**Advance Digital Signal Process**

**Final Tutorial**

**Introduction of Weiner Filter**

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Contents

**Abstract3**

1. **Introduction of Wiener Filter4**

**Weiner Filter**

1. **The Linear Optimal Filtering Problem** **6**
2. Linear Estimation with Mean-Square Error Criterion7
3. **The Real-Valued Case** **9**
4. Minimization of performance function 10
5. Error performance surface for FIR filtering 12
6. Explicit form of 12
7. Canonical form of 13
8. **Principle of Orthogonality** 15
9. See the Wiener-Hopf Equation byPrinciple of Orthogonality 16
10. More Of Principle of Orthogonality 17
11. **Normalized Performance Function** 19
12. **The Complex-Valued Case** 20
13. **Application** 22
14. Modelling 22
15. Inverse Modelling 23
16. **Implementation Issues** 28

**Summary** 30

**Reference** 31

**Abstract**

The **Wiener filter** was introduced by **Norbert Wiener** in the 1940's and published in 1949 in [signal processing](http://en.wikipedia.org/wiki/Signal_processing). A major contribution was the use of a statistical model for the estimated signal (the Bayesian approach!). And the Wiener filter solves the signal estimation problem for stationary signals. Because the theory behind this filter assumes that the inputs are [stationary](http://en.wikipedia.org/wiki/Stationary_process), a wiener filter is not an [adaptive filter](http://en.wikipedia.org/wiki/Adaptive_filter). So the filter is optimal in the sense of the MMSE. The wiener filter’s main purpose is to reduce the amount of [noise](http://en.wikipedia.org/wiki/Noise) present in a signal by comparison with an estimation of the desired noiseless signal.

As we shall see, the Kalman filter solves the corresponding filtering problem in greater generality, for non-stationary signals. We shall focus here on the discrete-time version of the Wiener filter.

1. **Introduction of wiener function**

Wiener filters are a class of optimum linear filters which involve linear estimation of a desired signal sequence from another related sequence. In the statistical approach to the solution of the linear filtering problem, we assume the availability of certain statistical parameters (e.g. mean and correlation functions) of the useful signal and unwanted additive noise. The goal of the Wiener filter is to filter out [noise](http://en.wikipedia.org/wiki/Noise) that has corrupted a signal. It is based on a [statistical](http://en.wikipedia.org/wiki/Statistical) approach. The problem is to design a linear filter with the noisy data as input and the requirement of minimizing the effect of the noise at the filter output according to some statistical criterion. A useful approach to this filter-optimization problem is to minimize the mean-square value of the error signal that is defined as the difference between some desired response and the actual filter output. For stationary inputs, the resulting solution is commonly known as the Weiner filter.

The Weiner filter is inadequate for dealing with situations in which nonstationarity of the signal and/or noise is intrinsic to the problem. In such situations, the optimum filter has to be assumed a time-varying form. A highly successful solution to this more difficult problem is found in the Kalman filter.

Now we summarize some Wiener filters characteristics

* Assumption:

Signal and (additive) noise are stationary linear [stochastic processes](http://en.wikipedia.org/wiki/Stochastic_process) with known spectral characteristics or known [autocorrelation](http://en.wikipedia.org/wiki/Autocorrelation) and [cross-correlation](http://en.wikipedia.org/wiki/Cross-correlation)

* Requirement:

We want to find the linear MMSE estimate of based on (all or part of) . So there are three versions of this problem:

1. The causal filter:



1. The non-causal filter:



1. The FIR filter:



And we consider in this tutorial the *FIR* case for simplicity.

1. **Wiener Filter － The Linear Optimal filtering problem**

There is a signal model, showed in Fig. 1,



Fig. 1 A signal model

Where *u*(*n*) is the measured value of the desired signal *d*(*n*). And there are some examples of measurement process:

* Additives noise problem *u*(*n*) = *d*(*n*) + *v*(*n*)
* Linear measurement *u*(*n*) = *d*(*n*)\* *h*(*n*) + *v*(*n*)
* Simple delay *u*(*n*) = *d*(*n* −*1*) + *v*(*n*)

=> It becomes a prediction problem

* Interference problem *u*(*n*) = *d*(*n*) + *i*(*n*) + *v*(*n*)

And the filtering main problem is to find an estimation of *d*(*n*) by applying a linear filter on *u*(*n*)



Fig. 2 The filtering goal

When the filter is restricted to be a linear filter => it is a linear filtering problem. Then the filter is designed to optimize a performance index of the filtering process, such as

🡺 Solving is a linear optimal filtering problem

The whole process can indicate Fig. 3



Fig. 3

where the output is

1. **Linear Estimation with Mean-Square Error Criterion**

Fig. 4 shows the block schematic of a linear discrete-time filter for estimating a desired signal based on an excitation *.* We assume that both and are random processes (discrete-time random signals). The filter output is and is the estimation error.

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Fig. 4

To find the optimum filter parameters, the cost function or performance function must be selected. In choosing a performance function the following points have to be considered：

1. The performance function must be mathematically tractable.
2. The performance function should preferably have a single minimum so that the optimum set of the filter parameters could be selected unambiguously.

