(q) = x((n-Q+q)\Delta_t)$ for $q \le 2Q$, $\quad x_1(q) = 0$ for $q > 2Q$

$O(TF)$

(2) Applying the recursive formula to calculate $X(n\Delta_t, m\Delta_f)$,

$F + N\log N + (T-1)(2F) \cong (T-1)(2F) \cong 2TF$

$n = n_0 + 1 \sim \max(n)$

$p = n-1-Q$

$$X(n\Delta_t, m\Delta_f) = X((n-1)\Delta_t, m\Delta_f) - x((n-Q-1)\Delta_t) e^{-j2\pi(n-Q-1)m/N} \Delta_t$$
$$+ x((n+Q)\Delta_t) e^{-j2\pi(n+Q)m/N} \Delta_t$$

T點 F點

$p = n+Q$

IV-D Method 4: Chirp Z Transform

$$\exp\left(-j2\pi\, pm\Delta_t\Delta_f\right) = \exp\left(-j\pi\, p^2\Delta_t\Delta_f\right)\exp\left(j\pi\,(p-m)^2\,\Delta_t\Delta_f\right)\exp\left(-j\pi m^2\Delta_t\Delta_f\right)$$

$p^2-2pm+m^2$

For the STFT

$y[n]= x[n]*h[n]$

$y[n] = \sum_m x[m]h[n-m]$

$Y(f)= X(f)H(f)$

$y[n]= IFT(FT(x(n))FT(h[n]))$

$$X\left(n\Delta_t, m\Delta_f\right) = \sum_{p=n-Q}^{n+Q} w((n-p)\Delta_t)x\left(p\Delta_t\right)e^{-j2\pi\, pm\Delta_t\Delta_f}\Delta_t$$

$$X\left(\underbrace{n}_{T}\Delta_t, m\Delta_f\right) = \Delta_t e^{-j\pi m^2\Delta_t\Delta_f}\sum_{p=n-Q}^{n+Q} \underline{w((n-p)\Delta_t)x\left(p\Delta_t\right)e^{-j\pi\, p^2\Delta_t\Delta_f}}\, e^{j\pi\,(p-m)^2\,\Delta_t\Delta_f}$$

Step 1 multiplication

Step 2 convolution

Step 3 multiplication

$T\cdot 3N\log N$ $\quad O(TN\log N$

Step 1 $x_1[p] = w((n-p)\Delta_t)x(p\Delta_t)e^{-j\pi p^2\Delta_t\Delta_f}$ $n\text{-}Q \le p \le n\text{+}Q$

Step 2 $X_2[n,m] = \sum_{p=n-Q}^{n+Q} x_1[p]c[m-p]$ $c[m] = e^{j\pi m^2\Delta_t\Delta_f}$

Step 3 $X(n\Delta_t, m\Delta_f) = \Delta_t\, e^{-j\pi m^2\Delta_t\Delta_f} X_2[n,m]$

Step 2 在計算上，需要用到 linear convolution 的技巧

Question: Step 2 要用多少點的 DFT?

• Illustration for the Question on Page 120

$$y[n]=\sum_k x[n-k]h[k]$$

• Case 1

When length($x[n]$) = N, length($h[n]$) = K, N and K are finite,

$\longrightarrow$ length($y[n]$) = $N+K-1$,

Using the ($N+K-1$)-point DFTs (學信號處理的人一定要知道的常識)

• Case 2

$x[n]$ has finite length but $h[n]$ has infinite length ????

$$y[n]=\sum_k x[n-k]h[k]$$

- Case 2

$x[n]$ has finite length but $h[n]$ has infinite length

$x[n]$ 的範圍為 $n \in [n_1, n_2]$，範圍大小為 $N = n_2 - n_1 + 1$

$h[n]$ 無限長

$y[n]=\sum_k x[n-k]h[k]$ $y[n]$ 每一點都有值 (範圍無限大)

但我們只想求出 $y[n]$ 的其中一段

希望算出的 $y[n]$ 的範圍為 $n \in [m_1, m_2]$，範圍大小為 $M = m_2 - m_1 + 1$

$h[n]$ 的範圍？

要用多少點的 FFT？

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

改寫成 $y[n]=x[n]*h[n]=\sum_{s=n_1}^{n_2}x[s]h[n-s]$

$$y[n]=x[n_1]h[n-n_1]+x[n_1+1]h[n-n_1-1]+x[n_1+2]h[n-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[n-n_2]$$

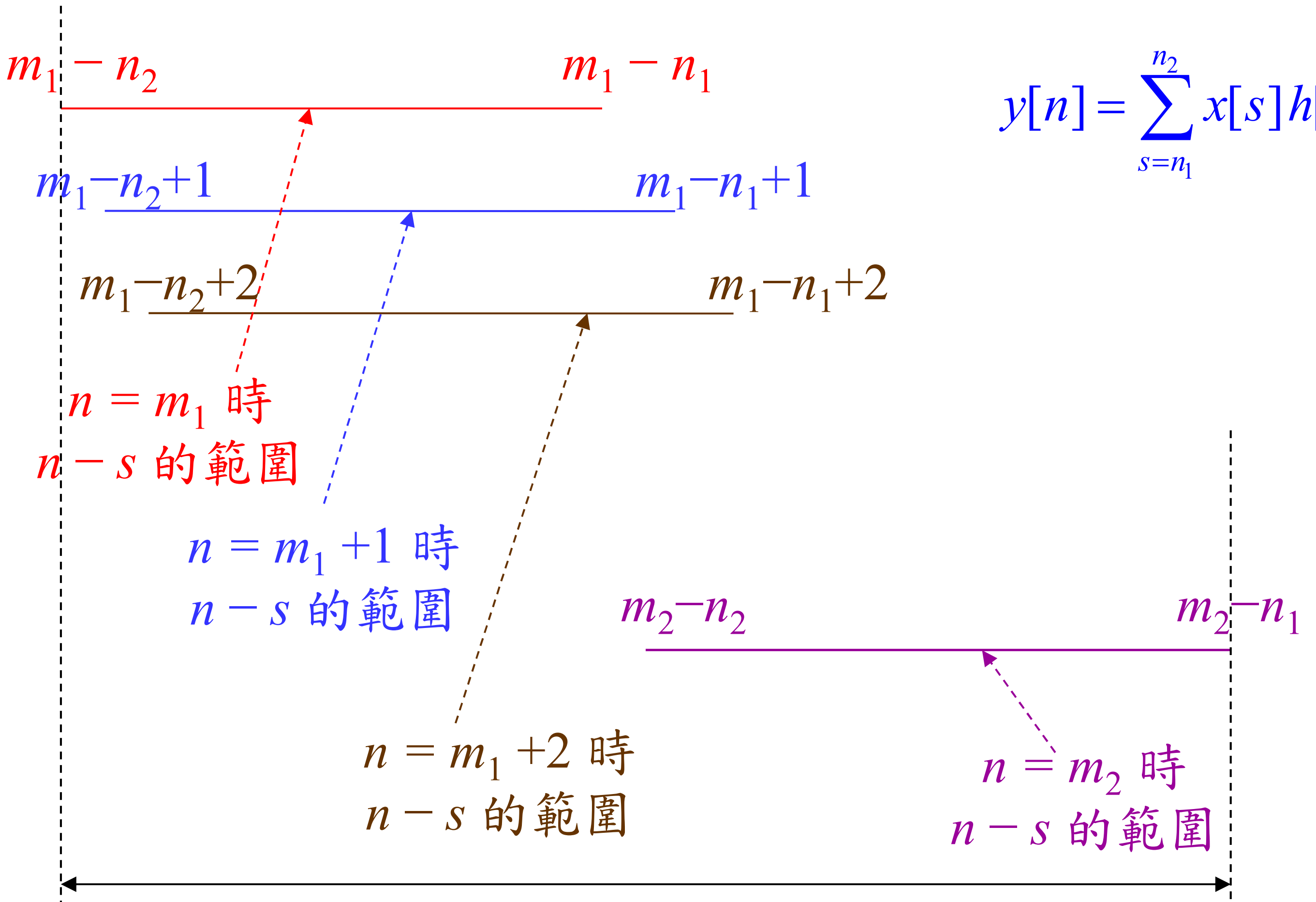
當$n=m_1$

$$y[m_1]=x[n_1]h[m_1-n_1]+x[n_1+1]h[m_1-n_1-1]+x[n_1+2]h[m_1-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_1-n_2]$$

當$n=m_2$

$$y[m_2]=x[n_1]h[m_2-n_1]+x[n_1+1]h[m_2-n_1-1]+x[n_1+2]h[m_2-n_1-2]$$
$$+\cdots\cdots+x[n_2]h[m_2-n_2]$$

$$y[n]=\sum_{s=n_1}^{n_2} x[s]h[n-s]$$



有用到的 $h[k]$ 的範圍：$k \in [m_1-n_2, m_2-n_1]$

所以，有用到的 $h[k]$ 的範圍是 $k \in [m_1 - n_2, m_2 - n_1]$

範圍大小為 $m_2 - n_1 - m_1 + n_2 + 1 = N + M - 1$

FFT implementation for Case 2

$x_1[n] = x[n + n_1]$ for $n = 0, 1, 2, \ldots, N-1$

$x_1[n] = 0$ for $n = N, N+1, N+2, \ldots\ldots, L-1$ $L = N + M - 1$

$h_1[n] = h[n + m_1 - n_2]$ for $n = 0, 1, 2, \ldots, L-1$

$$y_1[n] = IFFT_L\left(FFT_L\{x_1[n]\}\, FFT_L\{h_1[n]\}\right)$$

$y[n] = y_1[n - m_1 + N - 1]$ for $n = m_1, m_1+1, m_1+2, \ldots, m_2$

IV-E Unbalanced Sampling for STFT and WDF

將 pages 111 and 115 的方法作修正

$$\boxed{X(t,f)=\int_{-\infty}^{\infty} w(t-\tau)\,x(\tau)\,e^{-j2\pi f\tau}\,d\tau}$$

$w(t-\tau)$
$= w(n\Delta_t - p\Delta_\tau)$
$= w(nS\Delta_\tau - p\Delta_\tau)$

$$\downarrow$$

$$\boxed{X(n\Delta_t, m\Delta_f)=\sum_{p=nS-Q}^{nS+Q} w\big((nS-p)\Delta_\tau\big)\,x(p\Delta_\tau)\,e^{-j2\pi\, pm\Delta_\tau\Delta_f}\,\Delta_\tau}$$

where $t = n\Delta_t$, $f = m\Delta_f$, $\tau = p\Delta_\tau$, $B = Q\Delta_\tau$ (假設 $w(t) \cong 0$ for $|t| > B$),

$S = \Delta_t / \Delta_\tau$ $\qquad$ $\Delta_t \neq \Delta_\tau$

Ex: $\Delta_\tau = \frac{1}{44100}$
$\Delta_t = \frac{1}{100}$, $S = 441$

註：Δ_τ (sampling interval for the input signal)

Δ_t (sampling interval for the output t-axis) can be different.

However, it is better that $S = \Delta_t / \Delta_\tau$ is an integer.

