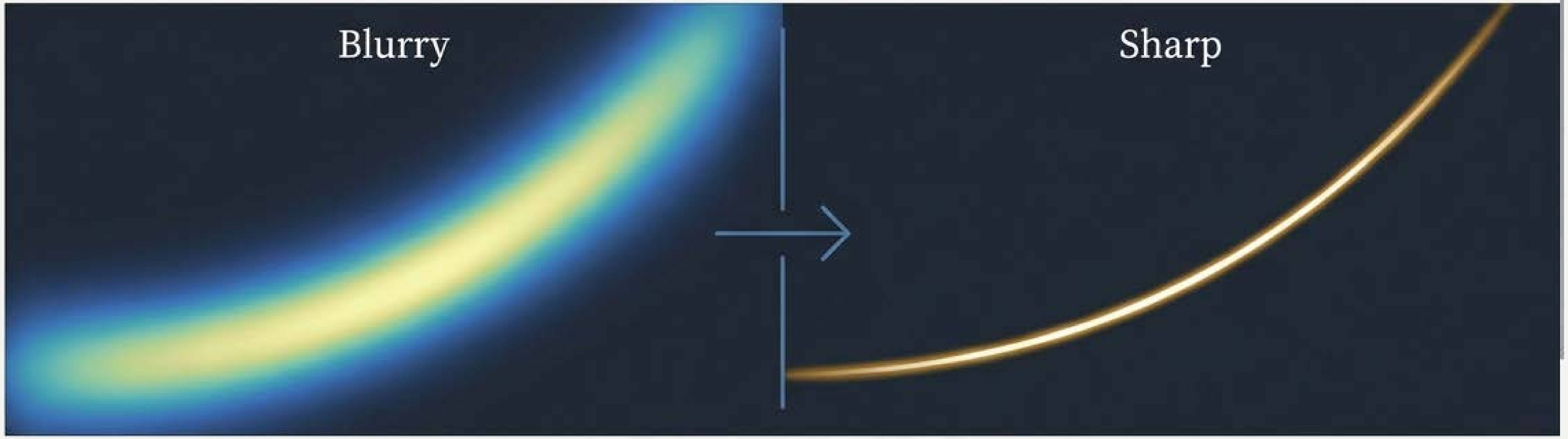
# From Blurry to Sharp: The Reassignment Method & Synchrosqueezing Transform

Advanced Techniques for Sharpening Time-Frequency Representations

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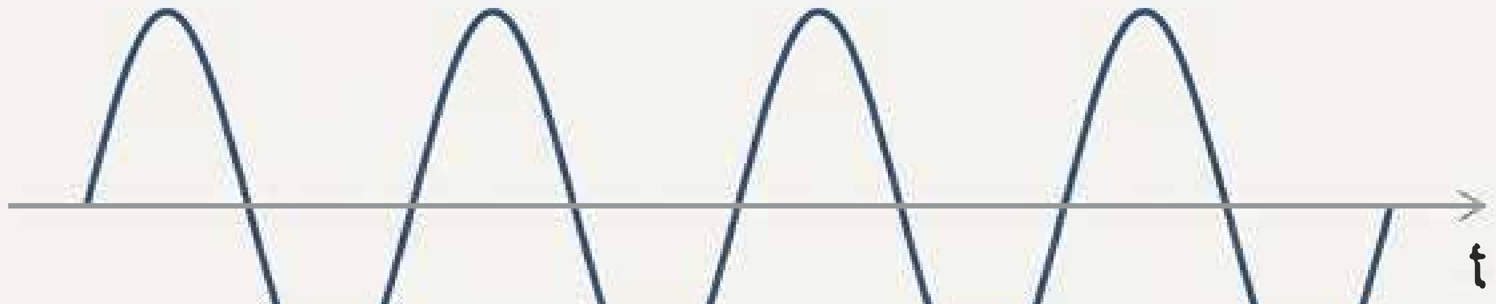
# Our Roadmap for Today

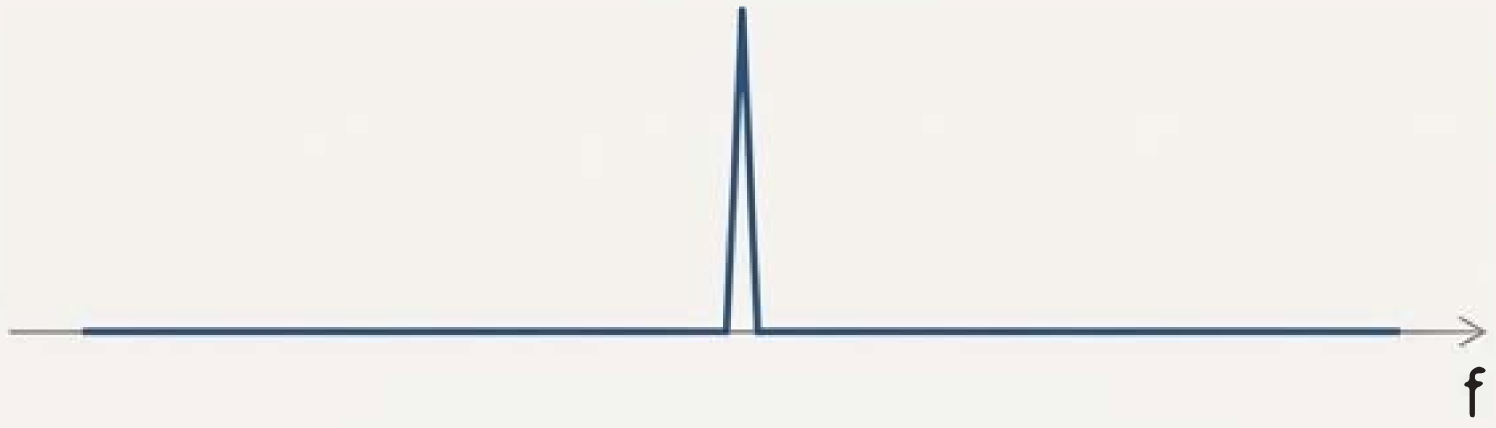
1. **Motivation:** The Limits of the Fourier Transform & the Uncertainty Principle
2. **The Reassignment Method (RM):** Correcting Position with Physical Intuition
3. **The Synchrosqueezing Transform (SST):** Achieving Sharpness with Invertibility
4. **Comparison:** A head-to-head analysis of STFT, RM, and SST
5. **Applications:** Real-world examples in biomedical signals and denoising
6. **Conclusion:** Key takeaways and summary

# The Two Worlds of Signals: Stationary vs. Non-Stationary

## Stationary Signals

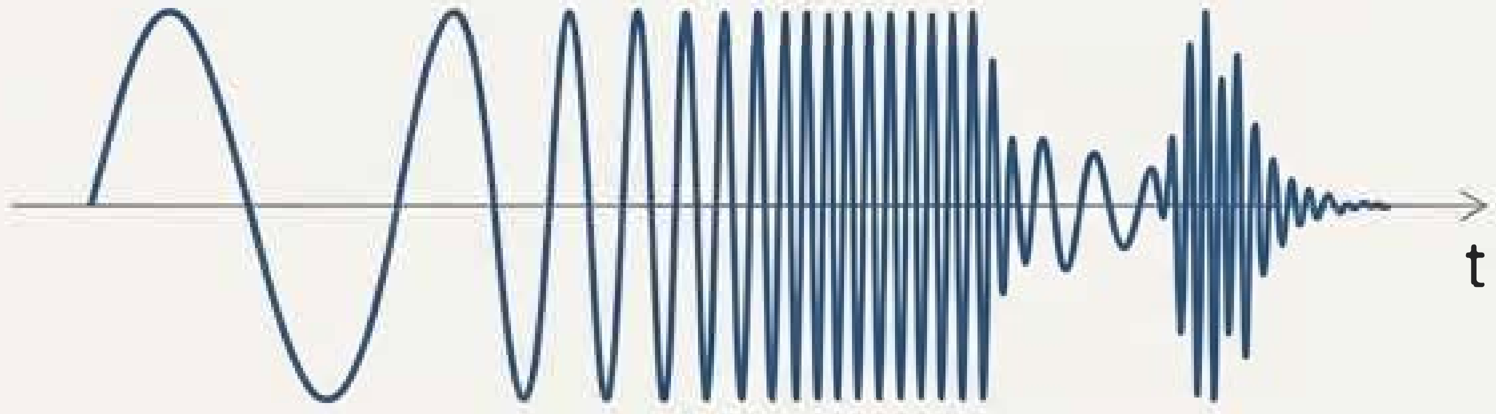



Fourier Transform




Frequency content is constant. The Fourier Transform is sufficient.

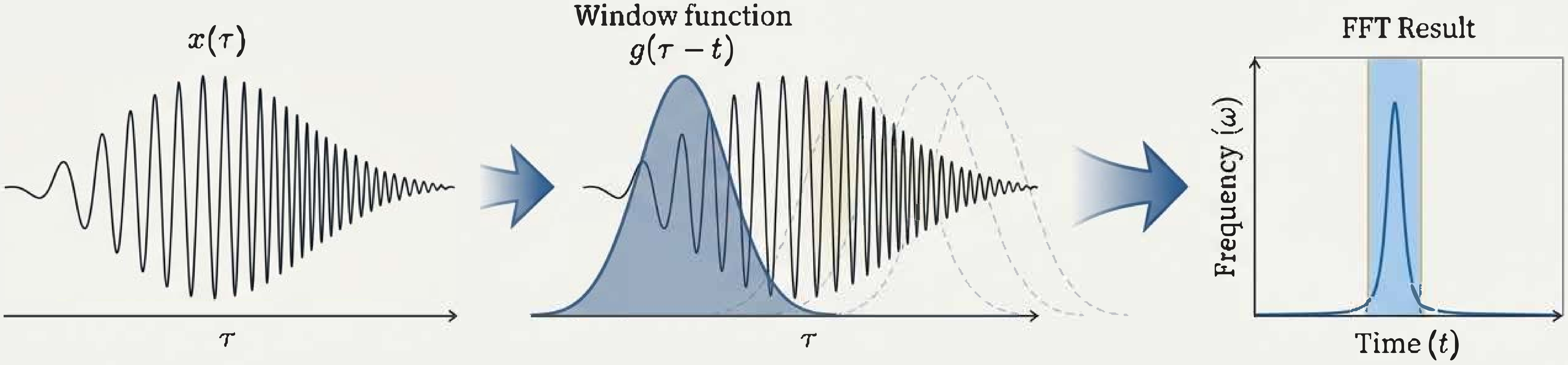
## Non-Stationary Signals




Frequency content changes over time (e.g., Speech, ECG, Radar). The critical question becomes: **Which frequency occurs at what time?**

# Rfc Qr_lb_pb ?nnpm_af8 ?l_jwxgle Qgel_jq Through a Sliding Window

**Concept:** Analyze a non-stationary signal as a series of short, “quasi-stationary” segments.




- **Mechanism:** Sliding Window Function
- **Output:** The Spectrogram, defined as $|STFT(t, \omega)|^2$

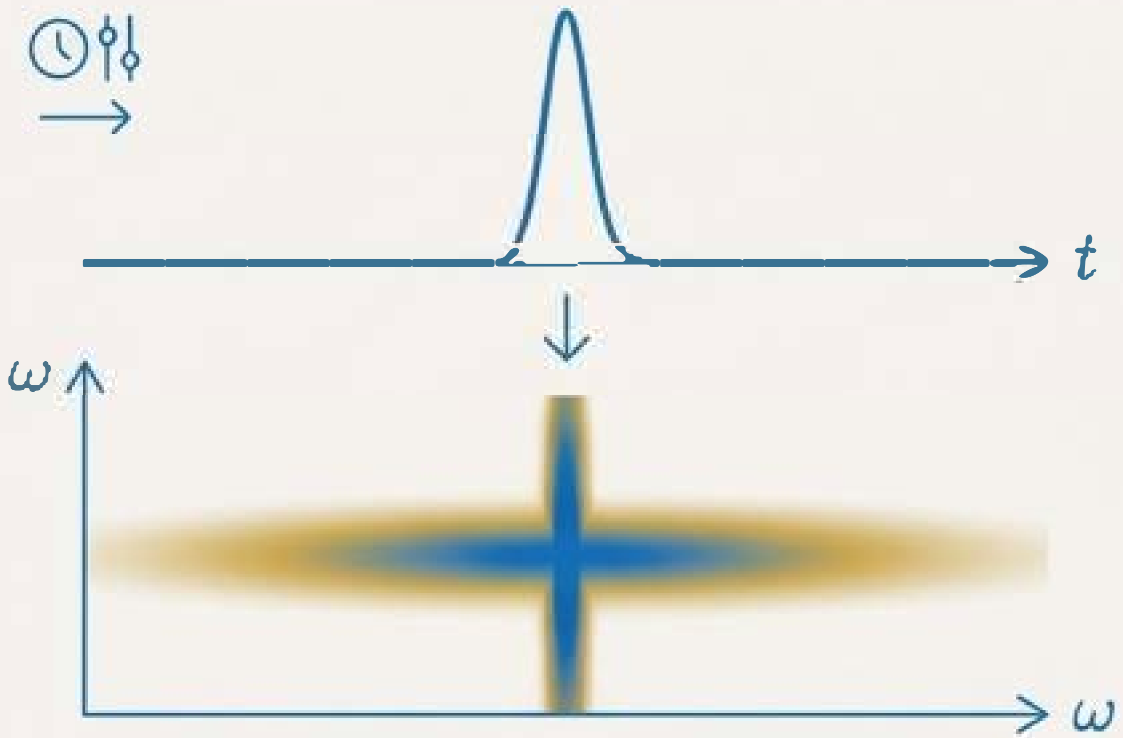
$$STFT_x(t, \omega) = \int_{-\infty}^{\infty} x(\tau)\, g(\tau - t)\, e^{-j\omega\tau}\, d\tau$$

# The Inescapable Barrier: Heisenberg's Uncertainty Principle

**Core Concept:** We cannot simultaneously achieve perfect resolution in both time ($\sigma_t$) and frequency ($\sigma_\omega$).

