XII. Wavelet Transform

小波
漣漪；碎波

Main References

[1] R. C. Gonzalez and R. E. Woods, *Digital Image Processing*, Chap. 7, 4th edition, Prentice Hall, New Jersey, 2017. (適合初學者閱讀)

[2] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 3rd edition, 2009. (適合想深入研究的人閱讀)
(若對時頻分析已經有足夠的概念，可以由這本書 Chapter 4 開始閱讀)

[3] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Comm. Pure Appl. Math.*, vol. 4, pp. 909-996, Nov. 1988.

[4] S. Mallat, "Multiresolution approximations and wavelet orthonormal bases of L2(R)," *Trans. Amer. Math. Soc.*, vol. 315, pp. 69-87, Sept. 1989.

[5] C. Heil and D. Walnut, "Continuous and discrete wavelet transforms," *SIAM Rev.*, vol. 31, pp. 628-666, 1989.

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[8] S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Chapter 4, Prentice-Hall, New Jersey, 1996.

[9] L. Debnath, *Wavelet Transforms and Time-Frequency Signal Analysis*, Birkhäuser, Boston, 2001.

[10] B. E. Usevitch, "A Tutorial on Modern Lossy Wavelet Image Compression: Foundations of JPEG 2000," *IEEE Signal Processing Magazine*, vol. 18, pp. 22-35, Sept. 2001.

(1) Conventional method for signal analysis

- Fourier transform : $X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$
- Cosine and Sine transforms: if $x(t)$ is even and odd
- Orthogonal Polynomial Expansion

傳統方法共通的問題：

(2) Time frequency analysis

例如， STFT

$$X(t,f)=\int_{-\infty}^{\infty}w(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Time frequency analysis 共通的問題：

12-A Haar Transform

wavelet 前身

original goal: replace the FT

一種最簡單又可以反應 time-variant spectrum 的 signal representation

number of multiplication = 0

8-point Haar transform

$$H[m,n]=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

low freq; high freq; width 4; width 2

$\phi(t)$

$\psi\left(\frac{t-a}{b}\right)$

8-point Haar transform

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

y_1: low frequency component $y_2 \sim y_8$: high frequency component

different rows
= different scales
or different locations

$$y_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

$$y_2 = x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8$$

$$y_3 = x_1 + x_2 - x_3 - x_4$$

$$y_4 = x_5 + x_6 - x_7 - x_8$$

$$y_5 = x_1 - x_2$$

$N=2$

$e^{-j\frac{2\pi}{N}mn}$ $N=2$ $e^{-j\pi mn}$ $1=(-1)^{mn}$

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$H_2 : F_2$

$N=4$

$$\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$N=4$

$F_4 = e^{-j\frac{\pi}{2}mn} = (-j)^{mn}$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$N=8$

$$\mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

General way to generate the Haar transform:

$$\mathbf{H}_{2N} = \begin{bmatrix} \mathbf{H}_N \otimes [1,1] \\ \mathbf{I}_N \otimes [1,-1] \end{bmatrix}$$

where $\otimes$ means the Kronecker product

$$\mathbf{I}_N = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} H_2 \otimes [1,1] \\ I_2 \otimes [1,-1] \end{bmatrix} = \begin{bmatrix} 1\cdot[1,1] & 1\cdot[1,1] \\ 1\cdot[1,1] & (-1)[1,1] \\ 1\cdot[1,-1] & 0\cdot[1,-1] \\ 0\cdot[1,-1] & 1[1,-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \vdots & & & \vdots \\ a_{M1}B & a_{M2}B & \cdots & a_{MN}B \end{bmatrix}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}$

Ex: 4th row of the 16-point Haar transform

$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ -1\ -1\ -1\ -1]$

11th row of H_{16}

$[0\ 0\ 0\ 0\ 1\ -1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$

$N = 2^k$ 時

$$\mathbf{H} = \begin{bmatrix} \phi \\ h_{0,1} \\ h_{1,1} \\ h_{1,2} \\ \vdots \\ \vdots \\ h_{k-1,1} \\ h_{k-1,2} \\ \vdots \\ h_{k-1,2^{k-1}} \end{bmatrix}$$

H 除了第 1 個row 為 $\underbrace{\phi = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}_{N \text{ 個 } 1}$ 以外

第 $2^p + q$ 個row 為 $h_{p,q}[n]$

$$p = 0, 1, \ldots, k-1, \quad q = 1, 2, \ldots, 2^p$$

$$k = \log_2 N$$

$h_{p,q}[n] = 1$ when $(q\text{-}1)2^{k-p} < n \le (q\text{-}1/2)2^{k-p}$

$h_{p,q}[n] = -1$ when $(q\text{-}1/2)2^{k-p} < n \le q2^{k-p}$

- Inverse 2^k-point Haar Transform

$$\mathbf{H}^{-1} = \mathbf{H}^{\mathrm{T}}\mathbf{D}$$

$D[m, n] = 0$ if $m \neq n$

$D[1, 1] = 2^{-k}$, $D[2, 2] = 2^{-k}$,

$D[n, n] = 2^{-k+p}$ if $2^p < n \leq 2^{p+1}$

When $k = 3$,

$$\mathbf{D} = \begin{bmatrix} 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

12-B Characteristics of Haar Transform

(1) No multiplications

(2) Input 和 Output 點數相同

(3) 頻率只分兩種：低頻 (全為 1) 和高頻 (一半為 1，一半為 −1)

(4) 可以分析一個信號的 localized feature

(5) Very fast, but not accurate

Example:

$$\mathbf{H}\begin{bmatrix}1.2\\1.2\\1.8\\0.8\\2\\2\\1.9\\2.1\end{bmatrix}=\begin{bmatrix}13\\-3\\-0.2\\0\\0\\1\\0\\-0.2\end{bmatrix}$$

Transforms	Running Time	terms required for NRMSE $< 10^{-5}$
DFT	9.5 sec	43
Haar Transform	0.3 sec	128

References

- A. Haar, "Zur theorie der orthogonalen funktionensysteme ," *Math. Annal.*, vol. 69, pp. 331-371, 1910.
- H. F. Harmuth, *Transmission of Information by Orthogonal Functions*, Springer-Verlag, New York, 1972.

The Haar Transform is closely related to the Wavelet transform (especially the discrete wavelet transform).

12-C History of the Wavelet Transform

- 1910, Haar families.
- 1981, Morlet, wavelet concept.
- 1984, Morlet and Grossman, "wavelet".
- 1985, Meyer, "orthogonal wavelet".
- 1987, International conference in France.
- 1988, Mallat and Meyer, multiresolution.
- 1988, Daubechies, compact support orthogonal wavelet.
- 1989, Mallat, fast wavelet transform.
- 1990s, Discrete wavelet transforms
- 1999, Directional wavelet transform
- 2000, JPEG 2000

12-D Three Types of Wavelets

Wavelet 以 continuous / discrete 來分，有 3 種

	Input	Output	Name
Type 1	Continuous	Continuous	Continuous Wavelet Transform
Type 2	Continuous	Discrete	有時被稱為 discrete wavelet transform，但其實是continuous wavelet transform with discrete coefficients
Type 3	Discrete	Discrete	Discrete Wavelet Transform

比較：Fourier transform 有四種

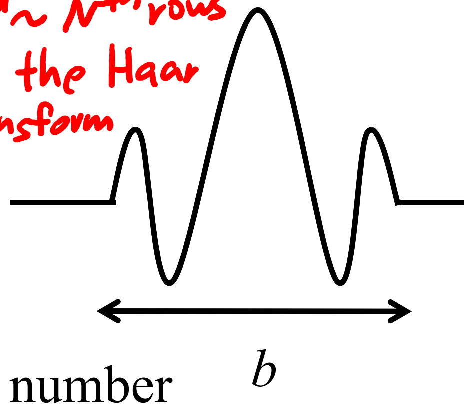
input	output	
C	C	Fourier transform
C	D	Fourier series
D	D	discrete Fourier transform
D	C	discrete-time Fourier transform

12-E Continuous Wavelet Transform (WT)

psi /saɪ/

analogous to 2nd ~ Nth rows of the Haar transform

Definition: $$X_w(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\, \psi\left(\frac{t-a}{b}\right) dt$$



$x(t)$: input, $\psi(t)$: mother wavelet

a: location, b: scaling

$a \in (-\infty, \infty)$ $b \in (0, \infty)$

a is any real number, b is any positive real number

Compare with time-frequency analysis:

$b\downarrow \Rightarrow$ fast variation $\Rightarrow$ high frequency ($f\uparrow$)

$b\uparrow \Rightarrow$ slow variation $\Rightarrow$ low frequency ($f\downarrow$)

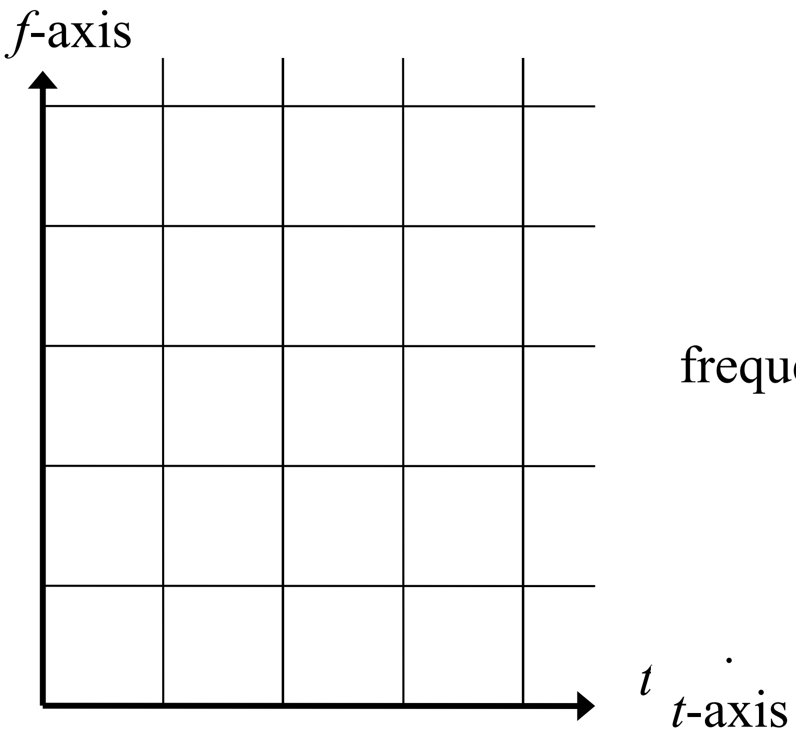
location + modulation

Gabor Transform $$G_x(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

$t \leftrightarrow a$, $f \leftrightarrow 1/b$

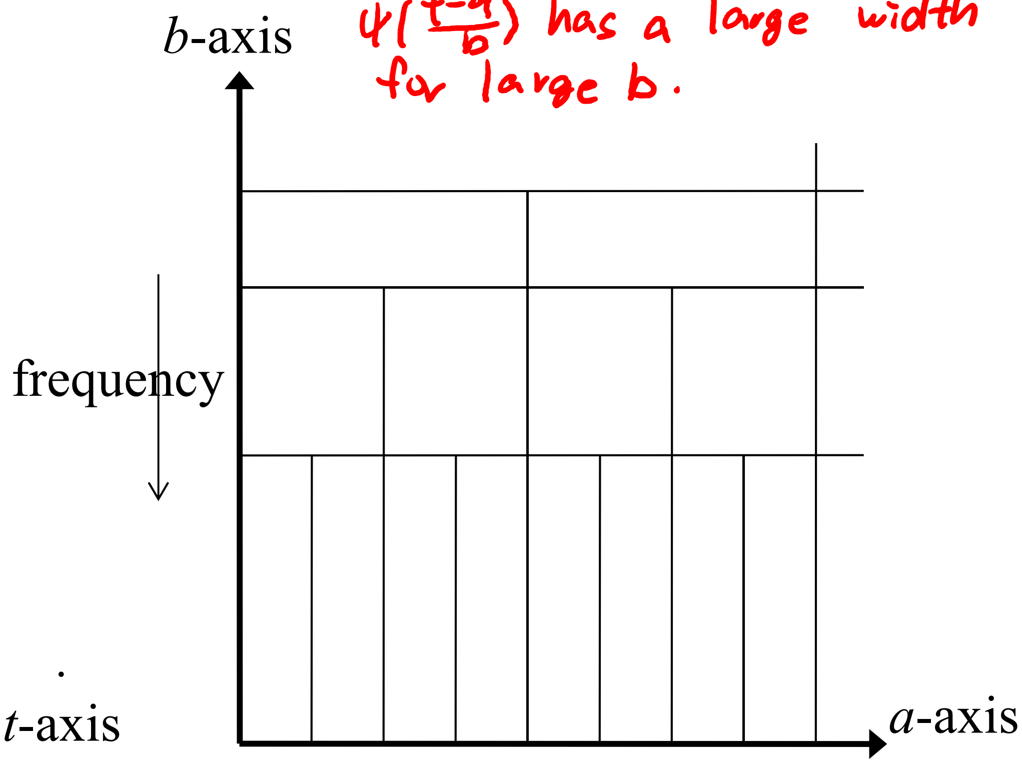
$e^{-\pi(\tau-t)^2}$

Gabor



Wavelet transform

ψ((t−a)/b) has a large width for large b.