When (1) $\Delta_\tau\Delta_f = 1/N$, (2) $N = 1/(\Delta_\tau\Delta_f) > 2Q+1$: ($\Delta_\tau\Delta_f$只要是整數的倒數即可)

(3) $\Delta_\tau < 1/2\Omega$, Ω is the bandwidth of $w(\tau - t)x(\tau)$

i.e., $|FT\{w(\tau-t)x(\tau)\}| = |X(t,f)| \approx 0$ when $|f| > \Omega$

$$\boxed{X(n\Delta_t, m\Delta_f) = \sum_{p=nS-Q}^{nS+Q} w((nS-p)\Delta_\tau)x(p\Delta_\tau)e^{-j\frac{2\pi pm}{N}}\Delta_\tau}$$

令 $q = p-(nS-Q) \rightarrow p = (nS-Q)+q$

$$\boxed{X(n\Delta_t, m\Delta_f) = \Delta_\tau e^{j\frac{2\pi(Q-nS)m}{N}} \sum_{q=0}^{N-1} x_1(q)e^{-j\frac{2\pi qm}{N}}}$$

不同一：$n \Rightarrow nS$

不同二：$\Delta_\tau\Delta_f = \frac{1}{N}$ (not Δ_t)

$x_1(q) = w((Q-q)\Delta_\tau)x((nS-Q+q)\Delta_\tau)$ for $0 \le q \le 2Q$,

$x_1(q) = 0$ for $2Q < q < N$.

If $w(t) = w(-t)$)

$x_1(q) = w(k\Delta_\tau)x((nS+k)\Delta_\tau)$ for $0 \le q \le 2Q$, $k = q-Q$ $\quad -Q \le k \le Q$

$x_1(q) = 0$ for $2Q < q < N$.

假設 $t = c_0\Delta_t,\ (c_0+1)\,\Delta_t,\ (c_0+2)\,\Delta_t,\ \cdots\cdots,\ (c_0+C\,-1)\,\Delta_t$

$= c_0S\Delta_\tau,\ (c_0S+S)\,\Delta_\tau,\ (c_0S+2S)\,\Delta_\tau,\ \cdots\cdots,\ [c_0S+(C\text{-}1)S]\,\Delta_\tau,$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ (m_0+F\text{-}1)\,\Delta_f$

$\tau\ = n_0\,\Delta_\tau,\ (n_0+1)\,\Delta_\tau,\ (n_0+2)\,\Delta_\tau,\ \cdots\cdots,\ (n_0+T\text{-}1)\,\Delta_\tau,$ $S = \Delta_t/\,\Delta_\tau$

Step 1: Calculate c_0, m_0, n_0, C, F, T, N, Q

Step 2: $n = c_0$

Step 3: Determine $x_1(q)$ compared with the original method, $n \Rightarrow nS$

Step 4: $X_1(m) = \text{FFT}[x_1(q)]$

Step 5: Convert $X_1(m)$ into $X(\,n\Delta_t,\ m\Delta_f)$

Step 6: Set $n = n+1$ and return to Step 3 until $n = c_0+C\,-1$.

Complexity = ?

IV-F Non-Uniform Δ_t

(A) 先用較大的 Δ_t

(B) 如果發現 $\left|X\left(n\Delta_t, m\Delta_f\right)\right|$ 和 $\left|X\left((n+1)\Delta_t, m\Delta_f\right)\right|$ 之間有很大的差異

則在 $n\Delta_t$, $(n+1)\,\Delta_t$ 之間選用較小的 sampling interval Δ_{t1}

($\Delta_\tau < \Delta_{t1} < \Delta_t$, Δ_t / Δ_{t1} 和$\Delta_{t1} / \Delta_\tau$ 皆為整數)

再用 page 127 的方法算出

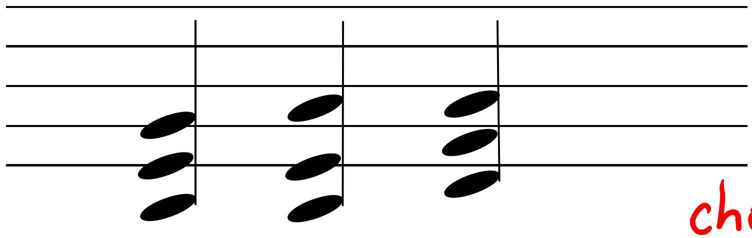
$$X\left(n\Delta_t + \Delta_{t1}, m\Delta_f\right),\quad X\left(n\Delta_t + 2\Delta_{t1}, m\Delta_f\right),\quad \cdots\cdots,\quad X\left((n+1)\Delta_t - \Delta_{t1}, m\Delta_f\right)$$

(C) 以此類推，如果 $\left|X\left(n\Delta_t + k\Delta_{t1}, m\Delta_f\right)\right|$, $\left|X\left(n\Delta_t + (k+1)\Delta_{t1}, m\Delta_f\right)\right|$

的差距還是太大，則再選用更小的 sampling interval Δ_{t2}

($\Delta\tau < \Delta t_2 < \Delta t_1$, $\Delta t_1 / \Delta t_2$ 和$\Delta t_2 / \Delta\tau$ 皆為整數)

Gabor transform of a music signal



chord.wav

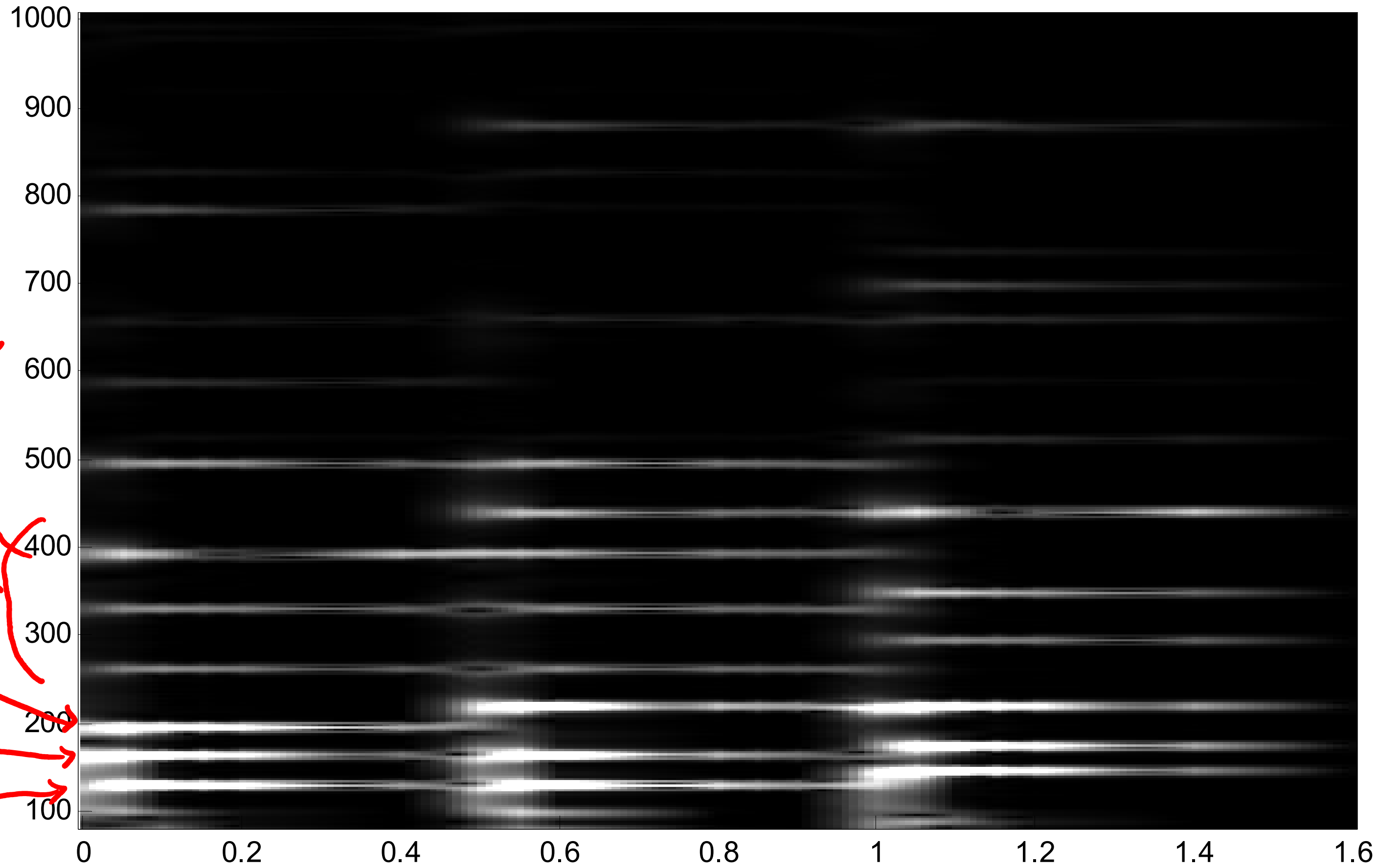
3倍頻

2倍頻

低So

低Mi

低Do



$\Delta_\tau = 1/44100$ (總共有 44100×1.6077 sec $+ 1 = 70902$ 點

(A) Choose $\Delta_t = \Delta_\tau$

running time = out of memory

(B) Choose $\Delta_t = 0.01 = 441\Delta_\tau$ (1.6/0.01 + 1 = 161 points)

running time = 1.0940 sec (2008年)

(C) Choose the sampling points on the *t*-axis as

t = 0, 0.05, 0.1, 0.15, 0.2, 0.4, 0.45, 046, 0.47, 0.48, 0.49, 0.5, 0.55, 0.6, 0.8,

