X. Other Applications of Time-Frequency Analysis

Applications

(1) Finding Instantaneous Frequency

(2) Signal Decomposition

(3) Filter Design

(4) Sampling Theory

(5) Modulation and Multiplexing

(6) Electromagnetic Wave Propagation

(7) Optics

(8) Radar System Analysis

(9) Random Process Analysis

(10) Music Signal Analysis

(11) Biomedical Engineering

(12) Accelerometer Signal Analysis

(13) Acoustics 聲學

(14) Data Compression

(15) Spread Spectrum Analysis

(16) System Modeling

(17) Image Processing

(18) Economic Data Analysis

(19) Signal Representation

(20) Seismology 地震學

(21) Geology

(22) Astronomy

(23) Oceanography

10-1 Sampling Theory

Number of sampling points == Sum of areas of time frequency distributions
+ the number of extra parameters

- How to make the area of time-frequency smaller?

(1) Divide into several components.

(2) Use chirp multiplications, chirp convolutions, fractional Fourier transforms, or linear canonical transforms to reduce the area.

[Ref] X. G. Xia, "On bandlimited signals with fractional Fourier transform," *IEEE Signal Processing Letters*, vol. 3, no. 3, pp. 72-74, March 1996.

[Ref] J. J. Ding, S. C. Pei, and T. Y. Ko, "Higher order modulation and the efficient sampling algorithm for time variant signal," *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

Analytic Signal Conversion

$$x(t) \rightarrow \underline{x_a(t)} = \underline{x(t) + jx_H(t)}$$

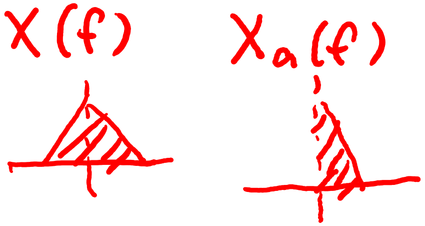
$$X_H(f) = X(f) \cdot \begin{cases} j & f<0 \\ -j & f>0 \end{cases} \quad (f=0) = -j\,\mathrm{sgn}(f)$$

$$X_a(f) = X(f) + j(-j\,\mathrm{sgn}(f))\,X(f)$$

$$= X(f) + \begin{cases} -1 & f<0 \\ 1 & f>0 \end{cases} X(f) \quad (f=0)$$

$$= \begin{cases} 2X(f) & \text{for } f>0 \\ 0 & \text{for } f<0 \\ X(0) & \text{for } f=0 \end{cases}$$

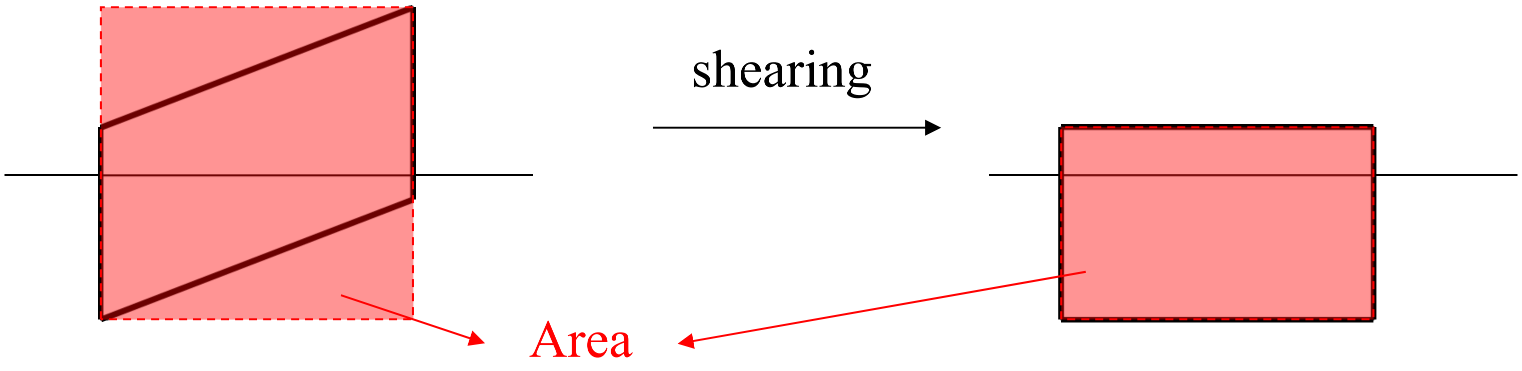
If $x(t)$ is real
$X(f) = X^*(-f)$



single sided band

Shearing

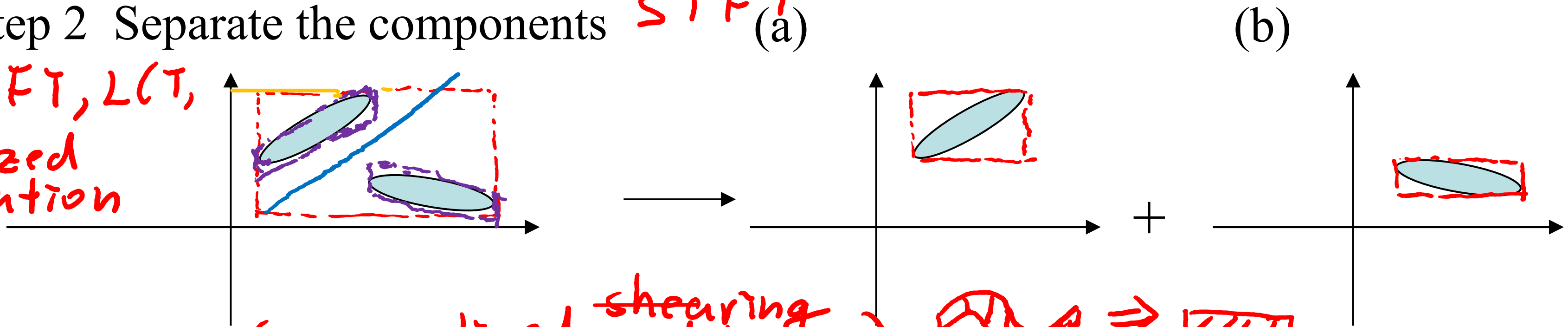
$$x(t) = \mathrm{Re}(x_a(t))$$



Step 1 Analytic Signal Conversion

Step 2 Separate the components STFT

FT, FrFT, LCT, generalized modulation



(generalized ~~shearing~~ modulation)

Step 3 Use shearing or rotation to minimize the “area” to each component

WDF

WDF

Step 4 Use the conventional sampling theory to sample each components

$\Delta t_1 < \frac{1}{F_1}$ $\Delta t_2 < \frac{1}{F_2}$

傳統的取樣方式

$$x_d[n] = x(n\Delta_t) \qquad \Delta_t < 1/F \qquad F=2B$$

重建： $$x(t) = \sum_n x_d[n]\,\mathrm{sinc}\left(\frac{t}{\Delta_t} - n\right)$$

新的取樣方式 $x_H(t)$: Hilbert transform of $x(t)$

(1) $x(t) \to x_a(t) = x(t) + jx_H(t)$

(2) $x_a(t) \to x_a(t) = x_1(t) + x_2(t) + \cdots + x_K(t)$

(3) $y_k(t) = \exp\left(j2\pi a_k t^2\right) x_k(t) \qquad k = 1, 2, \ldots, K$

(4) $$x_{d,k}[n] = y_k(n\Delta_{t,k}) \qquad k = 1, 2, \ldots, K$$

$$= \exp\left(j2\pi a_k n^2 \Delta_{t,k}^2\right) x_k(n\Delta_{t,k})$$

重建：

(1) $y_k(t) = \sum_n x_{d,k}[n] \operatorname{sinc}\left(\frac{t}{\Delta_{t,k}} - n\right)$

(2) $x_k(t) = \exp\left(-j2\pi a_k t^2\right) y_k(t)$

(3) $x_a(t) = x_1(t) + x_2(t) + \cdots + x_K(t)$

(4) $x(t) = \mathcal{R}e\left\{x_a(t)\right\}$

嚴格來說，沒有一個信號的 時頻分佈的「面積」是有限的。

Theorem:

If $x(t)$ is time limited ($x(t) = 0$ for $t < t_1$ and $t > t_2$)

then it is impossible to be frequency limited

If $x(t)$ is frequency limited ($X(f) = 0$ for $f < f_1$ and $f > f_2$)

then it is impossible to be time limited

但是我們可以選一個 "threshold" Δ

時頻分析 $|X(t, f)| > \Delta$ 或 的區域的面積是有限的

實際上，以「面積」來討論取樣點數，是犧牲了一些精確度。

只取 $t \in [t_1, t_2]$ and $f \in [f_1, f_2]$ 犧牲的能量所佔的比例

$$err = \frac{\int_{-\infty}^{t_1} |x(t)|^2 dt + \int_{t_2}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{f_1} |X_1(f)|^2 df + \int_{f_2}^{\infty} |X_1(f)|^2 df}{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

$\boxed{X_1(f) = FT[x_1(t)],}$ $x_1(t) = x(t)$ for $t \in [t_1, t_2]$, $x_1(t) =$ 0 otherwise

- For the Wigner distribution function (WDF)

$$|x(t)|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,df, \qquad |X(f)|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,dt$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,df dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad = \quad \text{energy of } x(t).$$

$$\left|x(t)\right|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,df \qquad \left|X(f)\right|^2 = \int_{-\infty}^{\infty} W_x(t,f)\,dt$$

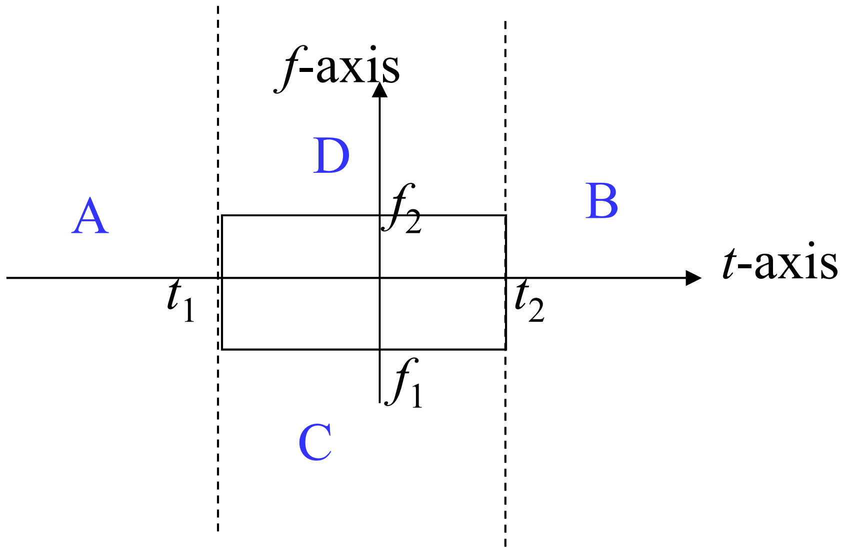
$$\int_{-\infty}^{t_1}\left|x(t)\right|^2 dt + \int_{t_2}^{\infty}\left|x(t)\right|^2 dt + \int_{-\infty}^{f_1}\left|X_1(f)\right|^2 df + \int_{f_2}^{\infty}\left|X_1(f)\right|^2 df$$

$$= \int_{-\infty}^{t_1}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{t_2}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{-\infty}^{\infty}\int_{-\infty}^{f_1} W_{x_1}(t,f)\,dfdt + \int_{-\infty}^{\infty}\int_{f_2}^{\infty} W_{x_1}(t,f)\,dfdt$$

$$= \int_{-\infty}^{t_1}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{t_2}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt + \int_{t_1}^{t_2}\int_{-\infty}^{f_1} W_{x_1}(t,f)\,dfdt + \int_{t_1}^{t_2}\int_{f_2}^{\infty} W_{x_1}(t,f)\,dfdt$$

$$\cong \underbrace{\int_{-\infty}^{t_1}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt}_{A} + \underbrace{\int_{t_2}^{\infty}\int_{-\infty}^{\infty} W_x(t,f)\,dfdt}_{B} + \underbrace{\int_{t_1}^{t_2}\int_{-\infty}^{f_1} W_x(t,f)\,dfdt}_{C} + \underbrace{\int_{t_1}^{t_2}\int_{f_2}^{\infty} W_x(t,f)\,dfdt}_{D}$$

$$err \cong 1 - \frac{\int_{t_1}^{t_2}\int_{f_1}^{f_2} W_x(t,f)\,dfdt}{\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt}$$



10-2 Modulation and Multiplexing

With the aid of

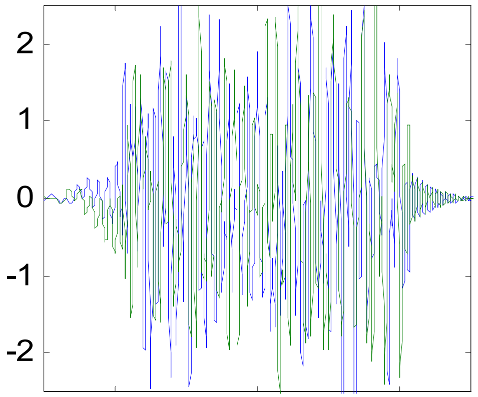
(1) the Gabor transform (or the Gabor-Wigner transform)

(2) horizontal and vertical shifting, dilation, shearing, generalized shearing, and rotation.

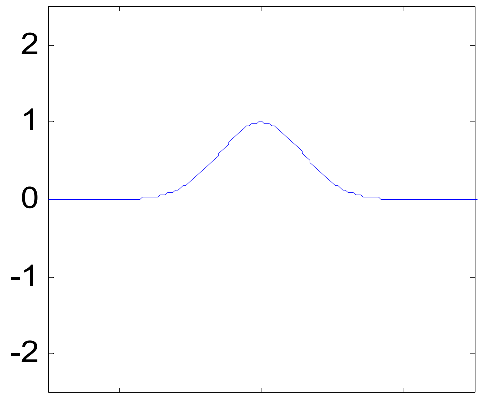
[Ref] C. Mendlovic and A. W. Lohmann, "Space-bandwidth product adaptation and its application to superresolution: fundamentals," *J. Opt. Soc. Am. A*, vol. 14, pp. 558-562, Mar. 1997.

[Ref] S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," vol. 55, issue 10, pp. 4839-4850, *IEEE Trans. Signal Processing*, 2007.