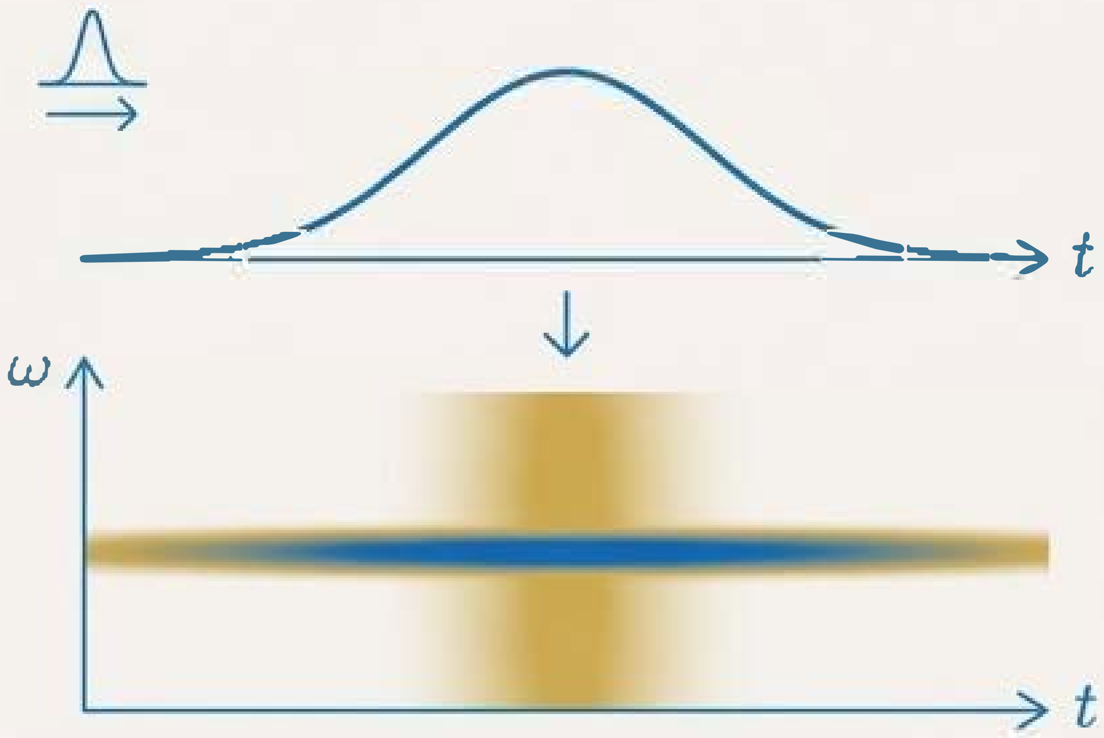
$$\sigma_t \cdot \sigma_\omega \geq 1/2$$

Excellent Time Resolution, Poor Frequency Resolution




Excellent Time Resolution, Poor Frequency Resolution

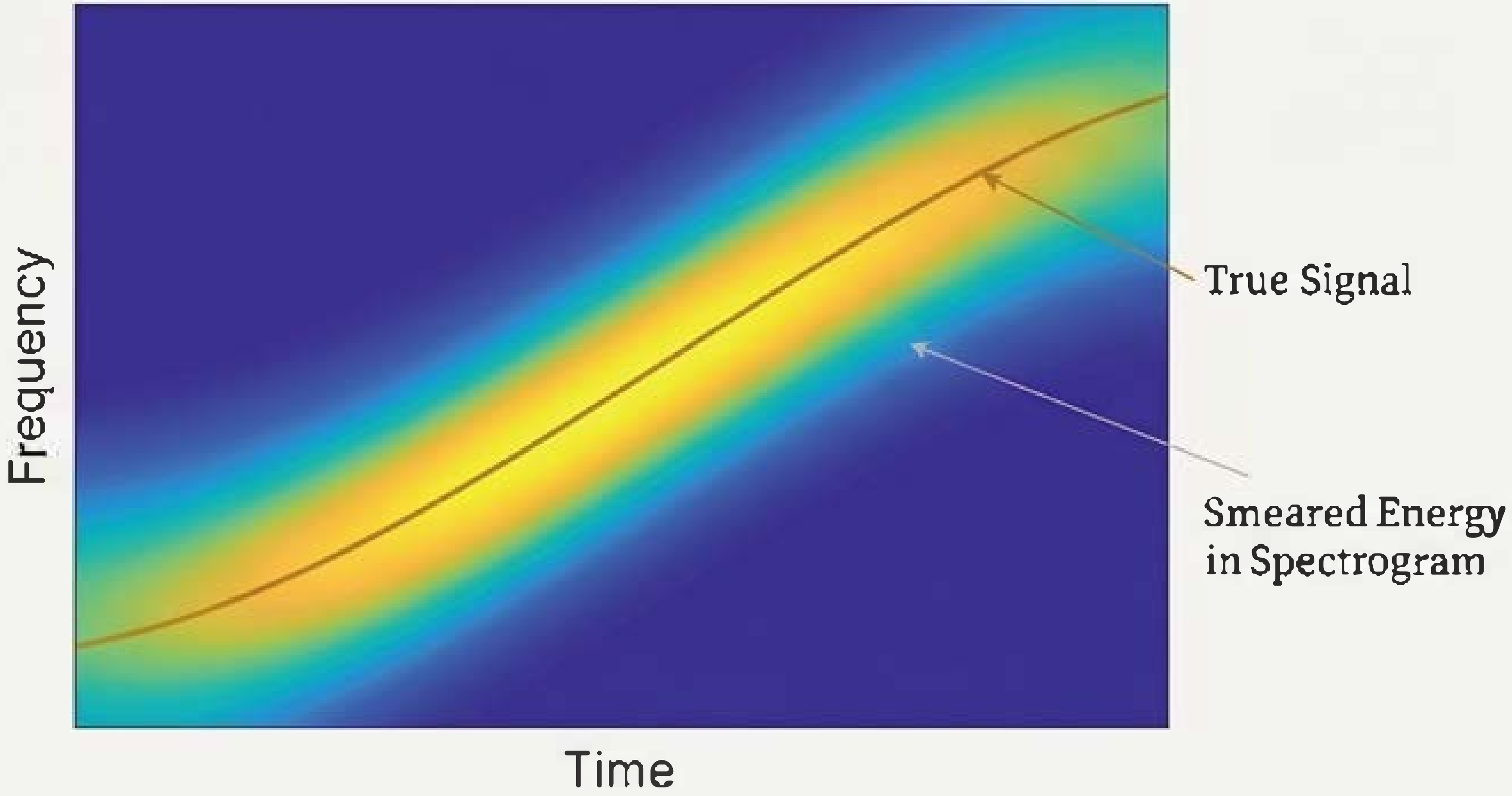
Poor Time Resolution, Excellent Frequency Resolution

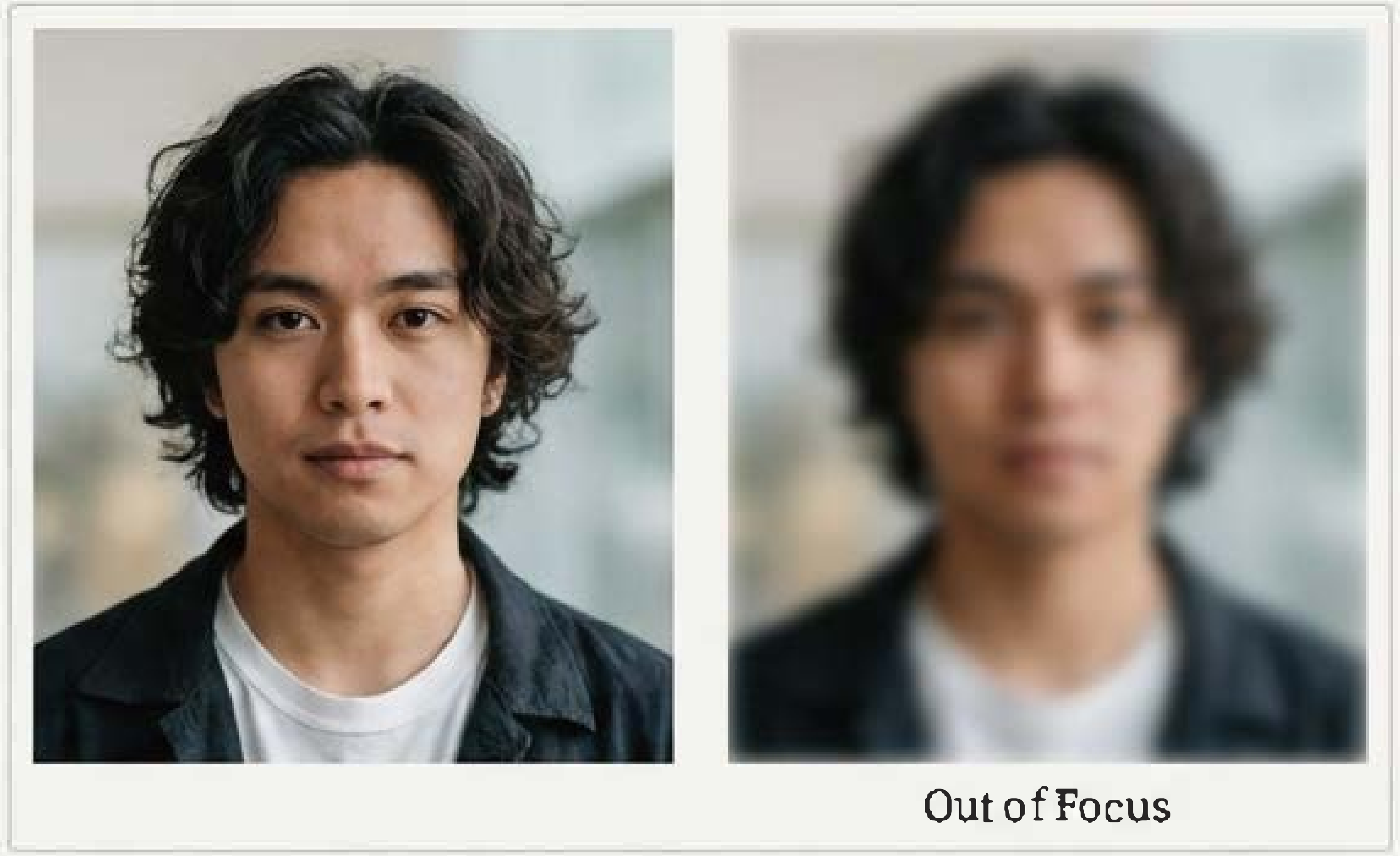



Poor Time Resolution, Excellent Frequency Resolution

**The Consequence:** This creates the **'Smearing Effect'**: energy in the spectrogram is always blurred. This is a physical limit, not a software bug.