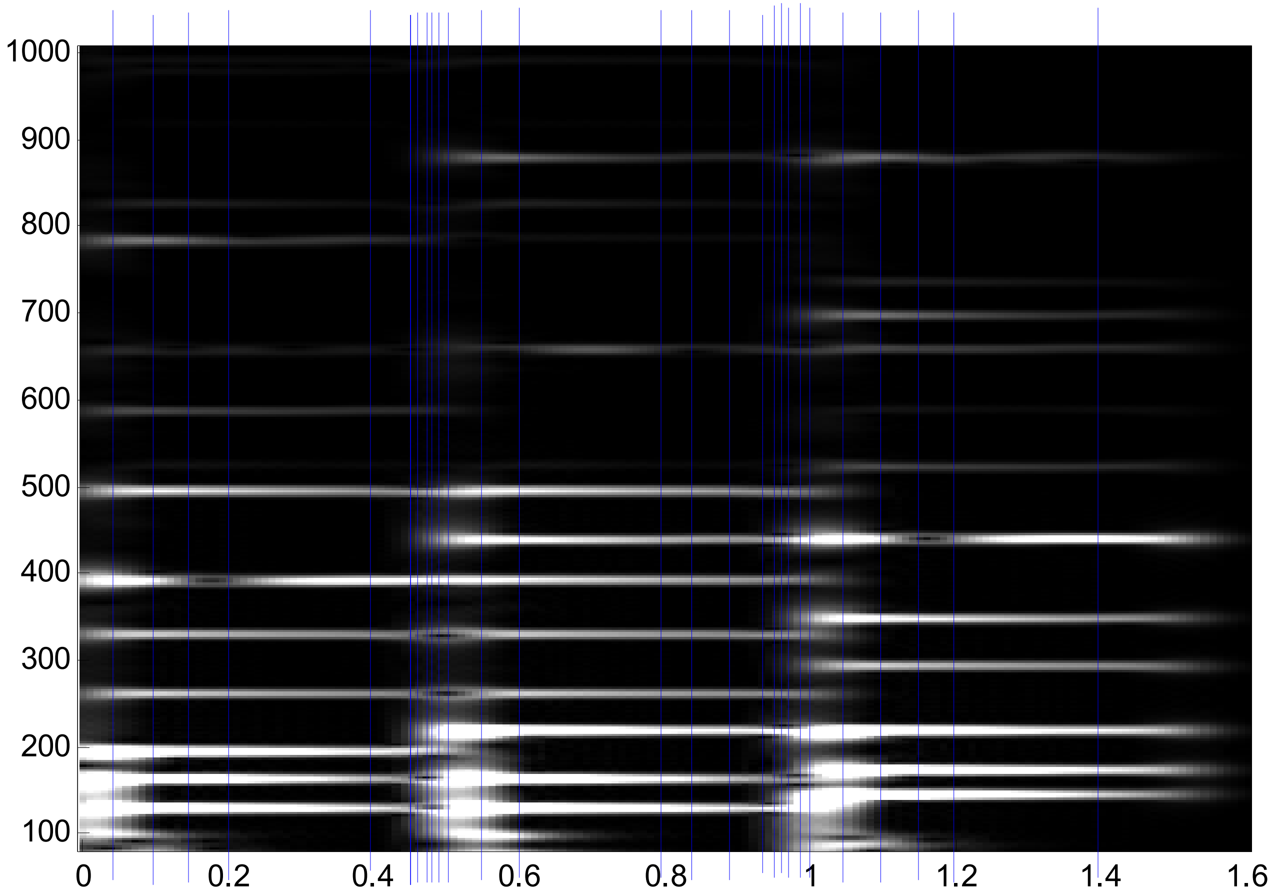
0.85, 0.9, 0.95, 0.96, 0.97, 0.98, 0.99, 1, 1.05, 1.1, 1.15, 1.2, 1.4, 1.6

(29 points)

running time = 0.2970 sec

Δ_t: 0.2 $\Rightarrow$ Δ_t = 0.05 $\Rightarrow$ Δ_t = 0.01

with adaptive output sampling intervals

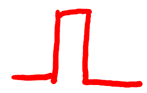


附錄七 和 Dirac Delta Function 相關的常用公式

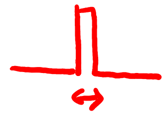
(1) $\int_{-\infty}^{\infty} e^{-j2\pi t f} dt = \delta(f)$

$\int \delta(t)\, dt = 1$

δ(t)



δ(at)



|a|倍

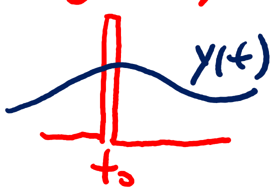
(2) $\delta(t) = |a|\delta(at)$ (scaling property)

(3) $\int_{-\infty}^{\infty} e^{-j2\pi t g(f)} dt = \delta(g(f)) = \sum_n |g'(f_n)|^{-1} \delta(f - f_n)$

where f_n are the zeros of $g(f)$

(4) $\int_{-\infty}^{\infty} \delta(t - t_0)\, y(t, \cdots\cdots)\, dt = y(t_0, \cdots\cdots)$ (sifting property I)

(5) $\delta(t - t_0)\, y(t, \cdots\cdots) = \delta(t - t_0)\, y(t_0, \cdots\cdots)$ (sifting property II)



V. Wigner Distribution Function

V-A Wigner Distribution Function (WDF)

Definition 1: $$W_x(t,f) = \int_{-\infty}^{\infty} \underline{x(t+\tau/2)\cdot x^*(t-\tau/2)}e^{-j2\pi\tau f}d\tau$$

Definition 2: $$W_x(t,\omega) = \int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j\omega\tau}d\tau$$

Another way for computation from the frequency domain

(output is the same)

Definition 1: $W_x(t,f)=\int_{-\infty}^{\infty} X(f+\eta/2)\cdot X^*(f-\eta/2)e^{j2\pi\eta t}d\eta$

where $X(f)$ is the Fourier transform of $x(t)$

Definition 2: $W_x(t,\omega)=\int_{-\infty}^{\infty} X(\omega+\eta/2)\cdot X^*(\omega-\eta/2)e^{j\eta t}d\eta$

Main Reference

[Ref] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chap. 5, Prentice Hall, N.J., 1996.

Other References

[Ref] E. P. Wigner, "On the quantum correlation for thermodynamic equilibrium," *Phys. Rev.*, vol. 40, pp. 749-759, 1932.

[Ref] T. A. C. M. Classen and W. F. G. Mecklenbrauker, "The Wigner distribution–A tool for time-frequency signal analysis; Part I," *Philips J. Res.*, vol. 35, pp. 217-250, 1980.

[Ref] F. Hlawatsch and G. F. Boudreaux–Bartels, "Linear and quadratic time-frequency signal representation," *IEEE Signal Processing Magazine*, pp. 21-67, Apr. 1992.

[Ref] R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley-Interscience, NJ, 2004.

The operators that are related to the WDF:

比較: in random process auto-correlation:

(a) Signal auto-correlation function:

$$C_x(t,\tau) = x(t+\tau/2)\cdot x^*(t-\tau/2)$$

$R(t,\tau) = E(x(t)x^*(t+\tau))$

(b) Spectrum auto-correlation function:

$\mathcal{F}_{\tau\to f}(R(t,\tau)) = S(t,f)$

$$S_x(\eta,f) = X(f+\eta/2)\cdot X^*(f-\eta/2)$$

power spectral density (PSD)

(c) Ambiguity function (AF):

$$A_x(\tau,\eta) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j2\pi t\eta}\,dt$$

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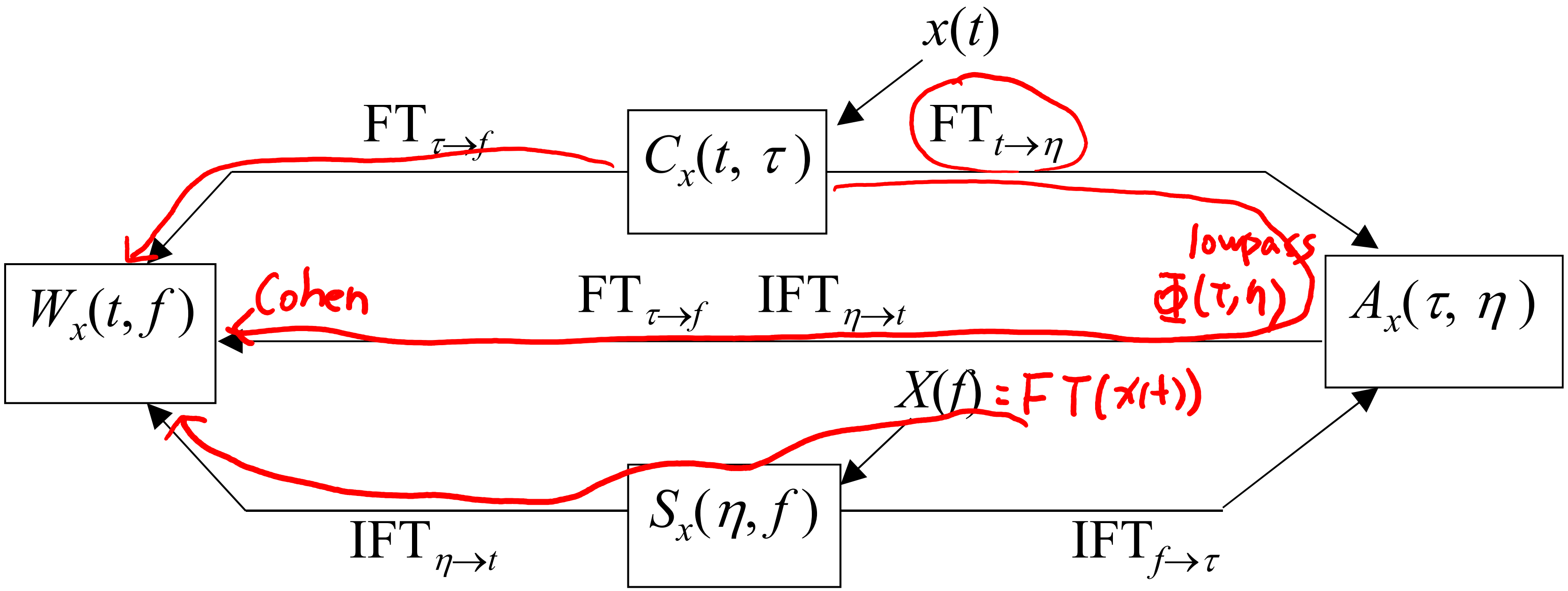
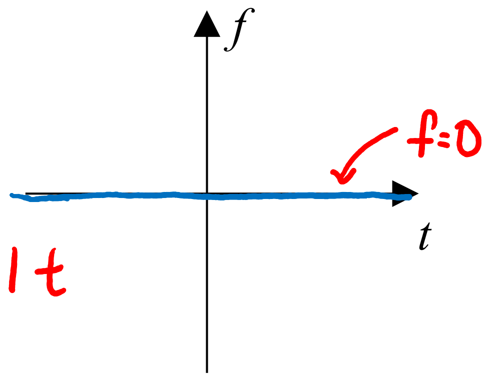


V-B Why the WDF Has Higher Clarity?

Due to signal auto-correlation function

(1) If $x(t) = 1$ $h=0$



(2) If $x(t) = \exp(j2\pi h\, t)$

$\frac{\phi'(t)}{2\pi} = h$ for all t

$x(t+\frac{\tau}{2})\, x^*(t-\frac{\tau}{2})$

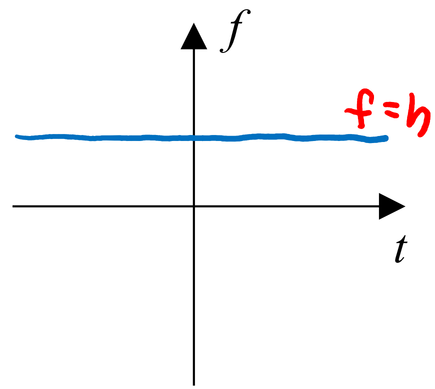
$$W_x(t,f) = \int_{-\infty}^{\infty} e^{j2\pi h(t+\tau/2)} e^{-j2\pi h(t-\tau/2)} \cdot e^{-j2\pi\tau f}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{j2\pi h\tau} \cdot e^{-j2\pi\tau f}\, d\tau$$

