Selected Topics in Engineering Mathematics: Advanced Operations in Linear Algebra

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Reference

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   [HJ2013]
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Outline

Scalars

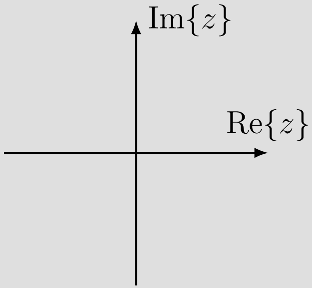
- Scalar: Lowercase letters (e.g., x or z)

Real Scalars

- $x \in \mathbb{R}$
- $\mathbb{R}$ is the set of real numbers.

Complex Scalars

- $z \in \mathbb{C}$
- $\mathbb{C}$ is the set of complex numbers.



Vectors

- Vector
  - Boldface lowercase letters (e.g., $\mathbf{x}$ or $\mathbf{z}$)
  - Single-underlined letters (e.g., $\underline{x}$ or $\underline{z}$)
- (By default) Column vectors

Real Column Vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N. \tag{1}$$

Complex Column Vectors

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{C}^N. \tag{2}$$

Matrices

- Matrix
  - Boldface uppercase letters (e.g., $\mathbf{A}$ or $\mathbf{B}$)
  - Double-underlined letters (e.g., $\underline{\underline{A}}$ or $\underline{\underline{B}}$)
- An M-by-N real-valued matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} \end{bmatrix} \in \mathbb{R}^{M\times N}. \tag{3}$$

- The (m,n)th entry (element) of $\mathbf{A}$: $a_{m,n}$ or $[\mathbf{A}]_{m,n}$
- An M-by-N complex matrix $\mathbf{B} \in \mathbb{C}^{M\times N}$

Structured Matrices

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} \end{bmatrix}.$$

- Square matrices: $M = N$
- Row vectors: $M = 1$
- Column vectors: $N = 1$
- Scalars: $M = N = 1$

Square Matrices ($M = N$) with Additional Structures

Diagonal matrices

$$\begin{bmatrix} a_{1,1} & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{N,N} \end{bmatrix}.$$

The identity matrix

$$\mathbf{I} = \mathbf{I}_N = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Upper triangular matrices

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ 0 & a_{2,2} & \dots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{N,N} \end{bmatrix}.$$

Lower triangular matrices

$$\begin{bmatrix} a_{1,1} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \dots & a_{N,N} \end{bmatrix}.$$

Permutation Matrices

- A permutation matrix is constructed from the identity matrix with reordered rows.
- An example of a permutation matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{P}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix}.$$

Examples of Permutation Matrices [GVL2013, pp. 20]

- The exchange permutation $\mathbf{E}_N$ is

$$\mathbf{E}_N = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{N\times N}.$$

- The downshift permutation matrix $\mathbf{D}_N$ is

$$\mathbf{D}_N = \begin{bmatrix} \mathbf{0}_{1\times(N-1)} & 1 \\ \mathbf{I}_{N-1} & \mathbf{0}_{(N-1)\times 1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{C}^{N\times N}.$$

Circulant Matrices [HJ2013, pp. 33]

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{N-1} & a_N \\ a_N & a_1 & a_2 & a_3 & \cdots & a_{N-2} & a_{N-1} \\ a_{N-1} & a_N & a_1 & a_2 & \cdots & a_{N-3} & a_{N-2} \\ a_{N-2} & a_{N-1} & a_N & a_1 & \cdots & a_{N-4} & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & a_5 & a_6 & \cdots & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_N & a_1 \end{bmatrix} \in \mathbb{C}^{N \times N}.$$

Toeplitz Matrices [HJ2013, pp. 34]

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_N \\ a_{-1} & a_0 & a_1 & a_2 & \cdots & a_{N-2} & a_{N-1} \\ a_{-2} & a_{-1} & a_0 & a_1 & \cdots & a_{N-3} & a_{N-2} \\ a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots & a_{N-4} & a_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{-N+1} & a_{-N+2} & a_{-N+3} & a_{-N+4} & \cdots & a_0 & a_1 \\ a_{-N} & a_{-N+1} & a_{-N+2} & a_{-N+3} & \cdots & a_{-1} & a_0 \end{bmatrix} \in \mathbb{C}^{(N+1)\times(N+1)}.$$

- The entries in $\mathbf{A}$ satisfies

$$[\mathbf{A}]_{m,n} = a_{n-m}.$$

Hankel Matrices [HJ2013, pp. 35]

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_N \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_N & a_{N+1} \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{N+1} & a_{N+2} \\ a_3 & a_4 & a_5 & a_6 & \cdots & a_{N+2} & a_{N+3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N-1} & a_N & a_{N+1} & a_{N+2} & \cdots & a_{2N-2} & a_{2N-1} \\ a_N & a_{N+1} & a_{N+2} & a_{N+3} & \cdots & a_{2N-1} & a_{2N} \end{bmatrix} \in \mathbb{C}^{(N+1)\times(N+1)}.$$

- The entries in $\mathbf{A}$ satisfies

$$[\mathbf{A}]_{m,n} = a_{m+n-2}.$$

Vandermonde Matrices [HJ2013, pp. 37]

$$\mathbf{A} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{N-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{N-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_N & \alpha_N^2 & \dots & \alpha_N^{N-1} \end{bmatrix} \in \mathbb{C}^{N\times N}.$$