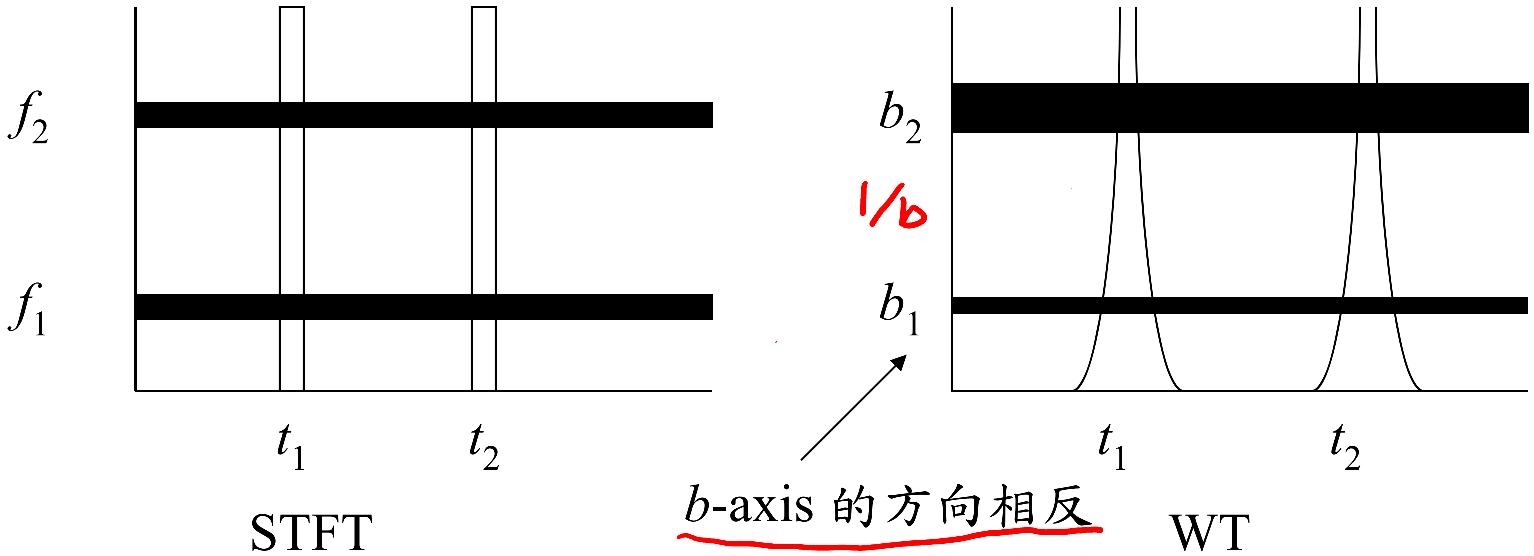
b↑ ⇒ f⇓ ⇒ time resolution ↓ ⇒ frequency resolution ↑

b↓ f↑ ↑ ↓

$$X_w(a,b) = \frac{1}{\sqrt{b}} \int_{-\infty}^{\infty} x(t)\psi\left(\frac{t-a}{b}\right)dt$$ *a*: location, *b*: scaling

• The resolution of the wavelet transform is invariant along *a* (location-axis) but variant along *b* (scaling axis).

If $x(t) = \delta(t - t_1) + \delta(t - t_2) + \exp(j2\pi f_1 t) + \exp(j2\pi f_2 t)$,

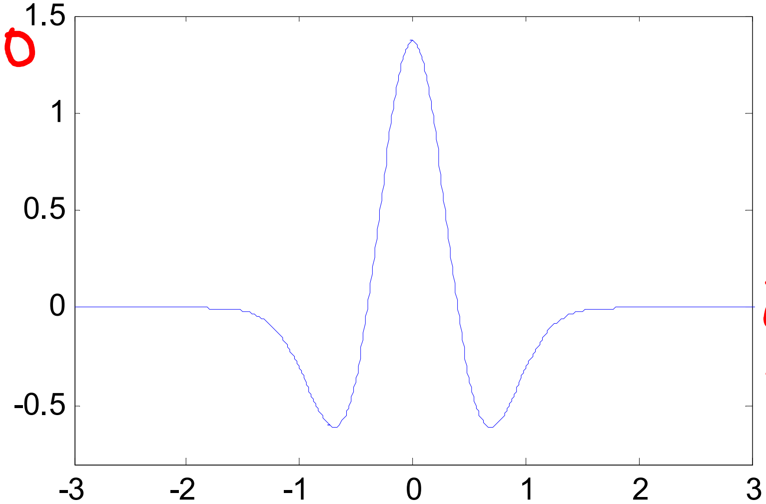


12-F Mother Wavelet

There are many ways to choose the mother wavelet. For example,

- Haar basis
- Mexican hat function $\psi(t) = \dfrac{2^{5/4}}{\sqrt{3}}\left(1-2\pi t^2\right)e^{-\pi t^2}$

In fact, the Mexican hat function is the 2nd order derivation of the Gaussian function.

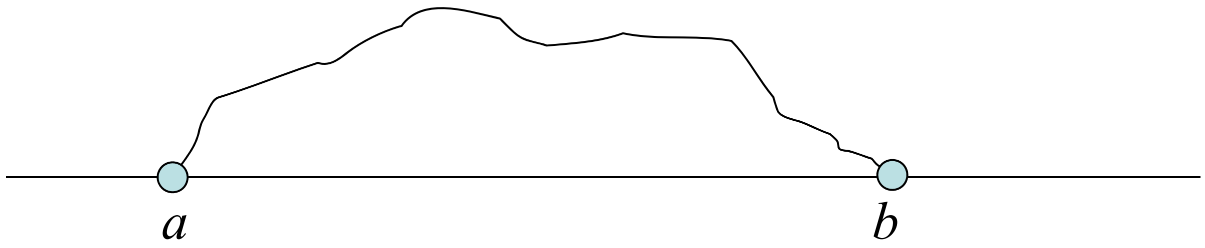


Constraints for the mother wavelet:

(1) Compact Support

support: the region where a function is not equal to zero

compact support: the width of the support is not infinite



(2) Real

(3) Even Symmetric or Odd Symmetric

(easy to compute)

(4) Vanishing Moments

Make $\psi(t)$ a high-frequency function.

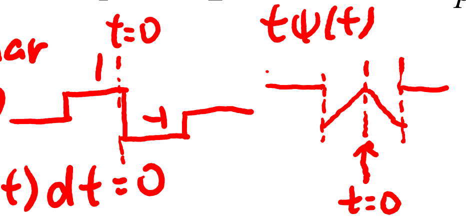
V.M. is about 5 in practice.

k^{th} moment: $m_k = \int_{-\infty}^{\infty} t^k \psi(t)\, dt$

t^k is a high frequency function if k is large.

If $m_0 = m_1 = m_2 = \ldots\ldots = m_{p-1} = 0$, we say $\psi(t)$ has p vanishing moments. ($m_p \neq 0$)

Ex: Haar



$m_0 = \int_{-\infty}^{\infty} \psi(t)\, dt = 0$

$m_1 = \int_{-\infty}^{\infty} t\psi(t)\, dt \neq 0$

vanish moment = 1

If $p\uparrow$, $\psi(t)$ is like a high-frequency signal.

If $x(t) = \sum_{n=0}^{p-1} c_n t^n$

$\int x(t)\psi(t)\, dt$

$= \sum_{n=0}^{p-1} c_n \int t^n \psi(t)\, dt$

$= \sum_{n=0}^{p-1} c_n m_n = 0$

Vanish moment 越高，經過內積後被濾掉的低頻成分越多

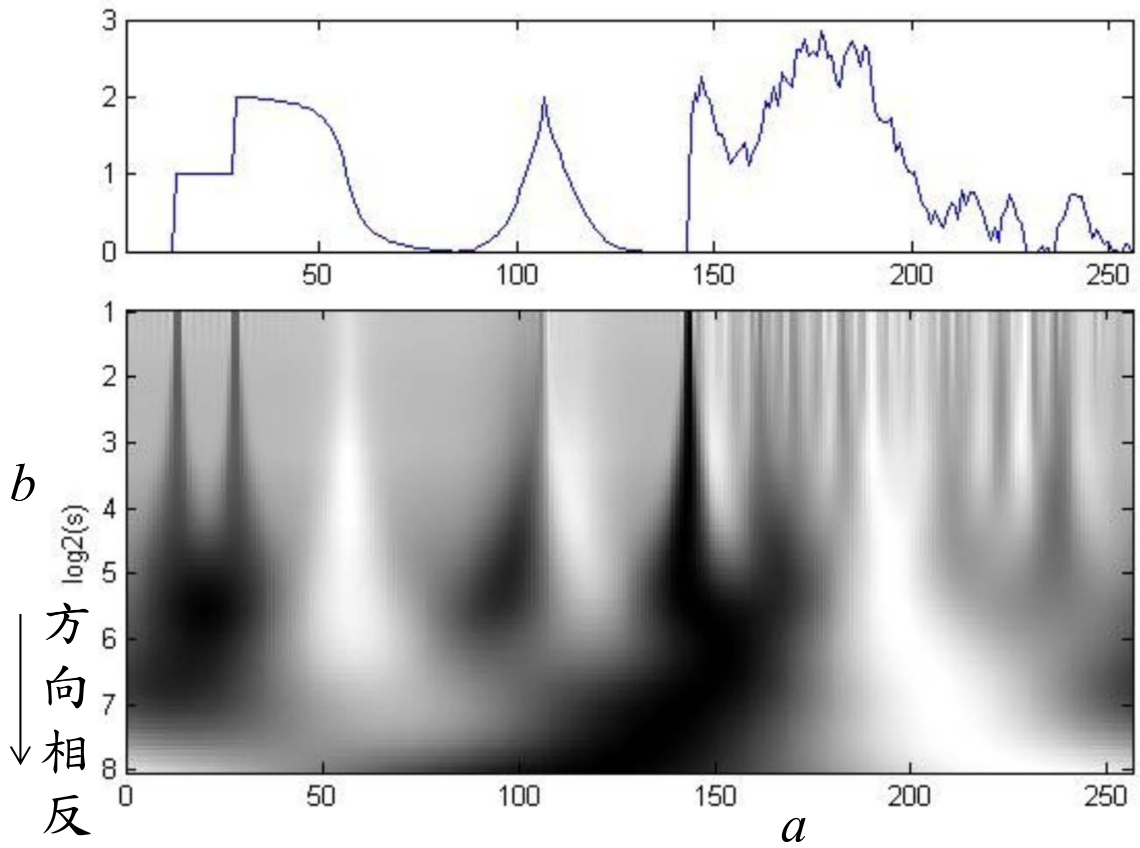
Question：為什麼要求 $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$? $\Rightarrow m_0 = 0$