# The Problem in Focus: The "Smearing Effect"



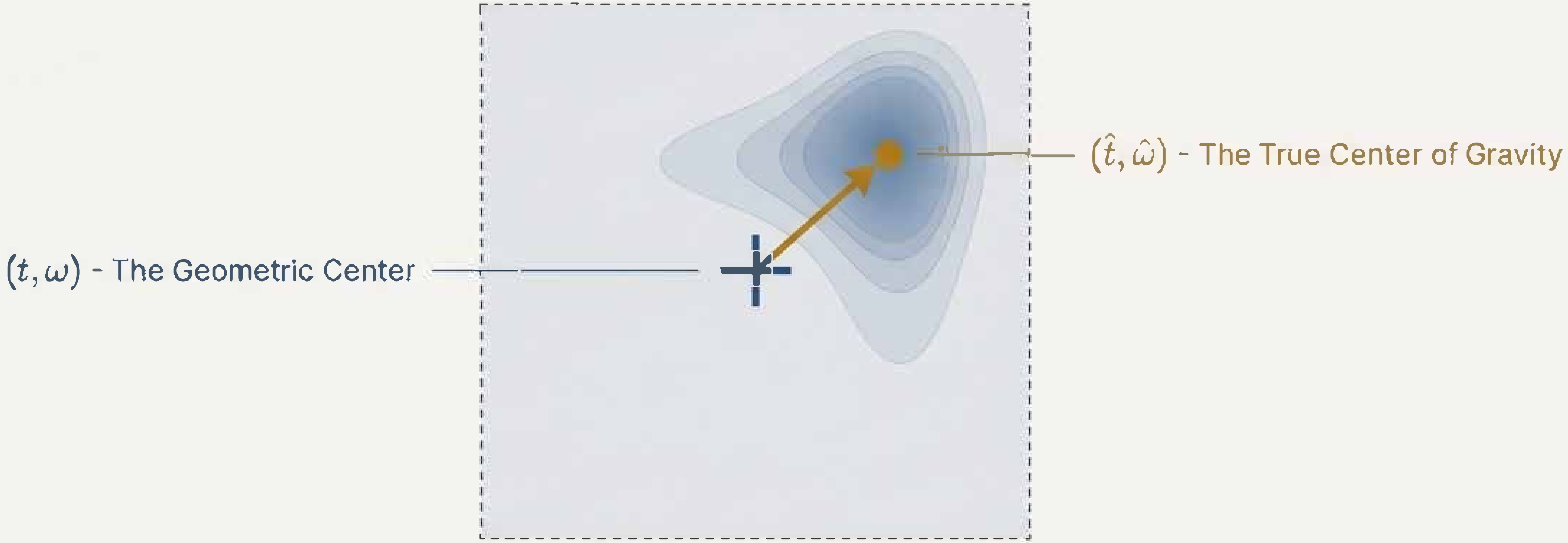




**The Issue:** Each point on the spectrogram does not represent a discrete event. It represents the *average* energy across the entire time-frequency analysis window.

**The Analogy:** This is conceptually identical to a photograph taken out of focus. The information is present, but it has been scattered.

**The Driving Question: Can we use post-processing to computationally "refocus" this scattered energy back to where it truly belongs?**

# The Reassignment Insight: Geometric Center vs. Center of Gravity



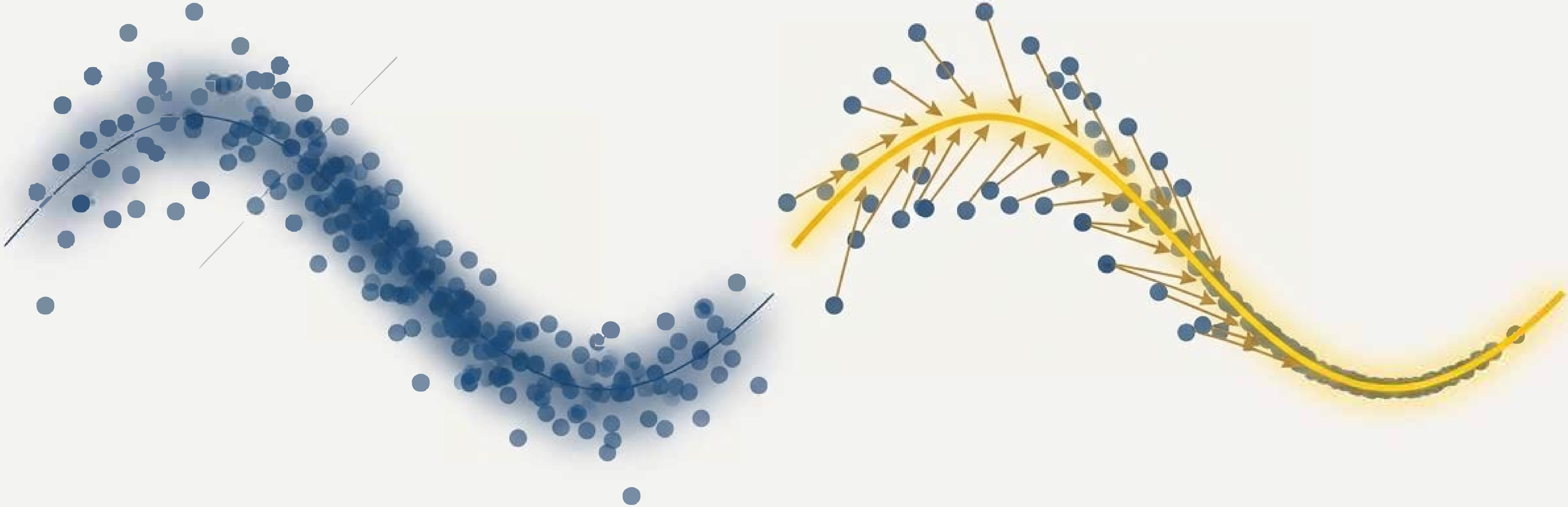

**STFT's Assumption:** The standard spectrogram places the calculated energy at the window's geometric center, $(t, \omega)$.

**Physical Reality:** However, the true "center of gravity" of the signal's energy within that window is often located elsewhere, at $(\hat{t}, \hat{\omega})$.

**The Mismatch:** This mismatch is the source of the blur, and it's most significant when the signal's frequency is changing.

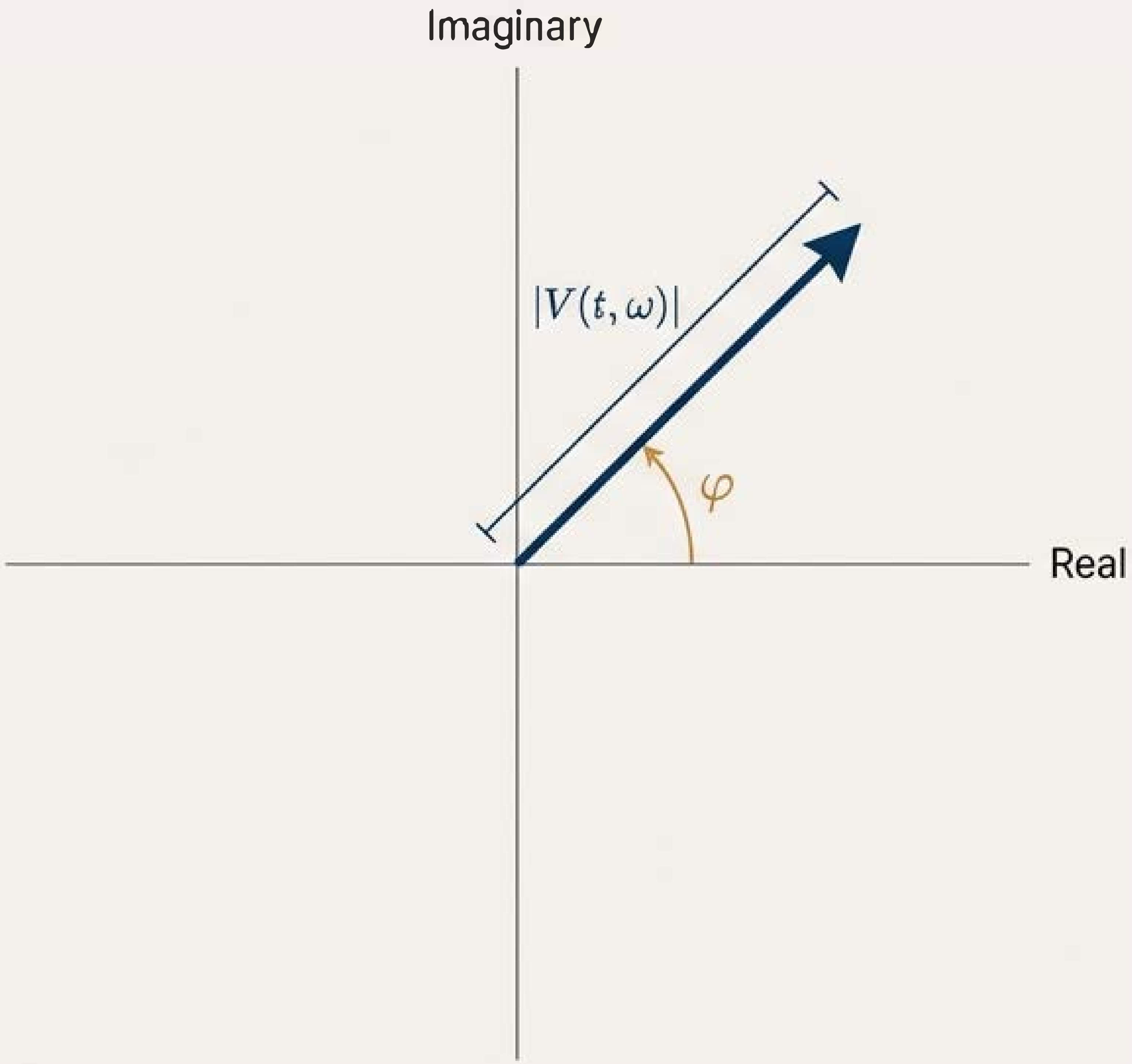
# The Solution: Move Energy to its True Center of Gravity

**The Idea:** Don't plot the energy at the window's center. Instead, **calculate the true center of gravity and move** the energy value from $(t, \omega)$ to its proper location $(\hat{t}, \hat{\omega})$.



**The Action:** Every point in the blurry spectrogram is **reassigned** to a new, more accurate coordinate. This is the principle of "Reassigning mass to its centroid."

# The Mathematics of Refocusing: Phase Unlocks Precision




The power of the Reassignment Method (RM) does not come from complex new integrations, but from leveraging the information already encoded in the STFT's phase $\varphi(t,\omega)$.