Example



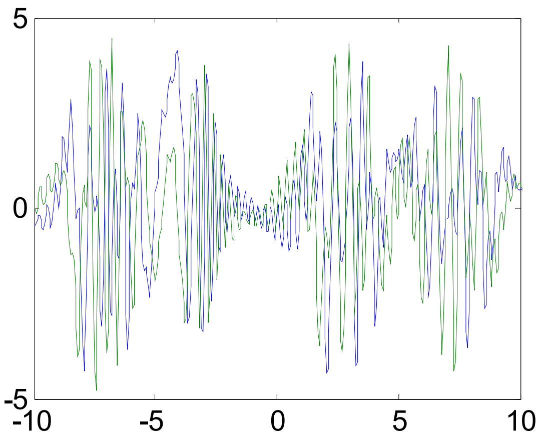
(a) G(u), consisted of 7 components



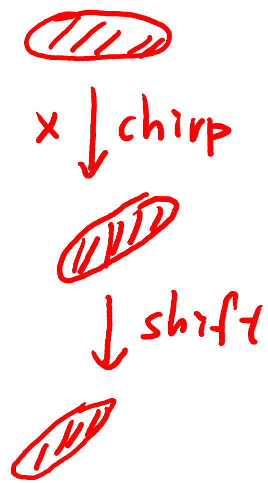
(b) f(t), the signal to be modulated

FT

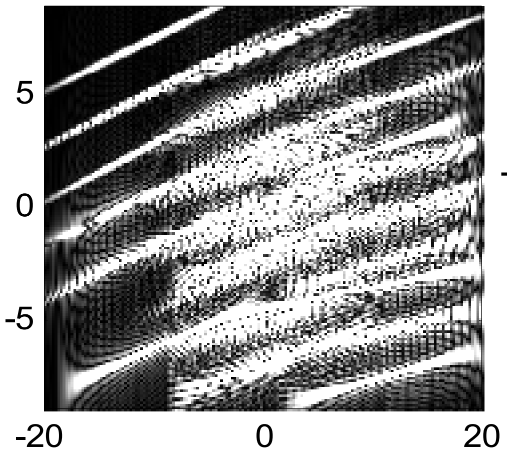
We want to add $f(t)$ into $G(u)$



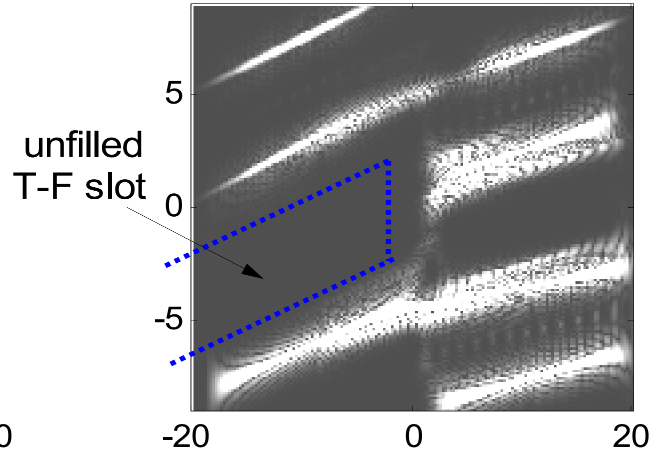
(no empty band)



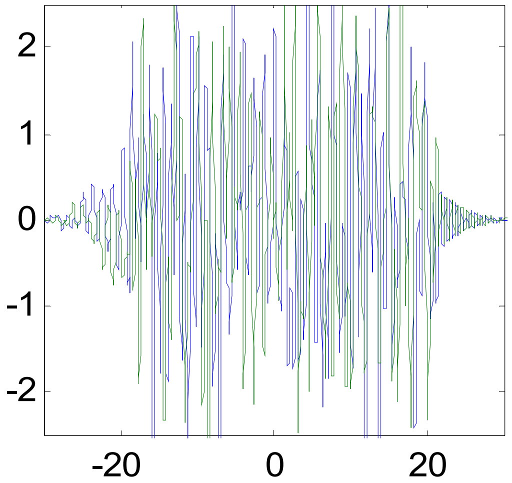
STFT



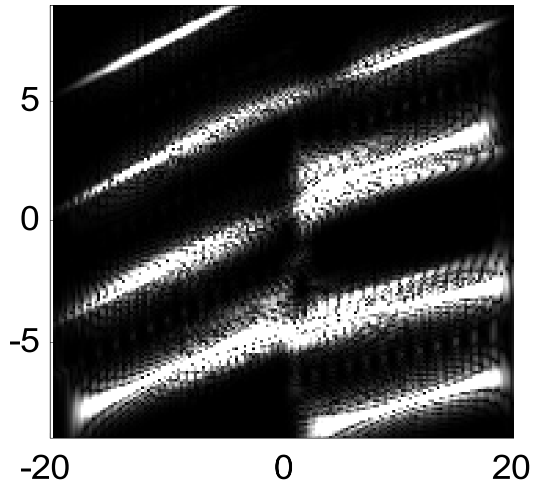
(c) WDF of G(u)



(d) GWT of G(u)



(e) multiplexing f(t) into G(u)



(f) GWT of (e)

◎ Conventional Modulation Theory

The signals $x_1(t)$, $x_2(t)$, $x_3(t)$, ……., $x_K(t)$ can be transmitted successfully if

$$\text{Allowed Bandwidth} \geq \sum_{k=1}^{K} B_k$$

B_k: the bandwidth (including the negative frequency part) of $x_k(t)$

◎ Modulation Theory Based on Time-Frequency Analysis

The signals $x_1(t)$, $x_2(t)$, $x_3(t)$, ……., $x_K(t)$ can be transmitted successfully if

$$\text{Allowed Time duration} \times \text{Allowed Bandwidth} \geq \sum_{k=1}^{K} A_k$$

A_k: the area of the time-frequency distribution of $x_k(t)$

- The interference is inevitable.

How to estimate the interference?

10-3 Electromagnetic Wave Propagation

Time-Frequency analysis can be used for

Wireless Communication

Optical system analysis

Laser

Radar system analysis

Propagation through the free space (Fresnel transform): chirp convolution

Propagation through the lens or the radar disk: chirp multiplication convolution with chirp

Fresnel Transform ：描述電磁波在空氣中的傳播 (See page 260-264)

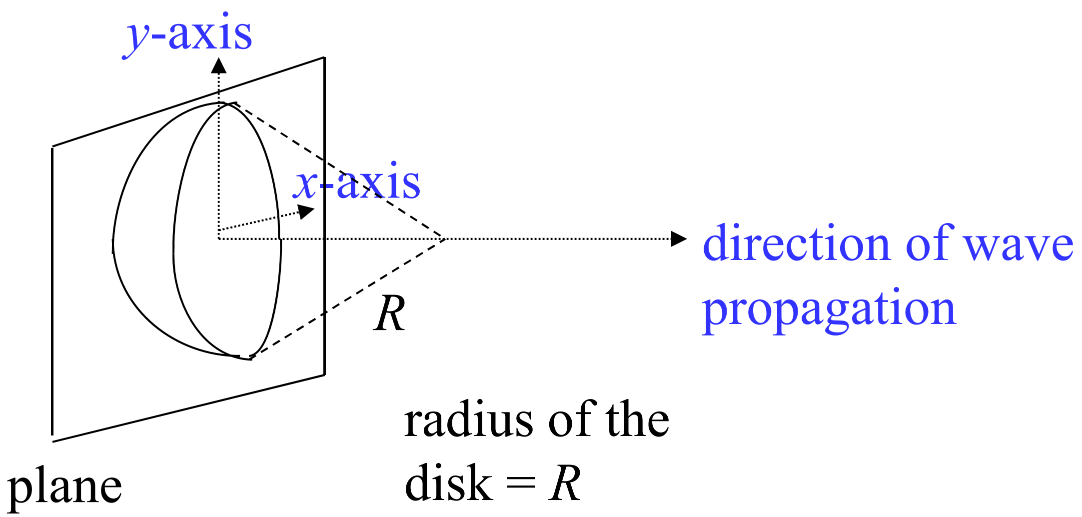
電磁波包括光波、雷達波、紅外線、紫外線………

Fresnel transform == LCT with parameters $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

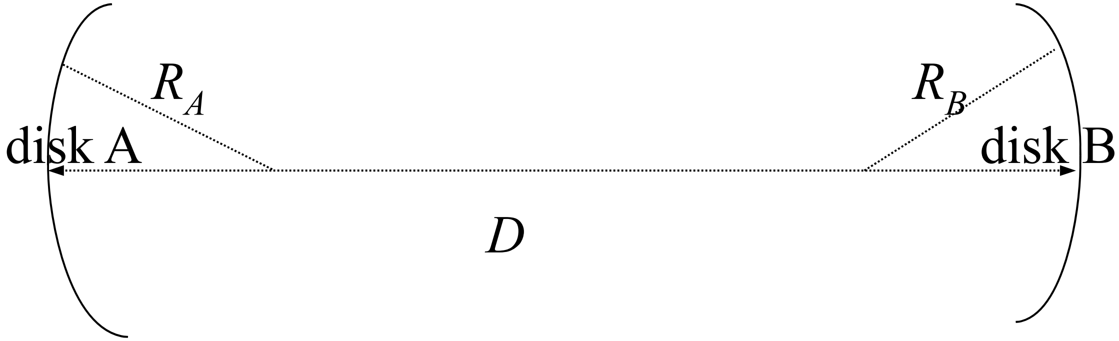
思考： (1) STFT 或 WDF 哪一個比較適合用在電磁波傳播的分析？

(2) 為何波長越短的電磁波，在空氣中散射的情形越少？

(4) Spherical Disk   x chirp



Disk 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\lambda R & 1 \end{bmatrix}$ 的情形



相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\lambda R_B & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda R_A & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 - D/R_A & -\lambda D \\ -\frac{1}{\lambda}\left(R_A^{-1} - R_B^{-1} + R_A^{-1} R_B^{-1} D\right) & 1 + D/R_B \end{bmatrix}$$

的情形

10-4 Music and Acoustic Signal Analysis

Music Signal Analysis

Acoustic

Voiceprint (Speaker) Recognition

Speech Signal ：

(1) 不同的人說話聲音頻譜不同 (聲紋 voiceprint)

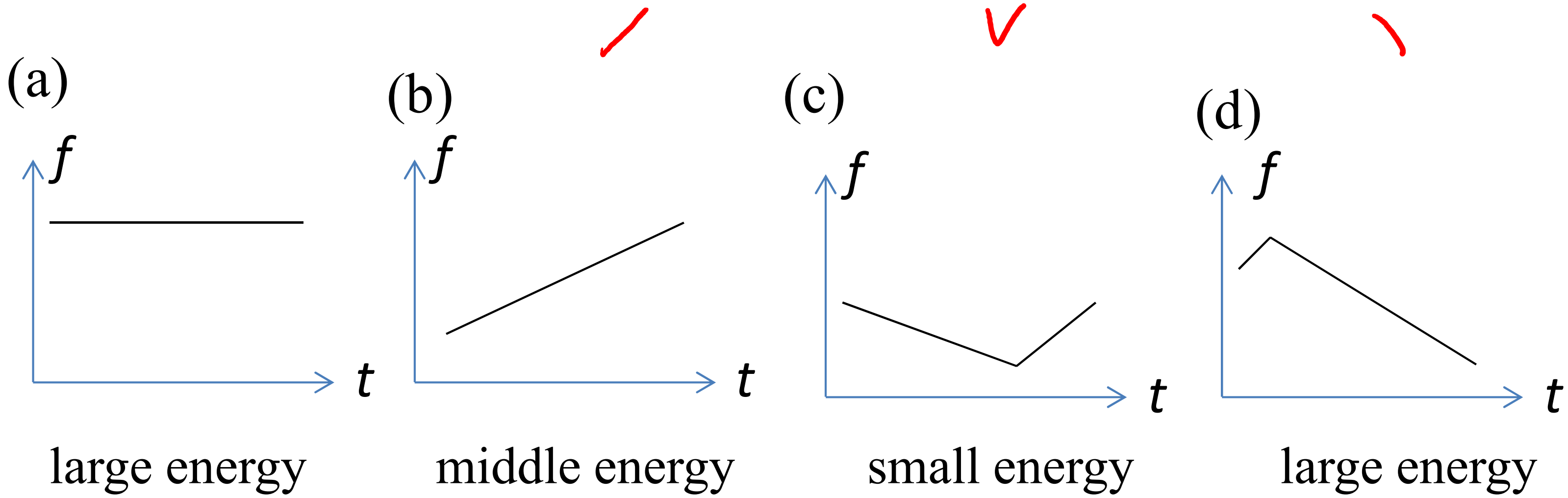
(2) 同一個人但不同的字音，頻譜不一樣

(3) 語調 (第一、二、三、四聲和輕聲) 不同，則頻譜變化的情形也不同

(4) 即使同一個字音，子音和母音的頻譜亦不相同

(5) 雙母音本身就會有頻譜的變化

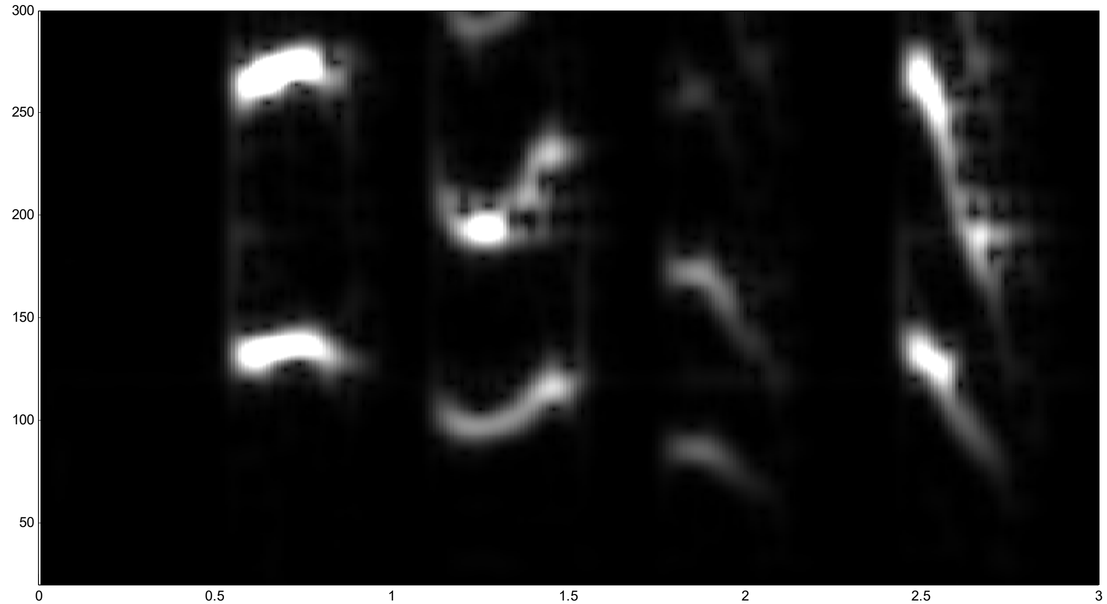
• 王小川，“語音訊號處理”，第二章，全華出版，台北，民國94年。



Typical relations between time and the instantaneous frequencies for (a) the 1st tone, (b) the 2nd tone, (c) the 3rd tone, and (d) the 4th tone in Chinese.