- The entries in $\mathbf{A}$ satisfies

$$[\mathbf{A}]_{m,n} = \alpha_m^{n-1}.$$

Block Matrices

- We assume that $p = 1, 2, \ldots, P$ and $q = 1, 2, \ldots, Q$.
- The matrix $\boldsymbol{\mathcal{A}}$ with submatrices $\mathbf{A}_{p,q}$ is

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \cdots & \mathbf{A}_{1,Q} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \cdots & \mathbf{A}_{2,Q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{P,1} & \mathbf{A}_{P,2} & \cdots & \mathbf{A}_{P,Q} \end{bmatrix}.$$

- An example:

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}.$$

Outline

Addition

- $\mathbf{A} \in \mathbb{C}^{M\times N}$ (with entries $a_{m,n}$)
- $\mathbf{B} \in \mathbb{C}^{M\times N}$ (with entries $b_{m,n}$)
- Matrix addition: $\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{C}^{M\times N}$.

$$\underbrace{\begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,N} \\ c_{2,1} & c_{2,2} & \dots & c_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M,1} & c_{M,2} & \dots & c_{M,N} \end{bmatrix}}_{\mathbf{C}} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & \dots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,N} \end{bmatrix}}_{\mathbf{A}} + \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,N} \\ b_{2,1} & b_{2,2} & \dots & b_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{M,1} & b_{M,2} & \dots & b_{M,N} \end{bmatrix}}_{\mathbf{B}},$$

$$c_{m,n} = a_{m,n} + b_{m,n}.$$

Multiplication

- $\mathbf{A} \in \mathbb{C}^{P \times Q}$ (with entries $a_{p,q}$)
- $\mathbf{B} \in \mathbb{C}^{Q \times R}$ (with entries $b_{q,r}$)
- Matrix multiplication: $\mathbf{C} = \mathbf{AB} \in \mathbb{C}^{P \times R}$.

$$\underbrace{\begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,R} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,R} \\ \vdots & \vdots & \ddots & \vdots \\ c_{P,1} & c_{P,2} & \cdots & c_{P,R} \end{bmatrix}}_{\mathbf{C}} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,Q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,Q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{P,1} & a_{P,2} & \cdots & a_{P,Q} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,R} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,R} \\ \vdots & \vdots & \ddots & \vdots \\ b_{Q,1} & b_{Q,2} & \cdots & b_{Q,R} \end{bmatrix}}_{\mathbf{B}},$$

$$c_{p,r} = \sum_{q=1}^{Q} a_{p,q} b_{q,r}.$$

Block Multiplication

- $\boldsymbol{\mathcal{A}}$ (with submatrices $\mathbf{A}_{p,q}$)
- $\boldsymbol{\mathcal{B}}$ (with submatrices $\mathbf{B}_{q,r}$)
- Matrix multiplication: $\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}$.

$$\underbrace{\begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} & \cdots & \mathbf{C}_{1,R} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \cdots & \mathbf{C}_{2,R} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{P,1} & \mathbf{C}_{P,2} & \cdots & \mathbf{C}_{P,R} \end{bmatrix}}_{\boldsymbol{\mathcal{C}}} = \underbrace{\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \cdots & \mathbf{A}_{1,Q} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \cdots & \mathbf{A}_{2,Q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{P,1} & \mathbf{A}_{P,2} & \cdots & \mathbf{A}_{P,Q} \end{bmatrix}}_{\boldsymbol{\mathcal{A}}} \underbrace{\begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} & \cdots & \mathbf{B}_{1,R} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} & \cdots & \mathbf{B}_{2,R} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{Q,1} & \mathbf{B}_{Q,2} & \cdots & \mathbf{B}_{Q,R} \end{bmatrix}}_{\boldsymbol{\mathcal{B}}},$$

$$\mathbf{C}_{p,r} = \sum_{q=1}^{Q} \mathbf{A}_{p,q}\mathbf{B}_{q,r}.$$

Scalar Multiplication

- $\alpha \in \mathbb{C}$
- $\mathbf{B} \in \mathbb{C}^{M \times N}$ (with entries $b_{m,n}$)
- Scalar multiplication: $\mathbf{C} = \alpha \mathbf{B} \in \mathbb{C}^{M \times N}$.

$$\underbrace{\begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,N} \\ c_{2,1} & c_{2,2} & \dots & c_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M,1} & c_{M,2} & \dots & c_{M,N} \end{bmatrix}}_{\mathbf{C}} = \alpha \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,N} \\ b_{2,1} & b_{2,2} & \dots & b_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{M,1} & b_{M,2} & \dots & b_{M,N} \end{bmatrix}}_{\mathbf{B}},$$

$$c_{m,n} = \alpha b_{m,n}.$$

Transpose and Hermitian (1/2) [GVL2013, pp. 18]

- We consider

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} \end{bmatrix} \in \mathbb{C}^{M \times N}.$$