The true "center of gravity" of the signal's energy, $(\hat{t},\hat{\omega})$, can be calculated directly from the partial derivatives of the phase:

$$\hat{t}(t,\omega) = t - \frac{\partial\phi}{\partial\omega}$$

Reassigned Time Coordinate (Group Delay)

$$\hat{\omega}(t,\omega) = \omega + \frac{\partial\phi}{\partial t}$$

Reassigned Frequency Coordinate (Instantaneous Frequency)

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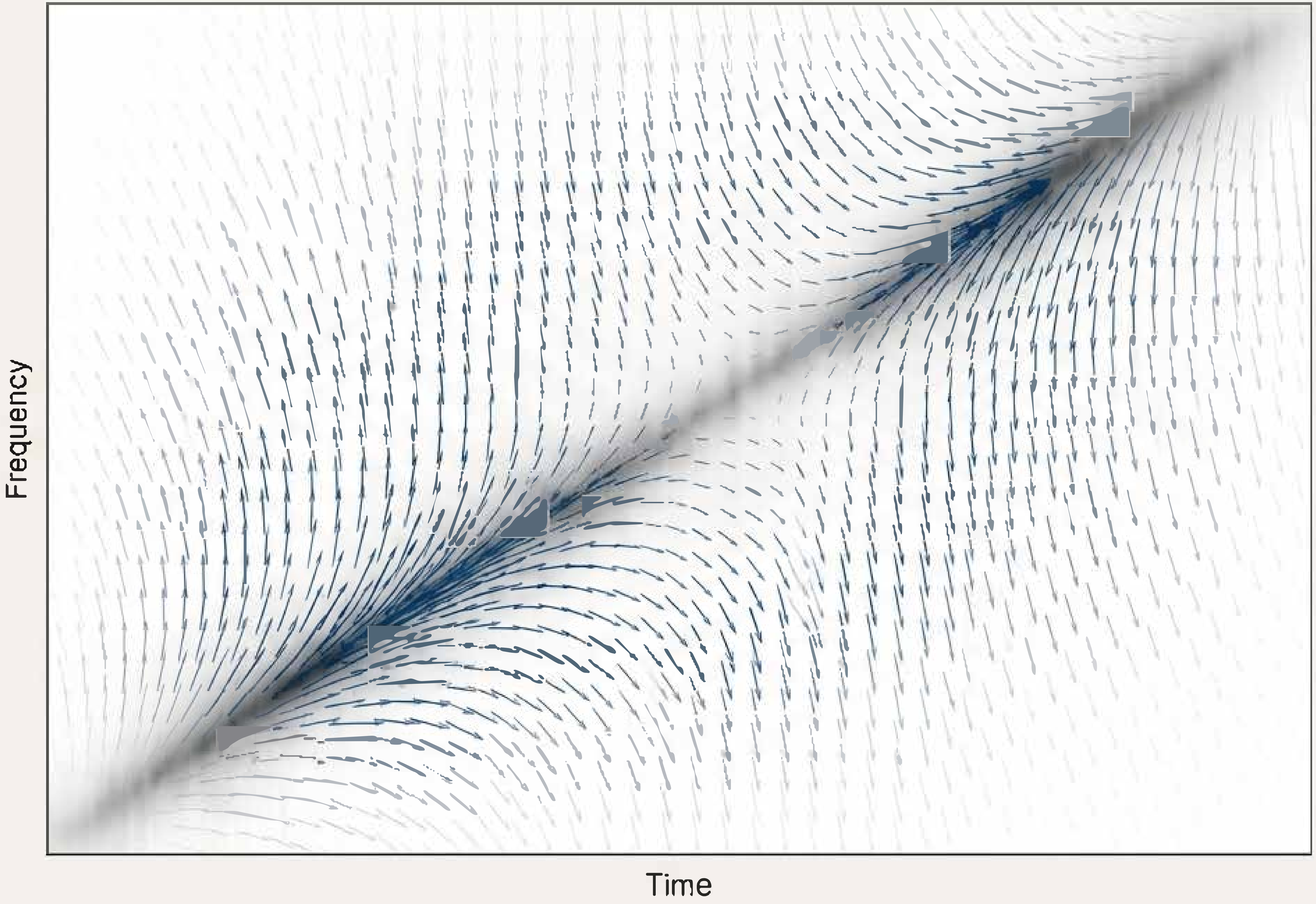
As derived by Kodera et al., we don't need to re-run the analysis. We simply use the phase gradient to calculate the precise coordinates where the energy for each point $(t,\omega)$ truly belongs.

# Visualizing the Correction: The Reassignment Vector Field



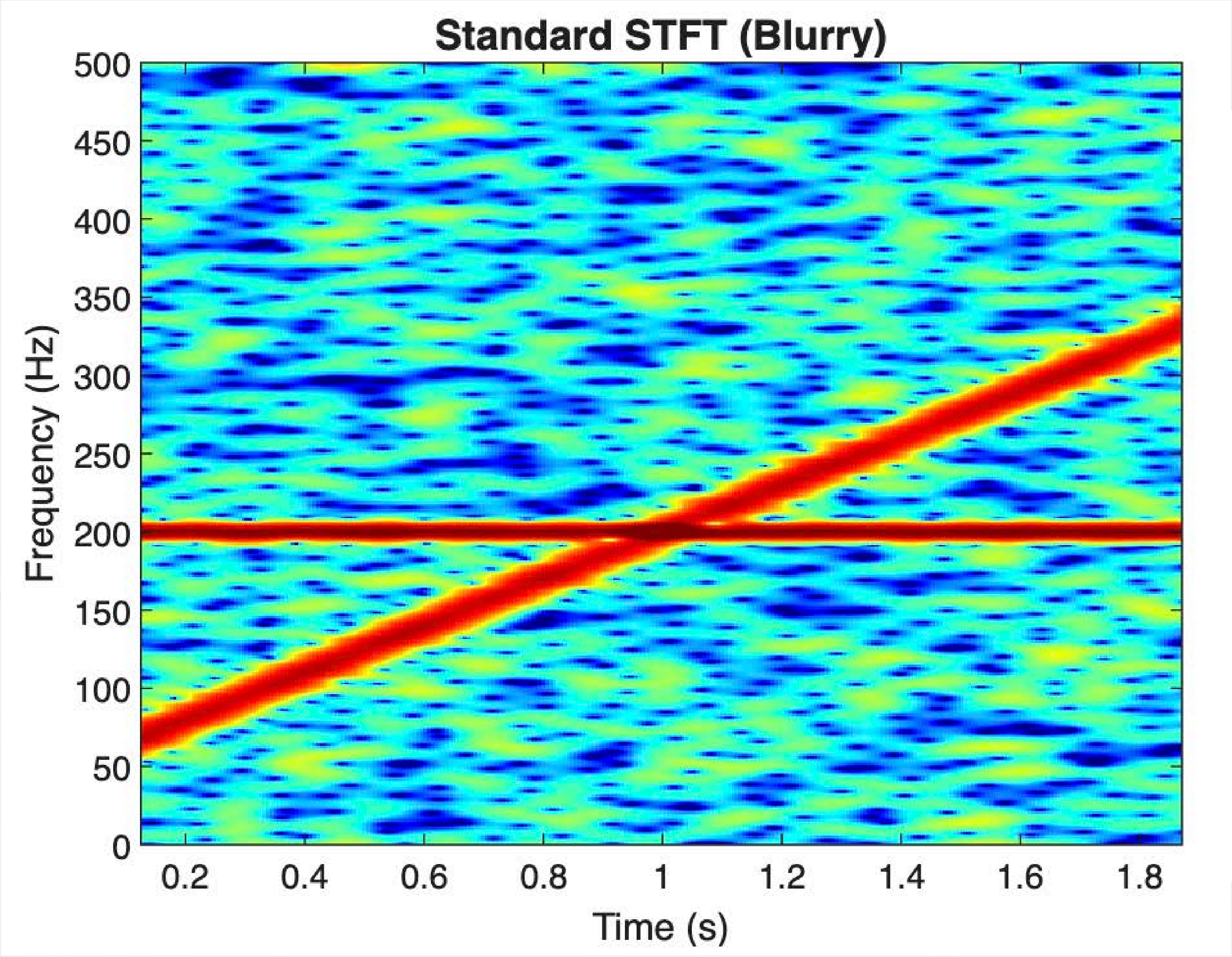

For every point on the time-frequency plane, we can define a correction vector that maps its measured location to its true energy centroid:

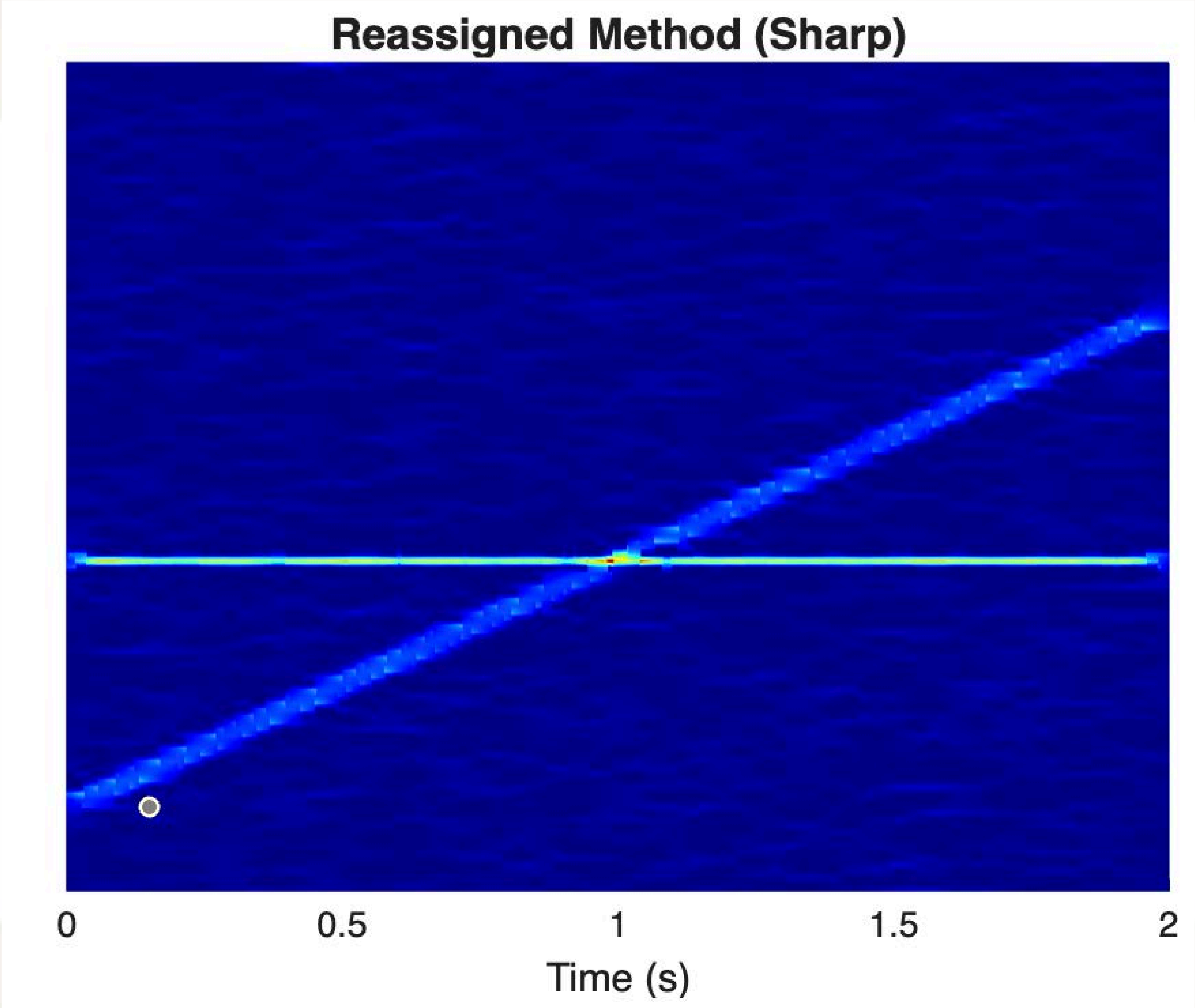
$$V(t, \omega) = (\hat{t} - t, \hat{\omega} - \omega)$$

This creates a vector field that directs every misplaced energy packet from where the STFT window placed it, to where it should have been. The vectors converge on the signal's "energy ridges" – the paths of its true instantaneous frequencies.