5th tone
(but the energy is even smaller)

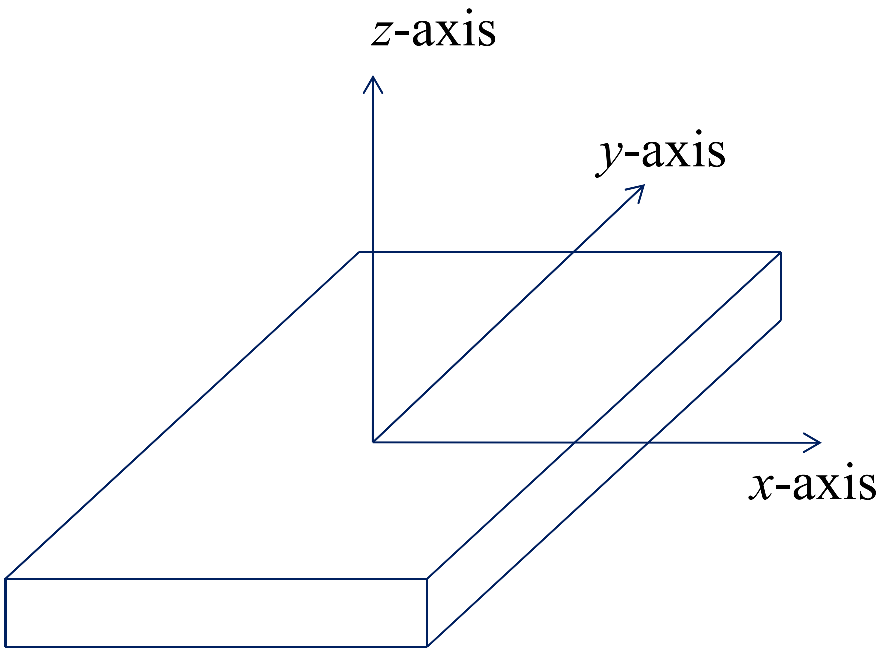
X. X. Chen, C. N. Cai, P. Guo, and Y. Sun, "A hidden Markov model applied to Chinese four-tone recognition," *ICASSP*, vol. 12, pp. 797-800, 1987.

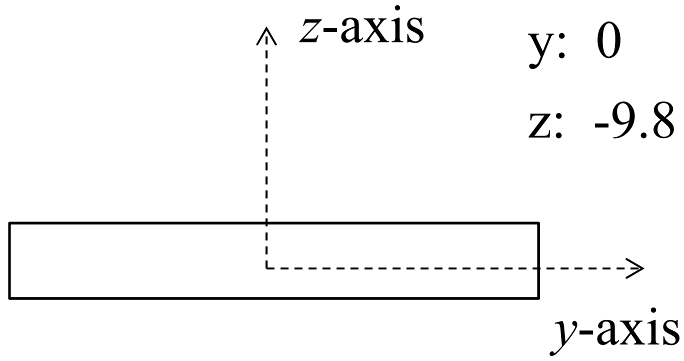


ㄚ1, ㄚ2, ㄚ3, ㄚ4

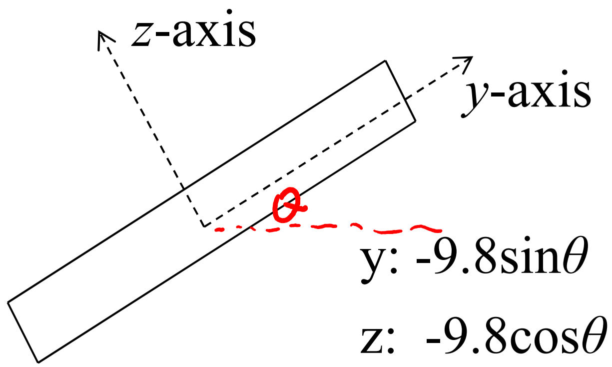
10-5 Accelerometer Signal Analysis

The 3-D Accelerometer (三軸加速規) can be used for identifying the activity of a person.





tilted by θ

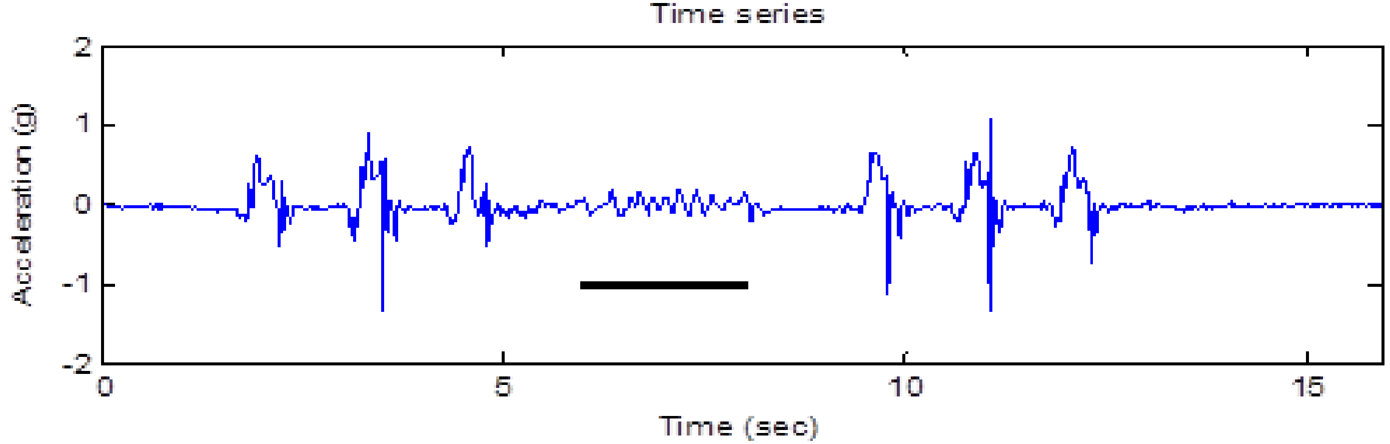


Using the 3D accelerometer + time-frequency analysis, one can analyze the activity of a person.

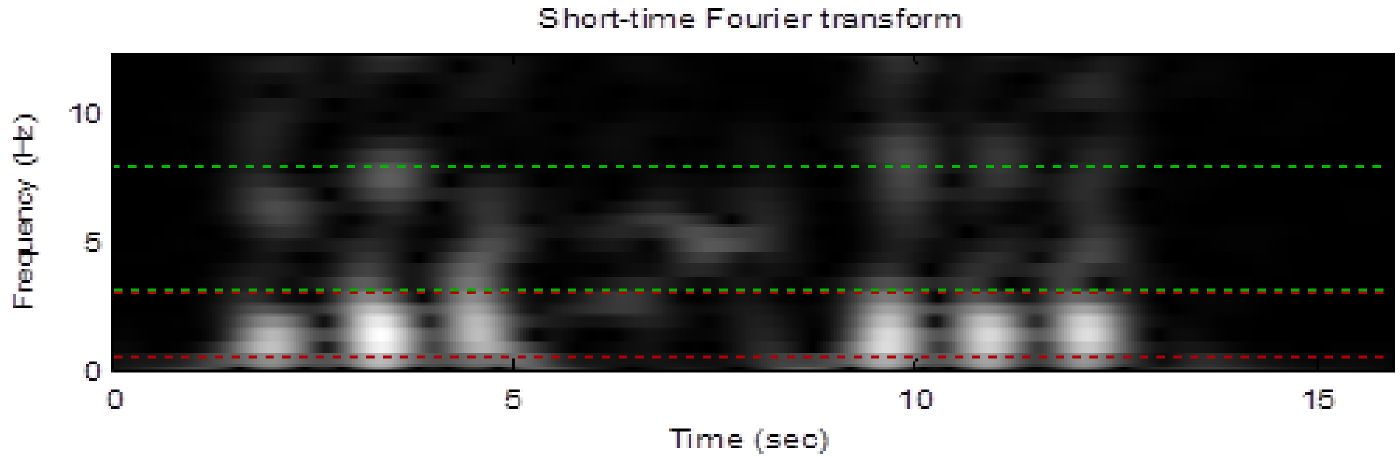
Walk, Run (Pedometer 計步器)

Healthcare for the person suffered from Parkinson’s disease

3D accelerometer signal for a person suffering from Parkinson's disease



The result of the short-time Fourier transform



Y. F. Chang, J. J. Ding, H. Hu, Wen-Chieh Yang, and K. H. Lin, "A real-time detection algorithm for freezing of gait in Parkinson's disease," *IEEE International Symposium on Circuits and Systems*, Melbourne, Australia, pp. 1312-1315, May 2014

10-6 Other Applications

時頻分析適用於頻譜會隨著時間而改變的信號

Biomedical Engineering (心電圖 (ECG), 肌電圖 (EMG), 腦電圖, ……)