- The transpose of $\mathbf{A}$ (denoted by $\mathbf{A}^{\mathsf{T}}$) and the Hermitian of $\mathbf{A}$ (denoted by $\mathbf{A}^{\mathsf{H}}$) are

$$\mathbf{A}^{\mathsf{T}} \triangleq \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{M,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{M,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,N} & a_{2,N} & \cdots & a_{M,N} \end{bmatrix} \in \mathbb{C}^{N \times M}, \quad \mathbf{A}^{\mathsf{H}} \triangleq \begin{bmatrix} a_{1,1}^* & a_{2,1}^* & \cdots & a_{M,1}^* \\ a_{1,2}^* & a_{2,2}^* & \cdots & a_{M,2}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,N}^* & a_{2,N}^* & \cdots & a_{M,N}^* \end{bmatrix} \in \mathbb{C}^{N \times M}.$$

- Complex conjugation $(*)$

Transpose and Hermitian (2/2)

- It can be shown that

$$\mathbf{A}^{\mathsf{H}} = \left(\mathbf{A}^{\mathsf{T}}\right)^{*} = \left(\mathbf{A}^{*}\right)^{\mathsf{T}}.$$

- A matrix $\mathbf{A}$ is a symmetric matrix if $\mathbf{A}^{\mathsf{T}} = \mathbf{A}$.
- A matrix $\mathbf{A}$ is a skew-symmetric matrix if $\mathbf{A}^{\mathsf{T}} = -\mathbf{A}$.
- A matrix $\mathbf{A}$ is a Hermitian matrix if $\mathbf{A}^{\mathsf{H}} = \mathbf{A}$.
- A matrix $\mathbf{A}$ is a skew-Hermitian matrix if $\mathbf{A}^{\mathsf{H}} = -\mathbf{A}$.

Determinant [GVL2013, pp. 66]

- The determinant maps an N-by-N (square) matrix to a scalar.
- Let $\mathbf{A} = [a_{i,j}] \in \mathbb{C}^{N \times N}$.
- If $N = 1$, then

$$\det(\mathbf{A}) = a_{1,1}.$$

- If $N \geq 2$, then

$$\det(\mathbf{A}) = \sum_{j=1}^{N} (-1)^{j+1} a_{1,j} \det(\mathbf{A}_{1,j}),$$

where $\mathbf{A}_{1,j}$ is a submatrix of $\mathbf{A}$ after removing the first row and the jth column.

Determinant with $N = 2$

- Assume that

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

- The determinant of $\mathbf{A}$ can be expressed as

$$\begin{aligned} \det(\mathbf{A}) &= \sum_{j=1}^{2} (-1)^{j+1} a_{1,j} \det(\mathbf{A}_{1,j}) \\ &= a_{1,1} \det(\mathbf{A}_{1,1}) - a_{1,2} \det(\mathbf{A}_{1,2}) \\ &= a_{1,1} a_{2,2} - a_{1,2} a_{2,1}. \end{aligned}$$

Determinant with $N = 3$

- Assume that

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}.$$

- The determinant of $\mathbf{A}$ can be expressed as

$$\begin{aligned} \det(\mathbf{A}) &= \sum_{j=1}^{3} (-1)^{j+1} a_{1,j} \det(\mathbf{A}_{1,j}) \\ &= a_{1,1} \det\left(\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}\right) - a_{1,2} \det\left(\begin{bmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{bmatrix}\right) + a_{1,3} \det\left(\begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix}\right) \\ &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ &\quad - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{aligned}$$

The Multiplicativity of Determinant [HJ2013, pp. 11]

- For $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N \times N}$, we have

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}).$$

Rank [HJ2013, pp. 12-13]

- Let $\mathbf{A} \in \mathbb{C}^{M \times N}$.
- $\text{rank}(\mathbf{A})$ is the length of the longest linearly independent list of columns of $\mathbf{A}$.
- Properties of rank: Section 0.4 in [HJ2013]

Example

- Let $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$, where

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- linearly independent lists: $\{\mathbf{a}_1\}, \{\mathbf{a}_2\}, \{\mathbf{a}_3\}, \{\mathbf{a}_1, \mathbf{a}_2\}, \{\mathbf{a}_1, \mathbf{a}_3\}, \{\mathbf{a}_2, \mathbf{a}_3\}$.
- linearly dependent lists: $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.
- $\text{rank}(\mathbf{A}) = 2$.

Trace

- Let $\mathbf{A} \in \mathbb{C}^{N\times N}$.
- The trace of $\mathbf{A}$ is the sum of the diagonal entries, defined as

$$\mathrm{tr}(\mathbf{A}) \triangleq \sum_{i=1}^{N} [\mathbf{A}]_{i,i}. \tag{4}$$

- For $\mathbf{A} \in \mathbb{C}^{M\times N}$ and $\mathbf{B} \in \mathbb{C}^{N\times M}$, then

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}).$$

Outline

Eigenvalues and Eigenvectors

Definition

Let $\mathbf{A} \in \mathbb{C}^{N\times N}$, $\lambda \in \mathbb{C}$, and $\mathbf{v} \in \mathbb{C}^N$. if

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \tag{5}$$

then λ is the eigenvalue of $\mathbf{A}$, and $\mathbf{v}$ is the eigenvector corresponding to λ.

Example: Finding the eigenvalue given $\mathbf{A}$ and $\mathbf{v}$

Let the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We obtain

$\mathbf{A}\mathbf{v} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\mathbf{v}$, which implies that 4 is an eigenvalue of $\mathbf{A}$.