$g(f) = f - h$

$$= \int_{-\infty}^{\infty} e^{-j2\pi\tau(f-h)}\, d\tau$$ page 133 (3)

$$= \underline{\delta(f-h)}$$

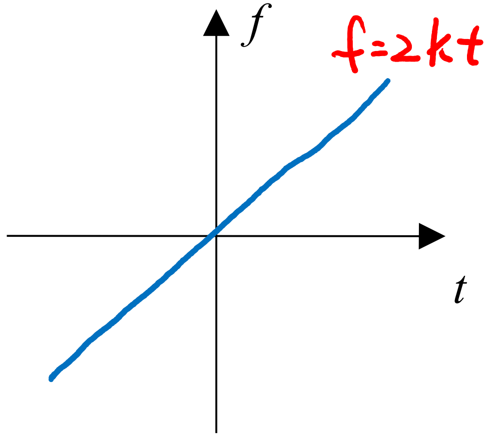


Comparing: for the case of the STFT

$\frac{\phi'(t)}{2\pi} = 2kt$

(3) If $x(t) = \exp(j2\pi k t^2)$

$$W_x(t,f) = \int x(t+\tfrac{\tau}{2})\, x^*(t-\tfrac{\tau}{2})\, e^{-j2\pi f\tau} d\tau$$
$$= \int e^{j2\pi k(t+\frac{\tau}{2})^2} e^{-j2\pi k(t-\frac{\tau}{2})^2} e^{-j2\pi f\tau} d\tau$$
$$= \int e^{j2\pi k(2t\tau)} e^{-j2\pi f\tau} d\tau$$
$$= \int e^{-j2\pi\tau(f-2tk)} d\tau$$
$$= \delta(f-2kt)$$ (page 133 (3))



(4) If $x(t) = \delta(t)$

$$W_x(t,f) = \int_{-\infty}^{\infty} \delta(t+\tau/2)\cdot\delta(t-\tau/2) e^{-j2\pi\tau f} d\tau$$

$t \to \tau;\ t_0 \to -2t$

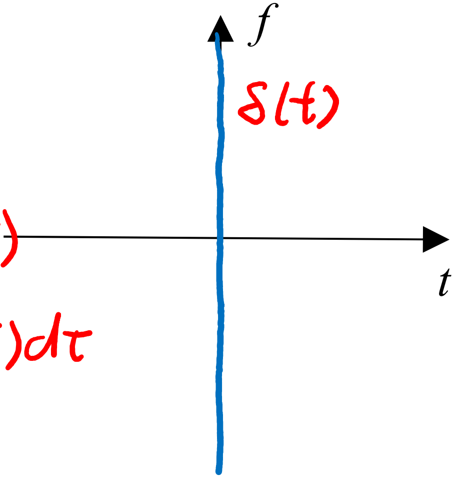
Page 133 公式(2)

$$= 4\int_{-\infty}^{\infty} \delta(2t+\tau)\cdot \underbrace{\boxed{\delta(2t-\tau) e^{-j2\pi\tau f} d\tau}}_{}$$ $y(\tau)$

$\int \delta(\tau-\tau_0)\, y(\tau)\, d\tau$

$$= 4\delta(4t) e^{j4\pi t f} = \delta(t) e^{j4\pi t f} = \delta(t)$$

Page 133 公式(4)　　Page 133 公式(2)　　Page 133 公式(5), $t_0 = 0$



V-C The WDF is not a Linear Distribution

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

If $h(t) = \alpha\, g(t) + \beta\, s(t)$

$$W_h(t,f)=\int_{-\infty}^{\infty} h(t+\tau/2)\cdot h^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty}\left[\alpha\, g(t+\tau/2)+\beta\, s(t+\tau/2)\right]\left[\alpha^*\, g^*(t-\tau/2)+\beta^*\, s^*(t-\tau/2)\right]e^{-j2\pi\tau f}\,d\tau$$

$$=\int_{-\infty}^{\infty}\Big[|\alpha|^2 g(t+\tau/2)g^*(t-\tau/2)+|\beta|^2 s(t+\tau/2)s^*(t-\tau/2)$$
$$+\alpha\beta^* g(t+\tau/2)s^*(t-\tau/2)+\alpha^*\beta\, g^*(t-\tau/2)s(t+\tau/2)\Big]e^{-j2\pi\tau f}\,d\tau$$

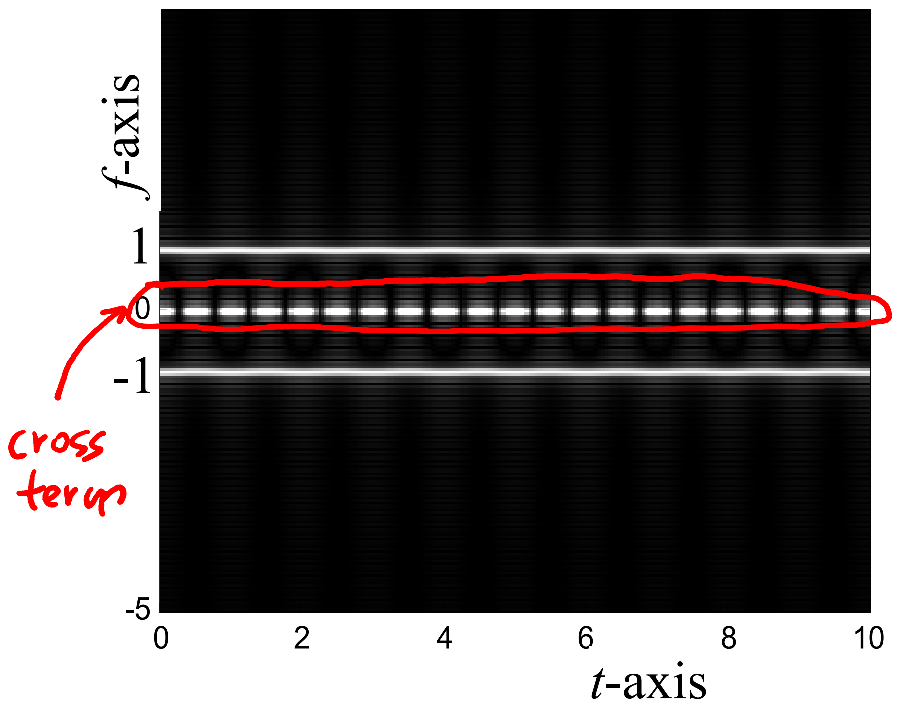
$$=\underbrace{|\alpha|^2 W_g(t,f)+|\beta|^2 W_s(t,f)}_{\text{auto terms}}$$
$$+\underbrace{\int_{-\infty}^{\infty}\left[\alpha\beta^* g(t+\tau/2)s^*(t-\tau/2)+\alpha^*\beta\, g^*(t-\tau/2)s(t+\tau/2)\right]e^{-j2\pi\tau f}\,d\tau}_{\text{cross terms}}$$

V-D Examples of the WDF

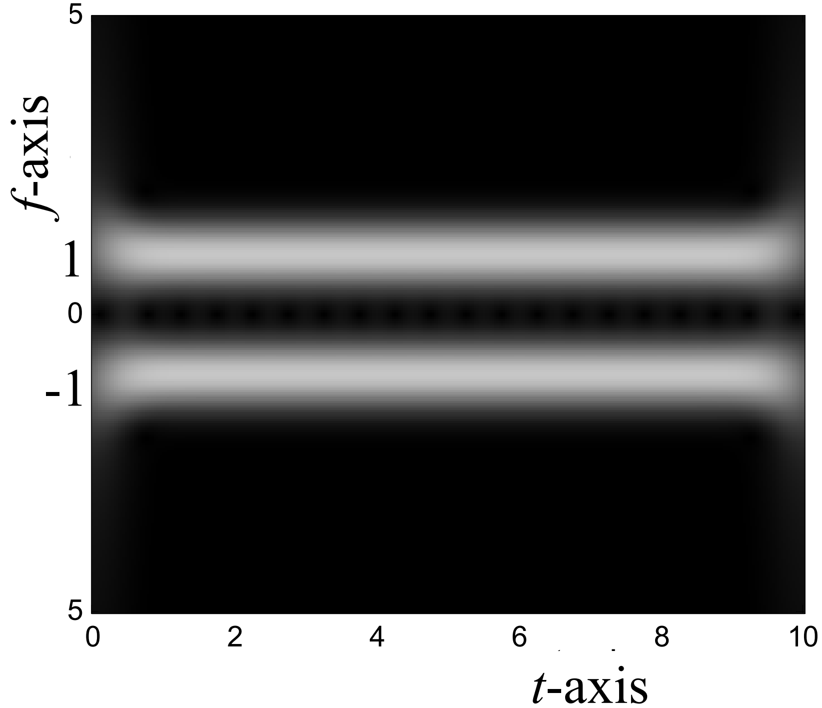
Simulations

$$x(t) = \cos(2\pi t) = 0.5[\exp(j2\pi t) + \exp(-j2\pi t)]$$

by the WDF (with a window)



by the Gabor transform



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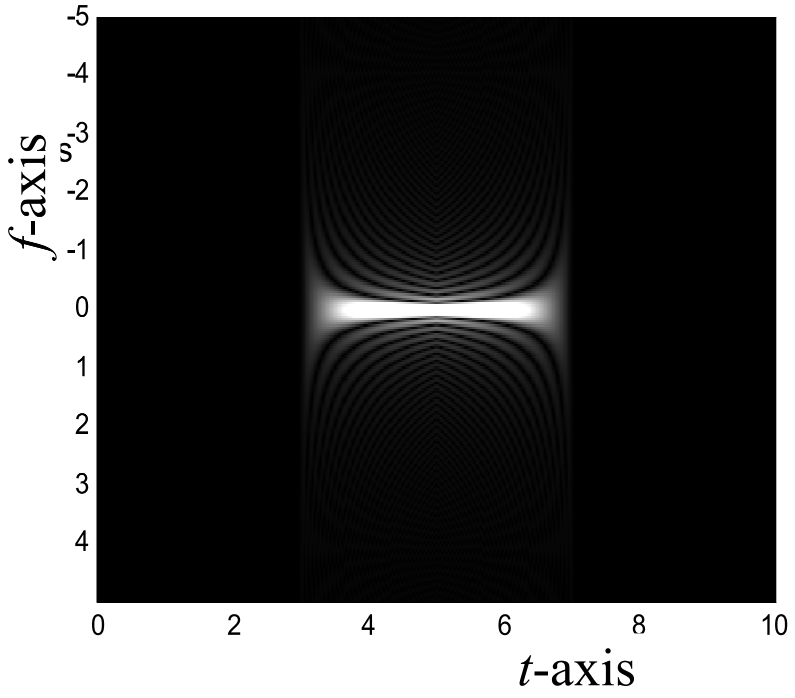
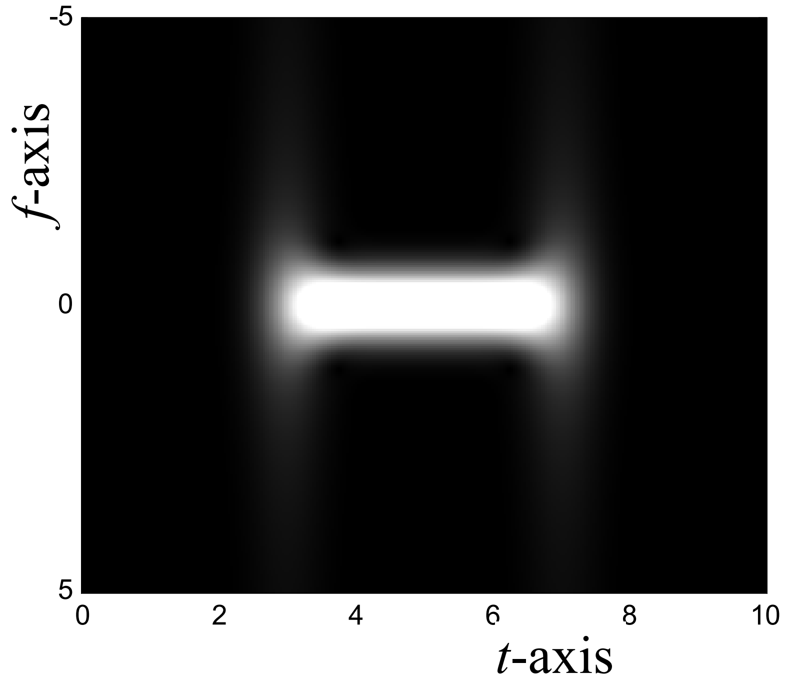
$$x(t) = \Pi((t-5)/4) \qquad \Pi\text{: rectangular function}$$