Communication and Spread Spectrum Analysis

Economic Data Analysis

Seismology

Geology

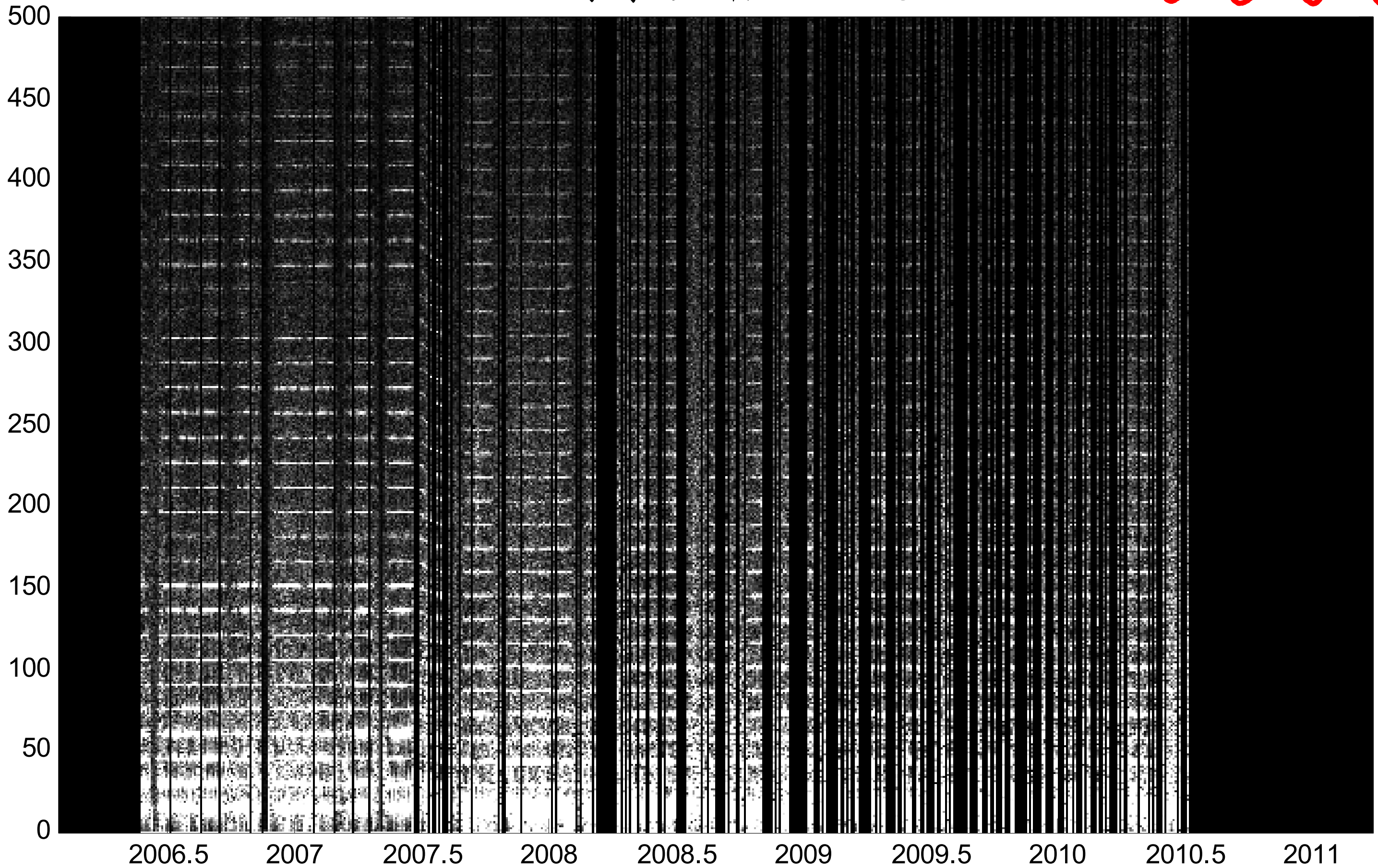
Astronomy

Oceanography

Satellite Signal

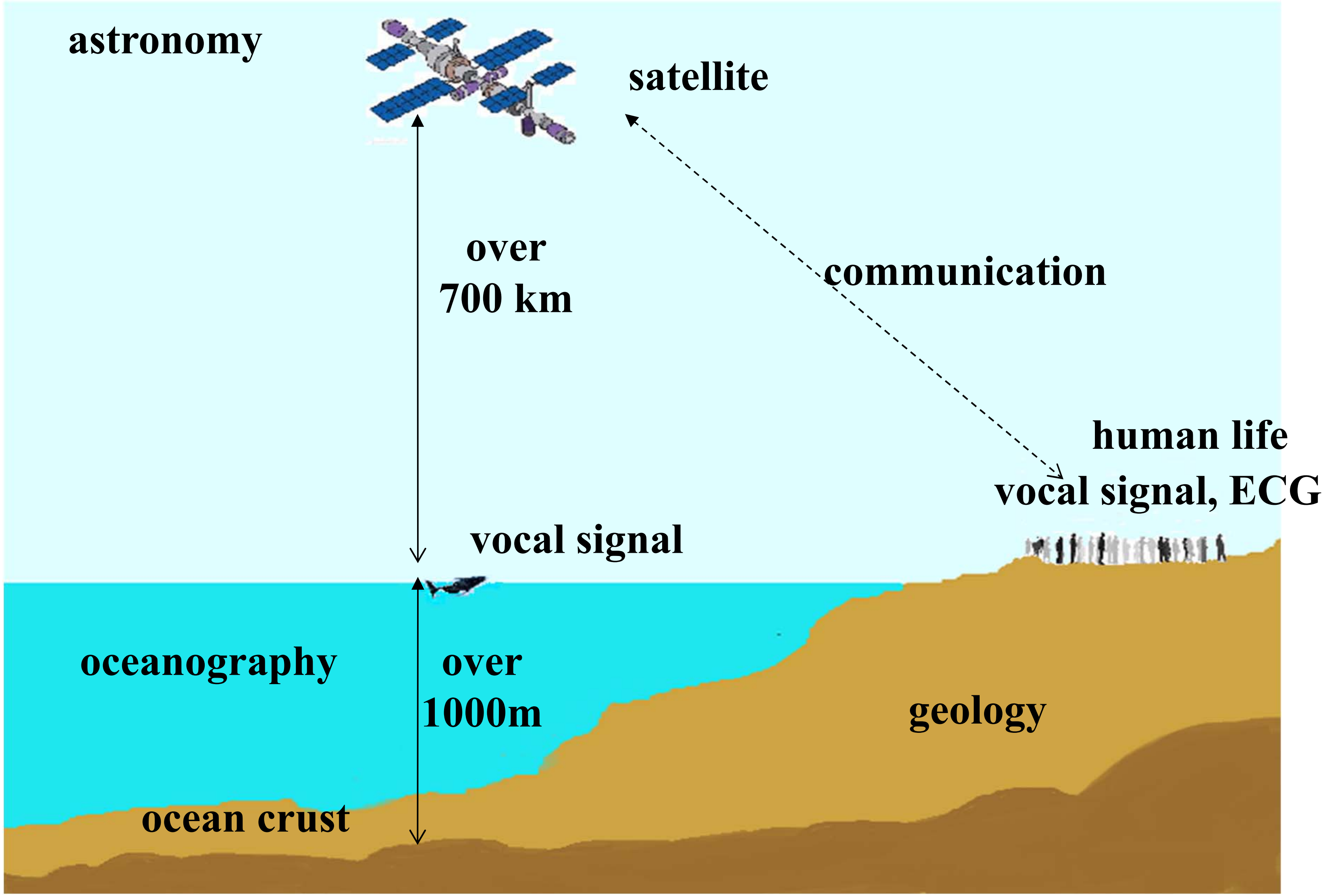
Short-time Fourier transform of the power signal from a satellite

福爾摩沙衛星三號



C. J. Fong, S. K. Yang, N. L. Yen, T. P. Lee, C. Y. Huang, H. F. Tsai, S. Wang, Y. Wang, and J. J. Ding, "Preliminary studies of the applications of HHT (Hilbert-Huang transform) on FORMOSAT-3/COSMIC GOX payload trending data," *6th FORMOSAT-3/COSMIC Data Users' Workshop*, Boulder, Colorado, USA, Oct. 2012

時頻分析的應用範圍


astronomy
satellite
over
700 km
communication
human life
vocal signal, ECG
vocal signal
oceanography
over
1000m
geology
ocean crust

附錄十：幾個常見的資料蒐尋方法

(1) Google 學術搜尋

http://scholar.google.com.tw/

(太重要了，不可以不知道) 只要任何的書籍或論文，在網路上有電子版，都可以用這個功能查得到

輸入關鍵字，或期刊名，或作者

再按「搜尋」，就可找到想要的資料

(2) 尋找 IEEE 的論文

http://ieeexplore.ieee.org/Xplore/guesthome.jsp

(3) Wikipedia

(4) Github (搜尋 code)

(5) 數學的百科網站

http://eqworld.ipmnet.ru/index.htm

有多個 tables，以及對數學定理的介紹

(6) 傳統方法：去圖書館找資料

台大圖書館首頁 http://www.lib.ntu.edu.tw/

或者去 http://www.lib.ntu.edu.tw/tulips

(7) 查詢其他圖書館有沒有我要找的書

「台大圖書館首頁」⟶「其他圖書館」

(8) 找尋電子書

「台大圖書館首頁」⟶「電子書」或「免費電子書」

(9) 查詢一個期刊是否為 SCI

Step 1: 先去 http://scientific.thomson.com/mjl/

Step 2: 在 Search Terms 輸入期刊全名

Search Type 選擇 “Full Journal Title”，再按 “Search”

Step 3: 如果有找到這期刊，那就代表這個期刊的確被收錄在 SCI

(10) 想要對一個東西作入門但較深入的了解:

看 journal papers 或 Wikipedia 會比看 conference papers 適宜

看書會比看 journal papers 或 Wikipedia 適宜

(11) 如果實在沒有適合的書籍，可以看 “review”， “survey”，或 “tutorial” 性質的論文

有了相當基礎之後，再閱讀 journal papers

(以 Paper Title， Abstract， 以及其他 Papers 對這篇文章的描述，

來判斷這篇 journal papers 應該詳讀或大略了解即可)

XI. Hilbert Huang Transform (HHT)

Proposed by 黃鍔院士 (AD. 1998)
黃鍔院士的生平可參考

http://djj.ee.ntu.edu.tw/%E9%BB%83%E9%8D%94%E9%99%A2%E5%A3%AB.pdf

References

[1] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N. C. Yen, C. C. Tung, and H. H. Liu, "The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis," *Proc. R. Soc. Lond. A*, vol. 454, pp. 903-995, 1998.

[2] N. E. Huang and S. Shen, *Hilbert-Huang Transform and Its Applications*, World Scientific, Singapore, 2005.

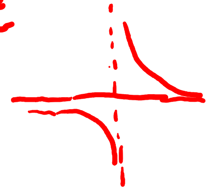
(PS: 謝謝 2007 年修課的趙逸群同學和王文阜同學)

11-A The Origin of the Concept

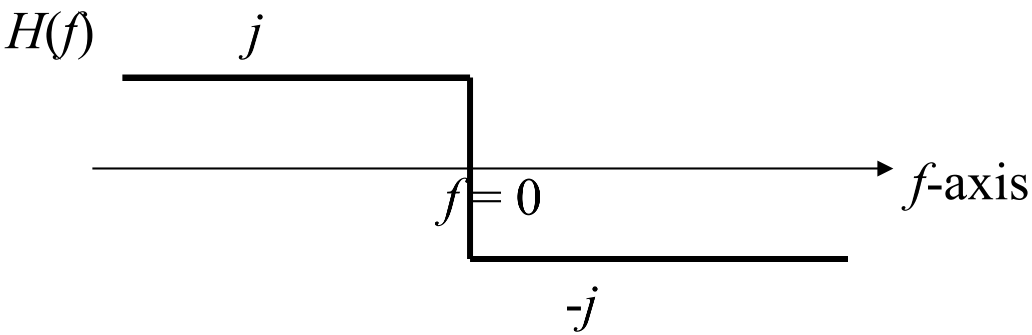
另一種分析 instantaneous frequency 的方式： Hilbert transform

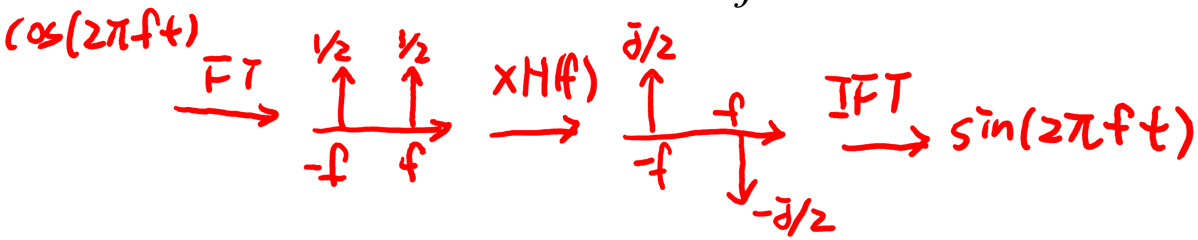
- Hilbert transform

$$x_H(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau \;=\; x(t) * \frac{1}{\pi t}$$



or $$x_H(t)=IFT\{FT[x(t)]H(f)\}$$





Applications of the Hilbert Transform

- analytic signal

$$x_a(t) = x(t) + jx_H(t)$$

- edge detection
- another way to define the instantaneous frequency:

$$\text{instantaneous frequency} = \frac{1}{2\pi}\frac{d}{dt}\theta$$

$$\text{where} \quad \theta = \tan^{-1}\frac{x_H(t)}{x(t)}$$

$-\cot(x) = \tan(x+\pi/2)$

Example:

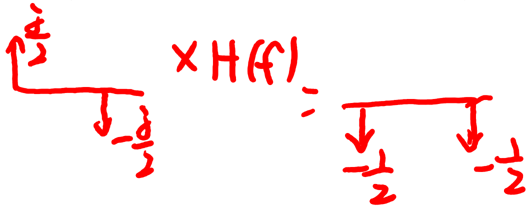
$$\cos(2\pi ft) \xrightarrow{\textit{Hilbert}} \sin(2\pi ft) \qquad \theta = 2\pi ft$$

$\theta = \tan^{-1}\left(\frac{\sin(2\pi ft)}{\cos(2\pi ft)}\right) = 2\pi ft$

$\frac{1}{2\pi}\frac{d\theta}{dt} = f$

$$\sin(2\pi ft) \xrightarrow{\textit{Hilbert}} -\cos(2\pi ft) \qquad \theta = 2\pi ft + \pi/2$$

$\frac{1}{2\pi}\frac{d\theta}{dt} = f$



$\theta = \tan^{-1}\left(\frac{-\cos(2\pi ft)}{\sin(2\pi ft)}\right) = \tan^{-1}(-\cot(2\pi ft))$

$= \tan^{-1}(\tan(2\pi ft + \frac{\pi}{2}))$

Problem of using Hilbert transforms to determine the instantaneous frequency:

This method is only good for cosine and sine functions with single component.

Not suitable for (1) complex function

(2) non-sinusoid-like function

(3) multiple components

Moreover, θ has multiple solutions.

signal $\xrightarrow{\text{decompose}}$ sinusoid-like components (IMFs)

Example:

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$

$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$

$$\cos(2\pi f_1 t)+\cos(2\pi f_2 t)\xrightarrow{Hilbert}\sin(2\pi f_1 t)+\sin(2\pi f_2 t)$$

$$\theta=\tan^{-1}\left(\frac{\sin(2\pi f_1 t)+\sin(2\pi f_2 t)}{\cos(2\pi f_1 t)+\cos(2\pi f_2 t)}\right)=\tan^{-1}\left(\frac{2\sin(\pi(f_1+f_2)t)\cos(\pi(f_1-f_2)t)}{2\cos(\pi(f_1+f_2)t)\cos(\pi(f_1-f_2)t)}\right)$$

$$=\pi(f_1+f_2)t$$

$$\frac{1}{2\pi}\frac{d\theta}{dt}=\frac{f_1+f_2}{2}$$

- Hilbert-Huang transform 的基本精神：

IMFs

先將一個信號分成多個 sinusoid-like components + trend

(和 Fourier analysis 不同的地方在於，這些 sinusoid-like components 的 period 和 amplitude 可以不是固定的)

再運用 Hilbert transform (或 STFT，number of zero crossings) 來分析每個 components 的 instantaneous frequency

完全不需用到 Fourier transform

潛在的

11-B Intrinsic Mode Function (IMF)

generalization of cos, sin

Amplitude and frequency can vary with time.

(iv) not IMF

但要滿足

local maximums & local minimums

(v) not IMF

(1) The number of extremes and the number of zero-crossings must either equal or differ at most by one.

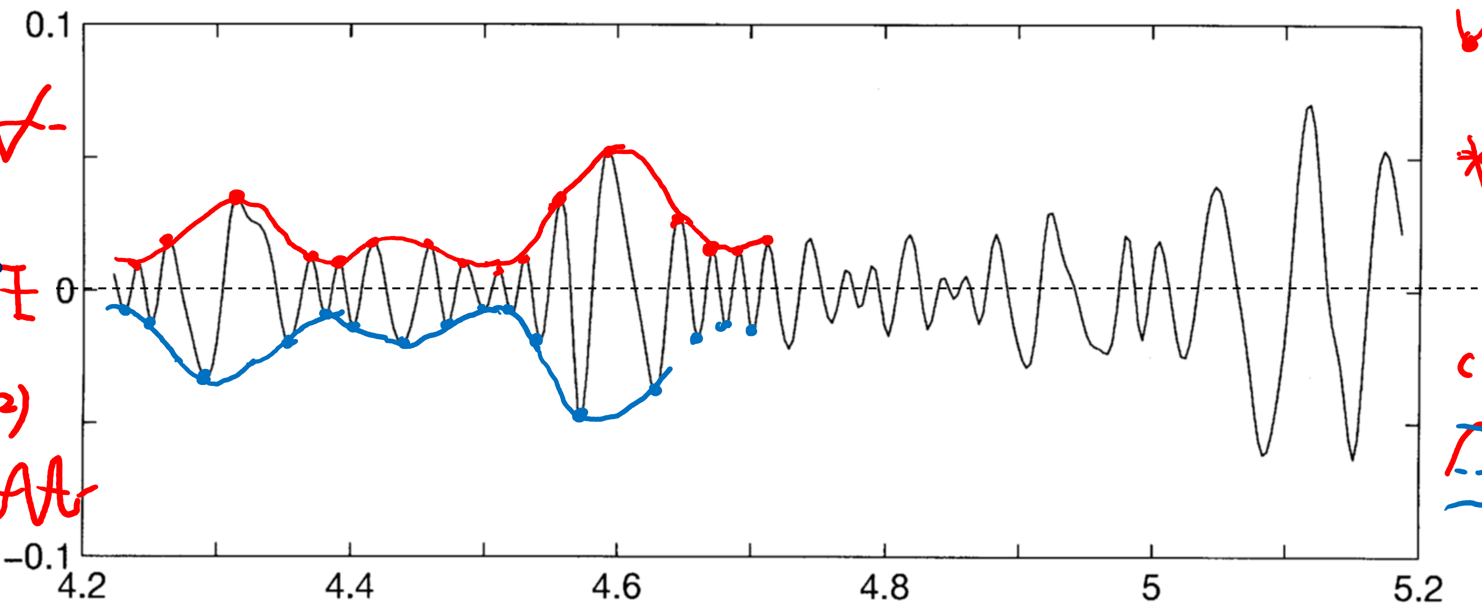
all local maximums > 0
all local minimums < 0

(2) At any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is near to zero.