Characteristic Equations

Eigenvalues from the Characteristic Equation

Let $\mathbf{A} \in \mathbb{C}^{N \times N}$. The eigenvalues are the solutions to the characteristic equation

$$\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = 0. \tag{6}$$

Example: Finding the eigenvalues given $\mathbf{A}$

Let the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The characteristic equation is

$$\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = \left(\lambda - 2\right)\left(\lambda - 4\right) = 0. \tag{7}$$

We have $\lambda = 2$ or $\lambda = 4$.

Eigenvectors Corresponding to λ

- Let λ be an eigenvalue of $\mathbf{A}$. Equation (5) can be rewritten as
$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}. \tag{8}$$
- We can solve for the eigenvector $\mathbf{v}$ from (8).

Example: Finding the eigenvector(s) given $\mathbf{A}$ and λ

Let the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. For the eigenvalue $\lambda = 2$ and the eigenvector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we have

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}, \quad \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{9}$$

We have $v_1 + v_2 = 0$. Therefore, $\mathbf{v} = \begin{bmatrix} 1 & -1 \end{bmatrix}^\mathsf{T}$ is an eigenvector of $\mathbf{A}$.

Eigenspace

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K$ be linearly independent eigenvectors corresponding to the eigenvalue λ.
- The eigenspace corresponding to λ is

$$\mathcal{V}_\lambda \triangleq \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_K\} \tag{10}$$

$$= \left\{ \sum_{k=1}^{K} c_k \mathbf{v}_k \;\middle|\; \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix} \in \mathbb{C}^K \right\}. \tag{11}$$

The Eigen-Decomposition of Square Matrices

- Let $\mathbf{A} \in \mathbb{C}^{N \times N}$.
- There exist N eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{N-1}, \lambda_N$.
- The eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-1}, \mathbf{v}_N$ are assumed to be linearly independent.
- The eigen-equations are $\mathbf{A}\mathbf{v}_n = \lambda_n \mathbf{v}_n$ for $n = 1, 2, \ldots, N$.
- Then $\mathbf{A}$ can be decomposed into

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}, \tag{12}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_{N-1} & \mathbf{v}_N \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{N-1} & 0 \\ 0 & 0 & \ldots & 0 & \lambda_N \end{bmatrix}. \tag{13}$$

The Eigen-Decomposition of Hermitian Matrices

- Let $\mathbf{A} \in \mathbb{C}^{N\times N}$ and $\mathbf{A}^{\mathsf{H}} = \mathbf{A}$ (Hermitian matrices).
- The eigenvectors of $\mathbf{A}$ form a complete and orthogonal set
- After normalization, the set of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-1}, \mathbf{v}_N\}$ is complete and orthonormal. Namely,

$$\mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}, \quad \Rightarrow \quad \mathbf{V}^{-1} = \mathbf{V}^{\mathsf{H}}. \tag{14}$$

- The eigen-decomposition of a Hermitian matrix $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{\mathsf{H}} \tag{15}$$

$$= \sum_{n=1}^{N} \lambda_n \mathbf{v}_n \mathbf{v}_n^{\mathsf{H}}. \tag{16}$$

Example: Eigen-Decomposition of a Hermitian Matrix (1/2)

- Let the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, which is a Hermitian matrix.
- According to Pages 30, 31, and 32, we have

$$\mathbf{A} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{u}_1} = \underbrace{(4)}_{\text{Eigenvalue } \lambda_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{A} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_2} = \underbrace{(2)}_{\text{Eigenvalue } \lambda_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{17}$$

- Normalization of the eigenvectors

$$\mathbf{v}_1 \triangleq \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|_2} = \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^{\mathsf{H}} \mathbf{u}_1}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{18}$$

$$\mathbf{v}_2 \triangleq \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|_2} = \frac{\mathbf{u}_2}{\sqrt{\mathbf{u}_2^{\mathsf{H}} \mathbf{u}_2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \tag{19}$$

Example: Eigen-Decomposition of a Hermitian Matrix (2/2)

- As a result, the Hermitian matrix $\mathbf{A}$ can be decomposed into

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{\mathsf{H}}. \tag{20}$$

- The matrices $\mathbf{V}$ and $\mathbf{D}$ are

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \tag{21}$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}. \tag{22}$$

Outline

The Kronecker Product

- Let $\mathbf{A} \in \mathbb{C}^{M\times N}$ and $\mathbf{B} \in \mathbb{C}^{P\times Q}$, where Namely,

$$\mathbf{A} \triangleq \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & \dots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,N} \end{bmatrix}. \tag{23}$$

- The Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} \triangleq \begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \dots & a_{1,N}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \dots & a_{2,N}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1}\mathbf{B} & a_{M,2}\mathbf{B} & \dots & a_{M,N}\mathbf{B} \end{bmatrix} \in \mathbb{C}^{(MP)\times(NQ)}. \tag{24}$$

- Reference [GVL2013]

An Example of the Kronecker Product

- We consider the matrices $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{25}$$