**Reassignment is a coherent, global flow of energy. It's a systematic process of guiding every value towards its rightful place on the time-frequency plane.**

# From Blurry to Brilliant: The Reassignment Method in Action

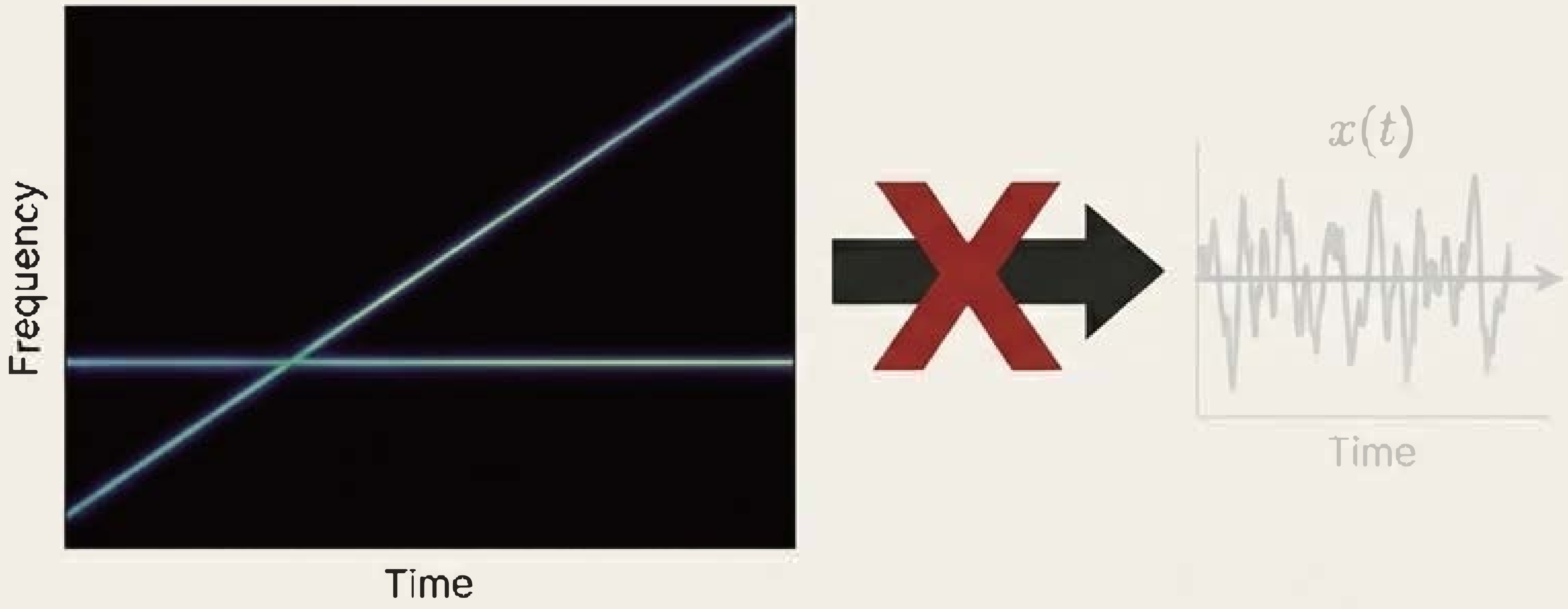



| Standard STFT | Reassigned Method (RM) |
|---|---|
| Blurry, smeared energy ("smearing effect") | Extremely sharp, focused energy |
| Ambiguous component boundaries | Clearly delineated individual components |
| Low effective resolution | Reveals the precise instantaneous frequency |

**RM transforms a fuzzy estimate into a high-fidelity representation, sharpening the time-frequency plane to a degree that reveals the signal's true underlying structure.**

# A Perfect View with a Critical Flaw: The Invertibility Problem



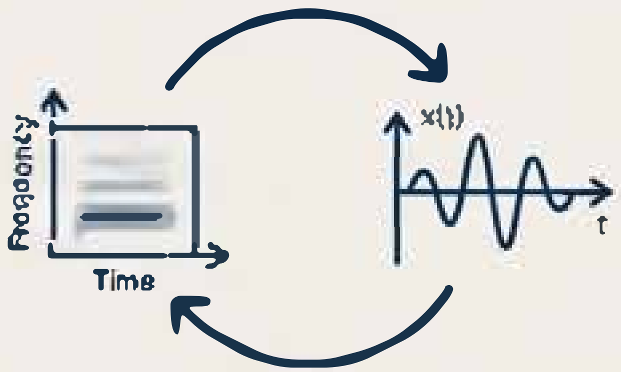

Reassignment achieves its incredible sharpness by moving energy values in **both time and frequency**: $(t, \omega) \rightarrow (\hat{t}, \tilde{\omega})$.

This unconstrained, two-dimensional shuffling fundamentally scrambles the geometric and phase structure required for a valid inverse transform. The phase information, essential for reconstruction, is effectively lost.

**The Consequence:** A reassigned spectrogram is a 'read-only' representation. It offers a crystal-clear view of the signal's structure, but we **cannot reconstruct the original signal** $x(t)$ from it. For most engineering applications requiring filtering or separation, this is a dead end.

# Synchrosqueezing: A Reformation of Reassignment

High Resolution




Invertibility

Mathematical Rigor

To overcome the limitations of RM, Ingrid Daubechies and her collaborators proposed the Synchrosqueezing Transform (SST).

**Core Philosophy:** A principled compromise. Can we achieve the sharpness of RM while guaranteeing the ability to reconstruct the original signal?

**The Solution:** Impose one critical constraint. Instead of reassigning energy freely, SST **only reassigns energy along the frequency axis**. The time coordinate is locked. This single change preserves the mathematical structure necessary for inversion.

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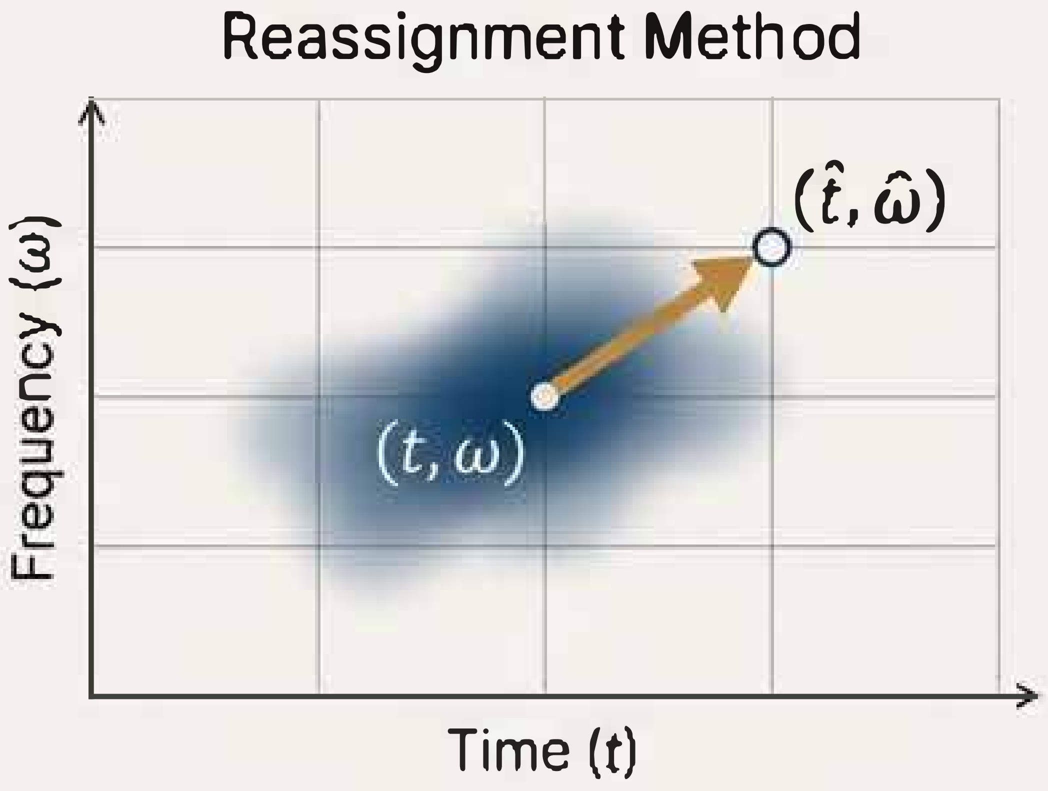
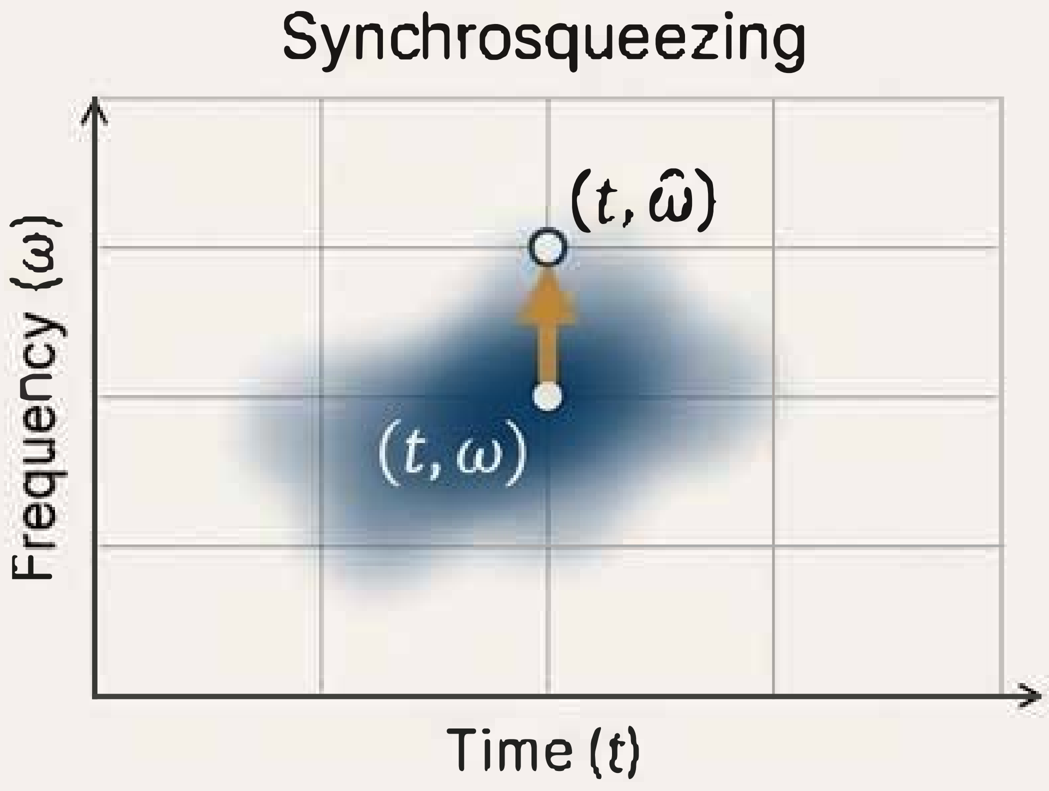
**KEY INSIGHT:** SST is designed from the ground up to provide the best of both worlds: a highly concentrated time-frequency representation that is also fully and robustly invertible.

# The Decisive Constraint: 1D Squeeze vs. 2D Shift




2D Reassignment ($t \to \hat{t}$, $\omega \to \hat{\omega}$)



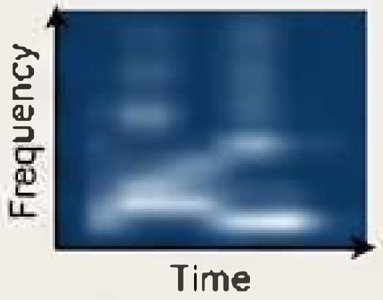

1D Reassignment ($t \to t$, $\omega \to \hat{\omega}$)

The fundamental difference lies in the direction of energy movement.

| **Reassignment Method (RM)** | **Synchrosqueezing Transform (SST)** |
|---|---|
| 2D movement in time and frequency. | 1D movement in frequency only. |
| **Result:** Maximum sharpness. | **Result:** High sharpness. |
| **Invertibility:** Lost. | **Invertibility:** Preserved. |

By locking the time axis, SST ensures that for any given time slice $t$, energy is simply squeezed from its measured frequencies to its true frequencies, never leaving that time slice. This preserves the signal's integrity for reconstruction.

# The Synchrosqueezing Algorithm: A Three-Step Process




**Step 1: Compute STFT**

Calculate the standard Short-Time Fourier Transform of the signal, $V(t,\omega)$. This provides the initial, blurry representation and the necessary phase information.

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**Step 2: Compute Instantaneous Frequency**

For every point (t, $\omega$), calculate the reassigned frequency coordinate $\hat{\omega}(t,\omega)$ using the phase derivative, just as in RM.

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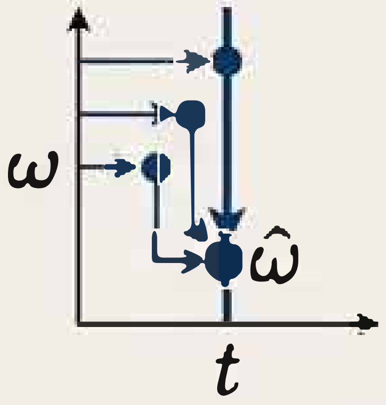


**Step 3: Squeeze**

For each time slice t, accumulate the STFT values into a new time-frequency representation. The value $V(t,\omega)$ is not moved, but rather added to the energy at its new target location, $(t,\hat{\omega}\})$.