center width

by the WDF

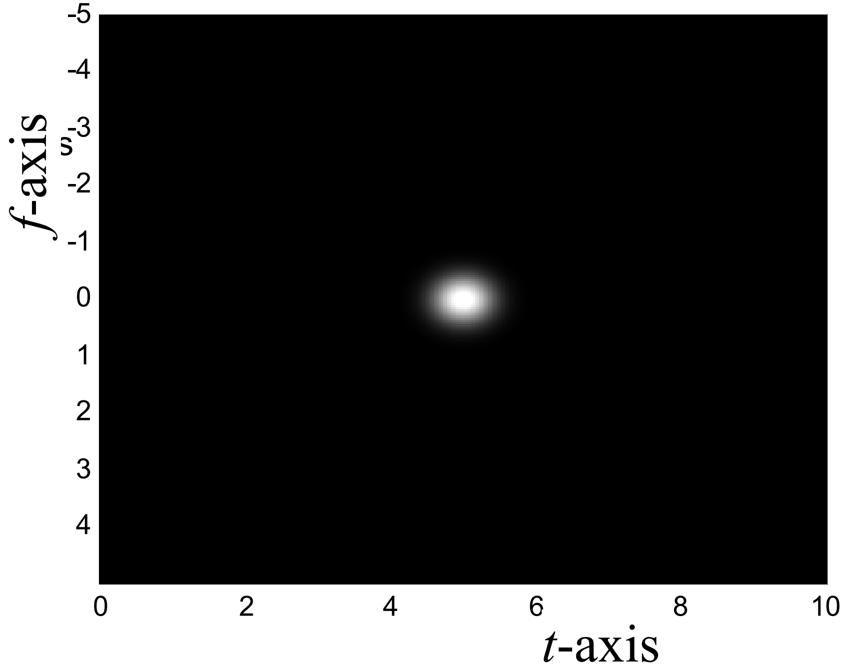


by the Gabor transform

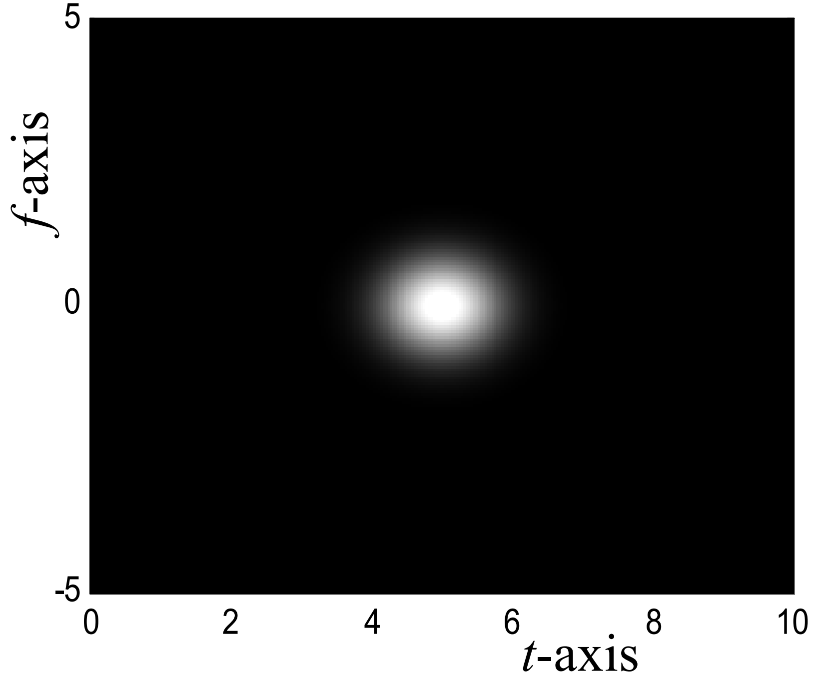


$$x(t) = \exp\left[-\pi(t-5)^2\right]$$

by the WDF



by the Gabor transform



Gaussian function: $e^{-\pi t^2} \xrightarrow{FT} e^{-\pi f^2}$

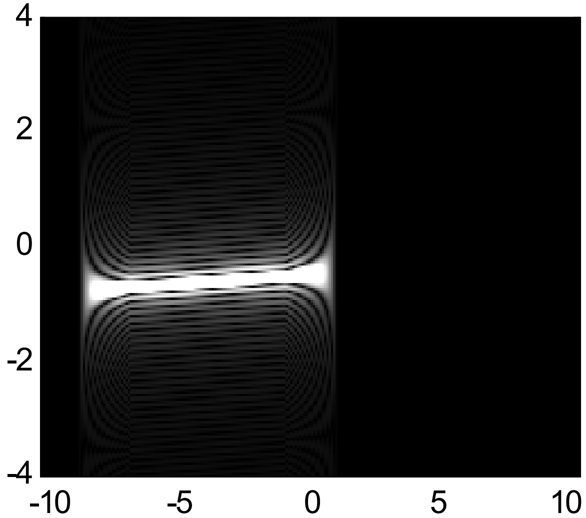
Gaussian function's T-F area is minimal.

$$s(t)=\exp\left(jt^2/10-j3t\right) \quad \text{for } -9 \le t \le 1,\ s(t)=0 \text{ otherwise,}$$

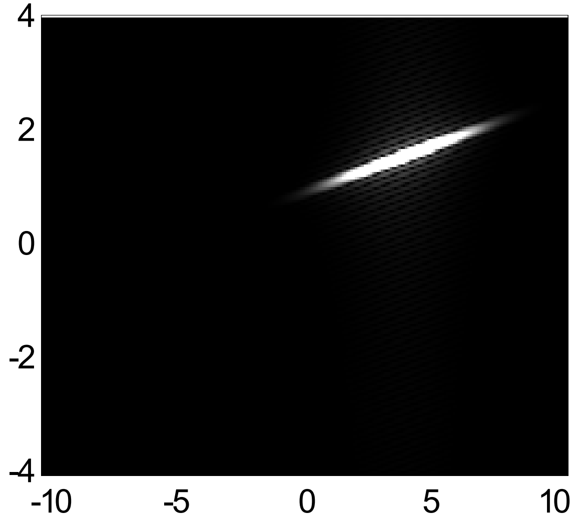
$$r(t)=\exp\left(jt^2/2+j6t\right)\exp\left[-(t-4)^2/10\right]$$

$$f(t) = s(t) + r(t)$$

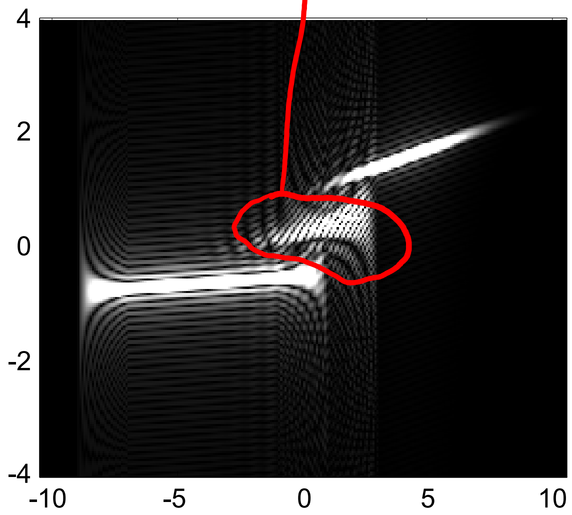
cross term



WDF of $s(t)$,



WDF of $r(t)$,



WDF of $s(t) + r(t)$

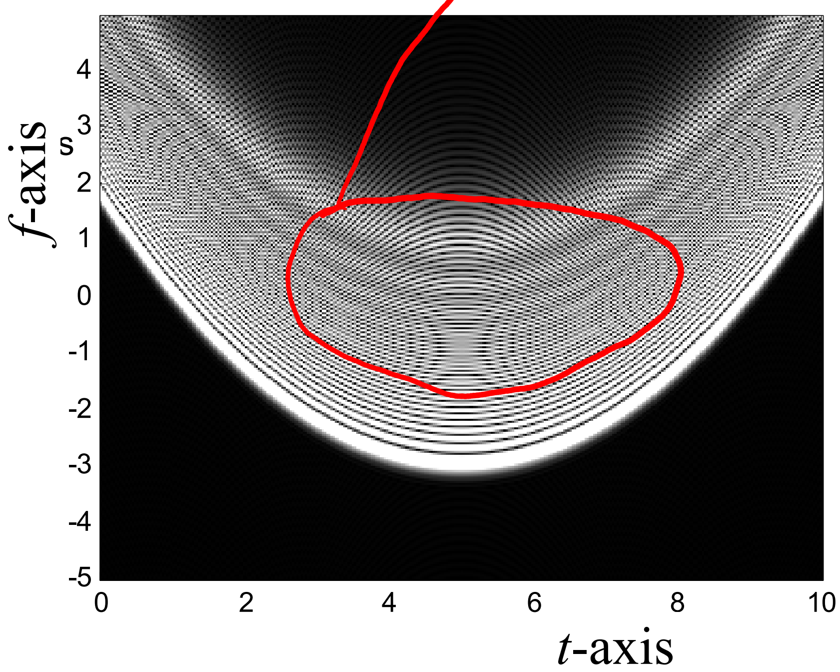
橫軸: t-axis, 縱軸: f-axis

$$\frac{\phi'(t)}{2\pi} = \frac{3}{2\pi}(t-5)^2 - 3$$

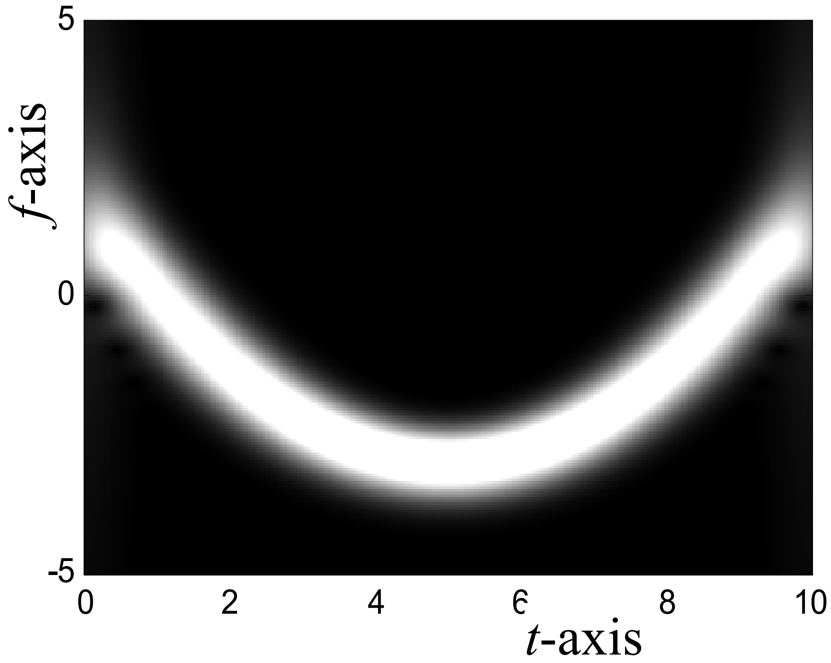
$$x(t) = \exp(j(t-5)^3 - j6\pi t)$$

by the WDF (with a window)

cross term



by the Gabor transform



V-E Digital Implementation of the WDF

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau\ ,$$

$$W_x(t,f)=2\int_{-\infty}^{\infty} x(t+\tau')\cdot x^*(t-\tau')e^{-j4\pi\tau' f}\cdot d\tau' \quad \text{(using } \tau' = \tau/2\text{)}$$