- Then the matrix $\mathbf{A} \otimes \mathbf{B}$ becomes

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} (1)\mathbf{B} & (2)\mathbf{B} & (3)\mathbf{B} \\ (4)\mathbf{B} & (5)\mathbf{B} & (6)\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 3 \\ -1 & 0 & -2 & 0 & -3 & 0 \\ 0 & 4 & 0 & 5 & 0 & 6 \\ -4 & 0 & -5 & 0 & -6 & 0 \end{bmatrix}, \tag{26}$$

Some Properties [Seber2008, pp. 234]

- If $\alpha \in \mathbb{C}$ and $\mathbf{A} \in \mathbb{C}^{M \times N}$, then

$$\alpha \otimes \mathbf{A} = \alpha \mathbf{A} = \mathbf{A} \otimes \alpha. \tag{27}$$

- If $\mathbf{u} \in \mathbb{C}^{N}$ and $\mathbf{v} \in \mathbb{C}^{M}$, then

$$\mathbf{u}^{\mathsf{T}} \otimes \mathbf{v} = \mathbf{v}\mathbf{u}^{\mathsf{T}} = \mathbf{v} \otimes \mathbf{u}^{\mathsf{T}}. \tag{28}$$

- If $\mathbf{A} \in \mathbb{C}^{M \times N}$ and $\mathbf{B} \in \mathbb{C}^{P \times Q}$, then

$$(\mathbf{A} \otimes \mathbf{B})^{\mathsf{T}} = \left(\mathbf{A}^{\mathsf{T}}\right) \otimes \left(\mathbf{B}^{\mathsf{T}}\right). \tag{29}$$

The Addition Property

- We consider

$$\mathbf{A} \in \mathbb{C}^{M\times N}, \qquad \mathbf{B} \in \mathbb{C}^{P\times Q}, \qquad \mathbf{C} \in \mathbb{C}^{M\times N}, \qquad \mathbf{D} \in \mathbb{C}^{P\times Q}. \tag{30}$$

- Then

$$(\mathbf{A} \otimes \mathbf{B}) + (\mathbf{C} \otimes \mathbf{B}) = (\mathbf{A} + \mathbf{C}) \otimes \mathbf{B}, \tag{31}$$

$$(\mathbf{A} \otimes \mathbf{B}) + (\mathbf{A} \otimes \mathbf{D}) = \mathbf{A} \otimes (\mathbf{B} + \mathbf{D})\,. \tag{32}$$

The Product Property

- We consider

$$\mathbf{A} \in \mathbb{C}^{M_1 \times N_1}, \qquad \mathbf{B} \in \mathbb{C}^{M_2 \times N_2}, \qquad \mathbf{C} \in \mathbb{C}^{M_3 \times N_3}, \qquad \mathbf{D} \in \mathbb{C}^{M_4 \times N_4}. \tag{33}$$

- We assume that $N_1 = M_3$ and $N_2 = M_4$.
- Then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}). \tag{34}$$

The Inverse Property

- Let $\mathbf{A} \in \mathbb{C}^{M\times M}$ and $\mathbf{B} \in \mathbb{C}^{N\times N}$.
- Assume that $\mathbf{A}$ and $\mathbf{B}$ are invertible.
- Then

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \left(\mathbf{A}^{-1}\right) \otimes \left(\mathbf{B}^{-1}\right). \tag{35}$$

The Eigenvector Property

- Let $\mathbf{A} \in \mathbb{C}^{M\times M}$ and $\mathbf{B} \in \mathbb{C}^{N\times N}$.
- We consider the eigenvalues and eigenvectors as follows:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \qquad \mathbf{B}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2. \tag{36}$$

- Then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{v}_1 \otimes \mathbf{v}_2) = (\lambda_1 \lambda_2)(\mathbf{v}_1 \otimes \mathbf{v}_2). \tag{37}$$

- Interpretations:
  - $(\mathbf{v}_1 \otimes \mathbf{v}_2)$ is an eigenvector of $(\mathbf{A} \otimes \mathbf{B})$.
  - $(\lambda_1 \lambda_2)$ is the corresponding eigenvalue of $(\mathbf{A} \otimes \mathbf{B})$.

The Orthogonal Property

- Let $\mathbf{A} \in \mathbb{C}^{M\times M}$ and $\mathbf{B} \in \mathbb{C}^{N\times N}$.
- We assume that $\mathbf{A}$ and $\mathbf{B}$ are unitary matrices, i.e.,

$$\mathbf{A}\mathbf{A}^{\mathsf{H}} = \mathbf{I}, \qquad \mathbf{B}\mathbf{B}^{\mathsf{H}} = \mathbf{I}. \tag{38}$$

- Let $\mathbf{C} \triangleq \mathbf{A} \otimes \mathbf{B}$.
- Then $\mathbf{C}$ is also an unitary matrix,

$$\mathbf{C}\mathbf{C}^{\mathsf{H}} = \mathbf{I}. \tag{39}$$

The Rank Property

- Let $\mathbf{A} \in \mathbb{C}^{M \times N}$ and $\mathbf{B} \in \mathbb{C}^{P \times Q}$.
- Then

$$\operatorname{rank}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{rank}(\mathbf{A}) \operatorname{rank}(\mathbf{B}). \tag{40}$$