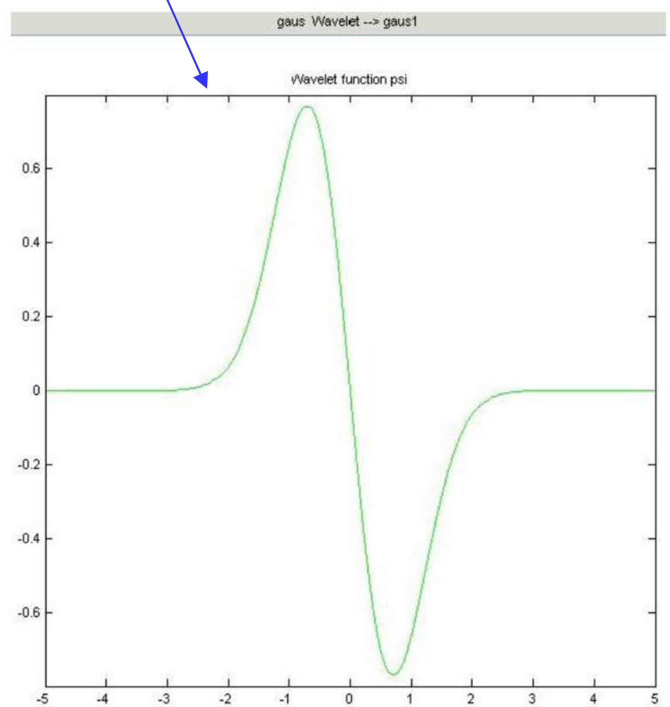
vanish moment is at least 1

註：感謝2006年修課的張育思同學

Vanish moment = 1

the 1st order derivation of the Gaussian function

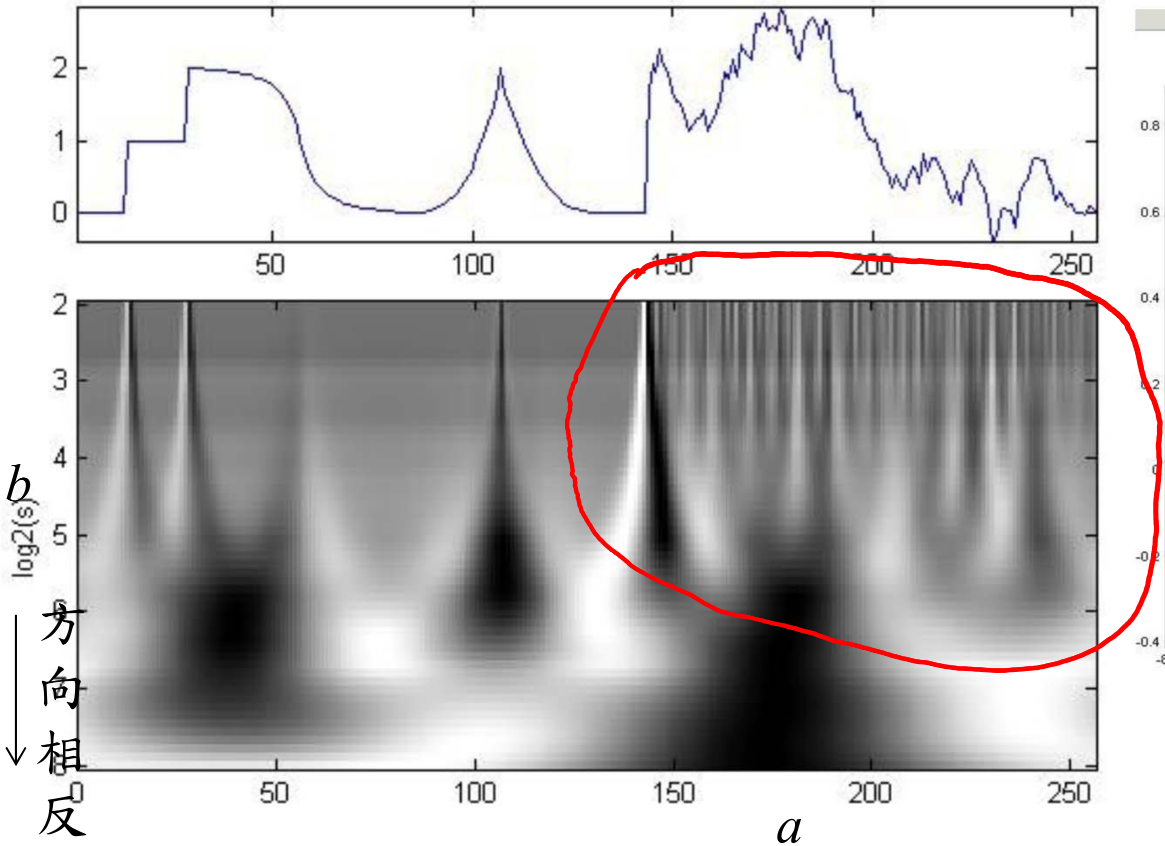


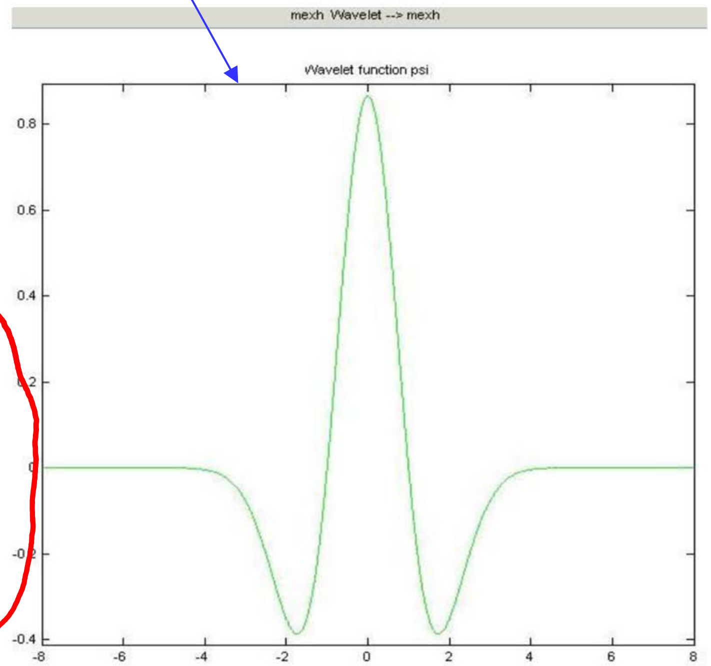


[Ref] S. Mallat, *A Wavelet Tour of Signal Processing*, 2nd Ed., Academic Press, San Diego, 1999.

Vanish moment = 2

the 2nd order derivation of the Gaussian function





Similarly, when

$$\psi(t) = \frac{d^p}{dt^p} e^{-\pi t^2}$$

the vanish moment is p

$m_k = \int t^k \psi(t)\,dt$ $\quad \Psi(f) = (j2\pi f)^p e^{-\pi f^2}$

$= FT(t^k \psi(t))\Big|_{f=0}$

$= \left(\frac{1}{-j2\pi}\right)^k \frac{d^k}{df^k} \Psi(f)\Big|_{f=0}$

$= \left(\frac{1}{-j2\pi}\right)^k \frac{d^k}{df^k} \left((j2\pi f)^p e^{-\pi f^2} \right)\Big|_{f=0}$

$= 0$ for $k = 0, 1, 2, \cdots p-1$

(5) Admissibility Criterion

$$C_{\psi} = \int_0^{\infty} \frac{|\Psi(f)|^2}{|f|} df < \infty$$,where $\Psi(f)$ is the Fourier transform of $\psi(t)$

For reversible

[Ref] A. Grossman and J. Morlet, "Decomposition of hardy functions into square integrable wavelets of constant shape," *SIAM J. Appl. Math.*, vol. 15, pp. 723-736, 1984.

12-G Inverse Wavelet Transform

$$x(t)=\frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\,\psi\left(\frac{t-a}{b}\right)da\,db$$

where $C_\psi=\int_0^\infty\frac{|\Psi(f)|^2}{|f|}df<\infty$

$\psi(t)\neq 0 \quad -t_0<t<t_0$

simplified $x(t)\simeq\frac{1}{C_\psi}\int_0^\infty\int_{t-bt_0}^{t+bt_0}\frac{1}{b^{5/2}}X_w(a,b)\,\psi\left(\frac{t-a}{b}\right)da\,db$ if $\psi(t)\cong 0$ for $t>t_0$

$-t_0<\frac{t-a}{b}<t_0$

$-bt_0<t-a<bt_0$

$t-bt_0<a<t+bt_0$

(Proof): Since $X_w(a,b)=x(t)*\frac{1}{\sqrt{b}}\psi\left(\frac{-t}{b}\right)$

if $$y(t)=\frac{1}{C_\psi}\int_0^\infty\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\,\psi\left(\frac{t-a}{b}\right)da\,db$$

then $$y(t)=\frac{1}{C_\psi}\int_0^\infty x(t)*\psi\left(\frac{-t}{b}\right)*\psi\left(\frac{t}{b}\right)\frac{db}{b^3}$$

$$y(t) = \frac{1}{C_\psi}\int_0^\infty x(t) * \psi\left(\frac{-t}{b}\right) * \psi\left(\frac{t}{b}\right)\frac{db}{b^3}$$

$$Y(f) = \frac{1}{C_\psi}\int_0^\infty X(f)\Psi(-bf)\Psi(bf)\frac{db}{b}$$

where

$$Y(f) = FT[y(t)]$$
$$X(f) = FT[x(t)]$$
$$\Psi(f) = FT[\psi(t)]$$

If $\psi(t)$ is real, $\Psi(-f) = \Psi^*(f)$, $\Psi(-bf)\,\Psi(bf) = \Psi^*(bf)\,\Psi(bf) = |\Psi(bf)|^2$

$$Y(f) = X(f)\frac{1}{C_\psi}\int_0^\infty \left|\Psi(bf)\right|^2 \frac{db}{b}$$

$$= X(f)\frac{1}{C_\psi}\int_0^\infty \left|\Psi(f_1)\right|^2 \frac{df_1}{bf} \qquad (f_1 = bf,\ df_1 = fdb)$$

$$= X(f)\frac{1}{C_\psi}\int_0^\infty \left|\Psi(f_1)\right|^2 \frac{df_1}{f_1}$$

$$= X(f)$$

Therefore, $y(t) = x(t)$.

12-H Scaling Function

low-frequency function (analogous to the 1st row of the Haar transform)

定義 scaling function 為

Simplify the inverse wavelet transform.

$$\phi(t) = \int_{-\infty}^{\infty} \Phi(f) e^{j2\pi f t} df$$

where $$|\Phi(f)|^2 = \int_{f}^{\infty} \frac{|\Psi(f_1)|^2}{|f_1|} df_1$$ for $f > 0$, $\Phi(-f) = \Phi^*(f)$

$|\Phi(f)|$ decreases with f

$\phi(t)$ is usually a lowpass filter (Why?)

修正型的 Wavelet transform

(1) $$X_w(a,b)=\frac{1}{\sqrt{b}}\int_{-\infty}^{\infty}x(t)\psi\left(\frac{t-a}{b}\right)dt$$

a is any real number, $0<b<b_0$

$b_0 < b < \infty$ is moved to the scaling function

(2) $$LX_w(a,b_0)=\frac{1}{\sqrt{b_0}}\int_{-\infty}^{\infty}x(t)\phi\left(\frac{t-a}{b_0}\right)dt$$

reconstruction:

$$x(t)=\frac{1}{C_\psi}\left[\int_0^{b_0}\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db+\int_{-\infty}^{\infty}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da\right]$$

由 b_0 至 ∞ 的積分被第二項取代

If $\psi(t)\cong 0$ for $|t|>t_0$, $\phi(t)\cong 0$ for $|t|>t_1$

$$x(t)\cong\frac{1}{C_\psi}\left[\int_0^{b_0}\int_{t-bt_0}^{t+bt_0}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db+\int_{t-b_0t_1}^{t+b_0t_1}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da\right]$$

(Proof): If $y_1(t) = \frac{1}{C_\psi}\int_0^{b_0}\int_{-\infty}^{\infty}\frac{1}{b^{5/2}}X_w(a,b)\psi\left(\frac{t-a}{b}\right)da\,db$

$$y_2(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{1}{b_0^{3/2}}LX_w(a,b_0)\phi\left(\frac{t-a}{b_0}\right)da$$

$$Y_1(f) = X(f)\frac{1}{C_\psi}\int_0^{b_0}\left|\Psi(bf)\right|^2\frac{db}{b}$$

$$= X(f)\frac{1}{C_\psi}\int_0^{b_0 f}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1}$$