(i) IMF

(ii) IMF

(iii) IMF $\cos(2\pi f t^2)$



cosine

11-C Procedure of the Hilbert Huang Transform

經驗法則

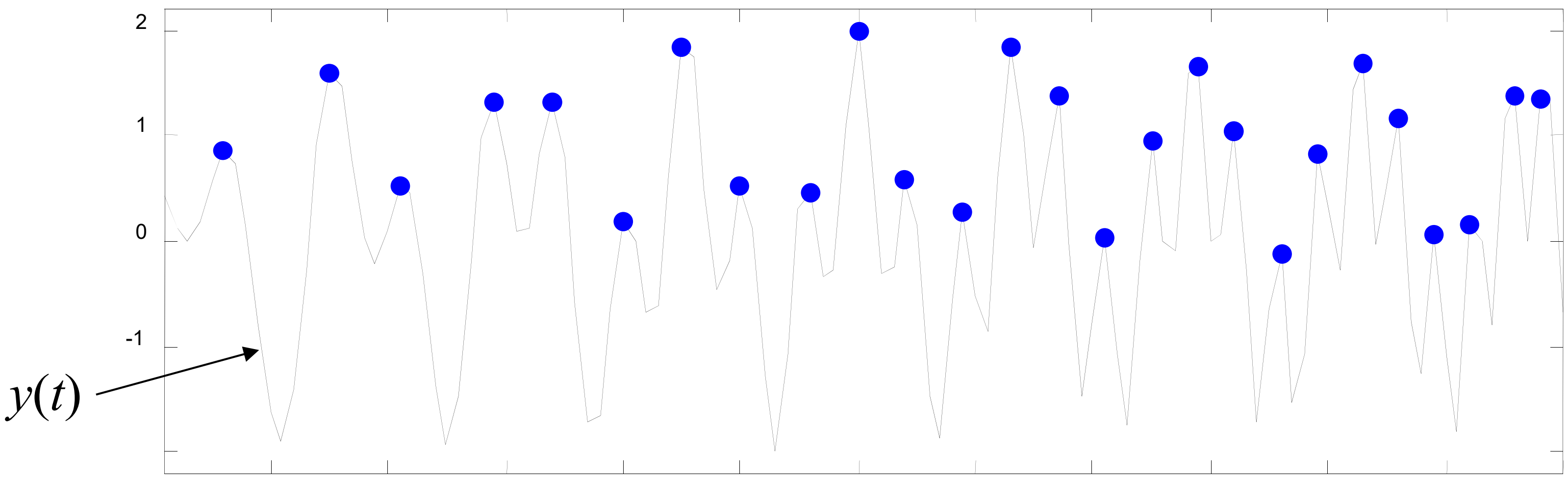
Steps 1~8 are called Empirical Mode Decomposition (EMD)

(Step 1) Initial: $y(t) = x(t)$, ($x(t)$ is the input) $n = 1$, $k = 1$

$\hat{y}[m] = y(m\Delta_t)$

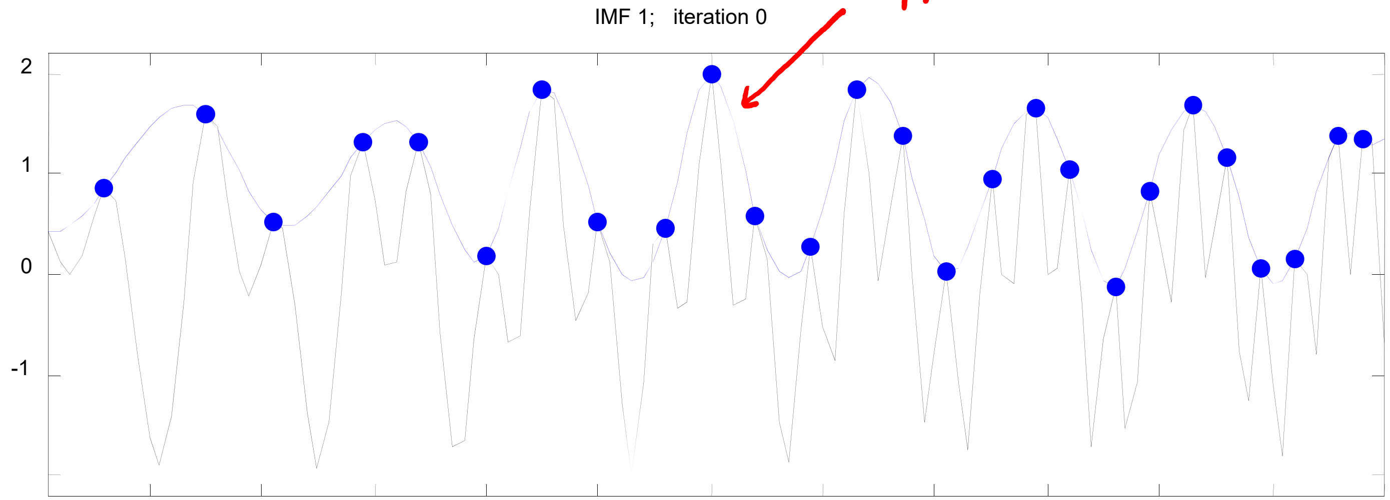
(Step 2) Find the local peaks

$(\hat{y}[m] > \hat{y}[m-1])$ && $(\hat{y}[m] > \hat{y}[m+1])$



(Step 3) Connect local peaks

upper envelope



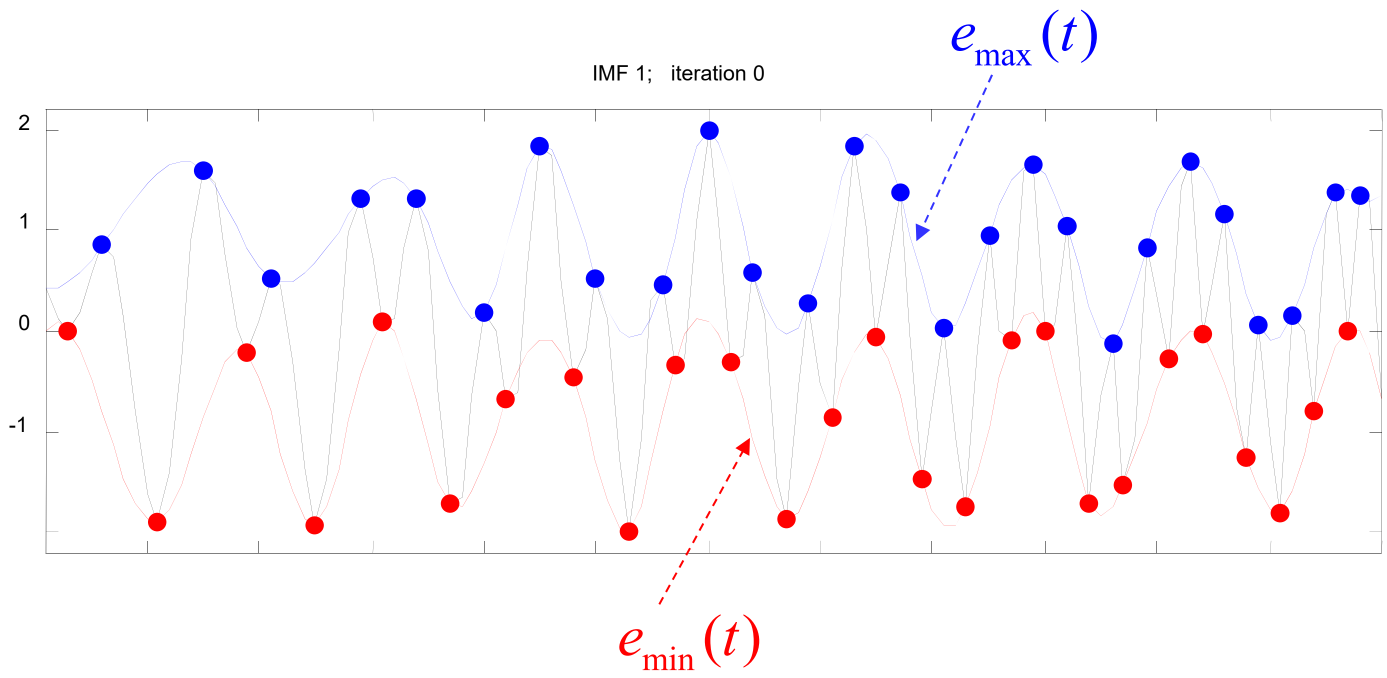
m=3

通常使用 B-spline，尤其是 cubic B-spline 來連接

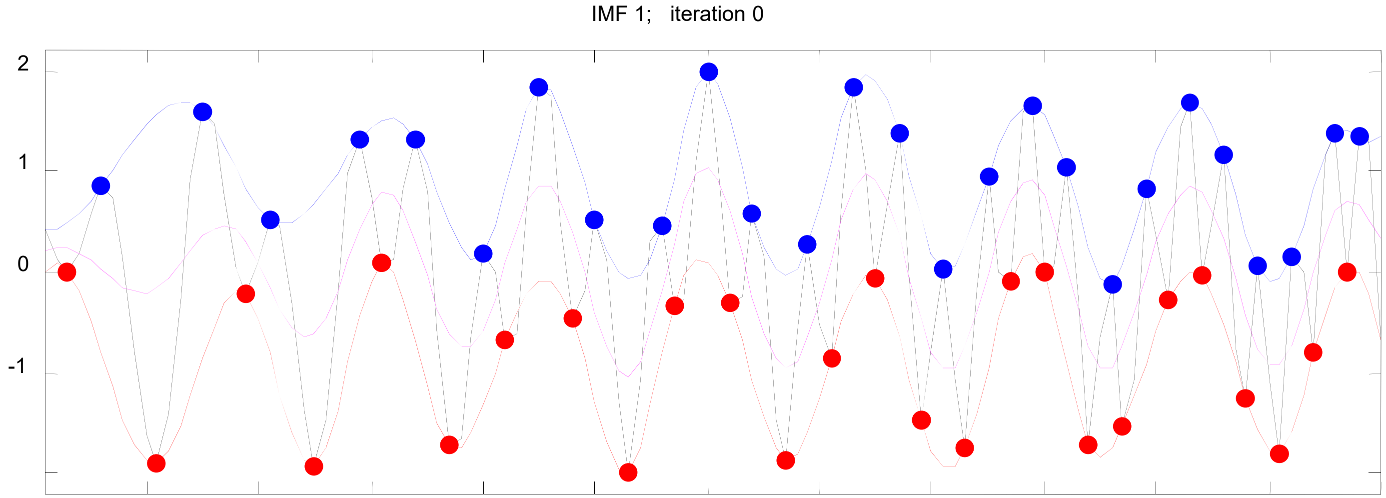
(參考附錄十一)

(Step 4) Find the local dips

(Step 5) Connect the local dips



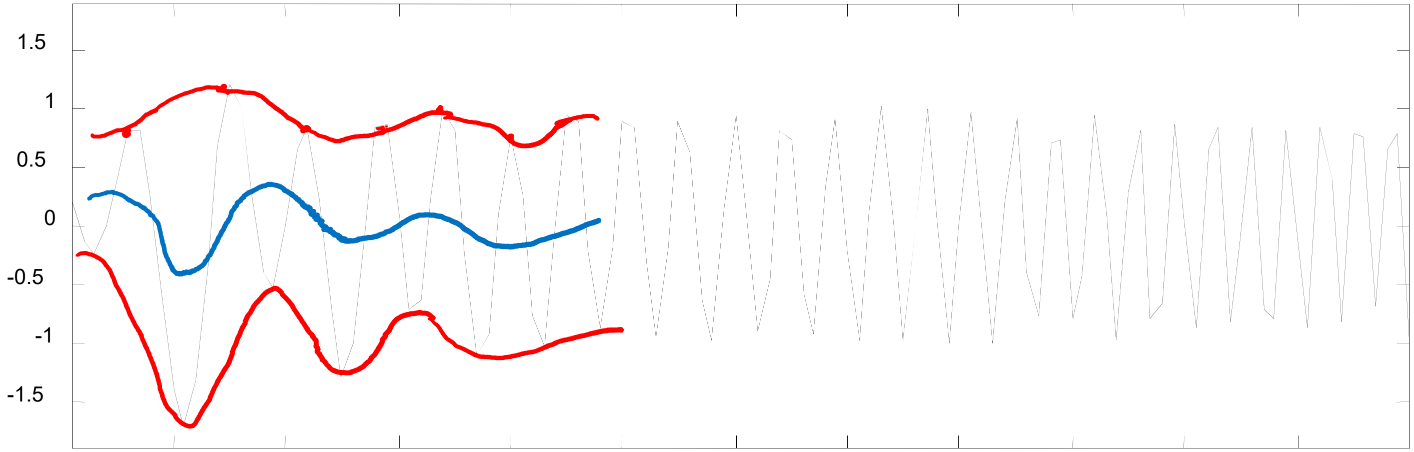
(Step 6-1) Compute the mean



$$z(t) = \frac{e_{\min}(t) + e_{\max}(t)}{2}$$

(pink line)

(Step 6-2) Compute the residue



$$h_k(t) = y(t) - z(t)$$

(Step 7) Check whether $h_k(t)$ is an intrinsic mode function (IMF)

(1) 檢查是否 local maximums 皆大於 0

local minimums 皆小於 0

(2) 上封包：$u_1(t)$，　下封包：$u_0(t)$

檢查是否 $\left|\dfrac{u_1(t)+u_0(t)}{2}\right| < threshold$ for all t

If they are satisfied (or $k \geqq K$), set $c_n(t) = h_k(t)$ and continue to Step 8

$c_n(t)$ is the n^{th} IMF of $x(t)$.