The number of minima points for a performance function is closely related to the filter structure. The recursive (IIR) filters, in general, result in performance function that may have many minima, whereas the non-recursive (FIR) filters are guaranteed to have a single global minimum point if a proper performance function is used.

In Weiner filter, the performance function is chosen to be

This is also called “mean-square error criterion”

1. **Wiener Filter － The Real-Valued Case**

Fig. 5 shows a transversal filter (FIR) with tap weights .



Fig. 5

Let

The output is

Thus we may write

The performance function, or cost function, is then given by

Now we define the **Nx1** **cross-correlation** vector

and the **NxN** **autocorrelation** matrix

Also we note that

Thus we obtain

Equation 4 is a quadratic function of the tap-weight vector with a single global minimum. We note that has to be a positive definite matrix in order to have a unique minimum point in the w-space.

1. **Minimization of performance function**

To obtain the set of tap weights that minimize the performance function,

we set

or

where is the gradient vector defined as the column vector

and zero vector is defined as N-component vector

Equation 4 can be expanded as

and can be expanded as

Then we obtain

By setting , we obtain

Note that

The symmetry property of autocorrelation function of real-valued signal, we have the relation

Equation 5 then becomes

In matrix notation, we then obtain

where is the optimum tap-weight vector.

Equation 6 is also known as the **Wiener-Hopf equation**, which has the solution

assuming that has inverse.

1. **Error performance surface for FIR filtering**

By , it can be deduced that , and is defined as the error performance surface over all possible weights .

1. **Explicit form of**



1. **Canonical form of**

can be written in matrix form as

And by

* The Hermitian property of has been used,

Then can be put into a perfect square-form as

The squared form can yield the minimum MSE explicitly as

is found, we have the minimum value of is

Equation 7 can also expressed as

And the squared form becomes

where

If is written in its similarity form, then can be put into a more informative form named Canonical Form.

Using the eigen-decomposition , we have

Where is the transformed vector of in the eigenspace of .

In eigenspace the in are the decoupled tape-errors, then we have

where is the error power of each coefficients in the eigenspace and is the weighting (ie. relative importance) of each coefficient error.

1. **Wiener Filter － Principle of Orthogonality**

For linear filtering problem, the wiener solution can be generalized to be a principle as following steps.

1. The MSE cost function (or performance function) is given by

By the chain rule,

where

Since is independent of , we get

Then we obtain

1. When the Wiener filter tap, which weights are set to their optimal values,

Hence, if is the estimation error when are set to their optimal values, then equation 8 becomes

For the ergodic random process,

On the other hands, the estimation error is orthogonal to input for all (Note: and are treated as vectors !).

That is, the estimation error is uncorrelated with the filter tap inputs, . This is known as “ **the principles of orthogonality**”.

1. We can also show that the optimal filter output is also uncorrelated with the estimation error. That is

This result indicates that the optimized Weiner filter output and the

estimation error are “orthogonal”.

The orthogonality principle is important in that it can be generalized to many complicated linear filtering or linear estimation problem.

1. **See the Wiener-Hopf Equation by Principle of Orthogonality**

Wiener-Hopf equation is a special case of the orthogonality principle, when which is the linear prediction problem. So we can derived Weiner-Hopf equation based on the principle of orthogonality

where

This is the Wiener-Hopf equation . For optimal FIR filtering problem, needs not to be , then the Wiener-Hopf equation is with being the cross correlation vector of and .

1. **More Of Principle of Orthogonality**

Since is a quadratic function of '*s* , i.e.,

J

it has a bowel shape in the hyperspace of , which has an unique extreme point at .

is also a sufficient condition for minimizing *J*



This is the principle of orthogonality.

For the 2D case , the error surface is showed in Fig. 6



Fig. 6 Error surface of 2D case

By

=>Both input and output of the filter are orthogonal to the estimation error .

The vector space interpretation

By and , i.e.,

=> ,i.e., is decomposed into two orthogonal components,



Fig. 7 Estimation value + Estimation error = Desired signal.

In geometric term, = projection of on . This is a reasonable result !

1. **Normalized Performance Function**
2. If the optimal filter tap weights are expressed by . The estimation error is then given by

and then

1. We may note that

and we obtain

1. Define as the normalized performance function, and
2. when

**5.** reaches its minimum value, when the filter tap-weights are chosen to achieve the minimum mean-squared error. This gives

and we have

1. **Wiener Filter － The Complex-Valued Case**

In many practical applications, the random signals are complex-valued. For example, the baseband signal of QPSK & QAM in data transmission systems. In the Wiener filter for processing complex-valued random signals, the tap-weights are assumed to be complex variables.

The estimation error, , is also complex-valued. We may write

The tap-weight is expressed by

The gradient of a function with respect to a complex variable is defined as

The optimum tap-weights of the complex-valued Wiener filter will be obtained from the criterion:

That is, and

Since , we have

Noting that

Applying the definition 9, we obtain

and

Thus, equation 10 becomes

The optimum filter tap-weights are obtained when . This givens

where is the optimum estimation error.