(from the similar process on pages 384 and 385)

$$y_2(t) = \frac{1}{b_0^2 C_\psi}x(t)*\phi\left(\frac{-t}{b_0}\right)*\phi\left(\frac{t}{b_0}\right)$$

$$Y_2(f) = X(f)\frac{1}{C_\psi}\Phi(-b_0 f)\Phi(b_0 f) = X(f)\frac{1}{C_\psi}\Phi^*(b_0 f)\Phi(b_0 f)$$

$$= X(f)\frac{1}{C_\psi}\left|\Phi(b_0 f)\right|^2$$

$$= X(f)\frac{1}{C_\psi}\int_{b_0 f}^{\infty}\frac{|\Psi(f_1)|^2}{|f_1|}df_1$$

關鍵

Therefore, if $y(t) = y_1(t) + y_2(t)$,

$$
\begin{aligned}
Y(f) &= Y_1(f) + Y_2(f) \\
&= X(f)\frac{1}{C_\psi}\int_0^{b_0 f}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1} + X(f)\frac{1}{C_\psi}\int_{b_0 f}^{\infty}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1} \\
&= X(f)\frac{1}{C_\psi}\int_0^{\infty}\left|\Psi(f_1)\right|^2\frac{df_1}{f_1} \\
&= X(f)
\end{aligned}
$$

$y(t) = x(t)$

12-I Property

(1) real input $\longrightarrow$ real output

(2) If $x(t) \longrightarrow X_w(a, b)$, then $x(t-\tau) \longrightarrow X_w(a-\tau, b)$,

(3) If $x(t) \longrightarrow X_w(a, b)$, then $x(t/\sigma) \longrightarrow \sqrt{\sigma}X_w(a/\sigma, b/\sigma)$

(4) Parseval's Theory:

$$\int |x(t)|^2\, dt = \frac{1}{C}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{1}{b^2}|X_w(a,b)|^2\, da\, db$$

12-J Scalogram

Scalogram 即 Wavelet transform 的絕對值平方

$$Sc_x(a,b) = \left|X_w(a,b)\right|^2 = \frac{1}{|b|}\left|\int_{-\infty}^{\infty} x(t)\psi\left(\frac{t-a}{b}\right)dt\right|^2$$

有時，會將 Scalogram 定義成

$$Sc_x(a,\zeta) = \left|X_w\left(a,\frac{\eta}{\zeta}\right)\right|^2$$

$$\eta = \frac{\int_0^{\infty} f\left|\Psi(f)\right|^2 df}{\int_0^{\infty}\left|\Psi(f)\right|^2 df}$$

$$b = \frac{\eta}{\zeta}, \quad \zeta = \frac{\eta}{b}$$

$$\Psi(f) = \int_{-\infty}^{\infty}\psi(t)e^{-j2\pi f t}dt$$

$f\uparrow \quad \zeta\uparrow$

$\downarrow \quad \downarrow$

12-K Problems

Problems of the continuous WT

(1) hard to implement

(2) hard to find $\phi(t)$

Continuous WT is good in mathematics.

In practical, the discrete WT and the continuous WT with discrete coefficients are more useful.

附錄十二 電機+資訊領域的中研院院士

王兆振 (電子物理學家，1968年當選院士)
葛守仁 (電子電路理論奠基者之一，1976年當選院士)
朱經武 (超導體，1987年當選院士)
田炳耕 (微波放大器，1987年當選院士)
崔琦 (量子霍爾效應，1992年當選院士，1998年諾貝爾物理獎)
王佑曾 (資料庫管理理論先驅，1992年當選院士)
高錕 (光纖通訊，1992年當選院士，2009年諾貝爾物理獎)
方復 (半導體，1992年當選院士)
厲鼎毅 (光電科技，1994年當選院士)
湯仲良 (光電科技，1994年當選院士)
施敏 (Non-volatile semiconductor memory 發明者，手機四大發明者之一，1994年當選院士)
張俊彥 (半導體，1996年當選院士)
薩支唐 (MOS and CMOS，1998年當選院士)
林耕華 (光電科技，1998年當選院士)
劉兆漢 (跨領域，電機與地球科學，1998年當選院士)
虞華年 (微電子科技，2000年當選院士)
蔡振水 (光電與磁微波，2000年當選院士)

王文一 (奈米與應用物理，2002年當選院士)
胡正明 (微電子科技，2004年當選院士)
黃鍔 (Hilbert Huang Transform，2004年當選院士)
胡玲 (奈米科技，2004年當選院士)
李德財 (演算法設計，2004年當選院士)
劉必治 (多媒體信號處理，2006年當選院士)
莊炳湟 (語音信號處理，2006年當選院士)
黃煦濤 (圖形辨識，2006年當選院士)
舒維都 (信號處理與人工智慧，2006年當選院士)
李雄武 (電磁學，2006年當選院士)
孟懷縈 (無線通信與信號處理，2010年當選院士)
李澤元 (電力電子，2012年當選院士)
馬佐平 (微電子，2012年當選院士)
張懋中 (電子元件，2012年當選院士)
林本堅 (積體電路與傅氏光學，2014年當選院士)
陳陽闓 (高速半導體，2016年當選院士)
王康隆 (自旋電子學，2016年當選院士)
李琳山 (語音訊號處理，2016年當選院士)
戴聿昌 (微積電系統與醫工，2016年當選院士)
張世富 (多媒體信號處理，2018年當選院士)
盧志遠 (半導體技術，2018年當選院士)

註：歷年中研院院士當中，屬於電機+資訊相關領域的有37人，佔了全部的 7.7 %

其中和通信、信號處理、影像處理相關的有9位，大多是2004年以後當選院士

XIII. Continuous WT with Discrete Coefficients

13-A Definition

The parameters a and b are not chosen arbitrarily.

$b \in (0, \infty) \Rightarrow b = 2^{-m},\ m \in (-\infty, \infty)$

For example,

$a \in (-\infty, \infty) \Rightarrow a = n2^{-m},\ n \in (-\infty, \infty)$

$a = n2^{-m}$ and $b = 2^{-m}$.

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

$\psi\left(\frac{t-a}{b}\right)$

$n \in Z, \quad n \in (-\infty, \infty)$

$m \in Z, \quad m \in (-\infty, \infty)$

註：某些文獻把這個式子稱作是 discrete wavelet transform，實際上仍然是 continuous wavelet transform 的特例

- Main reason for constrain a and b to be $n2^{-m}$ and 2^{-m} :

Easy to implementation

$X_w(n, m)$ can be computed from $X_w(n, m-1)$ by digital convolution.

13-B Inverse Wavelet Transform

$\psi\left(\frac{t-a}{b}\right)$

$$x(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m/2}\psi\left(2^m t-n\right)X_w(n,m)$$

$$x(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^{m/2}\psi\left(2^m t-n\right)2^{m/2}\int_{-\infty}^{\infty}x(t_1)\psi(2^m t_1-n)dt_1$$

$$=\int_{-\infty}^{\infty}\left\{\sum_{m_1=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^m\psi\left(2^m t-n\right)\psi(2^m t_1-n)\right\}x(t_1)dt_1$$

since $x(t)=\int_{-\infty}^{\infty}\delta\left(t-t_1\right)x(t_1)dt_1$

Constraint: $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2^m\psi\left(2^m t-n\right)\psi\left(2^m t_1-n\right)=\delta\left(t-t_1\right)$

⇓ duality

i.e., $2^m\int_{-\infty}^{\infty}\cancel{2^m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^m t-n\right)dt=\delta\left(m-m_1\right)\delta\left(n-n_1\right)$

should be satisfied.

$n\rightarrow$, $m\downarrow$ $A(m,n)$

$\sum_{m=1}^{N}A(m,n_1)A(m,n_2)=\begin{cases}1 & \text{if } n_1=n_2\\ 0 & \text{if } n_1\neq n_2\end{cases}$

$A^TA=I$

$A^T=A^{-1}$

$AA^T=I$

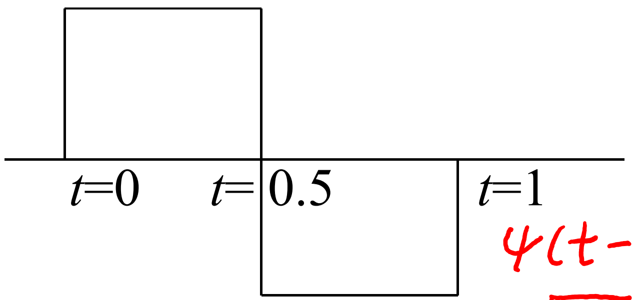
$\sum_{n=1}^{N}A(m_1,n)A(m_2,n)=\begin{cases}1 & \text{if } m_1=m_2\\ 0 & \text{if } m_1\neq m_2\end{cases}$

13-C Haar Wavelet

vanish moment = 1

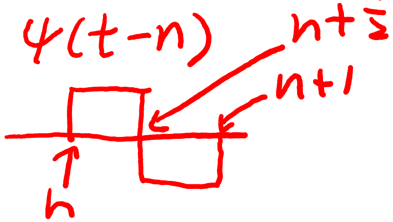
$\psi(t)$ mother wavelet
(wavelet function)

$\int \psi(t)\psi(2t)\,dt = 0$

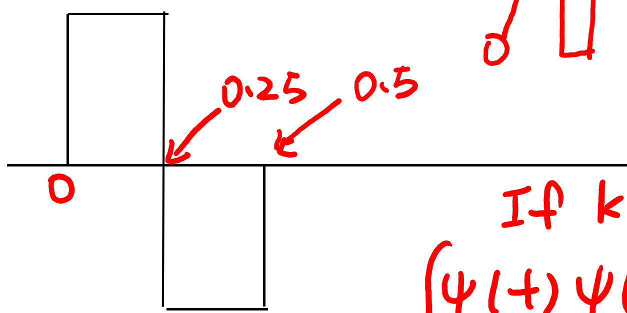


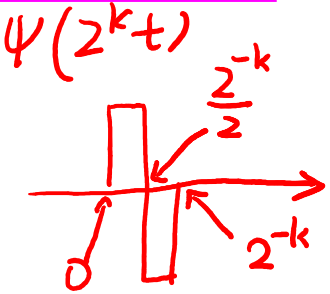
$(n \neq 0)$

$\int \psi(t)\psi(t-n)\,dt = 0$



$\psi(2t)$





If $k \geq 1$

$\int \psi(t)\psi(2^k t)\,dt$

$= \int 1 \cdot \psi(2^k t)\,dt = 0$

If $k \leq -1$

$\int \psi(t)\psi(2^k t)\,dt$

$= \int \psi(t) \cdot 1\,dt = 0$

The Haar wavelet satisfies

$$2^m \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

If $m_1 > m$

set $t_1 = 2^m t - n$ $\quad (dt_1 = 2^m dt)$

$2^{m_1} t - n_1 = 2^{m_1 - m} t_1 + 2^{m_1 - n} n - n_1$

$$2^m \int_{-\infty}^{\infty} \psi(2^{m_1} t - n_1)\, \psi(2^m t - n)\, dt$$

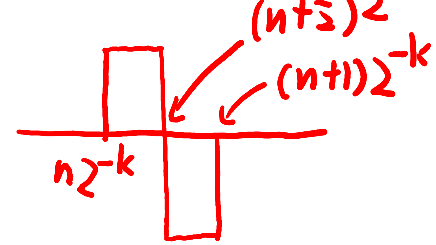
$$= \int_{-\infty}^{\infty} \psi(2^{m_1 - m} t_1 + 2^{m_1 - n} n - n_1)\, \psi(t_1)\, dt_1$$

We only have to prove

$\int_{-\infty}^{\infty} \psi(t)\, \psi(2^k t - n) = 0$ for any k, n

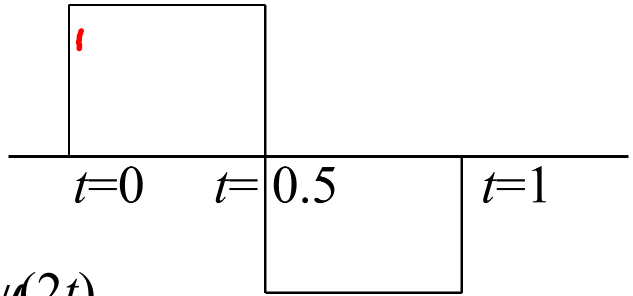
$k > 0$

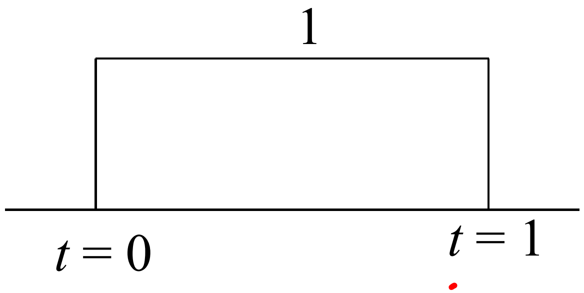
$\psi(2^k t - n)$



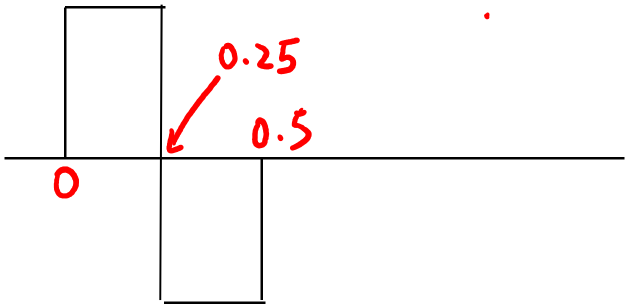
$\psi(t)$ mother wavelet
(wavelet function)