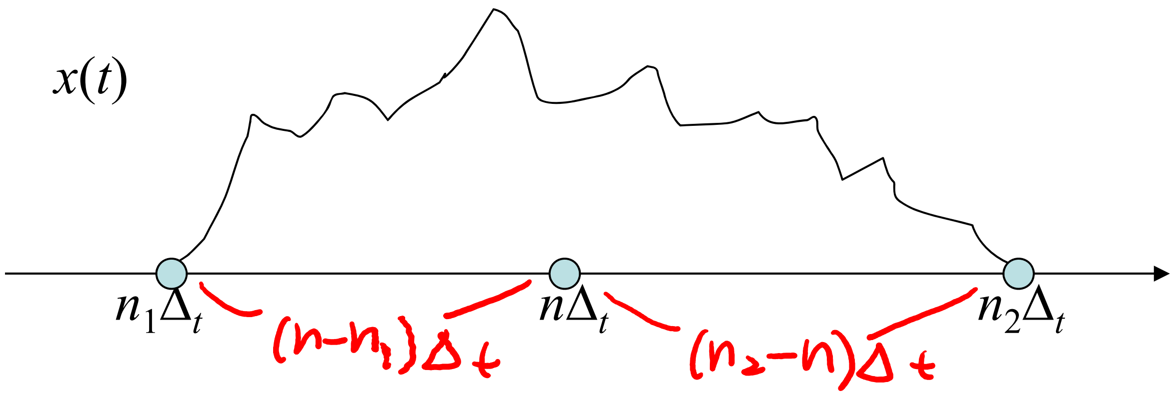
$d\tau' = d\tau/2$
$d\tau = 2d\tau'$

Sampling: $t = n\Delta_t$, $f = m\Delta_f$, $\tau' = p\Delta_t$

$$\boxed{W_x(n\Delta_t, m\Delta_f)=2\sum_{p=-\infty}^{\infty} x((n+p)\Delta_t)x^*((n-p)\Delta_t)\exp(-j4\pi mp\Delta_t\Delta_f)\Delta_t}$$

When $x(t)$ is not a time-limited signal, it is hard to implement.

Suppose that $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$



$x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right) = 0$ if $n+p \notin [n_1, n_2]$

or $n-p \notin [n_1, n_2]$

- ***p* 的範圍的問題 (當 *n* 固定時)**

$n_1 \le n+p \le n_2 \longrightarrow n_1 - n \le p \le n_2 - n$

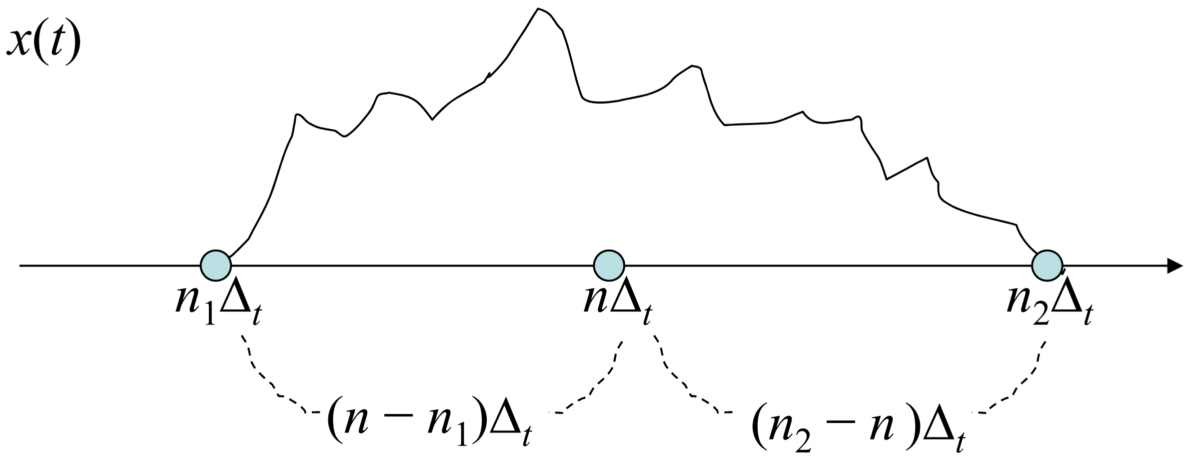
$n_1 \le n-p \le n_2 \longrightarrow n_1 - n \le -p \le n_2 - n, \quad n - n_2 \le p \le n - n_1$

$\max(n_1 - n\,, n - n_2) \le p \le \min(n_2 - n\,, n - n_1)$

$-\min(n_2 - n\,, n - n_1) \le p \le \min(n_2 - n\,, n - n_1)$

$\max(-3,-5)$

$= -\min(3,5)$



$$-\underbrace{\min(n_2-n\,,\,n-n_1)}_{-Q} \le p \le \underbrace{\min(n_2-n\,,\,n-n_1)}_{Q}$$

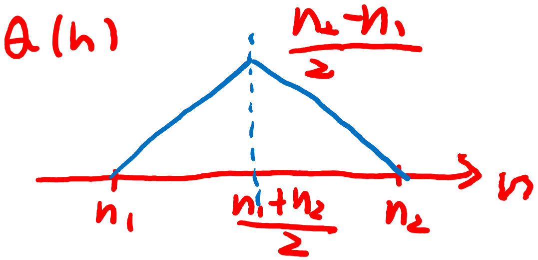
$(n_2-n)\Delta_t$, $(n-n_1)\Delta_t$： 離兩個邊界的距離

注意：當 $n>n_2$ 或 $n<n_1$ 時，

將沒有 p 能滿足上面的不等式

If $x(t) = 0$ for $t < n_1\Delta_t$ and $t > n_2\Delta_t$

$$\boxed{W_x\big(\underset{T點}{n\Delta_t}, \underset{F點}{m\Delta_f}\big) = 2\sum_{p=-Q}^{Q} x\big((n+p)\Delta_t\big)x^*\big((n-p)\Delta_t\big)\exp\big(-j4\pi mp\Delta_t\Delta_f\big)\Delta_t}$$



$Q = \min(n_2-n, n-n_1)$. (varies with n)

$p \in [-Q, Q]$, $n \in [n_1, n_2]$,

possible for implementation

Method 1: Direct Implementation (brute force method)

唯一的限制條件？

complexity: $T\,F \cdot \text{mean}(Q)$

$= TF\,\frac{n_2-n_1}{4} \cong \frac{T^2F}{4}$

$O(T^2F)$

Since $\text{WDF} = \underset{\tau\to f}{F}\left(x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})\right)$

If Ω_x is the bandwidth of $x(\tau)$

bandwidth of $x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})$

$= \frac{1}{2}$ bw of $x(2t+\tau)x^*(2t-\tau) = \frac{1}{2}(2\Omega_x)$

$= \Omega_x$ $\therefore \Delta_\tau < \frac{1}{2\Omega_x}$, $2\Delta_t < \frac{1}{2\Omega_x}$, $\Delta_t < \frac{1}{4\Omega_x}$

$\tau' = \frac{\tau}{2}$, $\Delta_{\tau'} = \frac{\Delta_\tau}{2} = \Delta_t$

Method 2: Using the DFT

3 大限制條件

When $\Delta_t\Delta_f = \frac{1}{2N}$ and $N \geq 2\text{Max}(Q)+1 = 2(n_2\text{-}n_1)/2+1 = n_2\text{-}n_1+1 = T$

$$\boxed{W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t \sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j\frac{2\pi mp}{N}}}$$

T點 F點

$q = p+Q \rightarrow p = q - Q$

complexity : $O(TN\cdot\log N) = O(T^2\log T)$

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{2Q} x\left((n+q-Q)\Delta_t\right)x^*\left((n-q+Q)\Delta_t\right)e^{-j\frac{2\pi mq}{N}}$$

$$\boxed{W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t e^{j\frac{2\pi mQ}{N}} \sum_{q=0}^{N-1} c_1\left(q\right)e^{-j\frac{2\pi mq}{N}}}$$

T

$Q = \min(n_2-n, n-n_1)$.

$n \in [n_1, n_2]$,

$c_1\left(q\right) = x\left((n+q-Q)\Delta_t\right)x^*\left((n-q+Q)\Delta_t\right)$ for $0 \leq q \leq 2Q$

i.e., $c_1\left(Q+k\right) = x\left((n+k)\Delta_t\right)x^*\left((n-k)\Delta_t\right)$ for $-Q \leq k \leq Q$ $(k = q\text{-}Q)$

$c_1\left(q\right) = 0$ for $2Q+1 \leq q \leq N-1$

假設 $t = n_0\Delta_t,\ (n_0+1)\,\Delta_t,\ (n_0+2)\,\Delta_t,\ \cdots\cdots,\ n_1\Delta_t$

$f = m_0\,\Delta_f,\ (m_0+1)\,\Delta_f,\ (m_0+2)\,\Delta_f,\ \cdots\cdots,\ m_1\,\Delta_f$

Step 1: Calculate n_0, n_1, m_0, m_1, N

Step 2: $n = n_0$

Step 3: Determine Q

Step 4: Determine $c_1(q)$

Step 5: $C_1(m) = \mathrm{FFT}[c_1(q)]$

Step 6: Convert $C_1(m)$ into $C(n\Delta_t, m\Delta_f)$

Step 7: Set $n = n+1$ and return to Step 3 until $n = n_1$.

Method 3: Using the Chirp Z Transform

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)\exp\left(-j4\pi\, mp\Delta_t\Delta_f\right)\Delta_t$$

T

$$W_x\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t\, e^{-j2\pi m^2\Delta_t\Delta_f}\sum_{p=-Q}^{Q} x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j2\pi p^2\Delta_t\Delta_f}e^{j2\pi(p-m)^2\Delta_t\Delta_f}$$

Step 1 $x_1\left(n,p\right) = x\left((n+p)\Delta_t\right)x^*\left((n-p)\Delta_t\right)e^{-j2\pi p^2\Delta_t\Delta_f}$

Step 2 $X_2\left[n,m\right] = \sum_{p=-Q}^{Q} x_1\left[n,p\right]c\left[m-p\right]$ $\qquad c\left[m\right] = e^{j2\pi m^2\Delta_t\Delta_f}$

Step 3 $X\left(n\Delta_t, m\Delta_f\right) = 2\Delta_t\, e^{-j2\pi m^2\Delta_t\Delta_f}X_2\left[n,m\right]$

complexity = ? $O($ $\qquad$ $)$

思考：Method 1 的複雜度為多少

思考：Method 2 的複雜度為多少

思考：Method 3 的複雜度為多少

The computation time of the WDF is more than those of the rec-STFT and the Gabor transform.