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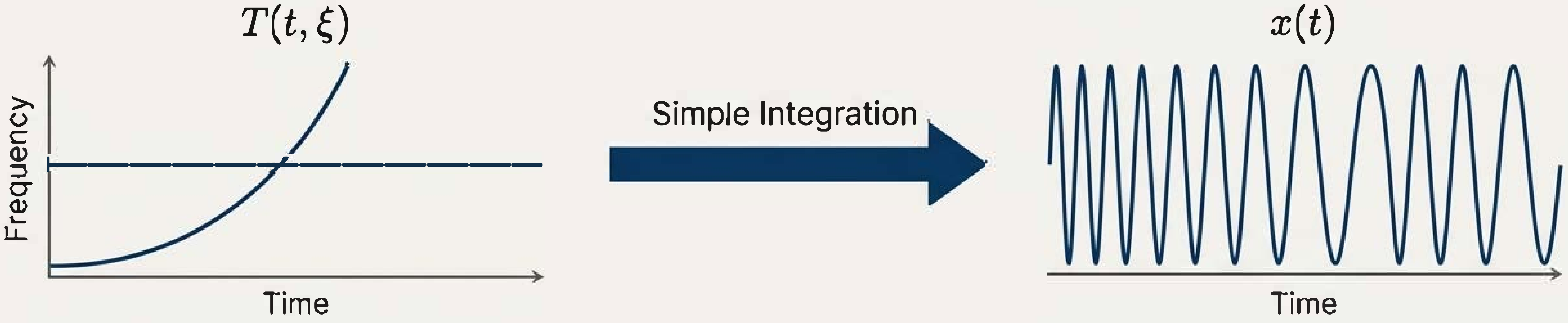
**THE SYNCHROSQUEEZING INTEGRAL**

This "squeezing and summing" operation is formally expressed as:

$$T(t,\xi) = \int V(t,\omega) \cdot \delta(\xi - \hat{\omega}(t,\omega))\, d\omega$$

*{This integrates all STFT energy $V(t,\omega)$ that maps to the new frequency bin $\xi$}*

# The Reward: Perfect Signal Reconstruction




Because SST preserves the energy within each time slice, the original signal can be recovered through a simple and exact integration.

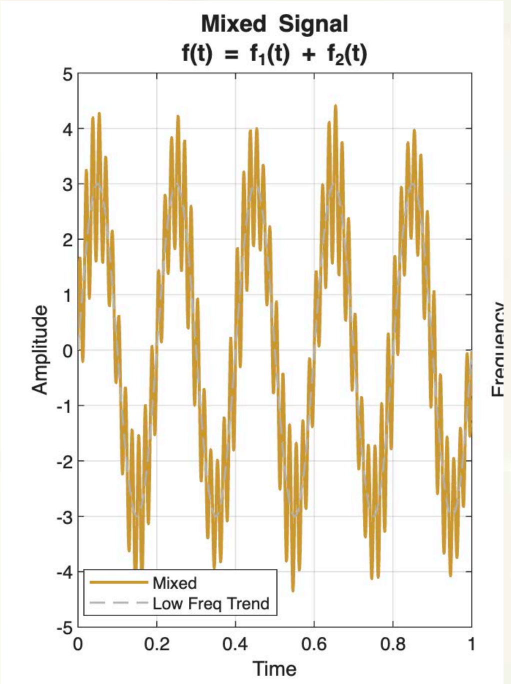
$$x(t) \approx C \cdot \mathrm{Re}\left[\int T(t, \xi)\, d\xi\right]$$

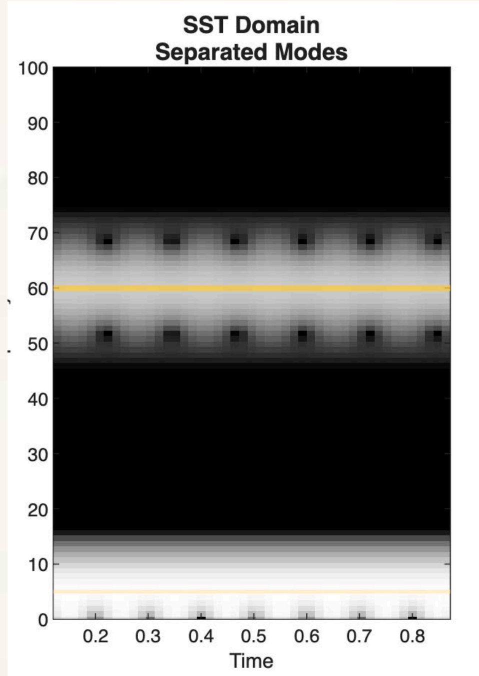
Reconstruction is achieved by:

1. For each point in time $t$, simply summing all the SST coefficients across the frequency axis $\xi$.
2. Taking the real part of the result.

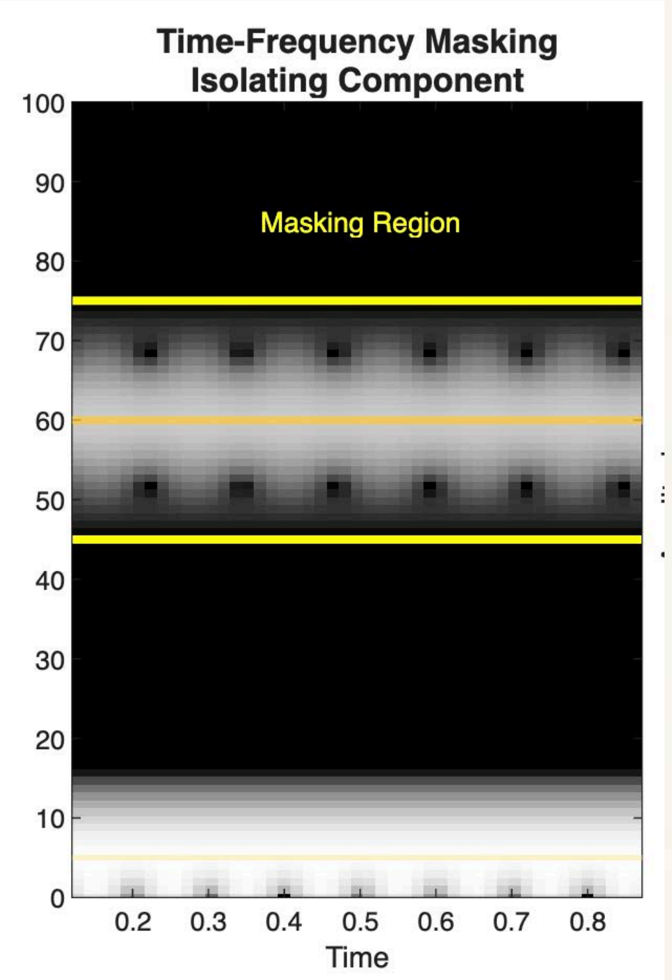
This robust invertibility is not merely a theoretical feature. It is the gateway to powerful applications like **mode decomposition** (separating a signal into its constituent parts) and advanced signal denoising.

# The True Power of Invertibility is Signal Decomposition

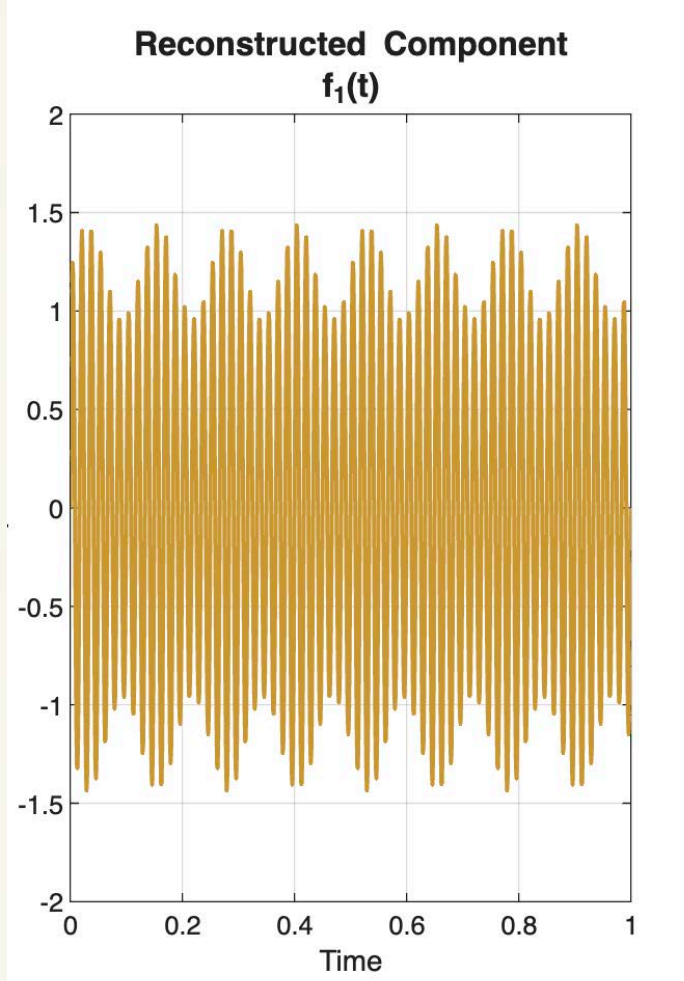



A mixed signal, $f(t) = f_1(t) + f_2(t)$, is transformed. Its components become clearly separated energy ridges in the SST domain.




By masking the time-frequency plane, we can isolate a single component of interest.

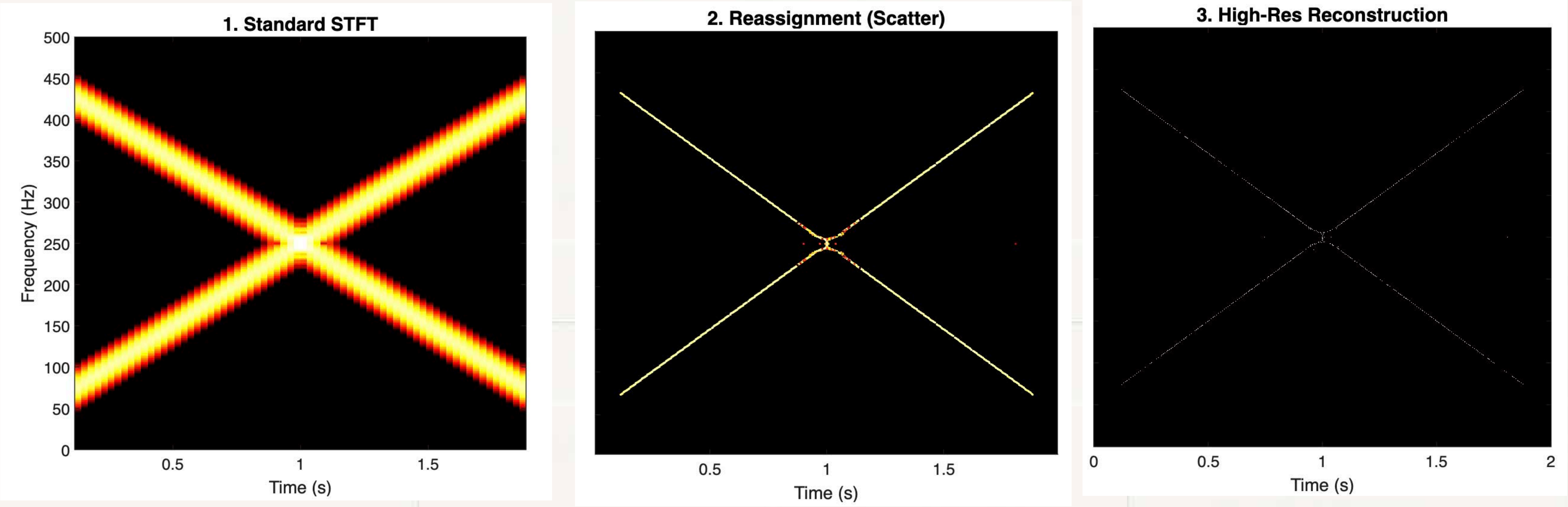



Applying the inverse SST to the masked region perfectly reconstructs the individual signal. This powerful mode decomposition is impossible with non-invertible methods.