$\phi(t)$ scaling function

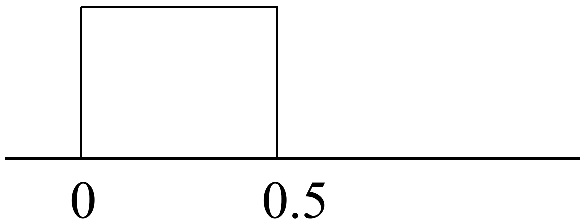




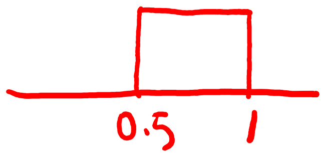
$\psi(2t)$



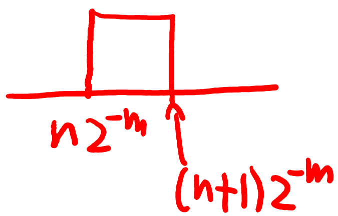
$\phi(2t)$ $\quad 2\phi(2t) = \phi(t) + \psi(t)$



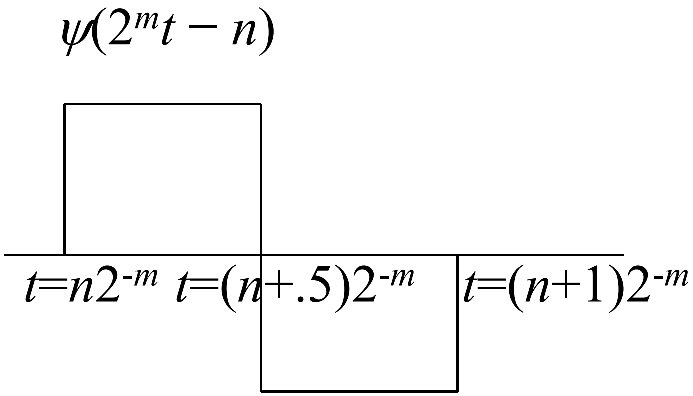
$\phi(2t-1)$

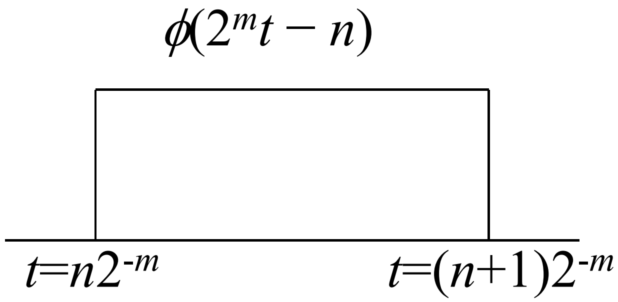


$\phi(2^m t - n)$



$$\begin{cases} \phi(t) = \phi(2t) + \phi(2t-1) \\ \psi(t) = \phi(2t) - \phi(2t-1) \end{cases}$$





- Advantages of Haar wavelet

  (1) Simple

  (2) Fast algorithm

  (3) Orthogonal →reversible

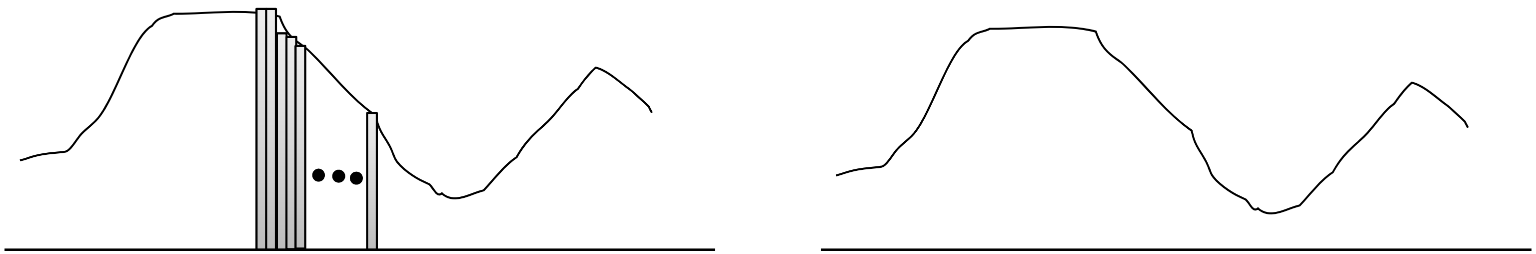
  (4) Compact, real, odd

- Disadvantages of Haar wavelet

  Vanish moment = 1

Properties

(1) 任何 function 都可以由 $\phi(t)$, $\phi(2t)$, $\phi(4t)$, $\phi(8t)$, $\phi(16t)$, ……….. 以及它們的位移所組成



(2) 任何平均為 0 的function 都可以由 $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ……….. 所組成

換句話說……. 任何 function 都可以由 constant, $\psi(t)$, $\psi(2t)$, $\psi(4t)$, $\psi(8t)$, $\psi(16t)$, ……….. 所組成

(4) 不同寬度 (也就是不同 m) 的 wavelet / scaling functions 之間會有一個關係

$$\phi(t) = \phi(2t) + \phi(2t-1)$$

$$\phi(t-n) = \phi(2t-2n) + \phi(2t-2n-1)$$

$$\phi(2^m t - n) = \phi(2^{m+1} t - 2n) + \phi(2^{m+1} t - 2n - 1)$$

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

$$\psi(t-n) = \phi(2t-2n) - \phi(2t-2n-1)$$

$$\psi(2^m t - n) = \phi(2^{m+1} t - 2n) - \phi(2^{m+1} t - 2n - 1)$$

(5) 可以用 m+1 的 coefficients 來算 m 的 coefficients

/kaɪ/

若 $\chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^m t - n)dt$

kai function

$$\chi_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n)dt + 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}}\left(\chi_w(2n, m+1) + \chi_w(2n+1, m+1)\right)$$

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t - n)dt$$

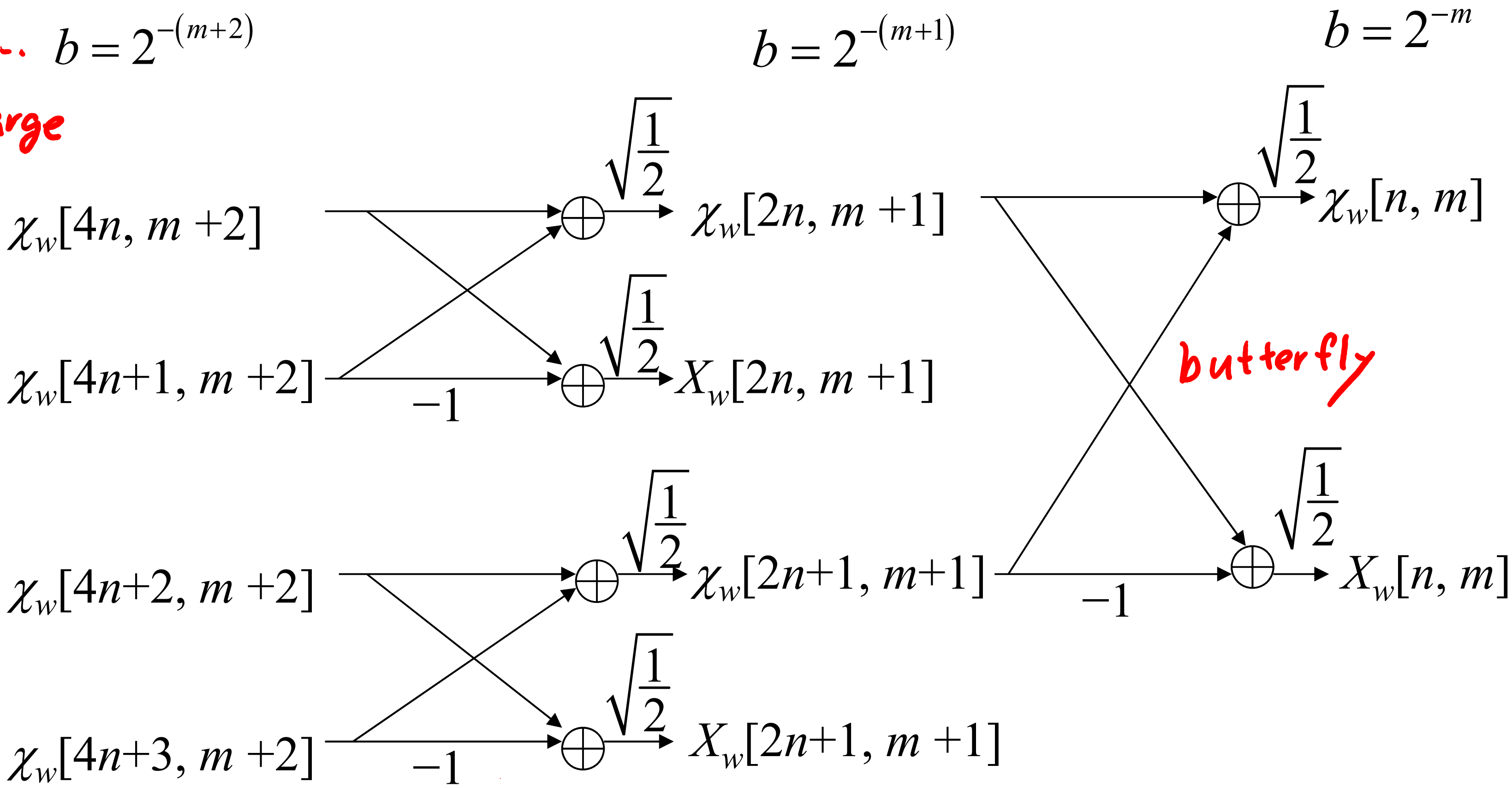
wavelet transform output

$$X_w(n,m) = 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n)dt - 2^{m/2}\int_{-\infty}^{\infty} x(t)\phi(2^{m+1} t - 2n - 1)dt$$

$$= \sqrt{\frac{1}{2}}\left(\chi_w(2n, m+1) - \chi_w(2n+1, m+1)\right)$$

layer: ← m↑ b↓ detail   m↓ b↑ → large scale features

initial: m is very large samples of the signal



structure of multiresolution analysis (MRA)

13-D General Methods to Define the Mother Wavelet and the Scaling Function

Constraints:
- (a) nearly compact support
- (b) fast algorithm
- (c) real
- (d) vanish moment
- (e) orthogonal

和 continuous wavelet transform 比較：

(1) compact support 放寬為 "near compact support"

(2) 沒有 even, odd symmetric 的限制 (replaced by fast algorithm)

(3) 由於只要是 complete and orthogonal, 必定可以 reconstruction

所以不需要 admissibility criterion 的限制

(4) 多了對 fast algorithm 的要求

13-E Fast Algorithm Constraints

Higher and lower resolutions 的 recursive relation 的一般化

$$\phi(t) = 2\sum_k g_k \phi(2t-k)$$

稱作 dilation equation

$$\psi(t) = 2\sum_k h_k \phi(2t-k)$$

比較: for Haar wavelet

$g_0 = g_1 = 1/2$, $g_k = 0$ otherwise

$h_0 = \frac{1}{2}$, $h_1 = \frac{-1}{2}$, $h_k = 0$ otherwise

$\psi(t)$: mother wavelet, $\phi(t)$: scaling function

這些關係式成立，才有fast algorithms

$$\phi(t)=2\sum_k g_k\phi(2t-k) \qquad \psi(t)=2\sum_k h_k\phi(2t-k)$$

If $\chi_w(n,m)=2^{m/2}\int_{-\infty}^{\infty}x(t)\phi(2^m t-n)dt$

then $\chi_w(n,m)=\sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty}x(t)g_k\phi(2^{m+1}t-2n-k)dt$

$$=2^{\frac{1}{2}}\sum_k g_k\chi_w(2n+k,m+1)$$

If $X_w(n,m)=2^{m/2}\int_{-\infty}^{\infty}x(t)\psi(2^m t-n)dt$

then $X_w(n,m)=\sum_k 2^{\frac{m}{2}+1}\int_{-\infty}^{\infty}x(t)h_k\phi(2^{m+1}t-2n-k)dt$

$$=2^{\frac{1}{2}}\sum_k h_k\chi_w(2n+k,m+1)$$

(Step 1) convolution

$$\tilde{\chi}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{g}_k \chi_w(n-k, m+1) \qquad \tilde{g}_k = g_{-k}$$