If not, set $y(t) = h_k(t)$,

$k = k + 1$, and repeat Steps 2~6

(為了避免無止盡的迴圈，可以定 k 的上限 K)

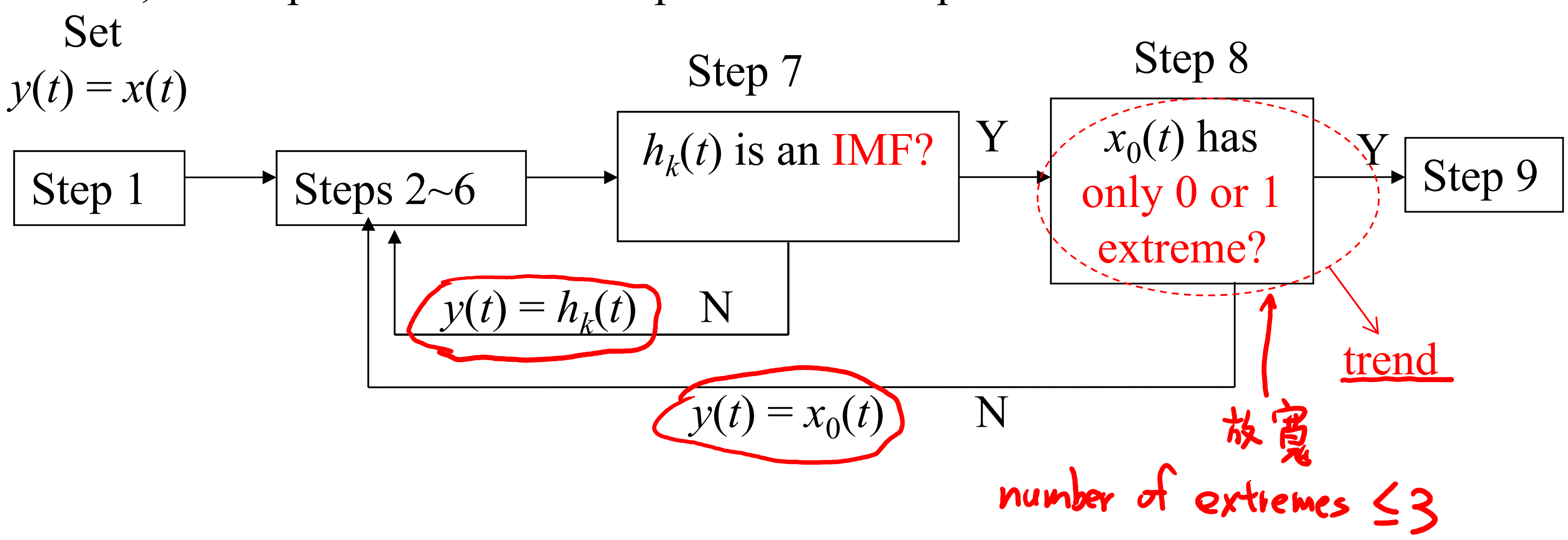
(Step 8) Calculate $x_0(t) = x(t) - \sum_{s=1}^{n} c_s(t)$

and check whether $x_0(t)$ is a function with no more than one extreme point.

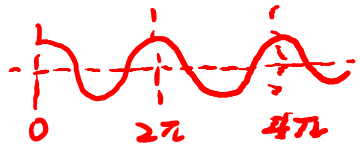
If not, set $n = n+1$, $y(t) = x_0(t)$

and repeat Steps 2~7

If so, the empirical mode decomposition is completed.



$$x(t) = x_0(t) + \sum_{s=1}^{n} c_s(t)$$



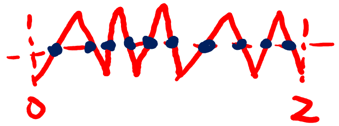
2 zero crossing: 1 period

(Step 9) Find the instantaneous frequency for each IMF $c_s(t)$ (s = 1, 2, …, n).

Method 1: Using the Hilbert transform

Method 2: Calculating the STFT for $c_s(t)$.

Method 3: Furthermore, we can also calculate the instantaneous frequency from the number of zero-crossings directly.



10 zero crossings
= 5 periods

instant. frequency $= \frac{5}{2} = 2.5$ Hz

instantaneous frequency $F_s(t)$ of $c_s(t)$

$$= \frac{\text{the number of zero-crossings of } c_s(t) \text{ between } t-B \text{ and } t+B}{4B}$$

Technique Problems of the Hilbert Huang Transform

(A) 邊界處理的問題：

目前尚未有一致的方法，可行的方式有

(1) 只使用非邊界的 extreme points

(2) 將最左、最右的點當成是 extreme points

(3) 預測邊界之外的 extreme points 的位置和大小

(4) 用邊界和最近的 extreme point 的距離來判斷是否邊界要當成 extreme points

local maximums: $p_1, p_2, \cdots p_N$

boundary: 0, B

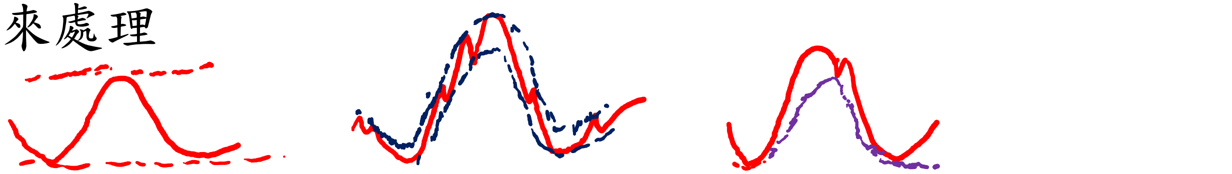
If $B - p_N > \alpha \, \mathrm{mean}(p_{n+1} - p_n)$

then B is a local maximum

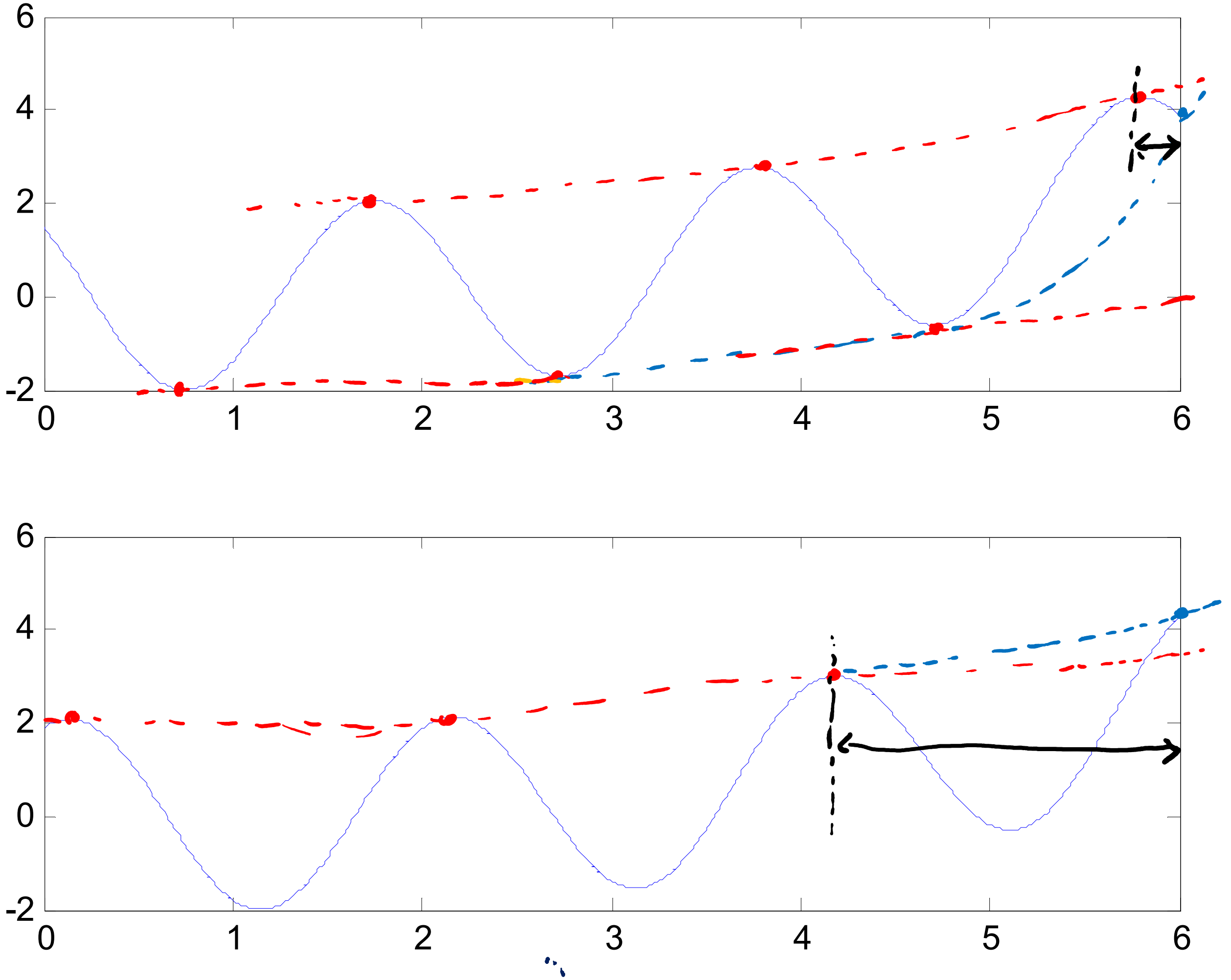
(B) Noise 的問題：

smoother

先用 pre-filter 來處理



最左、最右的點是否要當成是 extreme points



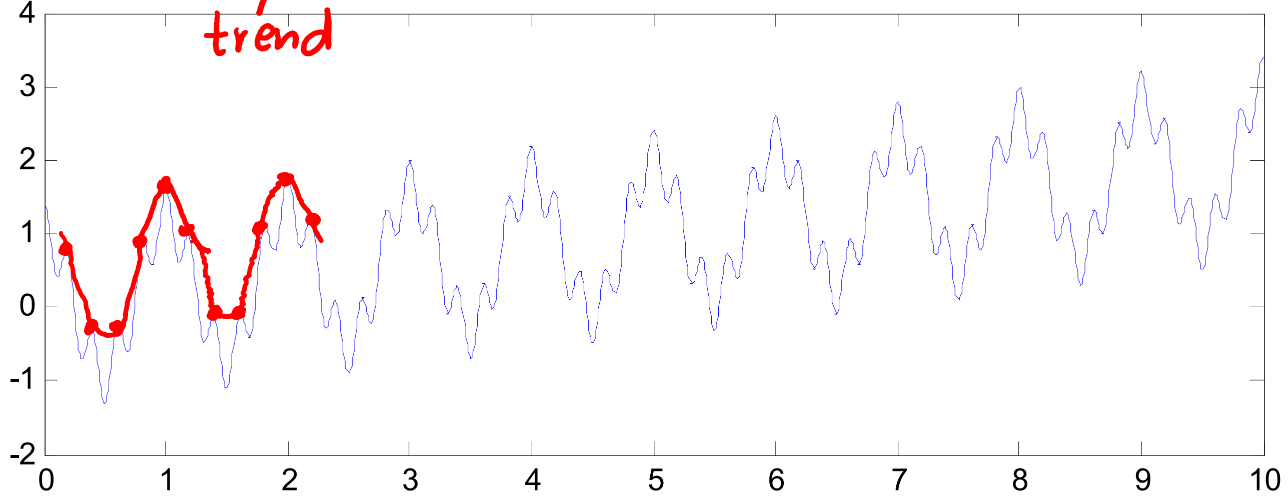
$1 \xrightarrow{FT} \delta(f) \qquad t \xrightarrow{FT} \frac{1}{j2\pi}\frac{d}{df}\delta(f) \qquad t^n \xrightarrow{FT} \left(\frac{j}{2\pi}\right)^n \delta^{(n)}(f)$