The Determinant Property

- Let $\mathbf{A} \in \mathbb{C}^{M \times M}$ and $\mathbf{B} \in \mathbb{C}^{N \times N}$.
- Then

$$\det(\mathbf{A} \otimes \mathbf{B}) = (\det(\mathbf{A}))^{N} (\det(\mathbf{B}))^{M}. \tag{41}$$

The Trace Property

- Let $\mathbf{A} \in \mathbb{C}^{M \times M}$ and $\mathbf{B} \in \mathbb{C}^{N \times N}$.
- Then

$$\operatorname{tr}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{tr}(\mathbf{A}) \operatorname{tr}(\mathbf{B}). \tag{42}$$

Outline

The Hadamard (Element-Wise) Product

- $\mathbf{A} \in \mathbb{C}^{M \times N}$ (with entries $a_{m,n}$)
- $\mathbf{B} \in \mathbb{C}^{M \times N}$ (with entries $b_{m,n}$)
- The Hadamard product: $\mathbf{C} = \mathbf{A} \circ \mathbf{B} \in \mathbb{C}^{M \times N}$.

$$\underbrace{\begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,N} \\ c_{2,1} & c_{2,2} & \dots & c_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M,1} & c_{M,2} & \dots & c_{M,N} \end{bmatrix}}_{\mathbf{C}} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & \dots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,N} \end{bmatrix}}_{\mathbf{A}} \circ \underbrace{\begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,N} \\ b_{2,1} & b_{2,2} & \dots & b_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{M,1} & b_{M,2} & \dots & b_{M,N} \end{bmatrix}}_{\mathbf{B}},$$

$$c_{m,n} = a_{m,n} \times b_{m,n}.$$

Examples

- Let the matrices $\mathbf{A}$ and $\mathbf{B}$ be

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}. \tag{43}$$

- Then the Hadamard product $\mathbf{A} \circ \mathbf{B}$ is

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} (5) \times (1) & (4) \times (-1) \\ (3) \times (2) & (2) \times (0) \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 6 & 0 \end{bmatrix}. \tag{44}$$

Properties (1/2)

- Let the matrices $\mathbf{A} \in \mathbb{C}^{M\times N}$, $\mathbf{B} \in \mathbb{C}^{M\times N}$, and $\mathbf{C} \in \mathbb{C}^{M\times N}$.
- The commutative property:

$$\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A}. \tag{45}$$

- The associative property:

$$(\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} = \mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}). \tag{46}$$

- The distributive property

$$\mathbf{A} \circ (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \circ \mathbf{B}) + (\mathbf{A} \circ \mathbf{C}), \tag{47}$$

$$(\mathbf{A} + \mathbf{B}) \circ \mathbf{C} = (\mathbf{A} \circ \mathbf{C}) + (\mathbf{B} \circ \mathbf{C}). \tag{48}$$

Properties (2/2)

- Let the matrices $\mathbf{A} \in \mathbb{C}^{M \times N}$, $\mathbf{B} \in \mathbb{C}^{M \times N}$, and $\mathbf{C} \in \mathbb{C}^{M \times N}$.
- Let $\mathbf{1}_M \triangleq \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^\mathsf{T}$ be a length-M vector of all ones.
- The trace function [Seber2008, pp. 252]:

$$\operatorname{tr}\left(\mathbf{A}\mathbf{B}^\mathsf{T}\right) = \mathbf{1}_M^\mathsf{T} \left(\mathbf{A} \circ \mathbf{B}\right) \mathbf{1}_N. \tag{49}$$

$$\operatorname{tr}\left(\left(\mathbf{A} \circ \mathbf{B}\right) \mathbf{C}^\mathsf{T}\right) = \operatorname{tr}\left(\left(\mathbf{A} \circ \mathbf{C}\right) \mathbf{B}^\mathsf{T}\right). \tag{50}$$

Outline

Definition of the Vectorization Operator

- Let the matrix $\mathbf{A}$ be

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & \dots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \dots & a_{M,N} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_N \end{bmatrix}, \tag{51}$$

  where $\mathbf{a}_n = \begin{bmatrix} a_{1,n} & a_{2,n} & \dots & a_{M,n} \end{bmatrix}^\mathsf{T} \in \mathbb{C}^M$.
- Then the vectorization of $\mathbf{A}$ is

$$\operatorname{vec}(\mathbf{A}) \triangleq \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_N \end{bmatrix}. \tag{52}$$

An Example of the Vectorization

- We consider the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}. \tag{53}$$

- The vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}. \tag{54}$$

- Then

$$\mathrm{vec}\left(\mathbf{A}\right) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{bmatrix}. \tag{55}$$

The Kronecker Product [Seber2008, pp. 240]

- Let the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{B}$ satisfy

$$\mathbf{A} \in \mathbb{C}^{M\times N}, \qquad \mathbf{B} \in \mathbb{C}^{N\times P}, \qquad \mathbf{C} \in \mathbb{C}^{P\times Q}, \tag{56}$$

Two Matrices

$$\begin{aligned} &\operatorname{vec}(\mathbf{AB}) \\ &= (\mathbf{I}_P \otimes \mathbf{A}) \operatorname{vec}(\mathbf{B}) \\ &= \left(\mathbf{B}^{\mathsf{T}} \otimes \mathbf{A}\right) \operatorname{vec}(\mathbf{I}_N) \\ &= \left(\mathbf{B}^{\mathsf{T}} \otimes \mathbf{I}_M\right) \operatorname{vec}(\mathbf{A}). \end{aligned}$$