V-F Properties of the WDF

(1) Projection property	$\left\|x(t)\right\|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,df \qquad \left\|X(f)\right\|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,dt$
(2) Energy preservation property	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dtdf = \int_{-\infty}^{\infty}\left\|x(t)\right\|^2 dt = \int_{-\infty}^{\infty}\left\|X(f)\right\|^2 df$
(3) Recovery property	$\int_{-\infty}^{\infty} W_x(t/2,f)e^{j2\pi ft}\,df = x(t)\cdot x^*(0)$ $x^*(0)$ 已知 $\int_{-\infty}^{\infty} W_x(t,f/2)e^{-j2\pi ft}\,dt = X(f)\cdot X^*(0)$
(4) Mean condition frequency and mean condition time	If $x(t) = \left\|x(t)\right\|\cdot e^{j2\pi\phi(t)}$, $X(f) = \left\|X(f)\right\|\cdot e^{j2\pi\Psi(f)}$ then $\phi'(t) = \left\|x(t)\right\|^{-2}\cdot\int_{-\infty}^{\infty} f\cdot W_x(t,f)\cdot df$ $-\Psi'(f) = \left\|X(f)\right\|^{-2}\int_{-\infty}^{\infty} t\cdot W_x(t,f)\cdot dt$
(5) Moment properties	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^n W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} t^n \left\|x(t)\right\|^2 dt$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^n W_x(t,f)\,dtdf = \int_{-\infty}^{\infty} f^n \left\|X(f)\right\|^2 df$

(6) $W_x(t,f)$ is bound to be real	$\overline{W_x(t,f)} = W_x(t,f)$
(7) Region properties	If $x(t) = 0$ for $t > t_2$ then $W_x(t,f) = 0$ for $t > t_2$ If $x(t) = 0$ for $t < t_1$ then $W_x(t,f) = 0$ for $t < t_1$
(8) Multiplication theory	If $y(t) = x(t)h(t)$, then $W_y(t,f) = \int_{-\infty}^{\infty} W_x(t,\rho) W_h(t, f-\rho) \cdot d\rho$
(9) Convolution theory	If $y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$, then $W_y(t,f) = \int_{-\infty}^{\infty} W_x(\rho,f) \cdot W_h(t-\rho,f) \cdot d\rho$
(10) Correlation theory	If $y(t) = \int_{-\infty}^{\infty} x(t+\tau)h^*(\tau)d\tau$, then $W_y(t,f) = \int_{-\infty}^{\infty} W_x(\rho,f) \cdot W_h(-t+\rho,f) \cdot d\rho$

(11) Time-shifting property	If $y(t) = x(t - t_0)$, then $W_y(t,f) = W_x(t - t_0, f)$
(12) Modulation property	If $y(t) = \exp(j2\pi f_0 t)x(t)$, then $W_y(t,f) = W_x(t, f - f_0)$
(13) Constant multiplication property	If $y(t) = cx(t)$, then $W_y(t,f) = \lvert c\rvert^2 W_x(t,f)$
(14) Conjugation property	If $y(t) = x^*(t)$, then $W_y(t,f) = W_x(t,-f)$
(15) Scaling property	If $y(t) = x(ct)$, then $W_y(t,f) = \frac{1}{\lvert c\rvert} W_x\left(ct, \frac{1}{c} f\right)$

The STFT (including the rec-STFT, the Gabor transform) does not have real region, multiplication, convolution, and correlation properties.

remember: If $x(t) = x^*(-t)$
then $X(f)$ is real

- Why the WDF is always real?

  What are the advantages and disadvantages it causes?

$$W_x(t,f) = \int x(t+\tfrac{\tau}{2})\, x^*(t-\tfrac{\tau}{2})\, e^{-j2\pi\tau f} d\tau$$

$$W_x^*(t,f) = \int_{-\infty}^{\infty} x^*(t+\tfrac{\tau}{2})\, x(t-\tfrac{\tau}{2})\, e^{j2\pi\tau f} d\tau$$

set
$\tau_1 = -\tau \quad d\tau_1 = -d\tau$

$$= \int_{\infty}^{-\infty} x^*(t-\tfrac{\tau_1}{2})\, x(t+\tfrac{\tau_1}{2})\, e^{-j2\pi\tau_1 f} (-d\tau_1)$$

$$= -\int_{-\infty}^{\infty} x(t+\tfrac{\tau_1}{2})\, x^*(t-\tfrac{\tau_1}{2})\, e^{-j2\pi\tau_1 f} d\tau_1 = W_x(t,f)$$

Note: If $y(t) = e^{j\phi} x(t)$, ϕ is a constant, then $W_y(t,f) = W_x(t,f)$

- Try to prove of the projection and recovery properties

$$W_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau/2) \cdot x^*(t-\tau/2) e^{-j2\pi\tau f} \cdot d\tau$$

- Proof of the region properties

If $x(t) = 0$ for $t < t_0$,

$x(t+\tau/2) = 0$ for $\tau < (t_0 - t)/2 = -(t - t_0)/2$,

$x(t-\tau/2) = 0$, for $\tau > (t - t_0)/2$,

Therefore, if $t - t_0 < 0$, the nonzero regions of $x(t+\tau/2)$ and $x(t-\tau/2)$ does not overlap and $x(t+\tau/2)\, x^*(t-\tau/2) = 0$ for all τ.

The importance of region property

V-G Advantages and Disadvantages of the WDF

Advantages: clarity

many good properties

suitable for analyzing the random process

Disadvantages: cross-term problem

more time for computation, especial for the signal with long time duration

not one-to-one

not suitable for $\exp\left(jt^{n}\right),\ n \neq 0,1,2$

V-H Windowed Wigner Distribution Function

When $x(t)$ is not time-limited, its WDF is hard for implementation

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$\Big\downarrow \text{ with mask}$$

$$W_x(t,f)=\int_{-\infty}^{\infty} w(\tau)x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$w(\tau)$ is real and time-limited

Advantages: (1) reduce the computation time

(2) may reduce the cross term problem

Disadvantages:

$\tau' = \frac{\tau}{2}, \quad \tau = 2\tau'$

$$W_x(t,f) = 2\int_{-\infty}^{\infty} \underline{\underline{w(2\tau')}}\, x(t+\tau') \cdot x^*(t-\tau')\, e^{-j4\pi\tau' f} \cdot d\tau'$$

$t = n\Delta_t, \ \tau' = p\Delta_t \quad f = m\Delta_f$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-\infty}^{\infty} w(2p\Delta_t)\, x((n+p)\Delta_t)\, x^*((n-p)\Delta_t)\, e^{-j4\pi mp\Delta_t\Delta_f}\Delta_t}$$

Suppose that $w(t) = 0$ for $|t| > B$

$w(2p\Delta_t) = 0$ for $p < -Q$ and $p > Q$

$$Q = \frac{B}{2\Delta_t}$$

$$\boxed{W_x(n\Delta_t, m\Delta_f) = 2\sum_{p=-Q}^{Q} w(2p\Delta_t)\, x((n+p)\Delta_t)\, x^*((n-p)\Delta_t)\, e^{-j4\pi mp\Delta_t\Delta_f}\Delta_t}$$

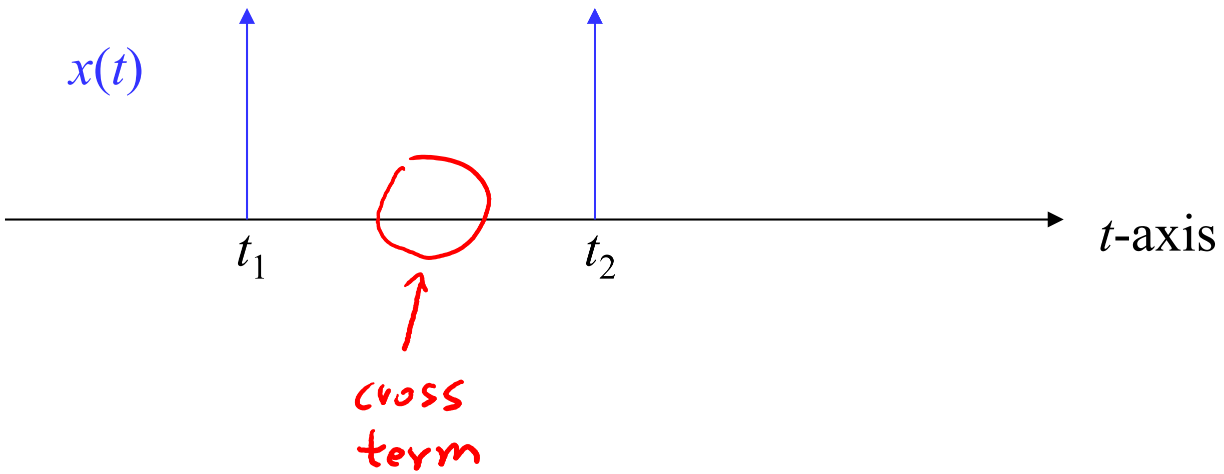
當然，乘上 mask 之後，有一些數學性質將會消失

(B) Why the cross term problem can be avoided ?

$$W_x(t,f) = \int_{-\infty}^{\infty} w(\tau)\, x(t+\tau/2) \cdot x^*(t-\tau/2)\, e^{-j2\pi\tau f} \cdot d\tau$$

$w(\tau)$ is real

Viewing the case where $x(t) = \delta(t - t_1) + \delta(t - t_2)$



理想情形：　$W_x(t,f)=0$　　for $t \neq t_1, t_2$

然而，當 mask function $w(\tau)=1$ 時　(也就是沒有使用 mask function)

$$y(t,\tau)=x(t+\tau/2) \qquad y^*(t,-\tau)=x^*(t-\tau/2)$$

$$W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$$

$$=\int_{-\infty}^{\infty}\left[\delta\left(t+\frac{\tau}{2}-t_1\right)+\delta\left(t+\frac{\tau}{2}-t_2\right)\right]\left[\delta\left(t-\frac{\tau}{2}-t_1\right)+\delta\left(t-\frac{\tau}{2}-t_2\right)\right]e^{-j2\pi\tau f}\cdot d\tau$$

from page 133, property 2

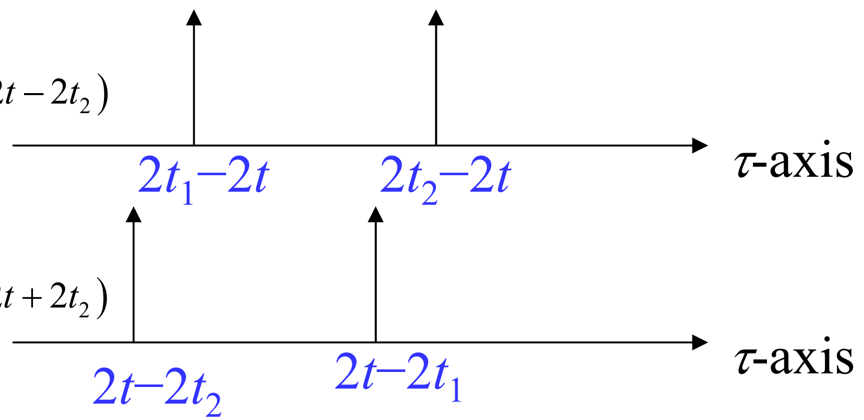
$$=4\int_{-\infty}^{\infty}\underbrace{\left[\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)\right]}_{\text{第一項}}\underbrace{\left[\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)\right]}_{\text{第二項}}e^{-j2\pi\tau f}\cdot d\tau$$