Equation 11 is the “principle of orthogonality” for the case of complex-valued signals in wiener filter.

The Wiener-Hopf equation can be derived as follows:

Define

and

We can also write

and

where *H* denotes complex-conjugate transpose or Hermitian.

Noting that

and

from equation 12, we have

and then

where

and

Equation 13 is the Wiener-Hopf equation for the case of complex-valued signals. The minimum performance function is then expressed as

Remarks:

In the derivation of the above Wiener filter we have made assumption that it is **causal** and **finite impulse response**, for both real-valued and complex-valued signals.

1. **Wiener Filter － Application**
2. **Modelling**

Consider the modeling problem depicted in Fig. 8



Fig. 8 The model of Modeling

, , are assumed to be stationary, zero-mean and uncorrelated with one another. The input to Wiener filter is given by

and the desired output is given by

where is the impulse response sample of the plant.

The optimum unconstrained Wiener filter transfer function

Note that

Taking Z-transform on both sides of equation 14, we get

To calculate , we must first find the expression for , We can show that

where is the plant output when the additive noise is excluded from that.

Moreover, we have

Thus

and we obtain

We note that is equal to only when is equal to zero. That is, when is zero for all values of *n*. The noise sequence may be thought of as introduced by a transducer that is used to get samples of the plant input. Replacing *z* by in equation 15, we obtain

Define

We obtain

With some mathematic manipulation, we can find the minimum mean-square error, min , expressed by

The best performance that one can expect from the unconstrained Wiener filter is

and this happens when .

The Wiener filter attempts to estimate that part of the target signal that is correlated with its own input and leaves the remaining part of (i.e. ) unaffected. This is known as “ the principles of correlation cancellation “.

1. **Inverse Modelling**

Fig. 9 depicts a channel equalization scenario.

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n(n)

Fig. 9 The model of inverse modeling

When the additive noise at the channel output is absent, the equalizer has the following trivial solution:

This implies that and thus for all *n*.

When the channel noise, , is non-zero, the solution provided by equation in modeling may not be optimal.

and

where is the impulse response of the channel, . From equation in modeling, we obtain

Also

With *z* 1, we may also write

And then

This is the general solution to the equalization problem when there is no constraint on the equalizer length and, also, it may be let to be non-causal.

Equation 16 can be rewritten as

Let and define the parameter

where and are the signal power spectral density and the noise power spectral density, respectively, at the channel output.

We obtain

We note that is a non-negative quantity, since it is the signal-to-noise power spectral density ratio at the equalizer input.

Also,

**Cancellation of ISI and noise enhancement**

Consider the optimized equalizer with frequency response given by

In the frequency regions where the noise is almost absent, the value of is very large and hence

The ISI will be eliminated without any significant enhancement of noise. On the other hand, in the frequency regions where the noise level is high, the value of is not large and hence the equalizer does not approximate the channel inverse well. This is of course, to prevent noise enhancement.

1. **Wiener Filter － Implementation Issues**

The Wiener (or MSE) solution exists in correlation domain, which needs to find the ACF and CCF in the Wiener-Hopf equation. This is the original theory developed by Wiener for the **linear prediction** case given if we have another chance.

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Fig. 10 The family of wiener filter in adaptive operation

The **existence of Wiener solution** depends on the availability of the desired signal , the requirement of may be avoided by using a model of or making it to be a constraint (e.g., LCMV). Another requirement for the existenceof Wiener solution is that the RP must be **stationary** to ensure the existence of ACF and CCF representation. Kalman (MV) filtering is the major theory developed for making the Wiener solution adapt to **nonstationary** signals.

For **real time** implementation, the requirement of signal statistics (ACF and CCF) must be avoided

=> search solution using LMS or estimate solution blockwise using LS.

Another concern for real time implementation is the **computation** issues in finding ACF, CCF and the inverse of ACM based on instead of . Recursive algorithms for finding Wiener solution were developed for the above cases.

* The Levinson-Durbin algorithm is developed for linear prediction filtering.
* Complicated recursive algorithms are used in Kalman filtering and RLS.
* LMS is the simplest recursive algorithm.
* SVD is the major technique for solving the LS solution.

So we do some conclusion, **existence** and **computation**are two major problems in finding the Wiener solution. And the existence problem includes: , we can find the computation problem is primarily for finding .

**Summary**

In this tutorial, we described the discrete-time version of Wiener filter theory, which has evolved from the pioneering work of Norbert Wiener on linear optimum filters for continuous-time signals. The importance of Wiener filter lies in the fact that it provides a frame of reference for the linear filtering of stochastic signals, assuming wide-sense stationarity. And the Wiener filter’s main purpose is that the output of filter can close to the desired response by filter out the noise which interference signal.

The Wiener filter has two important properties:

1. The principle of orthogonality:

The error signal (estimation error) produced by the Wiener filter is orthogonal to its tap inputs.

1. Statistical characterization of the error signal as white noise:

This condition is attained when the filter length matches the order if the multiple regression model describing the generation of the observable data (i.e., the desired response).

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