# A Tale of Three Transforms: A Comparative Summary

| | Short-Time Fourier Transform (STFT) | Reassignment Method (RM) | Synchrosqueezing Transform (SST) |
|---|---|---|---|
| Clarity | Blurry | Very Sharp | Sharp |
| Invertibility | Yes | No | Yes |
| Verdict | *The classic but limited approach.* | *Offers maximum sharpness but is non-invertible.* | *Provides the best of both worlds—high sharpness while retaining the crucial ability to reconstruct the signal.* |

# Visualizing the Difference with a Crossing-Chirp Test



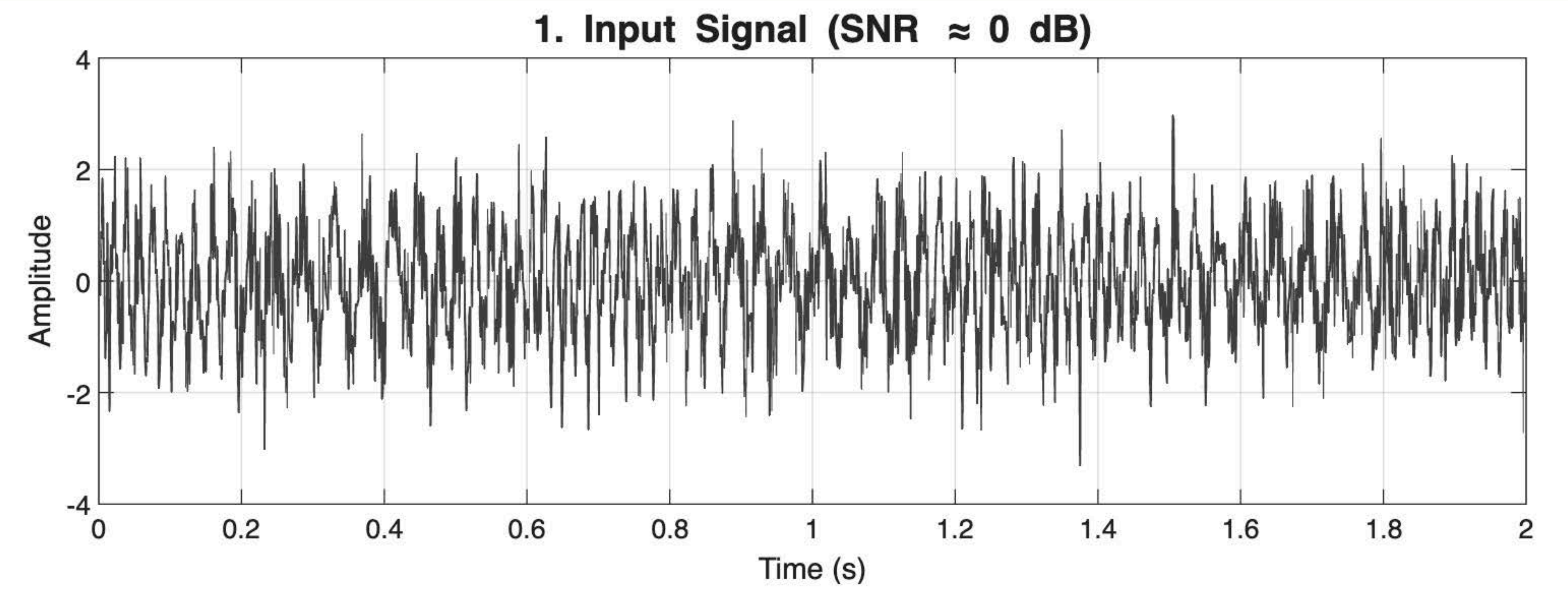

The spectrogram is blurry. At the crossover point, the energy from the two chirps is smeared into a single, indistinguishable blob.

While RM is the sharpest, it can break the continuity of signal components.

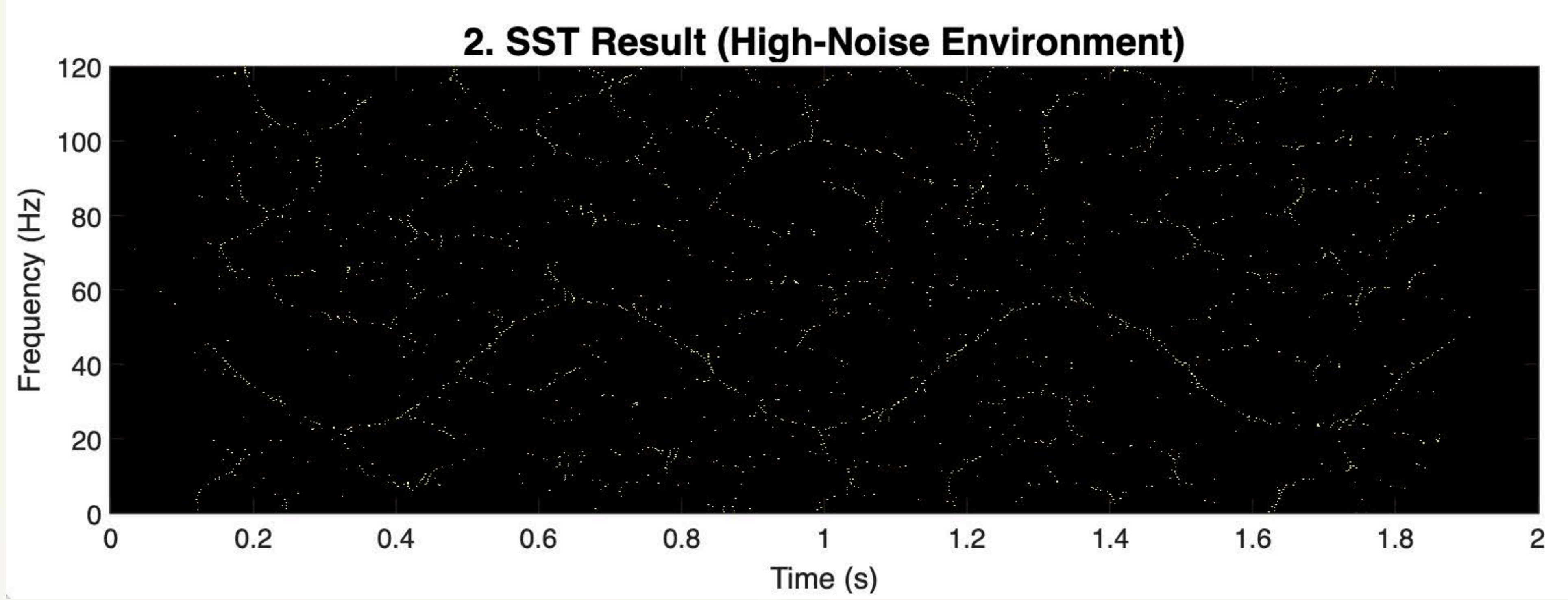
SST maintains high sharpness while preserving the integrity and continuity of individual components.

# SST's Performance Thrives in a High-Noise Environment

**Input Signal**
**(SNR = 0 dB)**




**SST Result**

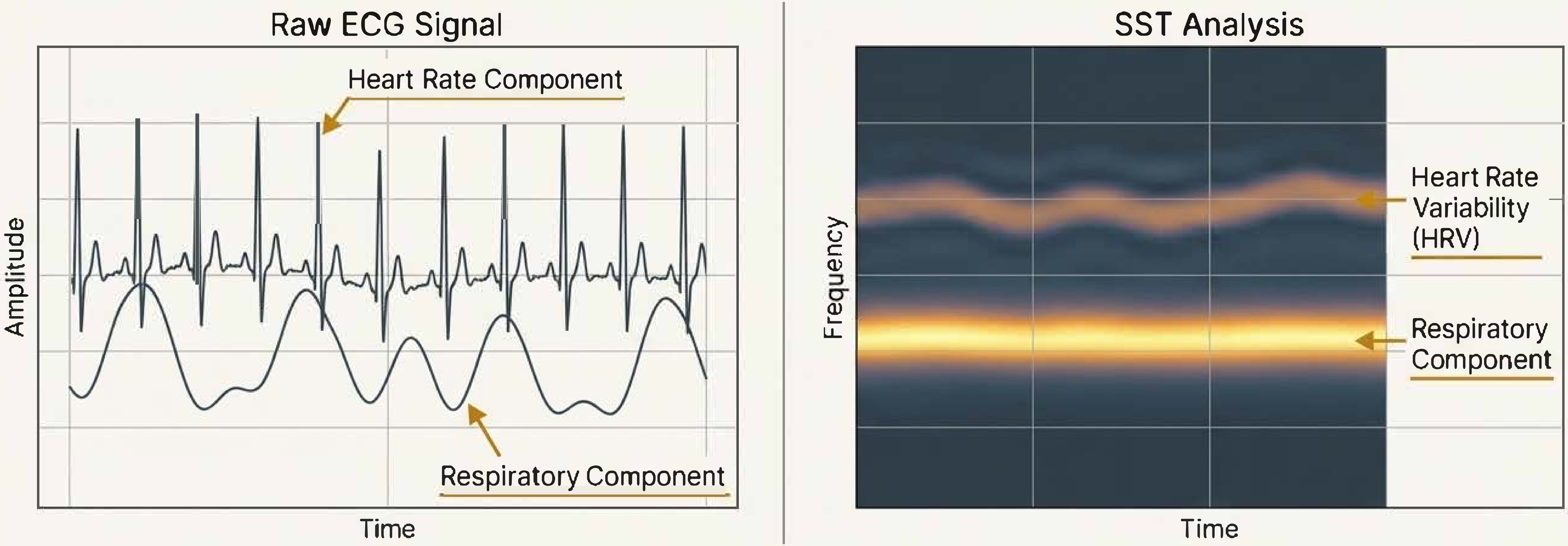



The 'squeezing' step of of the SST algorithm is an integration process. This has a natural averaging effect that suppresses random random noise and reveals the coherent signal structure.

# Application I: Separating Vital Signs in Biomedicine

**Context:** Analyzing an Electrocardiogram (ECG) to separate the heart rate from respiratory effects for more accurate diagnostics.

**Challenge:** The low-frequency baseline wander caused by breathing interferes with the analysis of Heart Rate Variability (HRV).