第一項

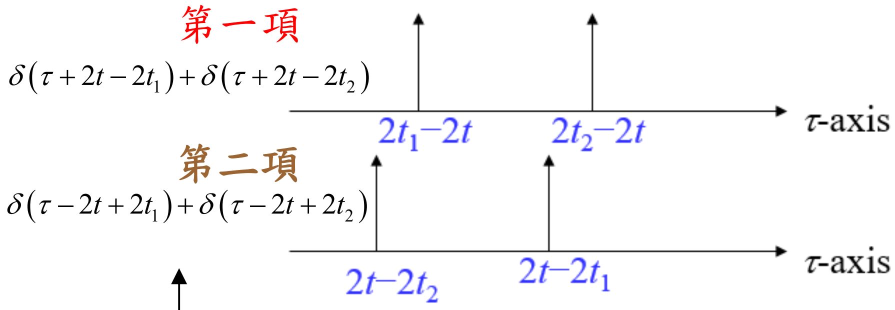
$\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)$

第二項

$\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)$

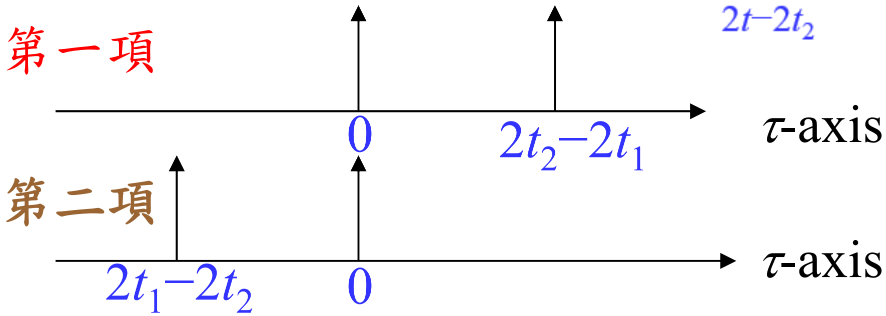


3種情形 $W_x(t,f) \neq 0$



(1) If $t = t_1$

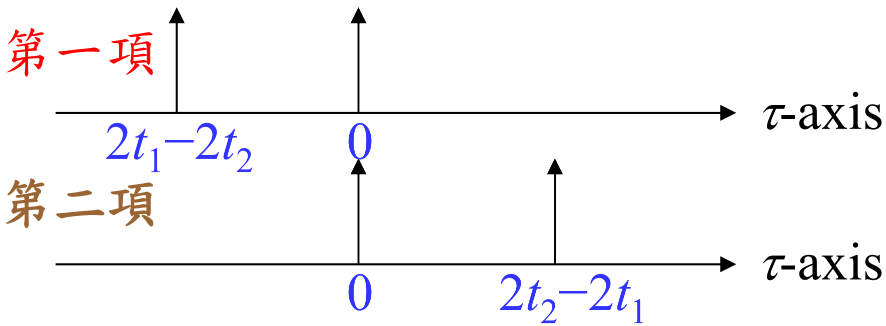
auto term



$2t_1-2t = 2t-2t_1$
$4t_1 = 4t$
$t_1 = t$

(2) If $t = t_2$

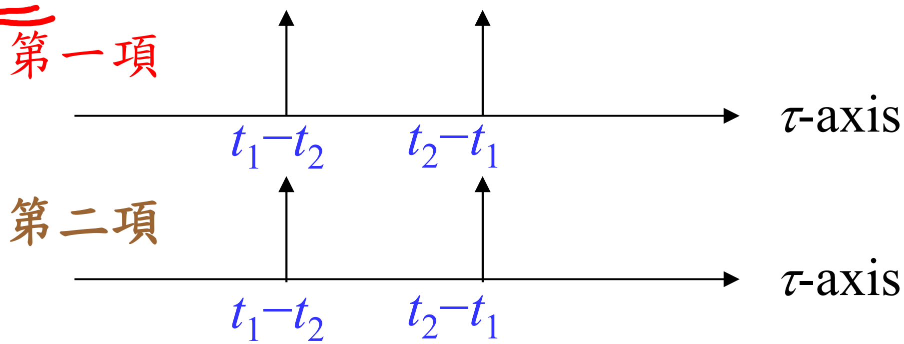
auto term



$2t_2-2t = 2t-2t_2$
$4t = 4t_2$
$t = t_2$

(3) If $t = (t_1 + t_2)/2$

cross term



$2t_1-2t=2t-2t_2$
$2t_1+2t_2=4t$
$t=\dfrac{t_1+t_2}{2}$

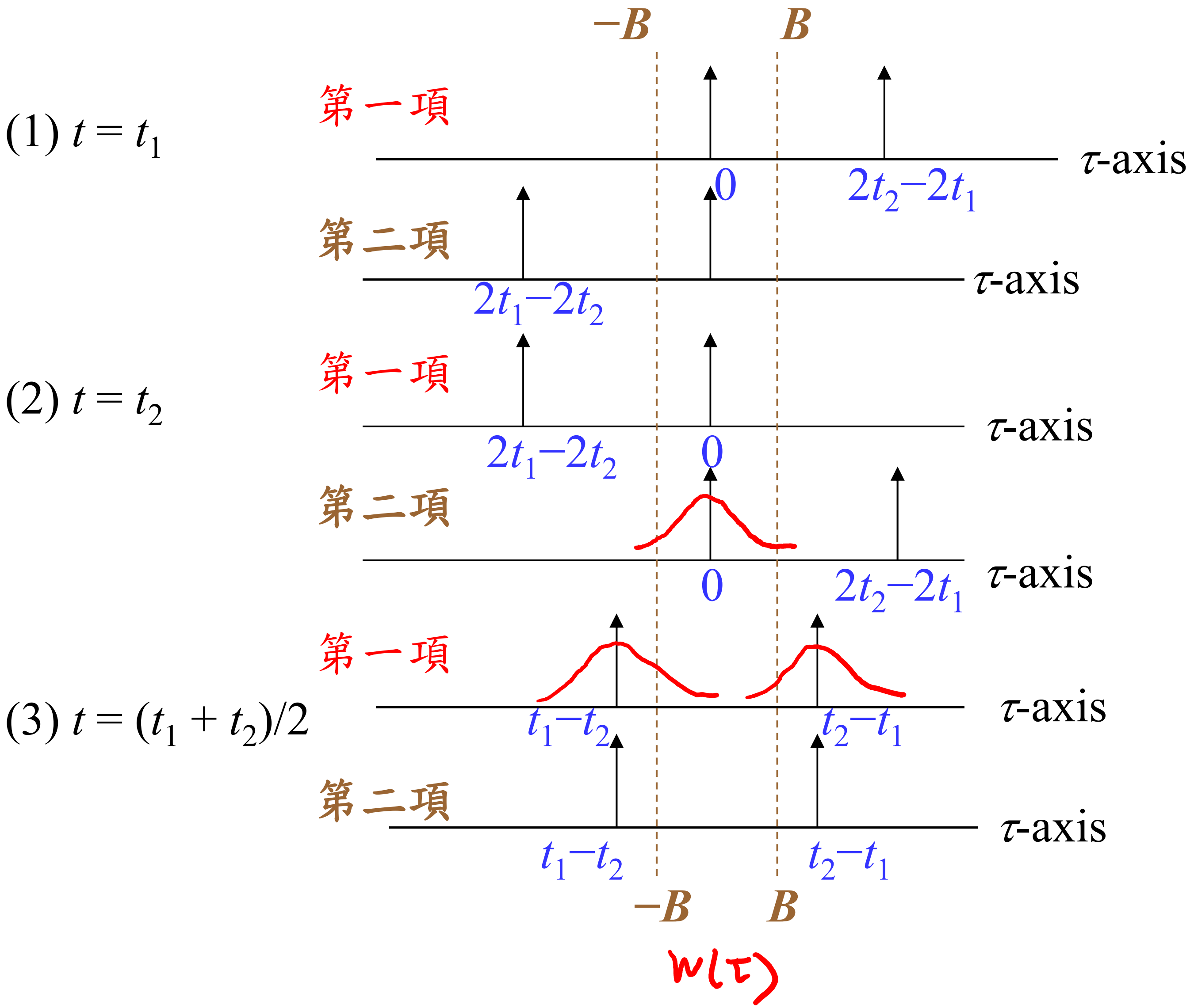
With mask function

$$W_x(t,f) = \int_{-\infty}^{\infty} w(\tau)\, x(t+\tau/2)\, x^*(t-\tau/2)\, e^{-j2\pi\tau f} \cdot d\tau$$

$$\begin{aligned} = \int_{-\infty}^{\infty} w(\tau) & \left[\delta(\tau+2t-2t_1)+\delta(\tau+2t-2t_2)\right] \\ & \times\left[\delta(\tau-2t+2t_1)+\delta(\tau-2t+2t_2)\right] e^{-j2\pi\tau f} \cdot d\tau \end{aligned}$$

Suppose that $w(\tau) = 0$ for $|\tau| > B$, B is positive.

If $B < t_2 - t_1$


−B
B
(1) t = t1
第一項
τ-axis
0
2t2−2t1
第二項
2t1−2t2
τ-axis
(2) t = t2
第一項
2t1−2t2
0
τ-axis
第二項
0
2t2−2t1
τ-axis
(3) t = (t1 + t2)/2
第一項
t1−t2
t2−t1
τ-axis
第二項
t1−t2
t2−t1
τ-axis
−B
B
w(τ)

附錄八： 研究所學習新知識把握的要點

(1) Concepts: 這個方法的核心概念、基本精神是什麼

(2) Comparison: 這方法和其他方法之間，有什麼相同的地方？
有什麼相異的地方

(3) Advantages: 這方法的優點是什麼

(3-1) Why? 造成這些優點的原因是什麼

(4) Disadvantages: 這方法的缺點是什麼

(4-1) Why? 造成這些缺點的原因是什麼

(5) Applications: 這個方法要用來處理什麼問題，有什麼應用

(6) Innovations: 這方法有什麼可以改進的地方
或是可以推廣到什麼地方

看過一篇論文或一個章節之後，若能夠回答 (1)-(5) 的問題，就代表你已經學通了這個方法

如果你的目標是發明創造出新的方法，可試著回答 (3-1), (4-1), 和 (6) 的問題

每個領域每個月至少都有100篇以上的新論文，閱讀能力再厲害的人，也不可能都像大學讀書一樣篇篇都逐字讀過，所以，要選讀哪幾篇論文，哪些要詳讀，哪些可以只抓重點，這種擇要的能力，是研究所學生們應該練習的。