Three Matrices

$$\begin{aligned} &\operatorname{vec}(\mathbf{ABC}) \\ &= \left(\mathbf{C}^{\mathsf{T}} \otimes \mathbf{A}\right) \operatorname{vec}(\mathbf{B}) \\ &= (\mathbf{I}_Q \otimes (\mathbf{AB})) \operatorname{vec}(\mathbf{C}) \\ &= \left((\mathbf{BC})^{\mathsf{T}} \otimes \mathbf{I}_M\right) \operatorname{vec}(\mathbf{A}). \end{aligned}$$

The Hadamard Product

- Let $\mathbf{A} \in \mathbb{C}^{M \times N}$ and $\mathbf{B} \in \mathbb{C}^{M \times N}$.
- Then

$$\mathrm{vec}\left(\mathbf{A} \circ \mathbf{B}\right) = \left(\mathrm{vec}\left(\mathbf{A}\right)\right) \circ \left(\mathrm{vec}\left(\mathbf{B}\right)\right). \tag{57}$$

The Trace Function [Seber2008, pp. 240]

- Let $\mathbf{A} \in \mathbb{C}^{M \times N}$ and $\mathbf{B} \in \mathbb{C}^{M \times P}$.
- Then

$$\operatorname{tr}\left(\mathbf{A}^{\mathsf{T}}\mathbf{B}\right) = \left(\operatorname{vec}\left(\mathbf{A}\right)\right)^{\mathsf{T}} \operatorname{vec}\left(\mathbf{B}\right). \tag{58}$$

Outline

Outline

The Vector p-Norm

Definition (The vector p-norm, the ℓ_p norm)

For a vector $\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^\mathsf{T} \in \mathbb{C}^N$ and $p \geq 1$, the vector p-norm is defined as

$$\|\mathbf{x}\|_p \triangleq \left(\sum_{i=1}^{N} |x_i|^p \right)^{1/p}. \quad (59)$$

- $p = 2$: The Euclidean norm or the ℓ_2 norm
- $p = 1$: The ℓ_1 norm
- $p \to \infty$: The ℓ_∞ norm

The ℓ_2 Norm $(p = 2)$

$$\|\mathbf{x}\|_2 \triangleq \left(\sum_{i=1}^{N} |x_i|^2\right)^{1/2} \tag{60}$$

$$= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_N|^2} \tag{61}$$

$$= \sqrt{\mathbf{x}^{\mathsf{H}}\mathbf{x}} = \sqrt{\operatorname{tr}\left(\mathbf{x}^{\mathsf{H}}\mathbf{x}\right)} = \sqrt{\operatorname{tr}\left(\mathbf{x}\mathbf{x}^{\mathsf{H}}\right)}. \tag{62}$$

- The Euclidean norm
- The ℓ_2 norm $\|\mathbf{u} - \mathbf{v}\|_2$ measures the distance between two vectors $\mathbf{u}$ and $\mathbf{v}$.
- Example: If $\mathbf{x} = \begin{bmatrix} 3 & -4 & 2 \end{bmatrix}^{\mathsf{T}}$, then

$$\|\mathbf{x}\|_2 = \sqrt{|3|^2 + |-4|^2 + |2|^2} = \sqrt{29}.$$

The ℓ_1 Norm $(p = 1)$

$$\|\mathbf{x}\|_1 \triangleq \left(\sum_{i=1}^{N} |x_i|^1 \right)^{1/1} \tag{63}$$

$$= |x_1| + |x_2| + \cdots + |x_N| . \tag{64}$$

- The sum of amplitudes
- Example: If $\mathbf{x} = \begin{bmatrix} 3 & -4 & 2 \end{bmatrix}^{\mathsf{T}}$, then

$$\|\mathbf{x}\|_1 = |3| + |-4| + |2| = 9.$$

The ℓ_∞ Norm $(p \to \infty)$

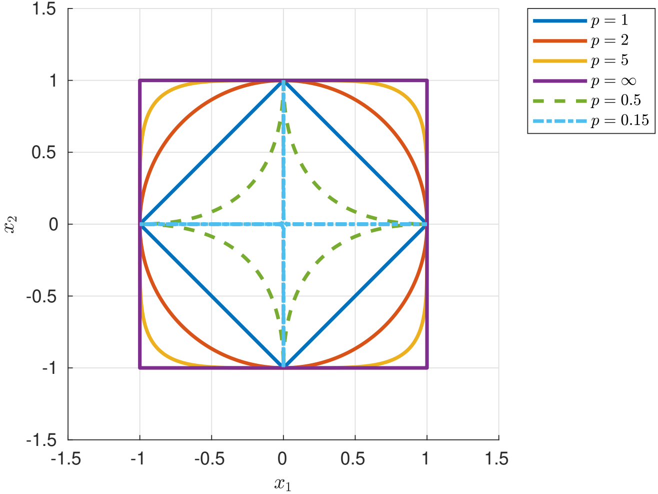
$$\|\mathbf{x}\|_\infty \triangleq \lim_{p\to\infty} \left(\sum_{i=1}^{N} |x_i|^p \right)^{1/p} \tag{65}$$

$$= \max_{i\in[N]} |x_i|. \tag{66}$$

- $[N] \triangleq \{1, 2, \ldots, N\}$.
- $\|\mathbf{x}\|_\infty$ represents the maximal amplitudes of $\mathbf{x}$.
- Example: If $\mathbf{x} = \begin{bmatrix} 3 & -4 & 2 \end{bmatrix}^\mathsf{T}$, then

$$\|\mathbf{x}\|_\infty = \max\{|3|, |-4|, |2|\} = 4.$$

Examples: Contour Plots of $\|\mathbf{x}\|_p = 1$ (The ℓ_p Ball)