11-D Example

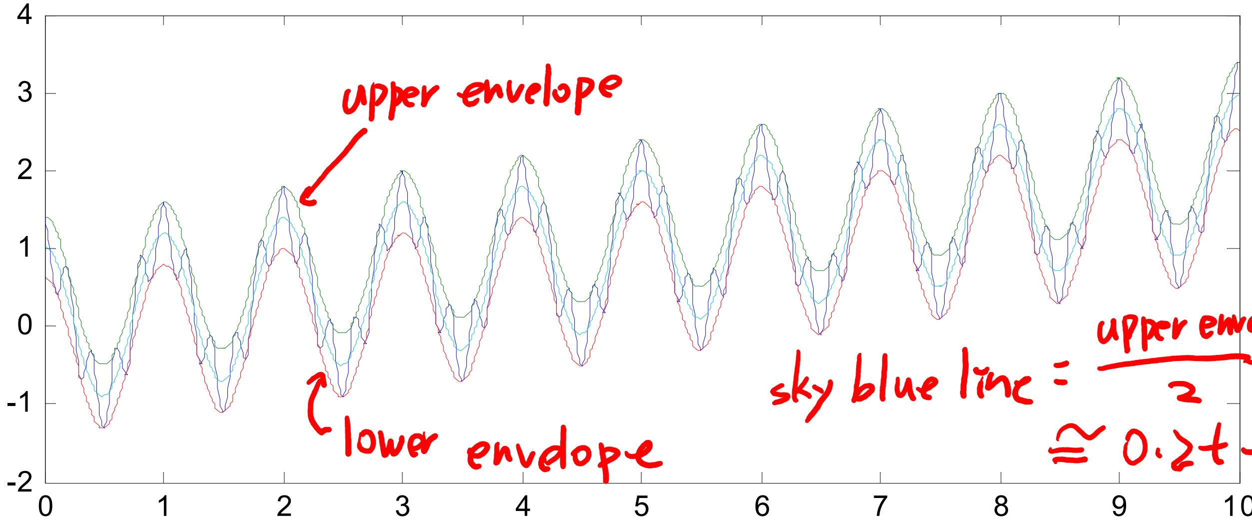
FT exists if $\int |x(t)|dt$ is finite

Example 1 $\quad x(t) = 0.2t + \cos(2\pi t) + 0.4\cos(10\pi t)$

$0.2t$ ← trend



After Step 6



sky blue line $= \frac{\text{upper envelope} + \text{lower envel}}{2}$

$\cong 0.2t + \cos(2\pi t)$

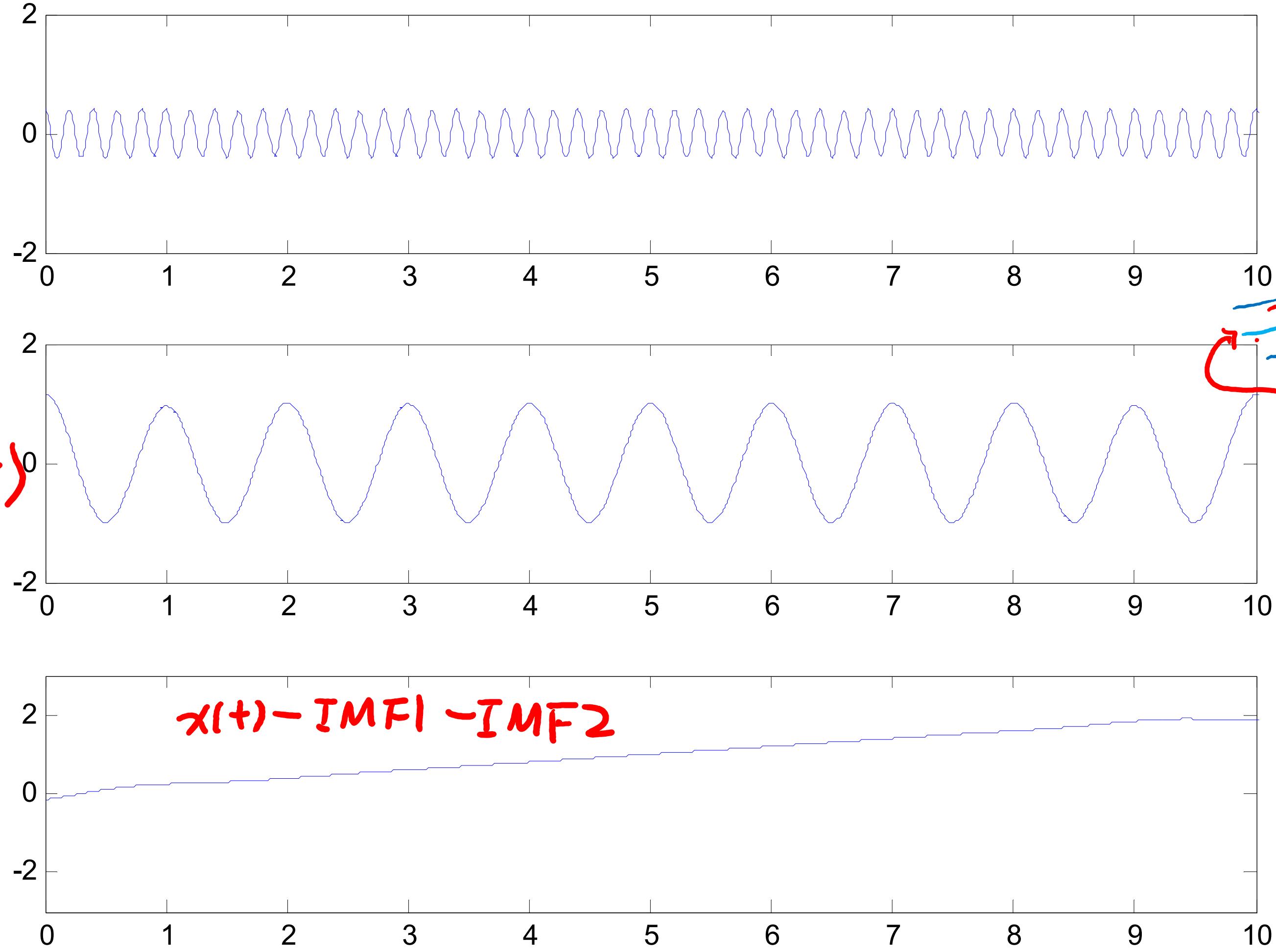
IMF1 = $x(t)$ − sky blue line $\cong 0.4\cos(10\pi t)$