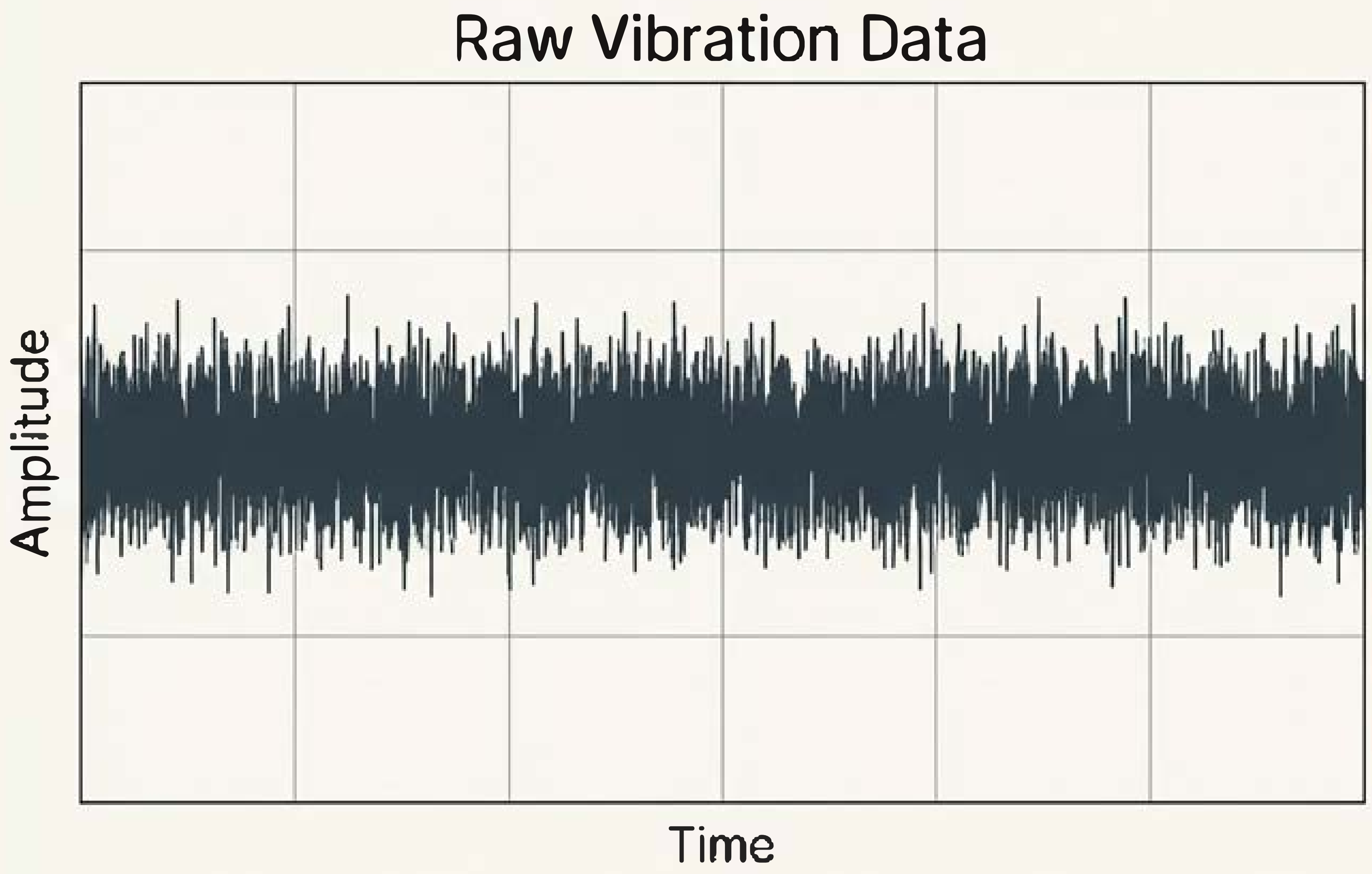


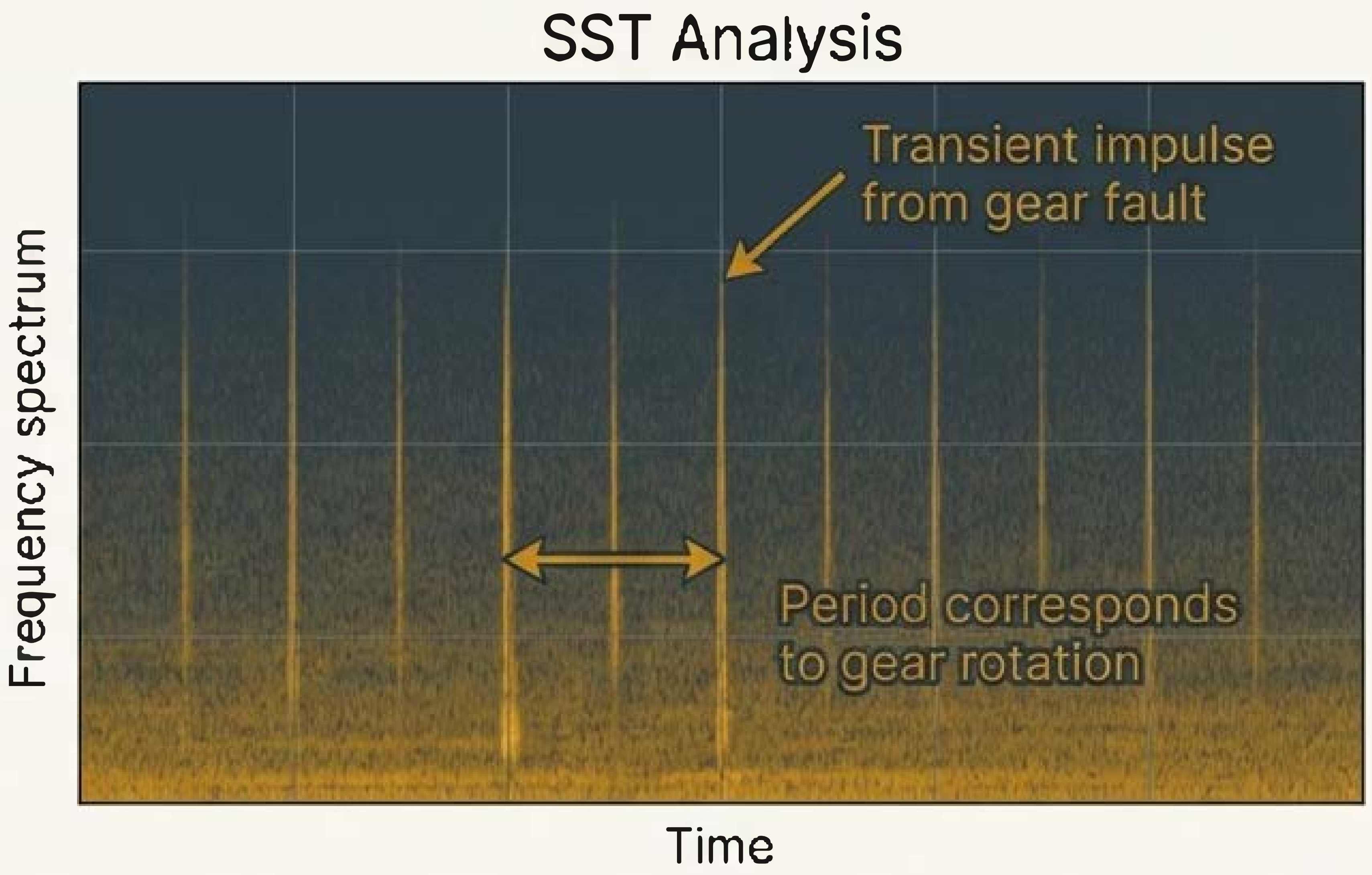

**Impact Statement:** SST enables the non-invasive separation of cardiac and respiratory signals from a single lead, improving the accuracy of patient monitoring and HRV analysis.

# Application II: Detecting Faults in Mechanical Systems

**Context:** Analyzing the vibration signature from a gearbox to detect a crack in a gear tooth.

**Challenge:** A microscopic crack produces brief, high-frequency transient impulses (bursts of energy) that are often buried in background noise.



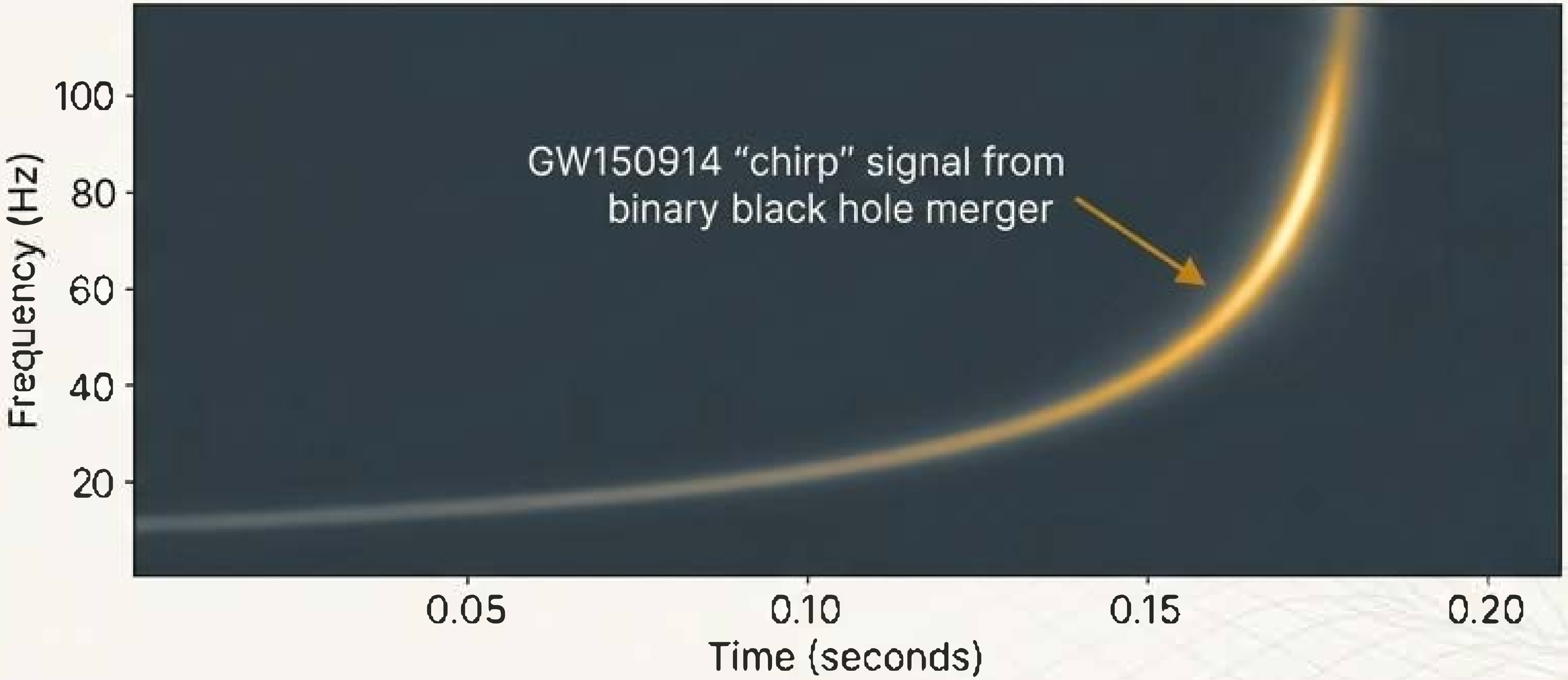




SST precisely identifies and localizes weak, transient fault signatures, enabling early detection for predictive maintenance and preventing catastrophic equipment failure.

# Application III: Listening to the Cosmos with Gravitational Waves

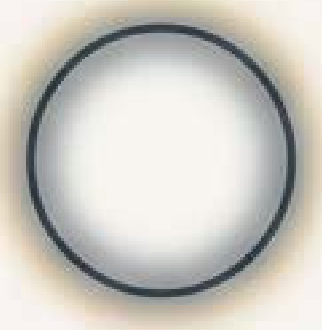
**Context**: Analysis of the GW150914 'chirp' signal detected by LIGO, the first direct observation of gravitational waves from a binary black hole merger.

**Significance:** This signal provided crucial evidence for Einstein's theory of general relativity.



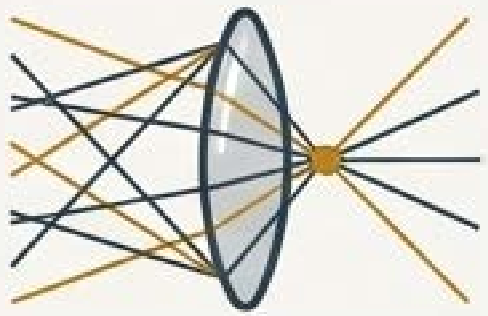

Time-frequency methods like SST are essential for extracting these incredibly faint cosmic signals from detector noise, providing a clear "fingerprint" that allows physicists to characterize the astronomical events that created them.

# From Blurry to Sharp: The Journey and the Payoff



## The Limit of Uncertainty

Traditional STFT is fundamentally limited by the Heisenberg Uncertainty Principle. This creates an unavoidable trade-off, resulting in 'smeared' energy and poor resolution in the time-frequency plane.



## Two Paths to Clarity

The **Reassignment Method (RM)** achieves extreme sharpness by moving energy to its true center of gravity, but sacrifices invertibility.

**Synchrosqueezing (SST)** offers a pragmatic solution: it sharpens only along the frequency axis, preserving the ability to reconstruct the signal.

## A Powerful Analytic Framework

By retaining invertibility, SST becomes more than just a visualization tool. It is a robust framework for practical engineering tasks like **signal decomposition**, **denoising**, and **feature extraction**.

# Core References for Further Study

The methods discussed in this presentation are primarily based on the following foundational works:


**On Synchrosqueezing**

Daubechies, I., Lu, J., & Wu, H. T. (2011). Synchrosqueezed wavelet transforms: A tool for empirical mode decomposition. *Applied and Computational Harmonic Analysis.*

**On The Reassignment Method**

Auger, F., & Flandrin, P. (1995). Improving the readability of time-frequency and time-scale representations by the reassignment method. *IEEE Transactions on Signal Processing.*

**On The Original Phase-Based Reassignment Concept**

Kodera, K., Gendrin, R., & de Villedary, C. (1978). Analysis of time-varying signals with small BT values. *IEEE Transactions on Acoustics, Speech, and Signal Processing.*