$= 2^{\frac{1}{2}} g_n * \chi_w(n, m+1)$ ← convolution along n

$$\tilde{X}_w(n) = 2^{\frac{1}{2}} \sum_k \tilde{h}_k \chi_w(n-k, m+1) \qquad \tilde{h}_k = h_{-k}$$

convolution:

$y[n] = x[n] * h[n]$

$= \sum_k h[k] x[n-k]$

(Step 2) down sampling

$$\chi_w(n, m) = \tilde{\chi}_w(2n)$$

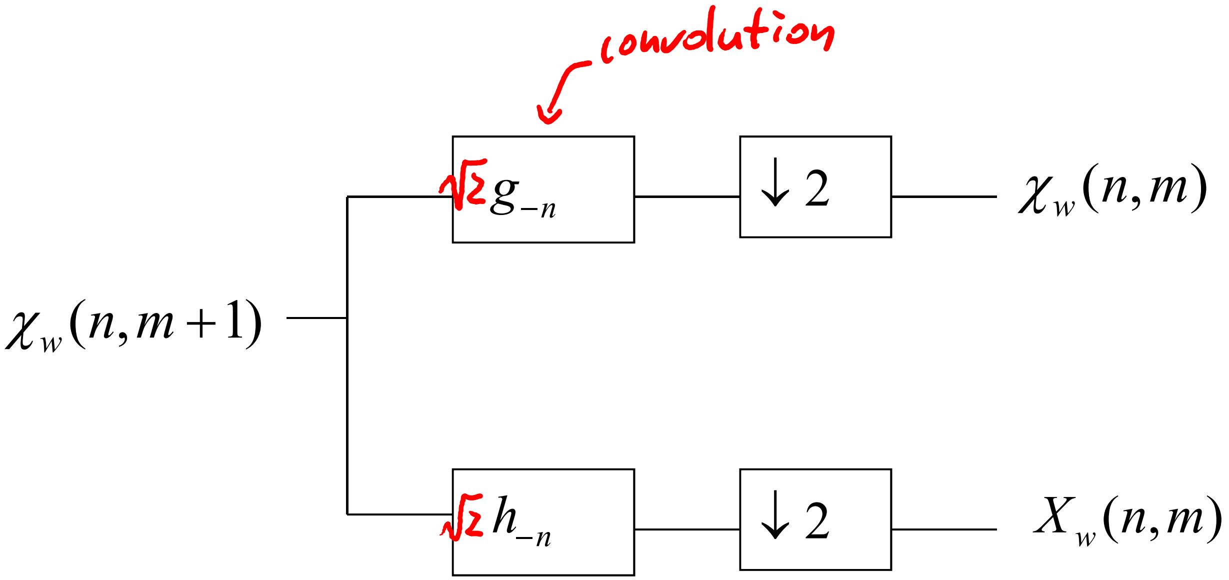
$$X_w(n, m) = \tilde{X}_w(2n)$$

$\hat{\chi}_w(n) = 2^{\frac{1}{2}} \sum_k g_{-k} \chi_w(n-k, m+1)$

$= 2^{\frac{1}{2}} \sum_k g_k \chi_w(n+k, m+1)$

$\chi_w(n, w) = \tilde{\chi}_w(2n)$

$= 2^{\frac{1}{2}} \sum_k g_k \chi_w(2n+k, m+1)$


convolution
$\sqrt{2}g_{-n}$
$\downarrow 2$
$\chi_w(n,m)$
$\chi_w(n,m+1)$
$\sqrt{2}h_{-n}$
$\downarrow 2$
$X_w(n,m)$

m 越大，越屬於細節

- To satisfy $\phi(t) = 2\sum_k g_k \phi(2t-k)$,

$$\phi(t/2) = 2\sum_k g_k \phi(t-k)$$

$$\downarrow \text{FT} \qquad \downarrow \text{FT}$$

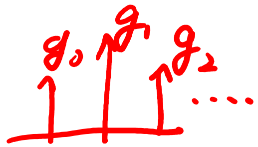
$$2\Phi(2f) = 2G(f)\Phi(f)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$\phi\left(\frac{t}{\sigma}\right) \to \sigma\Phi(\sigma f)$$

where $\Phi(f) = FT\left[\phi(t)\right] = \int_{-\infty}^{\infty} \phi(t) e^{-j2\pi f t} dt$

$$G(f) = FT\left[\sum_k g_k \delta(t-k)\right]$$



$$= \sum_k g_k \int_{-\infty}^{\infty} \delta(t-k) e^{-j2\pi f t} dt$$

$$= \sum_k g_k e^{-j2\pi f k}$$

$\Phi(f)$ 是 $\phi(t)$ 的 continuous Fourier transform

$G(f)$ 是 $\{g_k\}$ 的 discrete time Fourier transform

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$\Phi\left(\frac{f}{2}\right) = G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right)$$

$$\Phi(f) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)\Phi\left(\frac{f}{4}\right) = G\left(\frac{f}{2}\right)G\left(\frac{f}{4}\right)G\left(\frac{f}{8}\right)\Phi\left(\frac{f}{8}\right) = \cdots\cdots$$

$$\Phi(f) = \Phi\left(\frac{f}{2^\infty}\right)\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right) = \Phi(0)\prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$$

↑ 連乘

$$\Phi(0) = \int_{-\infty}^{\infty} \phi(t)\,dt$$ (可以藉由 normalization, 讓 $\Phi(0) = 1$)

$$\Phi(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t}\phi(t)\,dt$$

$$\boxed{\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)}$$

若 $G(f)$ 決定了，則 $\Phi(f)$ 可以被算出來

constraint 1

$G(f)$: 被稱作 generating function

- 同理

$$\psi(t) = 2\sum_k h_k \phi(2t-k) \qquad \Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi f t} dt$$

$$\psi(t/2) = 2\sum_k h_k \phi(t-k)$$

$2\Psi(2f) = 2H(f)\Phi(f)$

$$\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right) \qquad H(f) = \sum_k h_k e^{-j2\pi f k}$$

$$\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

constraint 2

- 另外，由於

$\Phi(0) = 1 \neq 0$, ∴ $G(0) = 1$

$$\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right) \qquad \Phi(0) = G(0)\Phi(0) \quad (f = 0 \text{ 代入})$$

$$G(0) = 1$$ 必需滿足

constraint 3

13-F Real Coefficient Constraints

Since $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

If $G(f) = G^*(-f)$ $H(f) = H^*(-f)$ are satisfied,

constraint 4 constraint 5

then $\Phi(f) = \Phi^*(-f)$, $\Psi(f) = \Psi^*(-f)$, and $\phi(t)$, $\psi(t)$ are real.

Note: If these constraints are satisfied, g_k, h_k on page 427 are also real.

13-G Vanish Moment Constraint

If $\psi(t)$ has p vanishing moments,

$$\int_{-\infty}^{\infty} t^k \psi(t)\,dt = 0 \qquad \text{for } k = 0, 1, 2, \ldots, p-1$$

Since $FT\left[t^k \psi(t)\right] = \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi(f)$

$$\int_{-\infty}^{\infty} x(t)\,dt = X(0) \quad \text{if} \quad X(f) = FT(x(t))$$

$$\int_{-\infty}^{\infty} t^k \psi(t)\,dt = 0 \quad \Longrightarrow \quad \left(\frac{j}{2\pi}\right)^k \frac{d^k}{df^k}\Psi(f)\Big|_{f=0} = 0$$

$X(f) = \int x(t)\, e^{-j2\pi ft}\,dt$

vanish moment in the frequecy domain

Therefore, $\left.\frac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$

Taking the conjugation on both sides, $\left.\frac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$

Since $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$

$$\frac{d^k}{df^k}\Psi(f) = \sum_{h=0}^{k}\binom{k}{h}\frac{d^h}{df^h}H\left(\frac{f}{2}\right)\frac{d^{k-h}}{df^{k-h}}\prod_{q=2}^{\infty}G\left(\frac{f}{2^q}\right)$$

$$\frac{d^k}{df^k}H\left(\frac{f}{2}\right) = \frac{1}{2^k}\frac{d^k}{df^k}H(f)$$

if $\left.\frac{d^k}{df^k}H(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$ is satisfied,

constraint 6

then $\left.\frac{d^k}{df^k}\Psi(f)\right|_{f=0} = 0$ for $k = 0, 1, 2, \ldots, p-1$ are satisfied

and the wavelet function has p vanishing moments.

13-H Orthogonality Constraints

• orthogonality constraint:

$$\int_{-\infty}^{\infty} 2^{m} \psi\left(2^{m_1} t - n_1\right) \psi\left(2^{m} t - n\right) dt = \delta\left(m - m_1\right) \delta\left(n - n_1\right)$$

$\psi(t)$: wavelet function

If the above equality is satisfied,

forward wavelet transform:

$$X_w(n,m) = 2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

inverse wavelet transform:

$$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w\left(m,n\right)$$

(much easier for inverse) C = mean of $x(t)$

(證明於後頁)

If $$x(t) = C + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2^{m/2} \psi\left(2^m t - n\right) X_w(m,n)$$

and $$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1} t - n_1\right) \psi\left(2^m t - n\right) dt = \delta(m - m_1)\delta(n - n_1),$$

then $$2^{m/2} \int_{-\infty}^{\infty} x(t) \psi(2^m t - n) dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} \left[C + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \psi\left(2^{m_1} t - n_1\right) X_w(m_1, n_1) \right] \psi(2^m t - n) dt$$

$$= 2^{m/2} \int_{-\infty}^{\infty} C \psi(2^m t - n) dt + 2^{m/2} \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} 2^{m_1/2} \int_{-\infty}^{\infty} \psi\left(2^{m_1} t - n_1\right) \psi(2^m t - n) dt X_w(m_1, n_1)$$

$$= 0 + \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \delta(m_1 - m)\delta(n_1 - n) X_w(m_1, n_1)$$

$$= X_w(m, n)$$

due to $\int_{-\infty}^{\infty} \psi(t) dt = 0$

Therefore, $2^{m/2}\int_{-\infty}^{\infty} x(t)\psi(2^m t-n)dt$ is the inverse operation of

$$C+\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} 2^{m/2}\psi\left(2^m t-n\right)X_w\left(m,n\right) \qquad \#$$

※ 要滿足

$$\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m_1}t-n_1\right)\psi\left(2^{m}t-n\right)dt=\delta\left(m-m_1\right)\delta\left(n-n_1\right)$$

之前，需要滿足以下三個條件

(1) $\int_{-\infty}^{\infty} \psi\left(t-n_1\right)\psi\left(t-n\right)dt=\delta\left(n_1-n\right)$ for mother wavelet

這個條件若滿足， $\int_{-\infty}^{\infty} 2^{m}\psi\left(2^{m}t-n_1\right)\psi\left(2^{m}t-n\right)dt=\delta\left(n-n_1\right)$

對所有的 m 皆成立

(2) $\int_{-\infty}^{\infty} \phi\left(t-n_1\right)\phi\left(t-n\right)dt=\delta\left(n_1-n\right)$ for scaling function

嚴格來說，這並不是必要條件，但是可以簡化 第 (3) 個條件的計算

(3) $\int_{-\infty}^{\infty} \psi(t-n_1)\psi\left(2^{-k}t-n\right)dt = 0$ for any n, n_1 if $k>0$

若 (1) 和 (3) 的條件滿足，則

$$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1}t-n_1\right)\psi\left(2^m t-n\right)dt = \delta(m-m_1)\delta(n-n_1)$$

也將滿足

(Proof): Set $t_1 = 2^m t$, $dt_1 = 2^m dt$

$$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1}t-n_1\right)\psi\left(2^m t-n\right)dt = \int_{-\infty}^{\infty} \psi\left(2^{m_1-m}t_1-n_1\right)\psi(t_1-n)dt_1$$