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2.$$

$$|x_1|^p + |x_2|^p = 1^p = 1.$$

The ℓ_0 Function (1/2)

- Notations
  - $[N] \triangleq \{1, 2, \dots, N\}$.
  - $\text{card}(\mathbb{S})$: The cardinality of a set $\mathbb{S}$.
  - $\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^\mathsf{T}$.

Definition (Support)

The support of a vector $\mathbf{x} \in \mathbb{C}^N$ is defined as $\text{supp}(\mathbf{x}) \triangleq \{i \in [N] : x_i \neq 0\}$

The support of a vector

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, then $\text{supp}(\mathbf{x}) = \{1, 2, 4\}$.

The ℓ_0 Function (2/2)

Definition (The ℓ_0 function)

For a vector $\mathbf{x} \in \mathbb{C}^N$, the ℓ_0 function is defined as

$$\|\mathbf{x}\|_0 \triangleq \mathrm{card}(\mathrm{supp}(\mathbf{x})). \tag{67}$$

- Precisely, $\|\mathbf{x}\|_0$ is not a norm.
- $\|\mathbf{x}\|_0$ is related to the following limit

$$\lim_{p \to 0} \sum_{i=1}^{N} |x_i|^p. \tag{68}$$

Remarks

- The ℓ_p norm and the ℓ_0 function are often used in the objective function of optimization problems.
- The ℓ_p norm with $p \geq 1$ is a convex function.
- The ℓ_0 function promotes sparse solutions.
- The ℓ_0 function is non-convex.
- Convex relexation in compressive sensing
    - Replacing the ℓ_0 function with the ℓ_1 norm
    - Basis pursuit

Outline

The Entry-Wise L_p Norm

- For $p \geq 1$, the entry-wise L_p norm is defined as

$$\|\mathbf{A}\|_p \triangleq \left(\sum_{m=1}^{M} \sum_{n=1}^{N} |a_{m,n}|^p \right)^{\frac{1}{p}} . \tag{69}$$

- It can be shown that

$$\|\mathbf{A}\|_p = \|\text{vec}\,(\mathbf{A})\|_p \,, \tag{70}$$

where the latter $\|\cdot\|_p$ denotes the vector p-norm.

The Frobenius Norm (The L_2 Norm)

Definition

For a matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$, the Frobenius norm $\|\mathbf{A}\|_F$ is defined as

$$\|\mathbf{A}\|_F \triangleq \sqrt{\sum_{m=1}^{M} \sum_{n=1}^{N} |a_{m,n}|^2}. \tag{71}$$

The Frobenius Norm and the Trace Function

- The Frobenius norm of $\mathbf{A}$ can be expressed as

$$\|\mathbf{A}\|_F = \sqrt{\operatorname{tr}\left(\mathbf{A}\mathbf{A}^{\mathsf{H}}\right)}. \tag{72}$$

- Equation (62), which is $\|\mathbf{x}\|_2 = \sqrt{\operatorname{tr}\left(\mathbf{x}\mathbf{x}^{\mathsf{H}}\right)}$, shares a similar form to (72).

Properties of the Frobenius Norm

- Let $\mathbf{A} \in \mathbb{C}^{M \times N}$.
- Let $\mathbf{U} \in \mathbb{C}^{M \times M}$ and $\mathbf{V} \in \mathbb{C}^{N \times N}$ be unitary matrices.
- Namely, $\mathbf{U}^{\mathsf{H}}\mathbf{U} = \mathbf{I}$ and $\mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}$.
- The Frobenius norm is unitarily invariant,

$$\|\mathbf{UAV}\|_F = \|\mathbf{A}\|_F . \tag{73}$$

Other Entry-Wise L_p Norms

- The entry-wise L_1 norm for $\mathbf{A} \in \mathbb{C}^{M \times N}$
  - $p = 1$.
  - Definition:

$$\|\mathbf{A}\|_1 \triangleq \sum_{m=1}^{M} \sum_{n=1}^{N} |a_{m,n}|. \tag{74}$$

- The entry-wise L_∞ norm for $\mathbf{A} \in \mathbb{C}^{M \times N}$
  - $p \to \infty$.
  - The max norm
  - Definition:

$$\|\mathbf{A}\|_\infty \triangleq \max_{(m,n)\in[M]\times[N]} |a_{m,n}|. \tag{75}$$

Remarks on the Matrix Norms

- The entry-wise L_p norm
  - The Frobenius norm (L_2)
  - The L_1 norm
  - The max norm (L_∞)
- Other matrix norms
  - Operator norms or induced norms
  - Nuclear norms
  - We will cover these matrix norms after the singular value decomposition.