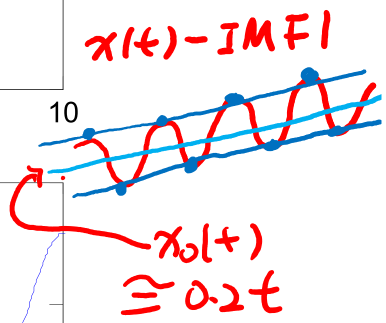
IMF1

IMF2 $\cong \cos(2\pi t)$

$x_0(t)$

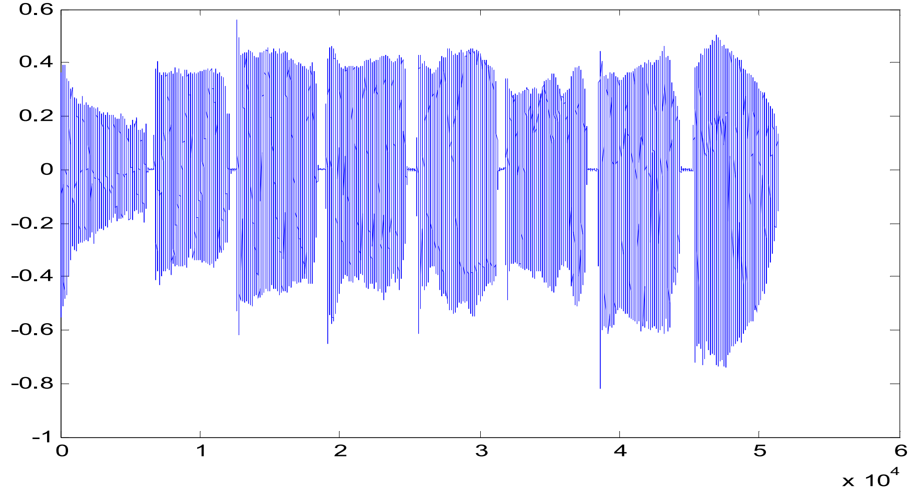
$0.2t$



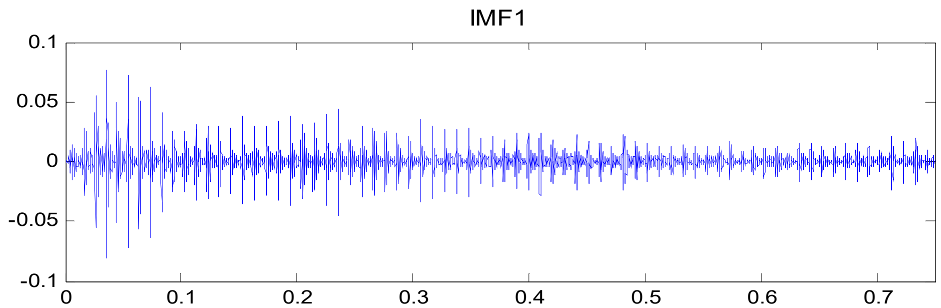


Example 2

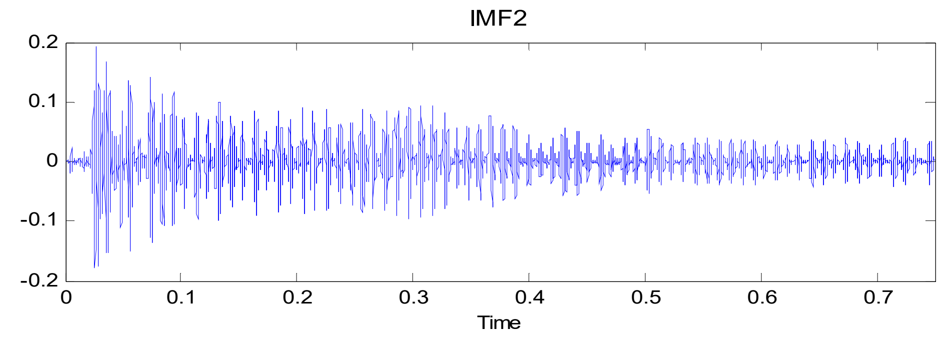
hum signal



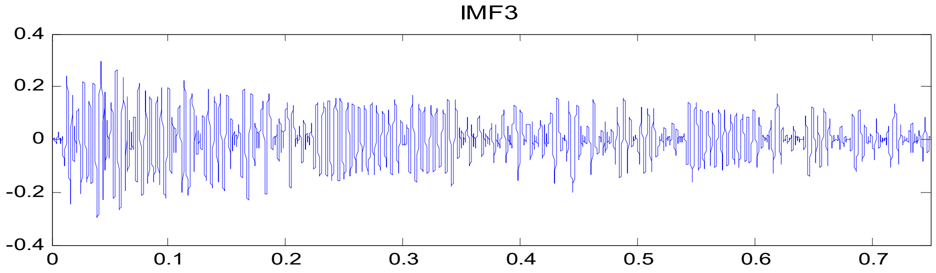
IMF1



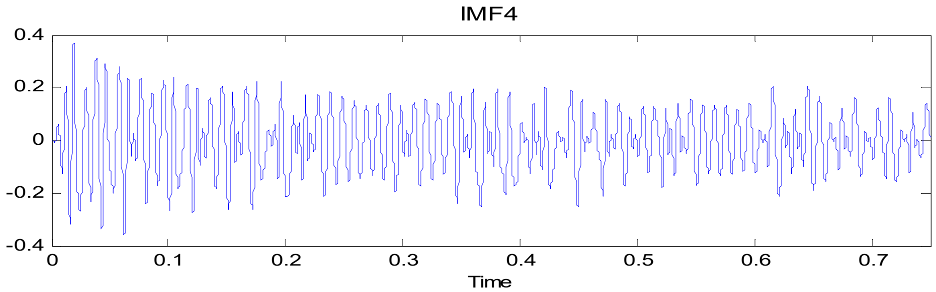
IMF2



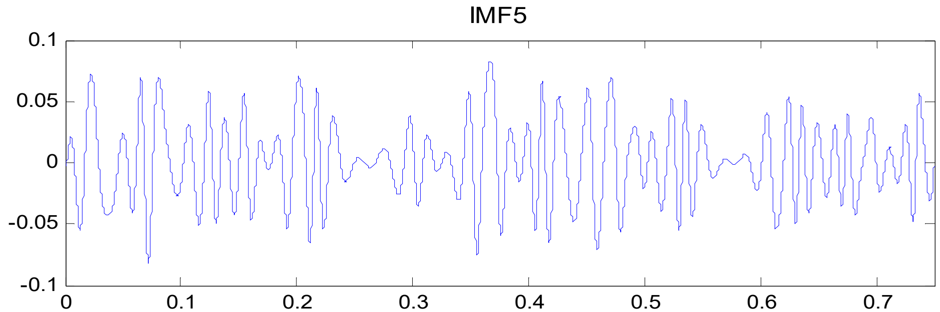
IMF3



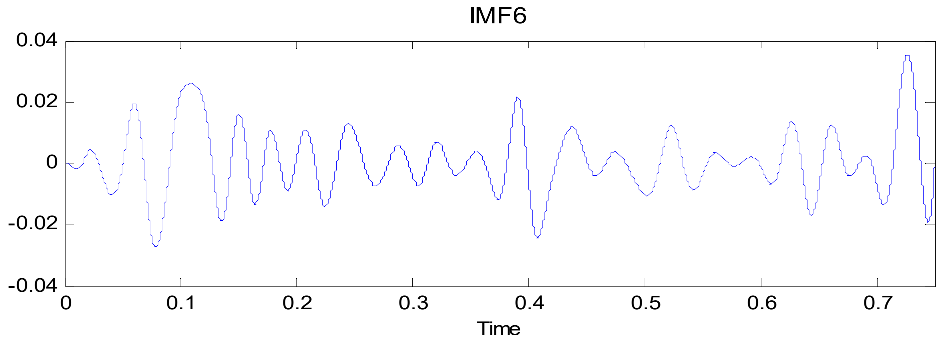
IMF4



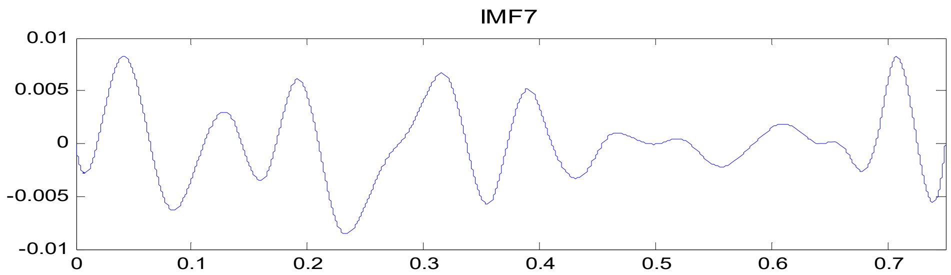
IMF5



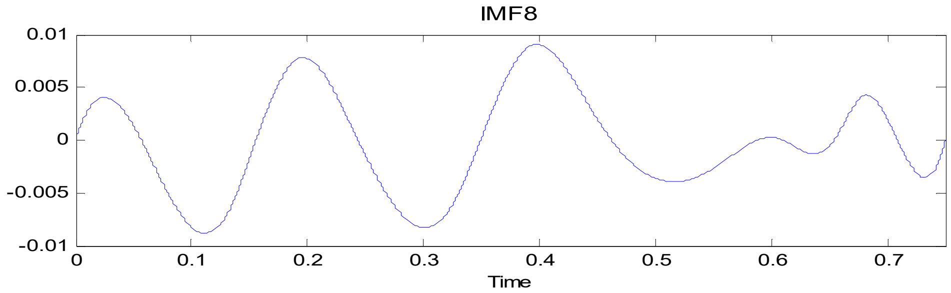
IMF6



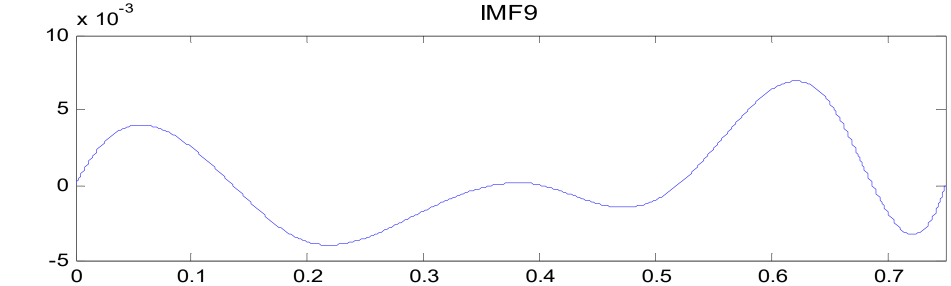
IMF7



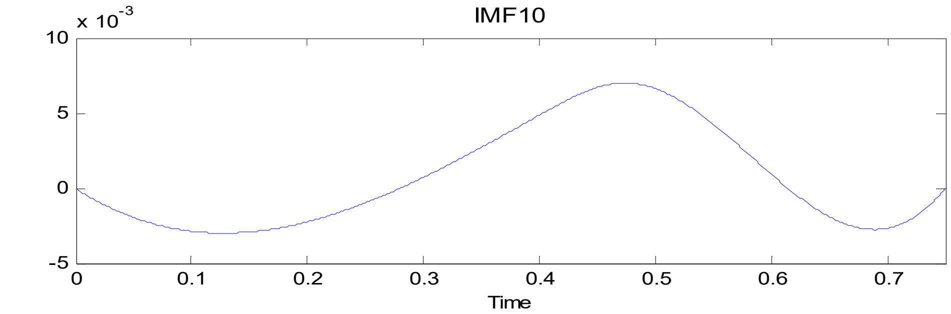
IMF8



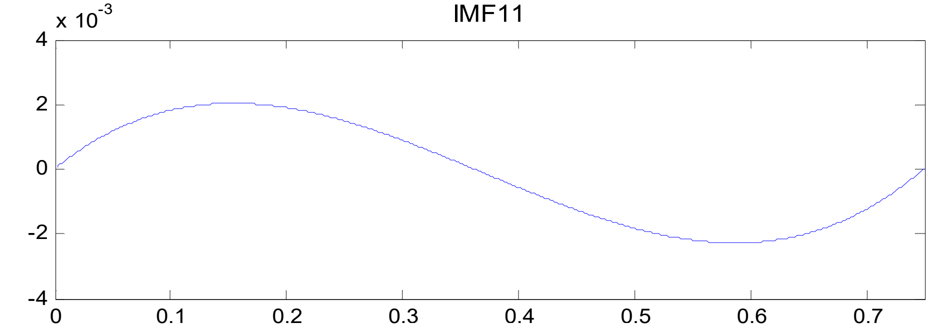
IMF9



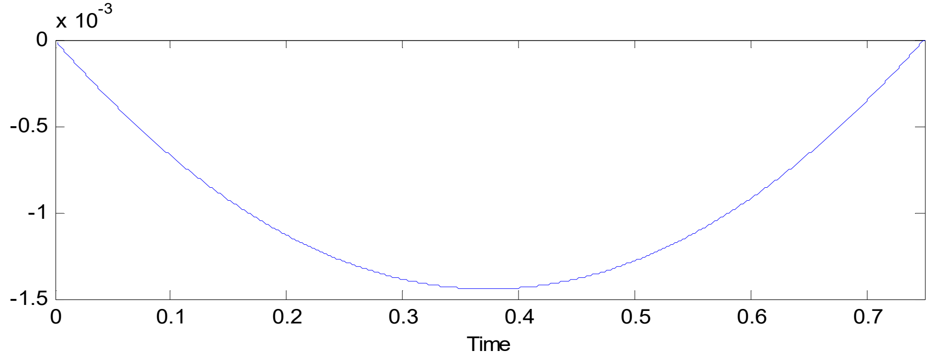
IMF10



IMF11


IMF11
x 10^-3
4
2
0
-2
-4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

$x_0(t)$


x 10^-3
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Time

11-E Comparison

(1) 避免了複雜的數學理論分析

(2) 可以找到一個 function 的「趨勢」

(3) 和其他的時頻分析一樣，可以分析頻率會隨著時間而改變的信號

(4) 適合於

- Climate analysis
- Economical data
- Geology
- Acoustics
- Music signal

- Conclusion

當信號含有「趨勢」

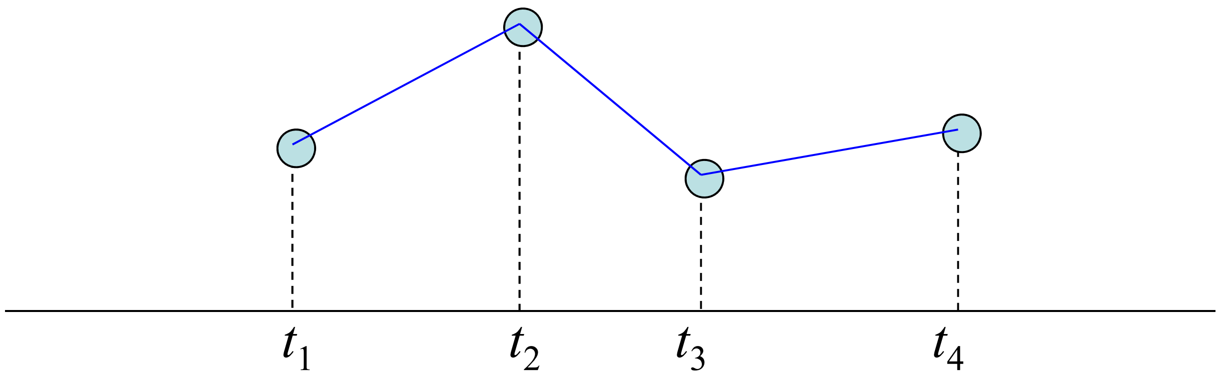
或是由少數幾個 sinusoid functions 所組合而成，而且這些sinusoid functions 的 amplitudes 相差懸殊時，可以用 HHT 來分析

附錄十一 Interpolation and the B-Spline

Suppose that the sampling points are t_1, t_2, t_3, …, t_N
and we have known the values of $x(t)$ at these sampling points.

There are several ways for interpolation.

(1) The simplest way: Using the straight lines (i.e., linear interpolation)



(2) Lagrange interpolation

when $t = t_m$
it is zero for $n \neq m$
it is equal to 1 if $n = m$.

$$x(t) = \sum_{n=1}^{N} \frac{\prod_{\substack{j=1 \\ j \neq n}}^{N} (t - t_j)}{\prod_{\substack{j=1 \\ j \neq n}}^{N} t_n - t_j} x(t_n)$$

$\prod$ 指的是連乘符號，

$$\prod_{j=1}^{N} h_j = h_1 h_2 h_3 \cdots\cdots h_N$$

$x(t)\big|_{t=t_m}$
$= x(t_m)$

(3) Polynomial interpolation

$$x(t) = \sum_{n=1}^{N} a_n t^{n-1},$$

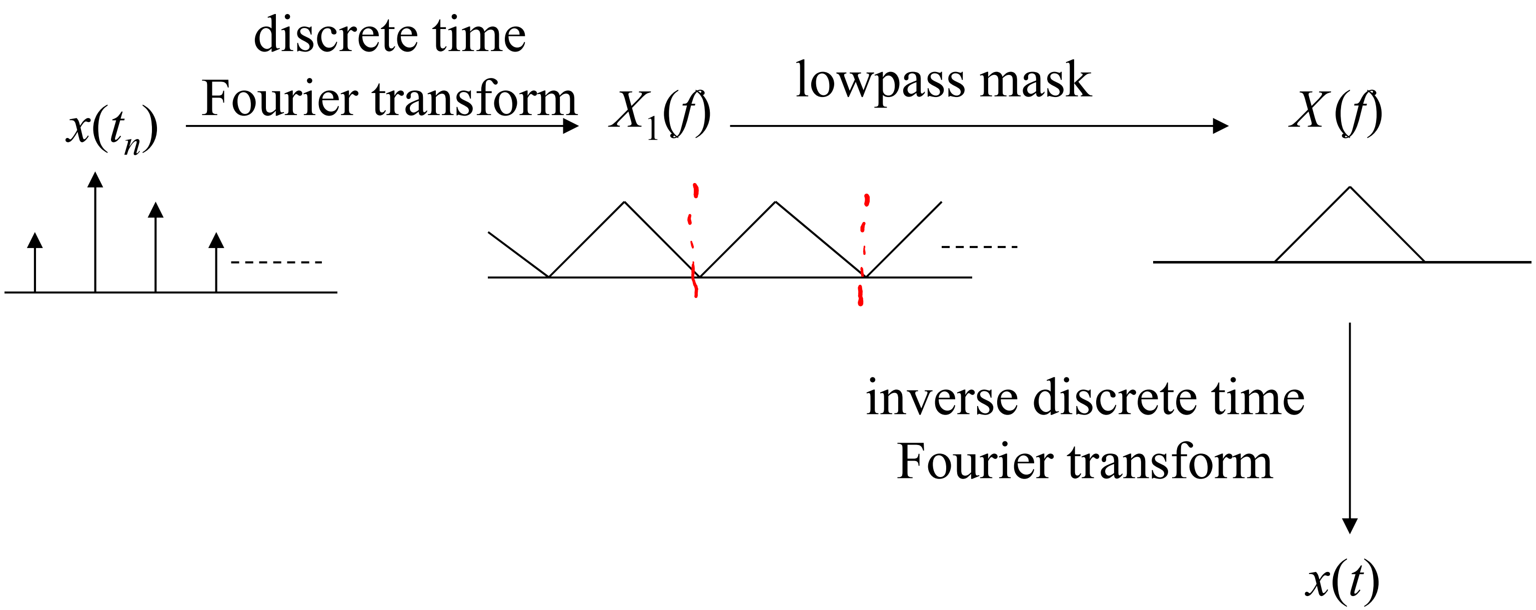
solve a_1, a_2, a_3, ……, $a_{N\text{-}1}$ from

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{N-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{N-1} \\ 1 & t_3 & t_3^2 & \cdots & t_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^{N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ x(t_3) \\ \vdots \\ x(t_N) \end{bmatrix}$$

(4) Lowpass Filter Interpolation

適用於 sampling interval 為固定的情形 $t_{n+1} - t_n = \Delta_t$ for all n

$$x(t) = \sum_{n=1}^{N} x(t_n)\,\mathrm{sinc}\left(\frac{t - t_n}{\Delta_t}\right)$$



(5) B-Spline Interpolation

B-spline 簡稱為 spline

$$B_{n,0}(t) = 1 \quad \text{for } t_n < t < t_{n+1}$$

$$B_{n,0}(t) = 0 \quad \text{otherwise}$$

$$B_{n,m}(t) = \frac{t - t_n}{t_{n+m} - t_n} B_{n,m-1}(t) + \frac{t_{n+m+1} - t}{t_{n+m+1} - t_{n+1}} B_{n+1,m-1}(t)$$

$$\boxed{x(t) = \sum_{n=1}^{N} x(t_n) B_{n,m}(t)}$$

$m = 1$: linear B-spline

$m = 2$: quadratic B-spline

$m = 3$: cubic B-spline (通常使用) $x(t)$, $x'(t)$, $x''(t)$ are continuous

In **Matlab**，the command “spline” can be used for spline interpolation.

(Note： In the command, the cubic B-spline is used)

Cubic B-Spline Interpolation by Matlab:

Generating a sine-like spline curve and samples it over a finer mesh:

```
x = 0:1:10;        % original sampling points
y = sin(x);
xx = 0:0.1:10;     % new sampling points
yy = spline(x,y,xx);
plot(x,y,'o',xx,yy)
```

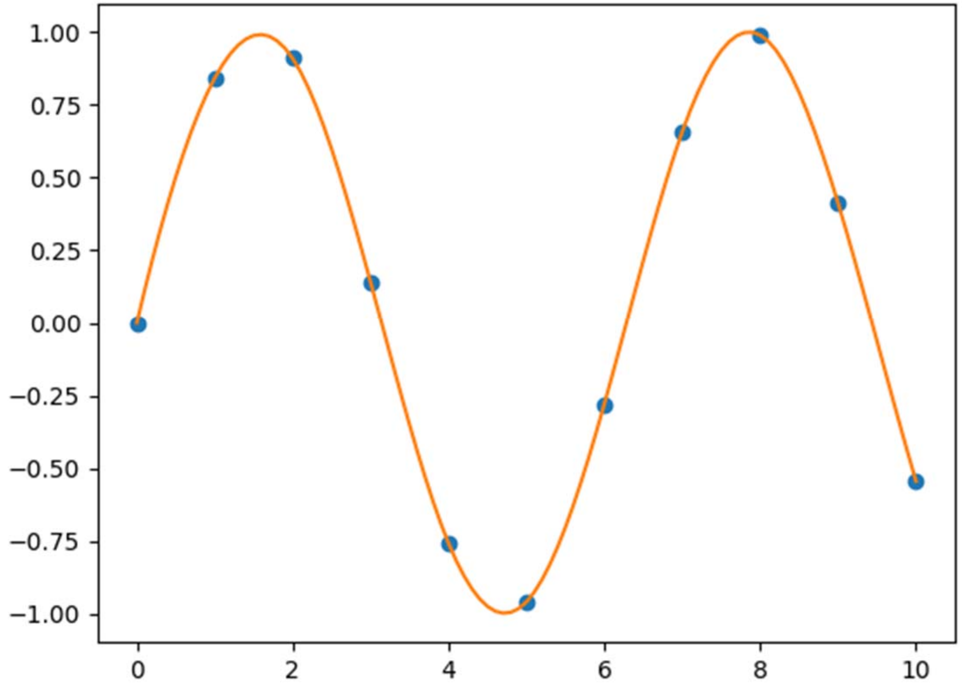
In **Python**, we can use the following way to perform cubic B-spline interpolation.

事前安裝模組

pip install numpy

pip install scipy

pip install matplotlib



感謝2021年擔任助教
的蔡昌廷同學

Reference :
https://docs.scipy.org/doc/scipy/reference/reference/generated/scipy.interpolate.interp1d.html#scipy.interpolate.interp1d

Cubic B-Spline Interpolation by Python


```
from scipy.interpolate import interp1d
import matplotlib.pyplot as plt
import numpy as np
x = np.arange(0, 11)          # original sample points, [0, 1, 2, …, 9, 10]
y = np.sin(x)
f = interp1d(x, y, kind=' cubic ') )   # Cubic means the cubic B-spline.
x_new = np.arange(0, 10.1, 0.1)
# new sample points, [0, 0.1, 0.2, ….., 9.9, 10]
y_new = f(x_new)
plt.plot(x,y,'o',x_new, y_new)
plt.show()
```