If (3) is satisfied,

$$\int_{-\infty}^{\infty} 2^m \psi\left(2^{m_1}t-n_1\right)\psi\left(2^m t-n\right)dt = 0 \quad \text{when } m \neq m_1$$

In the case where $m = m_1$, if (1) is satisfied, then

$$\int_{-\infty}^{\infty} 2^m \psi\left(2^m t-n_1\right)\psi\left(2^m t-n\right)dt = \int_{-\infty}^{\infty} \psi(t_1-n_1)\psi(t_1-n)dt_1 = \delta(n_1-n)$$

#

- 由 Page 420 的條件 (1)

$$\int_{-\infty}^{\infty}\psi(t-n_1)\psi(t-n)dt$$

$$=\int_{-\infty}^{\infty}e^{-j2\pi n_1 f}\Psi(f)e^{j2\pi n f}\Psi^*(f)df$$

Parseval's theorem

$$\int_{-\infty}^{\infty}x(t)y^*(t)dt=\int_{-\infty}^{\infty}X(f)Y^*(f)df$$

$$=\int_{-\infty}^{\infty}e^{j2\pi(n-n_1)f}\Psi(f)\Psi^*(f)df$$

$$=\sum_{p=-\infty}^{\infty}\int_{0}^{1}e^{j2\pi(n-n_1)(f'+p)}\Psi(f'+p)\Psi^*(f'+p)df'$$

$$=\int_{0}^{1}e^{j2\pi(n-n_1)f'}\sum_{p=-\infty}^{\infty}|\Psi(f'+p)|^2df'=\delta(n-n_1)$$

$$e^{j2\pi(n-n_1)(f'+p)}=e^{j2\pi(n-n_1)f'}$$

if p is an integer

Therefore,

$$\int_{0}^{1}e^{-j2\pi n_2 f'}\sum_{p=-\infty}^{\infty}|\Psi(f'+p)|^2df=\delta(-n_2)=\delta(n_2)$$

$$\boxed{\sum_{p=-\infty}^{\infty}|\Psi(f'+p)|^2=1}$$

for all f' should be satisfied

- 同理，由 Page 420 的條件 (2)

$$\int_{-\infty}^{\infty}\phi(t-n_1)\phi(t-n)dt=\delta(n_1-n)$$ for scaling function

↓ 推導過程類似 page 422

$$\boxed{\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2=1}$$ for all f should be satisfied

衍生的條件：將 $\Psi(f)=H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty}|\Psi(f+p)|^2=1$

(page 422)

$$\sum_{p=-\infty}^{\infty}\left|H\left(\frac{f}{2}+\frac{p}{2}\right)\Phi\left(\frac{f}{2}+\frac{p}{2}\right)\right|^2=1$$

$$\sum_{q=-\infty}^{\infty}\left|H\left(\frac{f}{2}+q\right)\Phi\left(\frac{f}{2}+q\right)\right|^2+\sum_{q=-\infty}^{\infty}\left|H\left(\frac{f}{2}+q+\frac{1}{2}\right)\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2=1$$

因為 h_k 是 discrete sequence, $H(f)$ 是 h_k 的 discrete-time Fourier transform

$$H(f)=H(f+1)=H(f+2)=\cdots\cdots$$

$$\left|H\left(\frac{f}{2}\right)\right|^2\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q\right)\right|^2+\left|H\left(\frac{f}{2}+\frac{1}{2}\right)\right|^2\sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2=1$$

$$|H\left(\frac{f}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2}+q\right)|^2 + |H\left(\frac{f}{2}+\frac{1}{2}\right)|^2 \sum_{q=-\infty}^{\infty} |\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)|^2 = 1$$

因為 $\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$ for all f

(page 422 的條件)

$$|H\left(\frac{f}{2}\right)|^2 + |H\left(\frac{f}{2}+\frac{1}{2}\right)|^2 = 1$$

$$|H(f)|^2 + |H\left(f+\frac{1}{2}\right)|^2 = 1$$

constraint 7

同理，將 $\Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ 代入 $\sum_{p=-\infty}^{\infty} |\Phi(f+p)|^2 = 1$

(page 422)

經過運算可得

$$|G(f)|^2 + \left|G\left(f+\frac{1}{2}\right)\right|^2 = 1$$

constraint 8

- Page 421 條件 (3) 的處理

由於

$\psi\left(2^{-k}t-n\right)$ 是 $\phi\left(2^{-k+1}t-n_1\right)$ 的 linear combination

$\phi\left(2^{-k+1}t-n_1\right)$ 是 $\phi\left(2^{-k+2}t-n_2\right)$ 的 linear combination

$\phi\left(2^{-k+2}t-n_2\right)$ 是 $\phi\left(2^{-k+3}t-n_3\right)$ 的 linear combination

$$\psi(t)=2\sum_k h_k\phi(2t-k)$$

$$\phi(t)=2\sum_k g_k\phi(2t-k)$$

⋮

⋮

$\phi\left(2^{-1}t-n_{k-1}\right)$ 是 $\phi\left(t-n_k\right)$ 的 linear combination

所以

$\psi\left(2^{-k}t-n\right)$ 必定可以表示成 $\phi\left(t-n_k\right)$ 的 linear combination

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k} b_{n_k}\phi\left(t-n_k\right)$$

$$\psi\left(2^{-k}t-n\right)=\sum_{n_k} b_{n_k}\phi\left(t-n_k\right)$$

所以，若 $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\phi\left(t-n_k\right)dt=0$ for any n_1, n_k 可以滿足

則 $\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\psi\left(2^{-k}t-n\right)dt=0$ for any n_1, n_k 必定能夠成立

Page 421 條件 (3) 可改寫成

$$\boxed{\int_{-\infty}^{\infty}\psi\left(t-n_1\right)\phi\left(t-n_k\right)dt=0}$$

$$\int_{-\infty}^{\infty}\psi\left(t\right)\phi\left(t-\tau\right)dt=0$$ (將 $t-n_1$ 變成 t, $\tau = n_k-n_1$)

$$\int_{-\infty}^{\infty}\Psi\left(f\right)\Phi^{*}\left(f\right)e^{j2\pi\tau f}\,df=0$$ (from Parseval's theorem)

$$\int_{-\infty}^{\infty} \Psi(f)\Phi^*(f)e^{j2\pi\tau f}\,df = 0$$

Since $\Psi(f) = H\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$ $\quad \Phi(f) = G\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$

$$\int_{-\infty}^{\infty} H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\left|\Phi\left(\frac{f}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

$$\sum_{p=-\infty}^{\infty}\int_0^1 H\left(\frac{f+p}{2}\right)G^*\left(\frac{f+p}{2}\right)\left|\Phi\left(\frac{f+p}{2}\right)\right|^2 e^{j2\pi\tau(f+p)}\,df = 0$$

$e^{j2\pi\tau(f+p)} = e^{j2\pi\tau f}$ (since from page 428, τ is an integer)

$$\sum_{q=-\infty}^{\infty}\int_0^1 H\left(\frac{f}{2}+q\right)G^*\left(\frac{f}{2}+q\right)\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\,df$$

$$+\sum_{q=-\infty}^{\infty}\int_0^1 H\left(\frac{f}{2}+q+\frac{1}{2}\right)G^*\left(\frac{f}{2}+q+\frac{1}{2}\right)\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

Since $H(f) = H(f+1) = H(f+2) = \cdots\cdots$

$G(f) = G(f+1) = G(f+2) = \cdots\cdots$

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right)\int_0^1 \sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q\right)\right|^2 e^{j2\pi\tau f}\,df$$

$$+H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right)\int_0^1 \sum_{q=-\infty}^{\infty}\left|\Phi\left(\frac{f}{2}+q+\frac{1}{2}\right)\right|^2 e^{j2\pi\tau f}\,df = 0$$

Since $\sum_{p=-\infty}^{\infty}|\Phi(f+p)|^2 = 1$ for all f (page 422)

$$H\left(\frac{f}{2}\right)G^*\left(\frac{f}{2}\right) + H\left(\frac{f}{2}+\frac{1}{2}\right)G^*\left(\frac{f}{2}+\frac{1}{2}\right) = 0$$

$$H(f)G^*(f) + H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right) = 0$$

constraint 9

13-I Nine Constraints

整理： 設計 mother wavelet 和 scaling function 的九大條件
(皆由 page 406 的 constraints 衍生而來)

(1) $\Phi(f)=\prod_{q=1}^{\infty}G\left(\frac{f}{2^q}\right)$ — for fast algorithm , page 412

(2) $\Psi(f)=H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty}G\left(\frac{f}{2^q}\right)$ — for fast algorithm , page 413

(3) $G(0)=1$ — for fast algorithm , page 413

(4) $H(f)=H^*(-f)$ — for real , page 414

(5) $G(f)=G^*(-f)$ — for real , page 413

(6) $\left.\frac{d^k}{df^k}H(f)\right|_{f=0}=0$ — for p vanish moments , page 416

for $k=0, 1, \ldots, p$-1

(7) $|H(f)|^2+|H\left(f+\frac{1}{2}\right)|^2=1$ for orthogonal , page 425

(8) $|G(f)|^2+|G\left(f+\frac{1}{2}\right)|^2=1$ for orthogonal , page 426

(9) $H(f)G^*(f)+H\left(f+\frac{1}{2}\right)G^*\left(f+\frac{1}{2}\right)=0$ for orthogonal , page 430

$G(f)$, $H(f)$ are the discrete-time Fourier transform of $\{g_k\}$, $\{h_k\}$ on page 407.

g_k: low frequency
h_k: high frequency

$G(f) = \sum_k g_k e^{-j2\pi fk}$

$G(f+1) = \sum_k g_k e^{-j2\pi(f+1)k} = \sum_k e^{-j2\pi k} g_k e^{-j2\pi fk} = G(f)$

- 條件的簡化

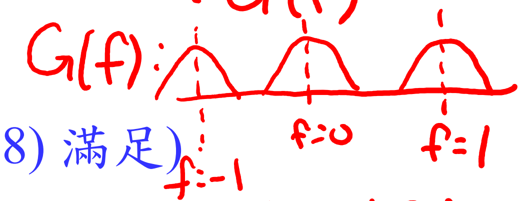
Specially, if we set that

$$H(f) = -e^{-j2\pi f} G^*(f + 1/2) \qquad h_k = (-1)^k g_{1-k}$$

when the following constraints are satisfied:

$$|G(f)|^2 + \left|G\left(f + \frac{1}{2}\right)\right|^2 = 1$$

$$G(f) = G^*(-f) \qquad \text{(條件 (5), (8) 滿足)}$$



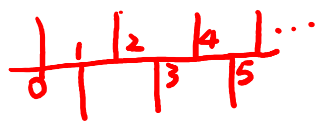
有效頻率：$-\frac{1}{2} < f < \frac{1}{2}$

high frequency $f \cong \pm\frac{1}{2}$

low frequency $f \cong 0$

then $\quad |H(f)|^2 + \left|H\left(f + \frac{1}{2}\right)\right|^2 = \left|G\left(f + \frac{1}{2}\right)\right|^2 + |G(f)|^2 = 1$

Ex:



$f = \frac{1}{2}$, period=2

$$H(f)G^*(f) + H\left(f + \frac{1}{2}\right)G^*\left(f + \frac{1}{2}\right)$$

$$= -e^{-j2\pi f} G^*\left(f + \frac{1}{2}\right)G^*(f) - e^{-j2\pi(f+\frac{1}{2})} G^*(f)G^*\left(f + \frac{1}{2}\right)$$

$g_k \Rightarrow G(f)$

$$= -e^{-j2\pi f} G^*\left(f + \frac{1}{2}\right)G^*(f) + e^{-j2\pi f} G^*(f)G^*\left(f + \frac{1}{2}\right) = 0$$

$(-1)^k g_k \Rightarrow G(f+\frac{1}{2})$ $\left(\because \sum_k (-1)^k g_k e^{-j2\pi fk} = \sum_k e^{-j\pi k} g_k e^{-j2\pi fk} = \sum_k g_k e^{-j2\pi(f+\frac{1}{2})k} = G(f+\frac{1}{2})\right)$

$$H^*(-f) = -e^{-j2\pi f} G(-f + 1/2) = -e^{-j2\pi f} G^*(f - 1/2) = H(f)$$

條件 (4), (7), (9) 也將滿足

整理： 設計 mother wavelet 和 scaling function 的幾個要求 (簡化版)

(1) $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm

(2) $\Psi(f) = H\left(\frac{f}{2}\right)\prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$ for fast algorithm

(3) $G(0) = 1$ for fast algorithm

(4) $G(f) = G^*(-f)$ for real

(5) $\left.\frac{d^k}{df^k}H(f)\right|_{f=0} = 0$ for p vanish moments

for $k = 0, 1, \ldots, p$-1

(6) $|G(f)|^2 + |G\left(f+\frac{1}{2}\right)|^2 = 1$ for orthogonal

(7) $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$

13-J Design Process

設計時，只要 $G(f)$ $(0 \le f \le 1/4)$ 決定了，mother wavelet 和scaling function 皆可決定

$G(f)$: 被稱作 generating function

Design Process (設計流程):

(Step 1): 給定 $G(f)$ $(0 \le f \le 1/4)$，滿足以下的條件

✓(a) $G(0)=1$

✓(b) $\left.\dfrac{d^k}{df^k}G(f)\right|_{f=\frac{1}{2}}=0$ for $k = 0, 1, 2, \ldots, p\text{-}1$

(Step 2) 由 $G(f) = G^*(-f)$ 　決定 $G(f)$ $(3/4 \le f < 1)$

$-\frac{1}{4} < f < 0$

(Step 3) 由 $|G(f)|^2 + \left|G\left(f + \frac{1}{2}\right)\right|^2 = 1$ 　決定 $G(f)$ $(1/4 < f < 3/4)$

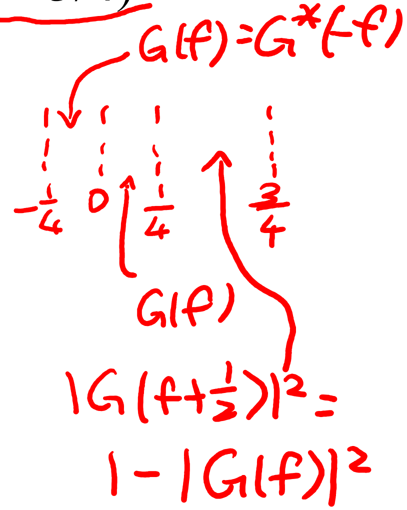
再根據 $G(f) = G(f+1)$，決定所有的 $G(f)$ 值

(Step 4) 由 $H(f) = -e^{-j2\pi f} G^*(f + 1/2)$ 　決定 $H(f)$

(Step 5) 由 $\Phi(f) = \prod_{q=1}^{\infty} G\left(\frac{f}{2^q}\right)$

$$\Psi(f) = H\left(\frac{f}{2}\right) \prod_{q=2}^{\infty} G\left(\frac{f}{2^q}\right)$$

決定 $\Phi(f)$, $\Psi(f)$



$G(f) = G(f+1)$

註：(1) 當 Step 1 的兩個條件滿足，由於$|G(f)|^2+|G(f+1/2)|^2=1$

$$\frac{d^k}{df^k}G(f)\bigg|_{f=1/2}=0 \quad \text{for } k = 0, 1, 2, \ldots, p\text{-}1$$

又由於 $H(f)=-e^{-j2\pi f}G^*(f+1/2)$

$$\frac{d^k}{df^k}H(f)\bigg|_{f=0}=0 \quad \text{for } k = 0, 1, 2, \ldots, p\text{-}1$$

(2) $|G(f)|^2+|G(f+1/2)|^2=1$ $\qquad |G(f)|^2=|G(-f)|^2$

所以當 $G(f)$ $(0\leq f\leq 1/4)$ 給定，$|G(f)|$ 有唯一解

(3) 對於離散信號而言，$G(f) = G(f+1)$

有意義的頻率範圍為 $-1/2 < f < 1/2$

$$G(f)=\sum_k g_k e^{-j2\pi f k}$$

13-K Several Continuous Wavelets with Discrete Coefficients

(1) Haar Wavelet — vanish moment = 1
= 2-point Daubechies wavelet

$g[0] = 1,\ g[1] = 1$ $\quad G(f) = 1 + \exp(-j2\pi f)$

$h[0] = 1,\ h[1] = -1$ $\quad H(f) = 1 - \exp(-j2\pi f)$

或 $\quad \sum g_k e^{-j2\pi f k}$

$g[0] = 1/2,\ g[1] = 1/2$ $\quad G(f) = [1 + \exp(-j2\pi f)]/2$

$h[0] = 1/2,\ h[1] = -1/2$ $\quad H(f) = [1 - \exp(-j2\pi f)]/2$

$G(\frac{1}{2}) = 0$

$G'(f) = -j\pi e^{-j2\pi f}$

$G'(\frac{1}{2}) = j\pi \neq 0$

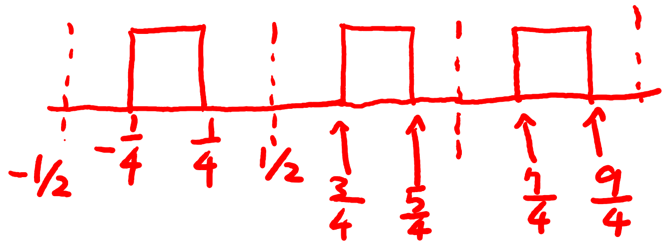
vanish moment = ? 1

Daubechies wavelet $k \to \infty$

(2) Sinc Wavelet

$$G(f) = 1 \qquad \text{for } |f| \leq 1/4$$

$$G(f) = 0 \qquad \text{otherwise}$$



$$\frac{d^k}{df^k} G(f) \Big|_{f=\frac{1}{2}} = \frac{d^k}{df^k} 0 \Big|_{f=\frac{1}{2}} = 0 \quad \text{for any } k$$

vanish moment = ? infinite

but g_k has infinite length

(3) 4-point Daubechies Wavelet (see pages 486-488)

$$g_k : \left[\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8} \right]$$

2k-point Daubechies wavelet : vanish moment = k

vanish moment = ?

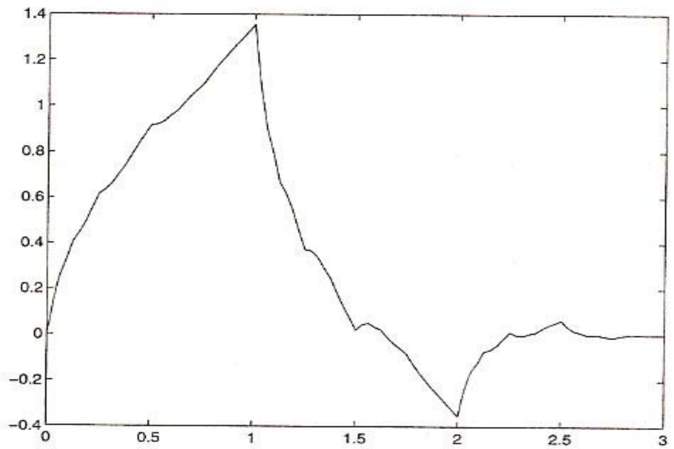
vanish moment VS the number of coefficients

Daubechies

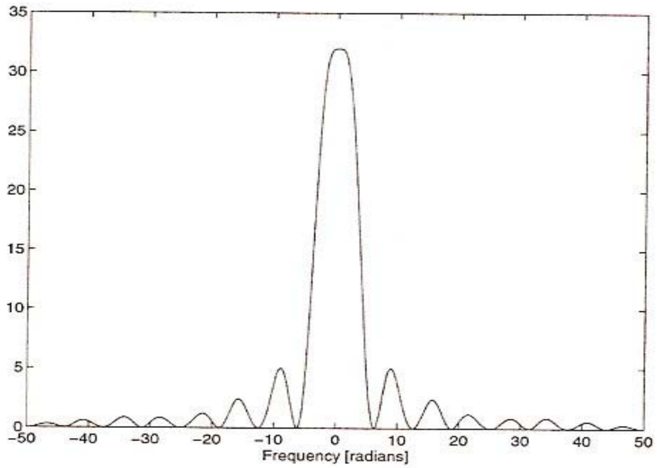
k=1 ⟵⟶ $k \to \infty$

Haar — sinc

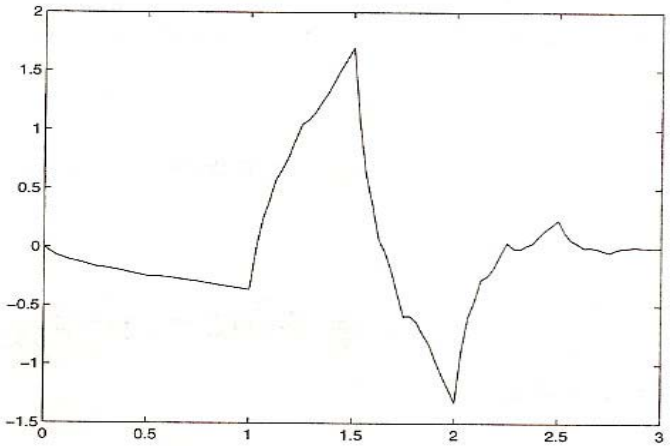
From: S. Qian and D. Chen, *Joint Time-Frequency Analysis: Methods and Applications*, Prentice Hall, N.J., 1996.



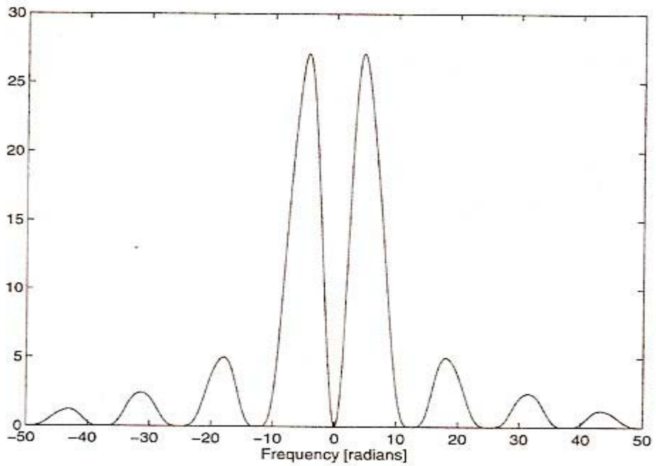
(a) Scaling function $\phi(t)$



(b) $|\Phi(\omega)|$



(c) Daubechies wavelet $\psi(t)$



(d) $|\Psi(\omega)|$

13-L Continuous Wavelet with Discrete Coefficients 優缺點

- Advantages:

(1) Fast algorithm for MRA

(2) Non-uniform frequency analysis

$$\psi\left(2^m t - n\right) \xrightarrow{\text{FT}} 2^{-m} e^{-j2\pi n 2^{-m} f} \Psi\left(2^{-m} f\right)$$

(3) Orthogonal

- Disadvantages:

(a) 無限多項連乘

(b) problem of initial

$\chi_w(n,m)$, $X_w(n,m)$ 皆由 $\chi_w(n,m+1)$ 算出

$\chi_w(n,m)\big|_{m\to\infty}$ 如何算

(c) 難以保證 compact support

(d) 仍然太複雜

附錄十三 幾種常見的影像壓縮格式

(1) JPEG: 使用 discrete cosine transform (DCT) 和 8×8 blocks
是當前最常用的壓縮格式 (副檔名為 *.jpg 的圖檔都是用 JPEG 來壓縮)

可將圖檔資料量壓縮至原來的 1/8 (對灰階影像而言) 或 1/16 (對彩色影像而言)

(2) JPEG2000: 使用 discrete wavelet transform (DWT)
壓縮率是 JPEG 的 5 倍左右

(3) JPEG-LS: 是一種 lossless compression
壓縮率較低，但是可以完全重建原來的影像

(4) JPEG2000-LS: 是 JPEF2000 的 lossless compression 版本

(5) JBIG: 針對 bi-level image (非黑即白的影像) 設計的壓縮格式

(6) GIF: 使用 LZW (Lempel–Ziv–Welch) algorithm (類似字典的建構)

適合卡通圖案和動畫製作，lossless

(7) PNG: 使用 LZ77 algorithm (類似字典的建構，並使用 sliding window)

lossless

(8) JPEG XR (又稱 HD Photo): 使用 Integer DCT，lossless

在 lossy compression 的情形下壓縮率可和 JPEG 2000 差不多

(9) TIFF: 使用標籤，最初是為圖形的印刷和掃描而設計的，lossless